An Overview of Dynamical Systems and Chaos

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Abstract:

This paper provides an introductory study of dynamical systems and chaos. The goal is to offer a succinct guide covering the concepts that are most fundamental to the analysis of dynamical systems and to the conditions that lead to chaos. Simple, one-dimensional maps are the easiest to characterize, and even these may demonstrate unpredictable behavior according to the values of certain parameters. The most relevant of these maps are the logistic maps $G(x) = ax(1 - x)$, where changes in the value of a produce different types of behaviors. Two-dimensional maps offer different analytical challenges, but the essential principles of understanding chaotic behavior are the same. The two-dimensional map that is most helpful for our study is the Hénon map $f(x, y) = (a - x^2 + by, x)$. The exploration of chaotic orbits involves discussion of topics like itineraries, fractal dimension, and manifolds. In addition to one-and two-dimensional maps, systems of differential equations are also conducive to the study of chaos, albeit with another set of analytical tools.

I. Introduction

This work is an overview of principles and methods involved in the analysis of dynamical systems and chaos. The content follows a basic structure. First, we build a conceptual foundation by reviewing topics of relevance to each aspect of our exploration of dynamical systems. Then, in the section that follows, we delve into how those topics help to explain chaotic behavior. We begin with "Basics in One Dimension" and then move to "Chaos in One Dimension." Next, we cover "Basics in Two Dimensions" and "Fractals" before proceeding to "Chaos in Two Dimensions." Lastly, we cover "Differential Equations" and "Chaos in Differential Equations." Throughout the overview, definitions of key terms are denoted in bold text. Citations reflect the sources that inspired the definitions, though no excerpts are reproduced in their exact form.

The project results from a semester-long course in Dynamical Systems and Chaos taught by Dora Matache at the University of Nebraska Omaha in the fall of 2017. *Chaos: An Introduction to Dynamical Systems* (1996) by Kathleen T. Alligood, Tim D. Sauer, and James A. Yorke, which served as the primary text for the course, is cited frequently here. Other texts and articles serve as supplemental references for specific topics of interest. Most graphics were created using MATLAB software and programs compiled for students by Dr. Matache.

II. Fundamental Terms

A. A **dynamical system** consists of a set of time-dependent variables and a rule that enables the precise determination of the system's state in terms of its initial conditions. (Zill, 2005).

B. A **map** (or function or transformation) is a rule that assigns to each element x of a set X a unique element $f(x)$ of a set Y. (Dangello and Seyfried, 2000).

C. An **orbit** is a set of points representing a sequence of repeated iteration of a map. The orbit generally takes the form of $\{f(x), f^2(x), f^3(x)\ldots\}$ or, alternatively, $\{f(x), f(f(x)), f(f(f(x)))\ldots\}$. (Alligood, Sauer, and Yorke, 1996).

D. The **initial condition** of an orbit, as described above, is x. Typically, this initial condition is represented as x_0.

E. The **stability** of points or orbits describes the tendency of nearby points/orbits to remain nearby or move away. (Holmes and Shea-Brown, 2006).

F. **Chaos** is a feature of some dynamical systems in which deterministic rules and unambiguous initial conditions produce unpredictable behavior.

III. <u>Basics in One Dimension</u>

A. Key Concepts: Fixed points; Periodic points; Bifurcations; Cobweb plots; Stability of fixed points and periodic orbits; Basins of attraction

1. Fixed points

Let X and Y be sets and let f be a map such that $f: X \rightarrow Y$. If $\exists x \in X$ such that $f(x) = x$, then x is a **fixed point** of the map f.

Example:

Let $f(x) = x^2$. This map has fixed point at $x = 0$ and $x = 1$ because $f(0) = 0^2 = 0$ and $f(1) = 1^2 = 1$.

If we list the elements of an orbit in sequence as $s_1s_2s_3\ldots$ and if some element s_n is a fixed point, then for every subsequent element of the form s_{n+k}, where k is a positive integer, $s_n = s_{n+k}$.

The procedure for determining the fixed points of any one-dimensional map is simple. We set $f(x) = x$ and solve for x. For the example above, since $f(x) = x^2$, we solve $x = x^2$. The equation has two solutions: $x = 0$ and $x = 1$, and thus, these are the fixed points.

2. Periodic points

Let X and Y be sets and let f be a map such that $f: X \rightarrow Y$. If $\exists x \in X$ such that $f^k(x) = x$, then x is a **period-*k* point** of the map f.

Example:

Let $f(x) = x^2 – 1$. This map has period-2 points at $x = 0$ and $x = -1$ because

$$f^2(0) = (f(0))^2 – 1 = -1^2 – 1 = 0$$

and

$$f^2(-1) = (f(-1))^2 – 1 = 0^2 – 1 = -1$$

Observe that the orbit for this map would alternate between 0 and -1 forever. The orbit $\{0, -1, 0, -1\ldots\}$ is called a **period-two orbit.**

Higher-period orbits can exist as well in the following patterns:

period-three	$s_1s_2s_3s_1s_2s_3s_1\ldots$
period-four	$s_1s_2s_3s_4s_1s_2s_3s_4s_1\ldots$
period-five	$s_1s_2s_3s_4s_5s_1s_2s_3s_4s_5s_1\ldots$
…	
period-*k*	$s_1s_2\ldots s_ks_1s_2\ldots s_ks_1\ldots$

Note that in addition to its period-two points, the function $f(x) = x^2 - 1$ has fixed points wherever $x^2 - 1 = x$. Using the quadratic formula, we find that the fixed points are $1/2 \pm \sqrt{5}/2$.

The procedure for determining the period-*k* points of a map is similar to the procedure for determining fixed points, albeit more computationally rigorous. We set $f^k(x) = x$ and solve for x. The case in which $k = 2$ is simple enough to demonstrate here. Using the above definition of *f*,

$$
\begin{aligned}
f^2(x) = x \quad & \Rightarrow f(x^2 - 1) = x \\
& \Rightarrow (x^2 - 1)^2 - 1 = x^4 - 2x^2 + 1 - 1 = x \\
& \Rightarrow x^4 - 2x^2 - x = 0 \\
& \Rightarrow x \bullet (x^3 - 2x - 1) = 0 \\
& \Rightarrow x = 0, -1
\end{aligned}
$$

As the value of *k* increases, the number of roots of $f^k(x)$ increases as well, making analysis unwieldy or impossible. For most systems that we can consider, computer assistance is required to find period-*k* points for $k > 1$.

3. Bifurcations

When we consider dynamical systems in terms of parameters, we see that changes in these parameters have qualitative effects on the behavior of the orbits. These changes are called **bifurcations.** **(**Wiens, N.D.) Reconsider the two functions already explored: $f(x) = x^2$ and $f(x) = x^2 - 1$. We can think of these functions as being part of the same dynamical system $f(x) = x^2 - a$, with the parameter *a* defined as 0 and 1, respectively. As we have seen, the system has different fixed points and period points for $a = 0$ and $a = 1$. Therefore, we know that at some point between $0 \leq a \leq 1$, the system undergoes a bifurcation.

In the case of $f(x) = x^2 - a$, the fixed point $x = 0$ at $a = 0$ is neither fixed nor periodic at $a = 1$. But this change is not always the result of bifurcation. For other systems, changes of parameters can have different types of impacts on the system's behavior. For example, they can turn fixed points into period points or turn stable fixed points into unstable fixed points. The simplest graphical understanding of bifurcations is the bifurcation diagram, which will be introduced in section III-B-1, "Logistic Map."

4. Cobweb plots

The simplest graphical tool for analyzing the orbits of a system is a **cobweb plot**. (Alligood, Sauer, and Yorke, 1996). Essentially, the cobweb plot features simultaneous graphs of two functions—some function *f* that is under consideration and *g*(x) = x. We can learn about the dynamics of *f* by observing its behavior in relationship to *g*.

Consider the cobweb plot for $f(x) = x^2$.

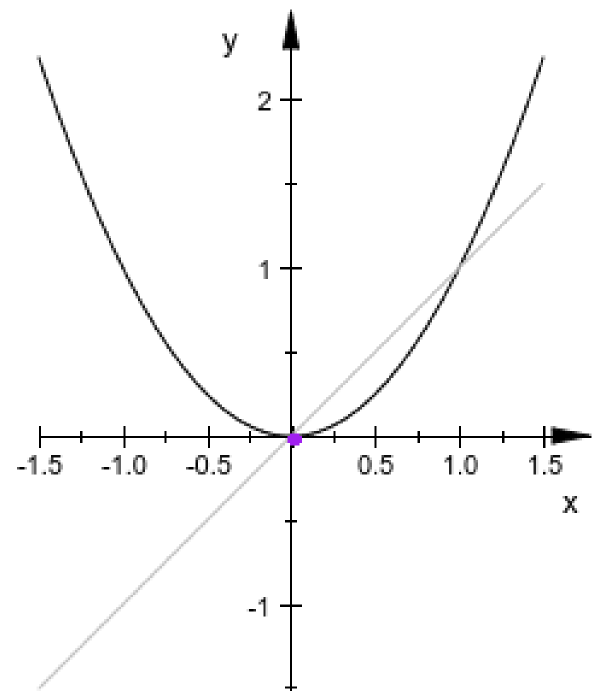

Figure 1 – Structure of the cobweb plot for $f(x) = x^2$.

First, we see that the graphs of *f* and *g* intersect at two points: x = 0 and x = 1. These intersections mean that at these two points, $f(x) = g(x)$. This confirms what we have already seen, because $f(x) = g(x) \Rightarrow x^2 = x$, and we know that this equation has solutions at 0 and 1.

We can also use cobweb plots to observe the dynamics of system as it undergoes multiple iterations. Begin by selecting an initial value x_0. Find $f(x_0)$ by drawing a vertical line from the x-axis to the curve $f(x)$. Then, draw a horizontal line from the $f(x)$ curve to meet the line $g(x)$.

From that point of intersection, draw another vertical line to reach the f(x) curve. Continue in this pattern.

Example:

Using the above definition of f, let $x_0 = 0.75$.

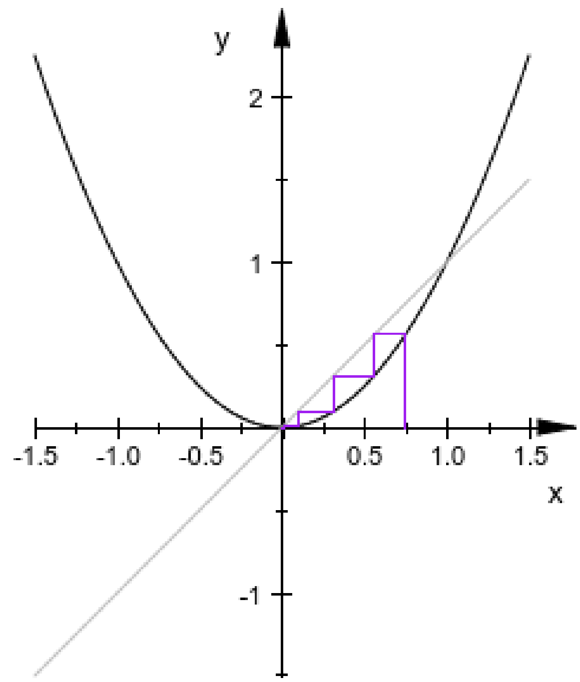

Figure 2 - Cobweb plot for $f(x) = x^2$, $x_0 = 0.75$.

We see that, following the pattern of horizontal and vertical lines described above, the cobweb plot illustrates the orbit of $x_0 = 0.75$ under multiple iterations of f. The vertical lines represent the value of x at repeated iterations:

$$f(0.75) = 0.5625$$
$$f^2(0.75) = f(0.5625) = 0.3164$$
$$f^3(0.75) = f^2(0.5625) = f(0.3164) = 0.1032$$

This pattern continues with further iterations, but changes are too small to be observable at this scale. Nonetheless, the cobweb plot illustrates an important feature of the map: the stability of the fixed point (0,0).

5. Stability of fixed points and periodic points

In Figure 2, we observed that the orbit of $f^k(x) = x^2$, initial condition $x_0 = 0.75$, appears to approach 0 as $k \to \infty$. We have also seen that

$f(0) = 0$, meaning that x = 0 is a fixed point for the system. Together, these two facts tell us something about the stability of the orbit. A fixed point is a **sink** if points near the fixed point move closer to it with each iteration of the function. If points move away from the fixed point, then they the fixed point is a **source**.

In other words, if $f(x) = x$ and if for all $\varepsilon > 0$ $\exists k$ such that $|f^k(p) - x| < \varepsilon$, and p is within the basin of attraction of x, then x is a sink. (Basins of attraction will be explored in section III-A-6).
If $f(x) = x$ and if $\exists \varepsilon > 0$ such that for all $p \in X$, $p \neq x$, $\exists k$ such that $|f^k(p) - x| > \varepsilon$, then x is a sink. (Alligood, Sauer, and Yorke, 1996).

The procedure for determining the stability of fixed point for a given system is straightforward. Since we are examining how maps change under different inputs, we can determine stability by examining the maps' derivatives of the fixed points.

Let x be a fixed point of a map $f(x)$. If $|f'(x)| < 1$, then x is a sink. If $|f'(x)| > 1$, then x is a source.

Proof

Assume $|f'(x)| < 1$, where x is a fixed point.

$$|f'(x)| < 1 \Rightarrow \lim_{x \to c} \frac{|f(x) - f(c)|}{|x - c|} < 1$$

Let $\varepsilon > 0$ such that $|x - c| = \varepsilon$, meaning that c is on the boundary of the ε-neighborhood of x.

Then $\frac{|f(x) - f(c)|}{|x - c|} < 1 \Rightarrow \frac{|f(x) - f(c)|}{\varepsilon} < 1$

$\Rightarrow |f(x) - f(c)| < \varepsilon$

Since x is a fixed point, $|f(x) - f(c)| < \varepsilon$

$\Rightarrow |x - f(c)| < \varepsilon$

$\Rightarrow f(c)$ is in the ε-neighborhood of the fixed point x.

So, f maps any point on the boundary of the ε-neighborhood of x to a point inside the ε-neighborhood of x. Observe that by the same reasoning, any point inside the ε-neighborhood of the fixed point x will be mapped to another point inside the ε-neighborhood, and even closer to x. Therefore, x is a sink.

A similar proof can be made to show that x is a source if

$$\lim_{x \to c} \frac{|f(x) - f(c)|}{|x - c|} > 1.$$

■

Example:

Again, consider $f(x) = x^2 - a$. Observe that $f'(x) = 2x$. For all fixed points $x \leq ½$, $f(x) \leq 1$ so x is a sink. For all $x \geq ½$, $f(x) \geq 1$ so x is a source.

We can analyze the fixed points we discovered previously for $a = 0$ ($x = 0, 1$) and $a = 1$ ($x_{1,2} = 1/2 \pm \sqrt{5}/2$). Of the four fixed points considered, only $x = 0 \Longrightarrow f(x) \leq ½$. So, $x = 0$, $a = 0$ is a sink. The other fixed points—$x = 1$, $a = 0$ and $x_{1,2} = 1/2 \pm \sqrt{5}/2$, $a = 1$—are sources.

The procedure for determining the stability of periodic points is similar. In this case, we evaluate $f'(x)$ at each period-*k* point x, find their products, and compare the resulting absolute value with 1. If $|f'(x_1) \bullet f'(x_2) \bullet \ldots \bullet f'(x_k)| < 1$, then the orbit is a sink, meaning that the orbit of all x_0 within some neighborhood of one of the period-*k* points will move towards the periodic orbit. On the other hand, if $|f'(x_1) \bullet f'(x_2) \bullet \ldots \bullet f'(x_k)| > 1$, then all $x_0 \neq x_1, x_2, \ldots, x_k$, will move away from the orbit.

Example

For $f(x) = x^2 - 1$, $f'(x) = 2x$, we have seen that $x = 0$ and $x = -1$ are period-two points. The cobweb plot in figure 3 illustrates the period-two orbit, with x alternating between 0 and -1.

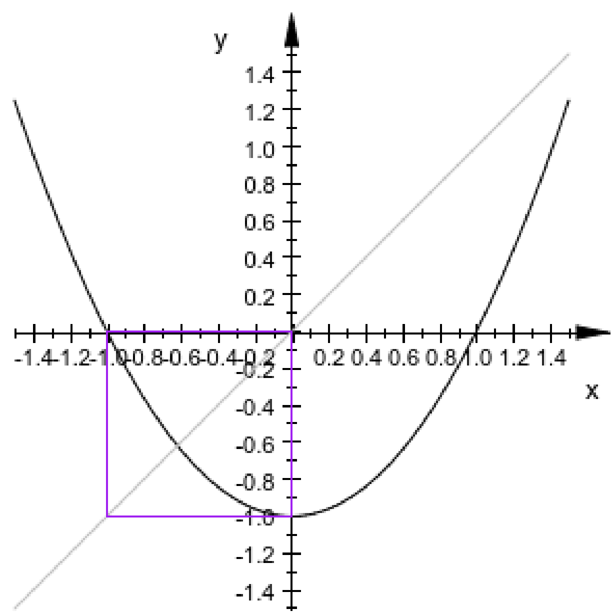

Figure 3 - Cobweb plot for $f(x) = x^2 - 1$, $x_0 = 0$.

We can determine the stability of the orbit analytically.
$|f'(x_1) \bullet f'(x_2)| = |f'(0) \bullet f'(-1)| = |0 \bullet -2| = 0 < 1$.
Thus, the orbit {0,-1,0,-1, 0…} is a periodic sink. Observe the cobweb plot for $x_0 = 0.5$.

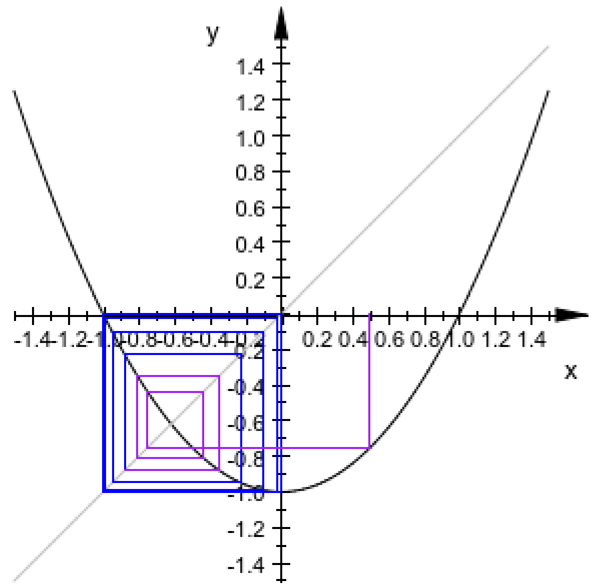

Figure 4 - Cobweb plot for $f(x) = x^2 - 1$, $x_0 = 0.5$.

As we would expect given our finding that {0,-1,0,-1,0…} is a sink, the orbit for $x_0 = 0.5$ moves towards the period-two points with each successive iteration of *f*.

6. Basins of attraction

Our definitions of fixed and periodic sinks leave unanswered the question of which initial conditions will move towards the sink and which will remain free to follow other orbits. All that is required for a fixed point or orbit to qualify as a sink is that initial conditions within some neighborhood of the sink will move towards the sink. The neighborhood could cover all real numbers (such as the fixed point $x = 0$ for the map $f(x) = 0$) or be very small. (Alligood, Sauer, and Yorke, 1996). The range of values that will move towards the sink is called its **basin of attraction.**

We can determine a sink's basin of attraction using algebraic methods. Let p be a fixed point for a function *f*. If $|f(x) - p| < 1$, then then x is within p's basin of attraction.

Example:

For $f(x) = x^2$, we have seen that $x = 0$ is a sink. Setting $p = 0$, we use the equation $|f(x) - p| = |x^2 - 0|| = x^2 < 1$. We find that the basin includes all values of x such that $-1 < x < 1$. We illustrate this basin in figure 5, showing via cobweb plot what happens when $x_0 = 1.05$.

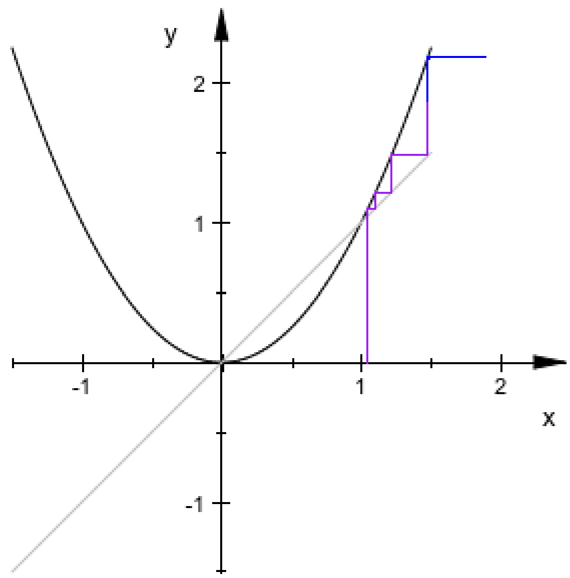

Figure 5 - Cobweb plot for $f(x) = x^2$, $x_0 = 1.05$.

Observe that with each iteration, the orbit increases exponentially. Because it does not move towards the sink at $x = 0$, $x_0 = 1.05$ is not with the sink's basin of attraction.

B. Maps

1. The logistic family: $G(x) = ax(1-x)$

Maps of the form $G(x) = ax(1-x)$ provide a particularly good view of the ways in which fixed points, periodic orbits, bifurcations; sinks, sources and basins of attraction describe a dynamical system. Observing the behavior of the system for different values of the parameter *a,* we can see instances where small changes in initial conditions can have consequences that make the system unpredictable.

Fixed points

Using the techniques explored in this section, we discover that for all *a,* we find the fixed points of G by evaluating

$$x = ax(1 - x) \Rightarrow x = ax - ax^2 \Rightarrow ax^2 - ax - x = 0$$

$$\Rightarrow x\bullet(ax - a - 1) = 0$$

$$\Rightarrow x = 0,\ x = \frac{a - 1}{a}$$

So, for any value of *a,* we can find fixed points at these values of x. To simplify the system, we will only look at values of x such that $0 \leq x \leq 1$. Observe that for $a < 1$, $\frac{a-1}{a} < 0$, the the only fixed point within our range is $x = 0$.

Periodic points

We can find period-k points analytically when $k = 2$. Otherwise, analytic methods will not suffice and we rely on computer assistance. When $k = 2$, we evaluate $x = a\bullet(ax\bullet(1-x))\bullet(1-(ax\bullet(1-x)))$. After some complicated algebra, we find that $x_{1,2} = \frac{(a+1) \pm \sqrt{(a^2 - 2a - 3)}}{2a}$

When $a < 3$, there are no period two points.

Proof

Observe that $a^2 - 2a - 3 = (a - 3)\bullet(a + 1)$.

$a < 3 \implies (a - 3)\bullet(a + 1) < 0.$

$\implies \sqrt{(a^2 - 2a - 3)}$ has no roots in R.

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Bifurcations

We can visualize the fixed points and periodic points for various values of *a* through a bifurcation diagram.

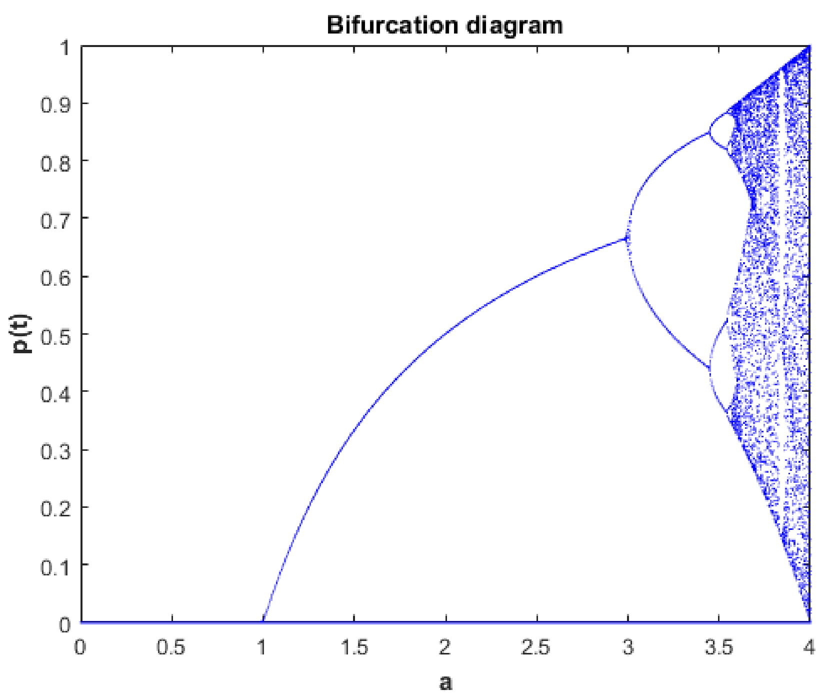


Figure 6 - Bifurcation diagram for the logistic map.

The blue lines in the diagram indicate values of fixed or periodic points. We see that for all values of a, x = 0 is a fixed point. At $a > 1$, a second fixed point emerges as a curve extending from a value close to x = 0 and increasing as a increases. Recall that we identified the fixed points analytically as x = 0 and x = $\underline{(a-1)}/a$. So, for $a = 1.5$, the fixed points are x = 0 and x = 0.5/1.5 = 1/3, as we see in figure 6.

The map undergoes another bifurcation at $a = 3$. While there continue to be two fixed points as previously described, at $a > 3$ there are also period-two points as illustrated by the blue lines in figure 6. At $a = 3.25$, for example, the period-two points are

$$\frac{3.25 + 1 \pm \sqrt{(3.25^2 - 6.5 - 3)}}{6.5} = \frac{4.25 \pm 1.0308}{6.5} = 0.495, 0.812.$$

This confirms what we see in the diagram.

Another bifurcation occurs at $a = 1 + \sqrt{6}$, where a period-four orbits emerges. From there, the rate of bifurcation increases rapidly for values of a such that $1 + \sqrt{6} \leq a \leq 4$. In very small subintervals, there periodic sinks for each period 2^n, n = 1,2,3…For much of the interval between $1 + \sqrt{6}$ and 4, the orbit appears to nearly fill the entire interval (0,1).

Sinks and Sources

We can determine the values of sinks by applying the principles explored in section III-A-5. To begin, note that $G'(x) = a - 2ax$.

First, we explore the stability of the fixed point x = 0. For all $a \in (0,4)$, $x = 0 \Longrightarrow 2ax = 0$

$\Longrightarrow a - 2ax = a$

$\Longrightarrow$ x = 0 is a sink for $a < 1$,

x = 0 is a source for a >1.

Next, we look at the fixed point $x = \dfrac{a-1}{a}$.

$\Longrightarrow G'(x) = a - 2(a - 1) = -a + 2$

$\Longrightarrow$ for all $a < 3$, this fixed point x is a sink,

for all $a > 3$, this fixed point x is a source.

In addition to the fixed points, we can learn something about the stability of periodic orbits in logistic maps. Recall that period-two points can be found using the formula

$$x = \frac{(a + 1) \pm \sqrt{(a^2 - 2a - 3)}}{2a}$$

Plugging this into G'(x), we get

$$G'(x) = a - (a + 1) \pm \sqrt{(a^2 - 2a - 3)} = 1 \pm \sqrt{(a^2 - 2a - 3)}.$$

Recall that there are no period-two points for $a < 3$, so we restrict ourselves to $3 < a < 4$. The stability of the period-two orbits is determined by the product:

$$\begin{aligned} &|(1 + \sqrt{(a^2 - 2a - 3)})\bullet(1 - \sqrt{(a^2 - 2a - 3)})| \\ &\quad = |1 - a^2 + 2a + 3| \\ &\quad = |- a^2 + 2a + 4| \end{aligned}$$

To find period-two sinks, we want to know the values of a such that $|- a^2 + 2a + 4| < 1$. So, we evaluate

$$\begin{aligned} &-1 = - a^2 + 2a + 4 \text{ and } - a^2 + 2a + 4 = 1 \\ &\quad \Rightarrow 0 = a^2 - 2a - 5 \text{ and } a^2 - 2a - 3 = 0 \\ &\quad a^2 - 2a - 5 \Rightarrow a = 1 \pm \sqrt{6} \\ &\quad a^2 - 2a - 3 \Rightarrow a = -1, 3 \end{aligned}$$

As the only values in the range of a that concern us are 3 and $1 + \sqrt{6}$, we see that for $3 < a < 1 + \sqrt{6}$, the period-two orbit is a sink.

Cobweb plots

We can use cobweb plots of the logistic map at various values of a to illustrate stability.

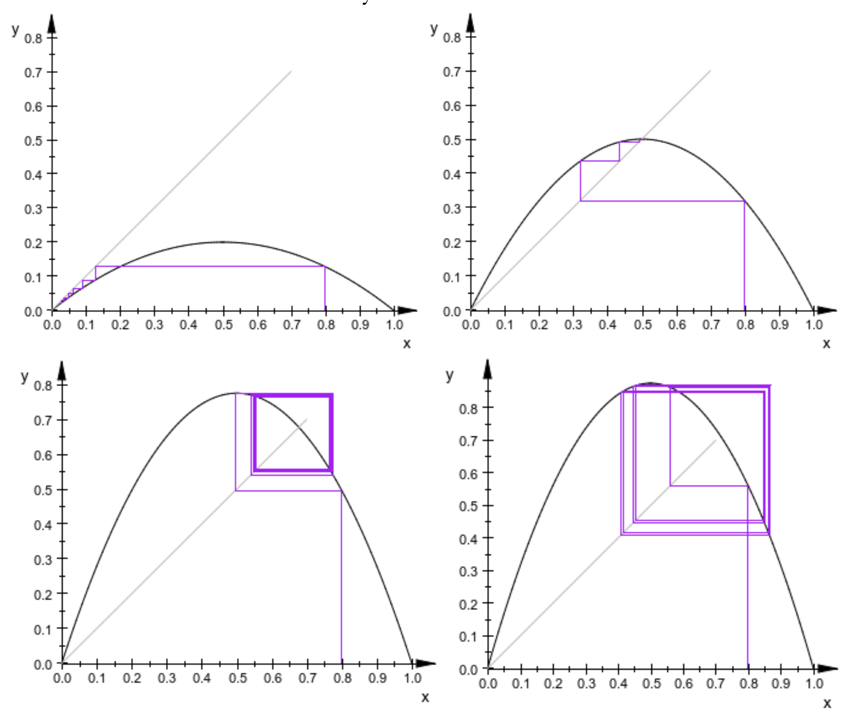


Figure 7 - Cobweb plots for $G(x) = ax(1 - x)$, $x_0 = 0.8$. From top left, going clockwise, the values of a are 0.8, 2, 3.1, and 3.5.

IV. Chaos in One Dimension

A. Key Concepts: Itineraries; Transition graphs; Sensitive dependence on initial conditions; Asymptotic periodicity; Lyapunov numbers and exponents; Chaotic orbits

1. Itineraries

Before examining the conditions that lead to chaos, we will explore a couple of tools that will help represent its characteristics: itineraries and transition graphs.

To introduce the concept of itineraries, we continue to utilize the example of the logistic map. When we graph the map on the x-y plane, we see a parabola in the first quadrant. We can bisect the graph to create equal halves corresponding to the intervals $0 \leq x \leq \frac{1}{2}$ and $\frac{1}{2} \leq x \leq 1$. We label these intervals *L* and *R*, respectively.

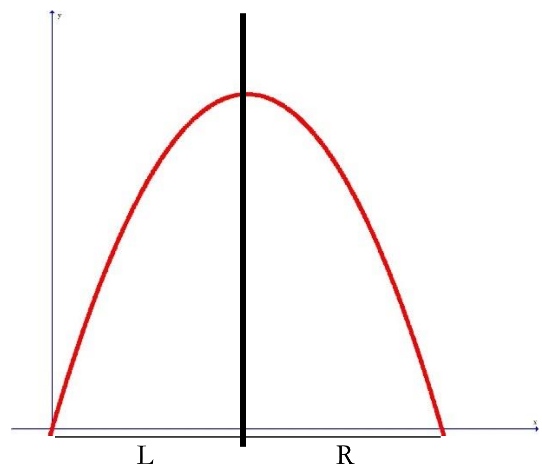


Figure 8 – The logistic map with L and R intervals demarcated. (Basic parabola from https://richbeveridge.files.wordpress.com/2009/10/general-parabola.jpg augmented by author).

Upon repeated iteration of the map, the orbit may move between the two intervals. An **itinerary** is a description of the orbit of a given initial condition using a sequence of symbols, in this case *L*s and *R*s, to show how often the orbit moves from one interval to another.

Example

Under the logistic map, let $a = 3.5$ and $x_0 = 2/3$. The orbit {2/3, 0.778, 0.605, 0.836, 0.479, 0.873, 0.387…} is represented by the itinerary RRRRLRL. Compare this to the itinerary for $a = 2.5$, $x_0 = 2/3$, with the orbit {2/3, 0.556, 0.617, 0.591, 0.604, 0.598, 0.601}, RRRRRRR. In the latter case, the orbit never leaves the right interval because it quickly moves towards the fixed sink at x = 0.6.

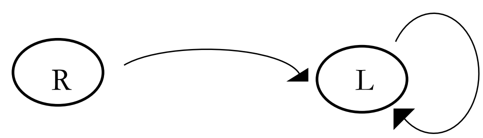
One helpful use of itineraries is to identify initial conditions that will have specific behavior. For instance, if we wanted to find an orbit that will start in the left interval and then move to the right, we could use G(x) and $G^2(x)$ to look for an interval of initial conditions that follow the itinerary LR.

2. Transition graphs

Any given map may have a limited number of possible itineraries. For example, let $f: (0,1) \to (0, \frac{1}{2})$, $f(x) = x/2$. In itineraries of this map, no R can ever follow an L because once an orbit enters the interval $(0, \frac{1}{2})$ it cannot leave. **Transition graphs** illustrate the types of movement between different intervals that is allowable under a given map.

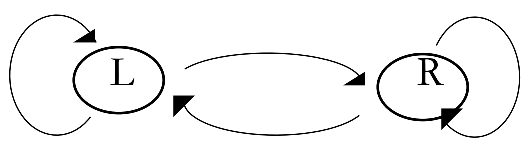
Example

For the function f defined above, the transition graph is:



Example

For the logistic map with a = 3.5, the transition graph is:



3. Sensitive dependence on initial conditions

In the discussion of the family of logistic maps in section III-B-1, we saw that between a = 3.4 and a = 4, the logistic map undergoes a series of rapid bifurcations producing periodic orbits of exponentially increasing orders. At a = 4, we observe no periodicity in the orbit. We cannot predict with any confidence the value of an orbit after several iterations.

Example

Figure 9 shows the beginnings of orbits for x_0 = 0.333, 0.336, and 0.339. We see that under the first few iterations, the orbits stay near to one another. By the eleventh and twelfth iterations, however, we see that the orbits seem to have no relation to one another.

0	0.333	0.336	0.339
1	0.888444	0.892416	0.896316
2	0.396445	0.384039	0.371735
3	0.957105	0.946212	0.934192
4	0.164218	0.20358	0.24591
5	0.549003	0.64854	0.741753
6	0.990395	0.911744	0.766223
7	0.038051	0.321869	0.716502
8	0.146413	0.873077	0.812508
9	0.499905	0.443254	0.609354
10	1	0.98712	0.952167
11	1.43E-07	0.050857	0.182181
12	5.74E-07	0.193084	0.595964
13	2.29E-06	0.623209	0.963164
14	9.18E-06	0.939278	0.141917
15	3.67E-05	0.228139	0.487105
16	0.000147	0.704367	0.999335
17	0.000587	0.832936	0.002659
18	0.002347	0.556614	0.010607
19	0.009368	0.987179	0.041976
20	0.03712	0.050625	0.160856

Figure 9 – Table showing first twenty elements of the orbit of $G(x) = 4x(1 - x)$ for $x_0 = 0.333$, 0.336, and 0.339.

This example shows **sensitive dependence on initial conditions**. Here is a more formal definition:

Let $f: R \rightarrow R$. $x_0 \in R$ has sensitive dependence on initial conditions if for all $\varepsilon, \delta > 0$, $\exists k$ such that $|x_0 - x| < \varepsilon$ and $|f^k(x_0) - f^k(x)| > \delta$.

4. Asymptotic periodicity

In section IV-A-1, we saw that periodic orbits can be represented by repeating itineraries like RRRLRRRL…If the periodic orbit is a sink, it may take many iterations before an initial condition in the sink's basin of attraction reaches the periodic orbit. In that case, the itinerary will begin with a number of non-repeating elements, possibly obscuring the periodic nature of the orbit. Therefore, it is helpful to understand the concepts of eventual periodicity and, more broadly, **asymptotic periodicity.**

Let $f: R \rightarrow R$. The orbit $\{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots, f^n(x_0), \ldots\}$ has asymptotic periodicity if as $n \rightarrow \infty$, the orbit approaches a periodic orbit $\{p_1, p_2, p_3, \ldots, p_k, p_1, p_2, p_3, \ldots\}$ such that $\lim_{n\rightarrow\infty} |f^n(x_0) - p_n| = 0$. (Alligood, Sauer, and Yorke, 1996).

If an initial condition lies within the basin of attraction of a fixed or periodic sink, its orbit will be asymptotically periodic, but these are not the only instances of asymptotic periodicity. The important idea is that if an orbit is asymptotically periodic, then after some number of iterations, we should be able to predict its behavior using itineraries.

5. Lyapunov numbers and Lyapunov exponents

In the discussion on stability of fixed and periodic points in section III-A-5, we found that derivatives are used to understand the behavior of a given point or orbit. By their definition, derivates measure the rate of change at a given point, which is exactly what we hope to know in our study of stability. For fixed points, the process is simple. If $|f'(x)| < 1$, then the point is a sink. If $|f'(x)| > 1$, the point is a source.

By the same reasoning, we can determine the stability of periodic orbits by finding the value of $|f'(x_1) \bullet f'(x_2) \bullet \ldots \bullet f'(x_k)|$. If the product is less than 1, then the orbit is a sink. If the product is greater than 1, it is a source. To relate this idea to sensitive dependence on initial conditions, the value $|f'(x_1) \bullet f'(x_2) \bullet \ldots \bullet f'(x_k)|$ represents the total change that a point very near the initial value would experience through k iterations. If the number is less than 1, then the point will move even closer to the initial value, and will continue to do so with further iterations. Thus, we have a periodic sink.

Since each element of the orbit is responsible for a kth of the total change, we can find the average rate of change for the whole orbit by evaluating $|f'(x_1) \bullet f'(x_2) \bullet \ldots \bullet f'(x_k)|^{1/k}$. This may not be incredibly useful for short periodic orbits, but for lengthy orbits or orbits that may never repeat it is very helpful. We call the limit of $|f'(x_1) \bullet f'(x_2) \bullet \ldots \bullet f'(x_n)|^{1/n}$ as $n \rightarrow \infty$ the **Lyapunov number** (**LyN)** of the orbit. If the product of the derivates is greater than 1, then so will be the Lyapunov number, the average rate of change. Therefore, LyN > 1 implies the orbit is a source. Likewise, LyN < 1 implies the orbit is sink.

Lyapunov exponents (**LyE)** represent the same principle, with an application of a natural log function:

$$\lim_{n\to\infty} \frac{\ln|f'(x_1)|+|f'(x_2)|+\ldots+|f'(x_n)|}{n}$$

LyN = 1 corresponds to LyE = 0, so if LyE is negative then the orbit is a sink and if LyE is positive then the orbit is a source.

If $|f'(x_1) \bullet f'(x_2) \bullet \ldots \bullet f'(x_n)| \neq 0$ and $\{f(x), f^2(x), f^3(x), \ldots, f^n(x)\ldots\}$ is asymptotically stable to an orbit $\{y_1, y_2, y_3, \ldots, y_n, \ldots\}$ and both orbits have Lyapunov exponents, then then the orbits will have equal Lyapunov exponents. (Alligood, Sauer, and Yorke, 1996).

Example

For the logistic map, a = 3.1, there is a period-two sink at $x_{1,2}$ = 0.558, 0.765. An orbit with x_0 = 0.558 would be {0.558, 0.765, 0.558, 0.765,…} and have a Lyapunov exponent of

$$\frac{\ln|f'(0.765)|+|f'(0.558)|}{2}$$

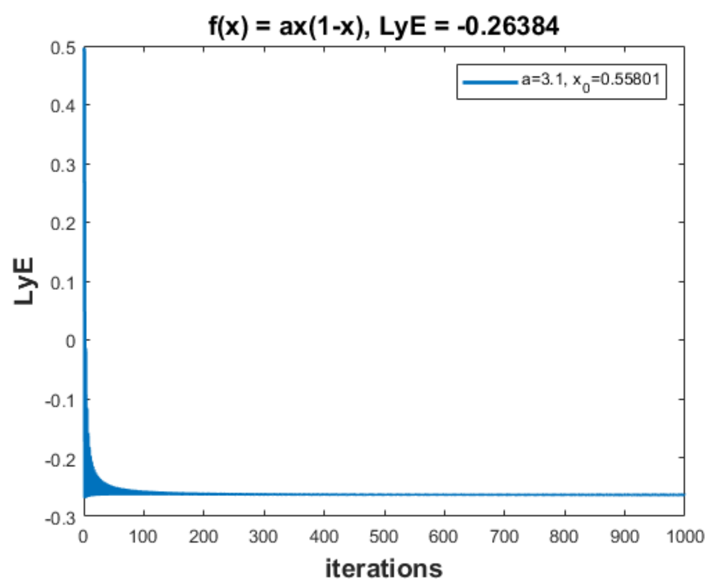


Figure 10 – Lyapunov exponent plot for a = 3.1, initial condition 0.765.

Our understanding of the relationship between Lyapunov exponents and asymptotic periodicity tells us that the orbit of any initial condition within the basin of attraction of the period-two sink represented in figure 10 will have a Lyapunov exponent of -0.26384.

Let x_0 = 0.1. At some point, the orbit of 0.1 will become arbitrarily close to the period-two sink, but we cannot immediately say how many iterations it will take. Ultimately, the orbit will take the form of $\{0.1, f(0.1), f^2(0.1),\ldots,f^j(0.1) = 0.558, f^{j+1}(0.1 = 0.765, f^{j+2}(0.1) = 0.558,\ldots,f^n(0.1)\ldots\}$. As $n\rightarrow\infty$, the number of elements of the orbit that are equal to 0.558 or 0.765 is infinite. In calculating the LyN or LyE of the orbit, the finite number of initial elements not equal to 0.558 or 0.765 have a negligible impact on the result, so the Lyapunov exponent equals -0.26384. Therefore, we should expect to see the LyE exponent of the orbit get closer to -0.26384 as the number of iterations increases.

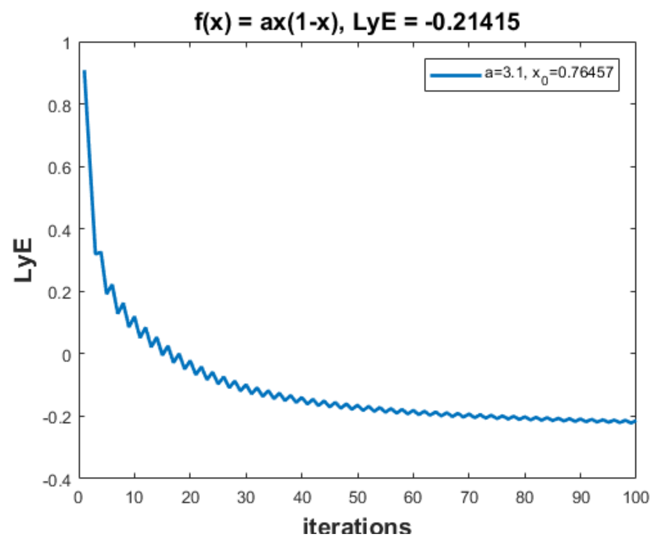


Figure 11 – Lyapunov exponent plot for a = 3.1, initial condition 0.1. 100 iterations.

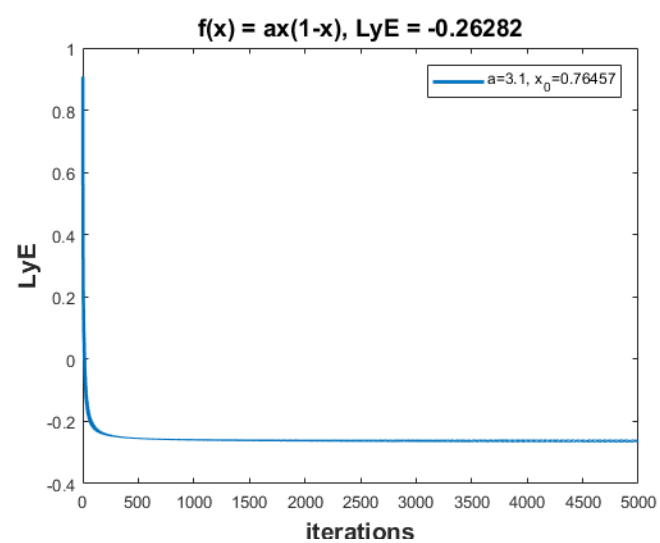


Figure 12 – Lyapunov exponent plot for a = 3.1, initial condition 0.1. 5000 iterations.

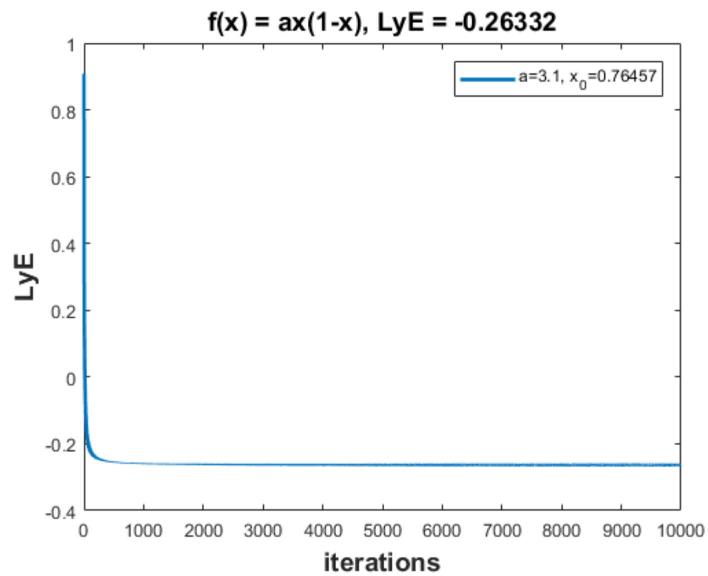


Figure 13 – Lyapunov exponent plot for a = 3.1, initial condition 0.1. 10000 iterations.

Notice in figures 11, 12, and 13, that as the number of iterations increases, the Lyapunov exponent of the period-two orbit gets closer to the value we found analytically.

6. Chaotic orbits

Let $\{f(x), f^2(x), f^3(x), \ldots, f^n(x) \ldots\}$ be an orbit under some map f. The orbit is chaotic if the following three conditions are met:

1) The orbit is bounded.
2) The LyE of the orbit does not equal the LyE of some periodic orbit, meaning that it is not asymptotically periodic.
3) The LyE of the orbit is greater than zero.

Together, these three conditions create behavior that shows sensitivity to initial conditions. First, since the orbit is bounded, we avoid the scenario whereby orbit blow up to infinity. Second, the orbit does not show periodicity for any number of iterations. And third, on average, the orbit does not shrink over time.

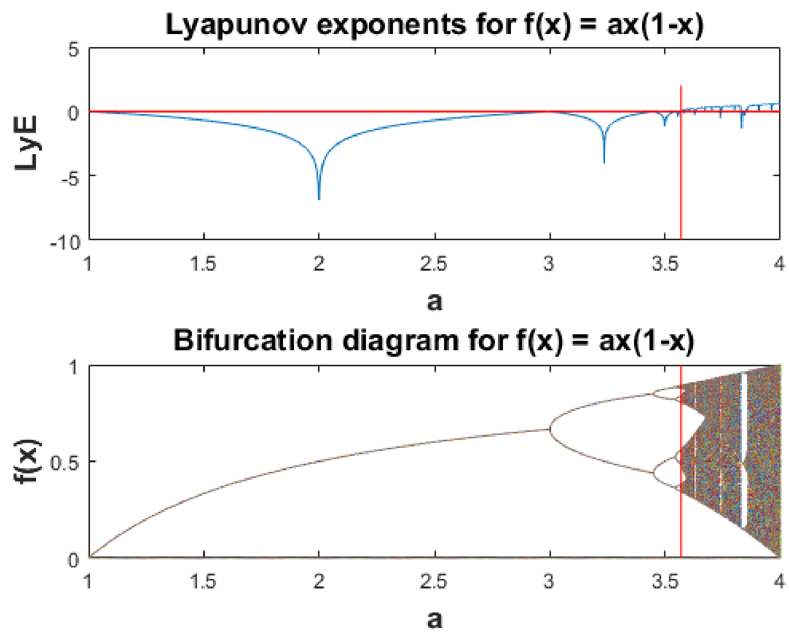


Figure 14 – Lyapunov exponent plot and bifurcation diagram for the logistic map for a = 3.1.

Example

We can look again at the sensitive dependence we see in figure 9 and understand that it is a product of the chaotic orbits that exist at $a = 4$.

1. The system is bound ($0 \leq x \leq 1$).
2. Findings from figure 9 and the cobweb plot in figure 15 support the claim that the orbit does not converge to a periodic orbit
3. Figure 14 shows us that at a =4, the LyE is positive.

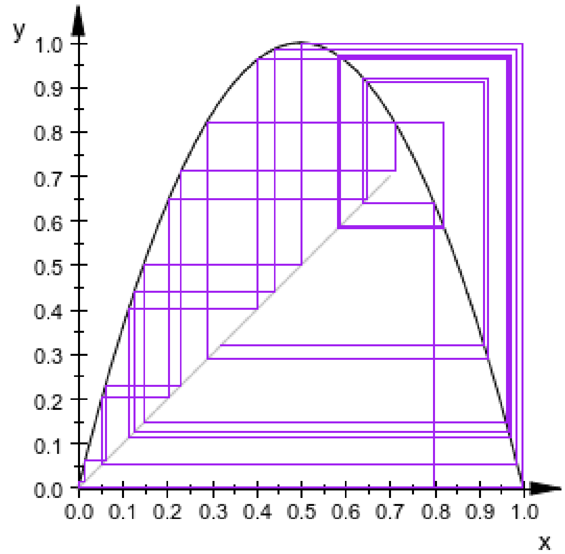


Figure 15 – Cobweb plot for a = 4.

Because the orbits never become periodic, the itinerary will never repeat.

B. Maps

1. Tent map: $T(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{2} \\ a(1 - x) & \text{ix } \frac{1}{2} \leq x \end{cases}$

Maps of the form T(x) as defined above enable us to explore chaotic orbits through describing the maps itineraries and Lyapunov exponents at $a = 2$.

First, we establish an intuition for how orbits behave under T at $a = 2$. For any initial condition x_0, compare $|x_0 - 0|$ to $|x_0 - 1|$. Choose the smaller quantity and multiply it by 2. Now we have $T(x_0)$. We can follow the same rule k times to find $T^k(x_0)$. No matter what x_0 we choose, the orbit will always be bound by the interval (0,1), so our first condition for chaotic orbits is met.

Next, we examine the schematic itineraries for the map. Let L represent values of $x \leq \frac{1}{2}$ and R represent values of $x \geq \frac{1}{2}$. Observe that $x \leq \frac{1}{4}$ maps to L and $\frac{1}{4} \leq x \leq \frac{1}{2}$ maps to R. Likewise, $\frac{1}{2} \leq x \leq \frac{3}{4}$ maps to L while $\frac{3}{4} \leq x \leq 1$ maps to R. Recognizing the pattern created when $a = 2$, we can draw simple subintervals by dividing the previous subinterval in two.

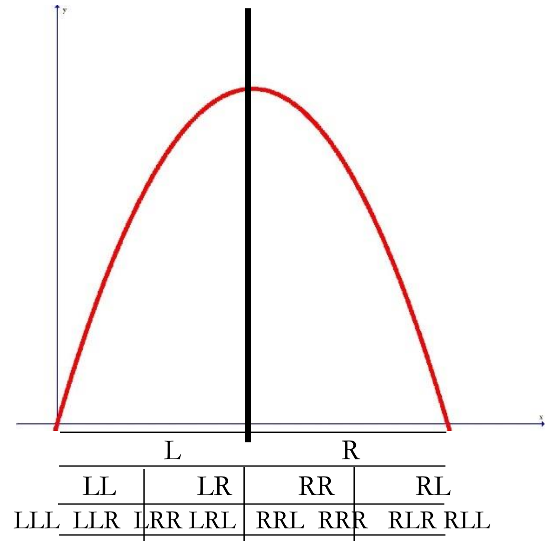
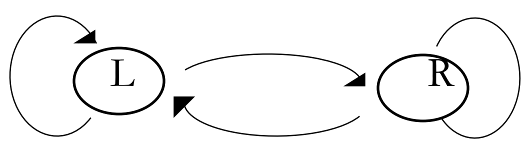


Figure 16 – The tent map with L and R intervals demarcated for the first three iterations.

The important realization is that subintervals decrease in length by a factor of 2 with each iteration. Therefore, we know that the length of the subintervals after k iterations is $1/2^k$. As $k \to \infty$, the length of the intervals approaches 0. But because R is dense there will always be infinite values of x inside every subinterval, however narrow it may be. Thus, for every itinerary we can write, every combination of k Ls and Rs, there is an associated subinterval of length $1/2^k$ such that if x_0 lies inside the subinterval, it will follow the prescribed orbit through k iterations of T.



Because any itinerary is possible, we can create nonrepeating strings of Ls and Rs of any length we choose. The corresponding orbits are neither periodic nor asymptotically periodic.

What would be the Lyapunov exponents of such orbits? Observe that for any point x Є (0,1), x ≠ ½, |T '(x)| = 2. This tells us a few helpful things:

- The slope of any periodic orbit (that doesn't include x = ½) is greater than 1. Thus. there are only periodic sources and no periodic sinks.
- The Lyapunov exponents for any such orbit will be greater than 0, because ln2 = 0.693.

We have now established that orbits of the tent map are bound, that we can find orbits that are not asymptotically periodic, and that the Lyapunov exponents for any such orbit would be positive. With that, we have met our three criteria for showing that chaotic orbits exist. In fact, because all orbits such that $x_i \neq ½$ are sources, there are infinitely many chaotic orbits of the tent map at $a = 2$.

V. Basics in Two Dimensions

Key Concepts: Sinks, Sources, and Saddles; Eigenvalues and Eigenvectors; the Jacobian Matrix; Manifolds; Lyapunov numbers and exponents in two dimensions

A.

1. Sinks, Sources, and Saddles

The concepts of fixed or periodic points in two dimensions is identical to the one-dimensional case.

Let $f: R^2 \rightarrow R^2$. If $\exists$x,y such that f(x,y) = (x,y), then (x,y) is a fixed point of the map f. If $\exists k$ such that f^k(x,y) = (x,y), then (x,y) is a period-k point of f.

The description of sinks and sources that we explored in section III-A-5 extends from one-dimensional cases to two-dimensional cases easily. The intuition is simple: a fixed point is a sink if there exists a surrounding neighborhood such that all points within a neighborhood move towards the fixed point from any direction. If all nearby points, however close to the fixed point, move away from the fixed point, then the fixed point is a source. Similarly, a periodic orbit is a sink if nearby points move towards it, but it is a sink if all nearby points move away.

Graphically, we can characterize the distance between a fixed point (x_1, y_1) and any other point (x_2, y_2) as the length of a vector **v**. For all $\varepsilon > 0$, if $|\mathbf{v}| < \varepsilon$, then we can say that (x_2, y_2) is in the ε-neighborhood of the fixed point (x_1, y_1). To say that (x_1, y_1) is a sink under f means that there exists an $\varepsilon > 0$ such that for all points connected to (x_1, y_1) by some vector **v**, if $|\mathbf{v}| < \varepsilon$, then (x_n, y_n) will move towards (x_1, y_1) upon repeated iterations of f.

In addition to sinks and sources, in two dimensions we have **saddles**. Once again, let $((x_1, y_1)$ be the fixed point of the two-dimensional map f. Depending on the rules of the map, we may find a case in which $\exists\varepsilon > 0$ such that if $|x_1 - x_2| < \varepsilon$, then $f^k(x_2) \rightarrow x_1$ as $k \rightarrow \infty$, but at the same time if $y_1 \neq y_2$, then for all $\varepsilon > 0$, $f^k(y_2)$ does not approach

y_1. In the case where orbits are attracted to a point along one axis and repelled along another axis, a fixed point is called a saddle.

The following sections focus on analytical methods for determining the stability of fixed points and periodic orbits. For these examples, we seek to establish an intuition for sinks, sources, and saddles.

Example (Fixed sink)

Let $f(x,y) = (x/2, y^2)$, with $0 \le x \le 1$, $0 \le y \le 1$.
We can identify fixed points by finding points where
$f(x,y) = (x,y)$.

$$f(x,y) = (x,y) \Rightarrow (x,y) = (x/2, y^2)$$
$$\Rightarrow x = x/2,\ y = y^2$$
$$\Rightarrow x = 0,\ y = 0,1$$

So, we have fixed points at (0,0) and (0,1). Now, consider the rules of f associated with x and y. Let k be the number of iterations of f. It is clear that for all $x \in (0,1)$ and $y \in (0,1)$, $x \to 0$ and $y \to 0$ as $k \to \infty$. Therefore, we should suspect that (0,0) is a sink.

Example (Periodic source)

Let $f(x,y) = (1-x^2, 2y)$, with $0 \le x \le 1$. There is a period-two orbit at (0,0), (1,0). Qualitative inspection of the rule suggests that the orbit is not a sink because it would be very unlikely for x to reach 0 or 1 from a different x_0. Similarly, y values clearly diverge from any $y_0 \neq 0$.

Example (saddle)

Let $f(x,y) = (x^2, y^2)$
The map f has fixed points at (0,0), (0,1), (1,0), and (1,1). For the points (0,1) and (1,0), one coordinate acts as a sink and the other acts as a source. Therefore, these two points are saddles.

2. Eigenvalues and Eigenvectors

To be able to determine the stability of points and orbits in two dimensions, we will review some concepts from matrix algebra.

For a matrix A, an **eigenvalue** is any scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some vector x. The vector corresponding to an eigenvalue λ is called an **eigenvector**. (Lay, 1994).

Calculation of eigenvalues for a given matrix relies upon the realizing that $A\mathbf{x} = \lambda\mathbf{x} \iff (A - \lambda I)\mathbf{x} = \mathbf{0}$, where I is the identity matrix and $\mathbf{0}$ is the zero vector. By the Invertible Matrix Theorem, this problems correlates to finding all λ such that $(A - \lambda I)$ is not invertible.

Example

Let $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Then $A - \lambda I = \begin{pmatrix} 2 - \lambda & 0 \\ 1 & 1 - \lambda \end{pmatrix}$

We want to find all λ such that $(A - \lambda I)$ is not invertible. If a matrix is not invertible, then its determinant equals 0. So, we can set $\text{Det}\,|(A - \lambda I)| = 0$ and solve.

$$\begin{aligned} \text{Det}\,|(A - \lambda I)| = 0 &\Rightarrow (2 - \lambda)\bullet(1 - \lambda) - (1)\bullet(0) = 0 \\ &\Rightarrow (2 - 3\lambda + \lambda^2) - 0 = 0 \\ &\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \\ &\Rightarrow \lambda = 1, 2 \end{aligned}$$

3. The Jacobian Matrix

Another principle that will be useful as we work to determine the stability of two-dimensional maps is the **Jacobian matrix**, which allows us to consider any map as a matrix representing the map's effects with respect to both variables. If the absolute values of all the eigenvalues of the Jacobian are less than 1, then the point is a sink. If all values are greater than 1, then it is a source. If some values are greater than 1 and others less than 1, then the point is a saddle.

Example

To create the Jacobian matrix, it's helpful to first rewrite the function as a system of rules. Using the example $f\,(x,y) = (x^2, y^2)$, we can say:

$x = x^2,\ y = y^2$

The matrix takes the form of **Df** (x, y) = $\begin{pmatrix} \frac{d'x}{x} & \frac{d'y}{x} \\ \frac{d'x}{y} & \frac{d'y}{y} \end{pmatrix}$

In our case, we have $\begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$.

To find the stability of any for the matrix we plug in the values from our fixed points,

(1,1) gives $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

And we find eigenvalues: $(2-\lambda)\bullet(2-\lambda) = 0$
$\Rightarrow (\lambda^2 - 4\lambda + 4) = 0$
$\Rightarrow \lambda = 2.$

Since $2 > 1$, the point (1,1) is a source.

Now, we try the fixed point (0,0). The resulting Jacobian matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Eigenvalues are $\lambda^2 = 0 \Rightarrow \lambda = 0$.
So (0,0) is a sink.

Now we try (1,0), which gives $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.
Eigenvalues are $(2-\lambda)\bullet(-\lambda) = 0 \quad \Rightarrow \lambda = 0, 2$.
Because $0 < 1$ and $2 > 1$, we know that (1,0) is a saddle. By the same reasoning, (0,1) is a saddle as well.

To determine the stability of a periodic orbit, we find the Jacobian matrix at each point in the orbit, and multiply them together to get $\mathbf{Df}^k(x_1, y_1) = \mathbf{Df}(x_1,y_1) \bullet \mathbf{Df}(x_2,y_2) \bullet \ldots \bullet \mathbf{Df}(x_k,y_k)$. Then we use the eigenvalues as we did above.

Example

We can use the map $f(x,y) = (1-x^2, 2y)$, which in the previous section we suspected of having a periodic source at (0,0), (1,0).

$$\mathbf{Df}^k(x_1, y_1) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \bullet \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

To find eigenvalues: $-\lambda(4-\lambda) = 0 \Rightarrow \lambda = 0, 4$, so the orbit is actually a saddle.

4. Manifolds

Let f be a map with two dimensions x and y with a fixed point (x_1, y_1) such that the fixed point is a saddle—that either of the following is true:

- if $\exists \varepsilon > 0$ such that if $|x_1 - x| < \varepsilon$, then $f^k(x) \to x_1$ as $k \to \infty$, but at the same time if $y_1 \neq y$, then for all $\varepsilon > 0$, $f^k(y)$ does not approach y_1.
- if $\exists \varepsilon > 0$ such that if $|y_1 - y| < \varepsilon$, then $f^k(y) \to y_1$ as $k \to \infty$, but at the same time if $x_1 \neq x$, then for all $\varepsilon > 0$, $f^k(x)$ does not approach y_1.

The dimension in which f^k converges to a fixed value is called the **stable manifold**. The dimension in which f^k does not converge is called the **unstable manifold**.

Example

Return to the map $f(x,y) = (x^2, y^2)$. In the previous section, we found that the points (1,0) and (0,1) are saddles. As $\lambda = 2$ is associated with the coordinate the equals 1, we know that this dimension is the unstable manifold. In the case of (1,0), for instance, x is the unstable manifold and y is stable manifold.

5. Lyapunov numbers and exponents in two dimensions

In section IV-A-5, we saw that Lyapunov numbers represent the average stretching at the points of an orbit. We found that this served as a valuable tool for analyzing the stability of periodic orbits in one dimension. Here, we explore their application to systems of higher dimension.

In a two-dimensional map f, stretching can take place in every direction along the two-dimensional plane. In determining the stability of the map at a given orbit, we pay closest attention to the direction of greatest stretching. We say that this is the first Lyapunov number of f and we denote it as L_1. Similar to the one-dimensional case, if $L_1 < 1$, then the orbit is a sink. If $L_1 > 1$, then the orbit is a source. The map has a second Lyapunov, L_2, which is the average stretching in the direction orthogonal to the direction of L_1.

We can generalize to m-dimensional space. Again, L_1 represents Lyapunov number in the direction of greatest average stretching. Now, L_2 represents the Lyapunov number in the direction of second great average stretching. We continue in this pattern until we reach L_m, the Lyapunov number in the direction of least average stretching.

B. Maps

1. Hénon Map: $f(x, y) = (a - x^2 + by, x)$

The Hénon Map presents an opportunity to discuss the issues related to stability of two-dimensional maps in a more compelling way.

To begin, we can apply what we know about fixed points and periodic orbits to characterize patterns for different values of the parameters a and b.

Fixed points

To find fixed points, we set $x = a - x^2 + by$ and $y = x$. We have a system of equation that we can solve through substitution.

$$x = a - x^2 + bx \quad \Rightarrow \quad x^2 + (1 - b)\cdot x - a = 0$$

By the quadratic equation, we have:

$$x_{1,2} = \frac{-(1 - b) \pm \sqrt{(-(1 - b)^2 + 4a)}}{2}$$

So, we can find fixed points for any values of a and b such that $-(1 - b)^2 < 4a$.

Example

Let $a = 2$ and $b = -0.5$. There are fixed points at

$$x = \frac{-(-0.5) \pm \sqrt{-(-0.5)^2 + 8)}}{2} = 0.25 \pm 1.39.$$

So, we have (1.64, 1.64) and (-1.14, -1.14).

Periodic orbits

To find period-k points of the map in terms of arbitrary parameters a and b, we need contract the map f^k. At higher values of k, this becomes analytically unwieldy. So, we restrict our exploration to $k = 2$.

To find period-two points, we set $x = a - (a - x^2 + by)^2 + bx$

$$y = a - x^2 + by = (a - x^2)/(1 - b)$$

Substituting this value of y into the equation for x, we get

$$x = a - (a - x^2 + b\cdot(a - x^2)/(1 - b))^2 + bx$$

$$\Rightarrow \; 0 = (x^2 - (1 - b) \cdot x - a + (1 - b)^2(x^2 + (1 - b)\cdot x - a)$$

As the factor on the right represents the fixed points already discovered, the factor on the left can be used to find period-two points.

Jacobian matrix

In order to understand the stability of the fixed points and periodic orbits, we need to create the Jacobian matrix in terms of arbitrary values of a and b.

$$\mathbf{Df}(x, y) = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}.$$

Our characteristics equation in terms of the parameters a and b is $(-2x - \lambda)\cdot(-\lambda) - b = \lambda^2 + 2x\lambda - b = 0$

$$\Rightarrow \quad \lambda_{1,2} = \frac{-2x \pm \sqrt{(4x^2 + 4b)}}{2}$$

$$\Longrightarrow \quad \lambda_{1,2} = -x \pm \frac{\sqrt{(4x^2 + 4b)}}{2}$$

So, we can determine the stability of any fixed points of the Hénon map by plugging in the values of x and *b* into this equation and comparing the resulting eigenvalues to 1. In the case of *a* = 2 and *b* = 0.5, we found fixed points at x = -1.14 and x = 1.64.

For x = -1.14, $\lambda_{1,2} = 1.14 \pm \frac{\sqrt{(4(-1.14)^2 + 4(0.5))}}{2} = 1.14 \pm \frac{1.82}{2}$

$$\Longrightarrow \lambda_{1,2} = 0.23,\ 2.05$$

For x = -1.14, $\lambda_{1,2} = -1.64 \pm \frac{\sqrt{(4(1.64)^2 + 4(0.5))}}{2} = 1.64 \pm \frac{2.17}{2}$

$$\Longrightarrow \lambda_{1,2} = 0.56,\ 2.72$$

Therefore, both fixed points are saddles.

Graphic interpretation

Determining the stability of periodic orbits becomes very complicated for any value of *k* > 1, so we rely on computer assistance for further inspection of the behavior of the Hénon map.

Figure 16 shows plots of the map for values of *a* at a fixed value of *b* = -0.5 and initial values of x = 0.3 and y = 0.3. In effect, we have isolated the parameter *a* to determine how increases in its value effect the behavior of the map.

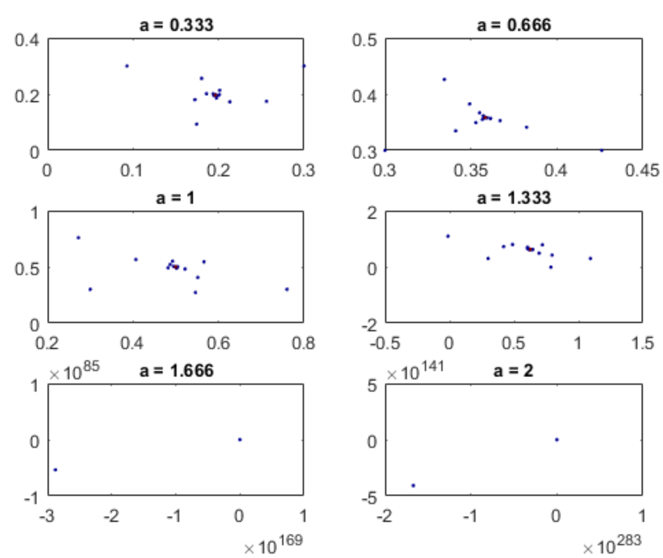


Figure 17 – The Hénon map at initial point (0.3,0.3), *b* = -0.5. 3000 iterations

At values of $a \leq 4/3$, we see that orbits congregate near fixed points. As a increases, the orbits blow up towards infinity.

We can inspect the case of $a = 1$, $b = -0.5$ more closely. Based on our understanding of the map, we can expect to find fixed points at $x_{1,2} = \frac{-(1 - b) \pm \sqrt{(b - 1)^2 + 4a}}{2} = -0.75 \pm 1.25$

Figure 18 shows a sink at (0.5,0.5).

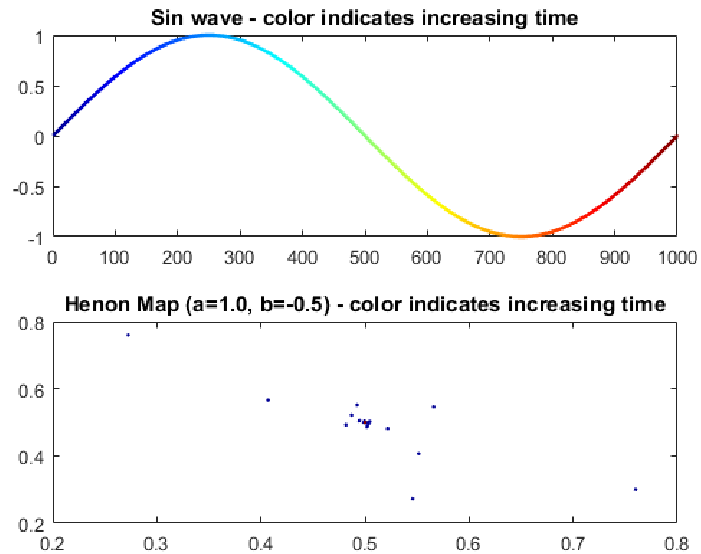


Figure 18 – At $a = 1$, $b = -0.5$, the map has a sink at (0.5,0.5).

VI. Fractals

A. Key Concepts: Cantor Sets; Sierpinski gasket; Sierpinski carpet; Fractal dimension

1. Cantor Sets

Before examining the conditions that lead to chaos in two dimensions, it is helpful to explore fractals. One accessible structure that shows fractal behavior is the middle-third **Cantor Set**. To build the Cantor Set, first remove the middle subinterval [1/3, 2/3] from the interval [0,1], leaving the disjoint set [0, 1/3] U [2/3,1]. Next, remove the middle thirds from each disjoint subinterval, giving:

[0, 1/3]/[1/9,2/9] U [2/3,1]/[7/9,8/9]
<==> [0,1/9] U [2/9,1/3] U [2/3,7/9] U [8/9, 1]

If we were to continue performing the removal of the middle thirds of all subintervals n times, the resulting set would be the union of 2^n intervals of length $1/3^n$. Any number that remains after the removal of all middle thirds is said to be in the Cantor Set. We find that ultimately all endpoints are preserved in the set.

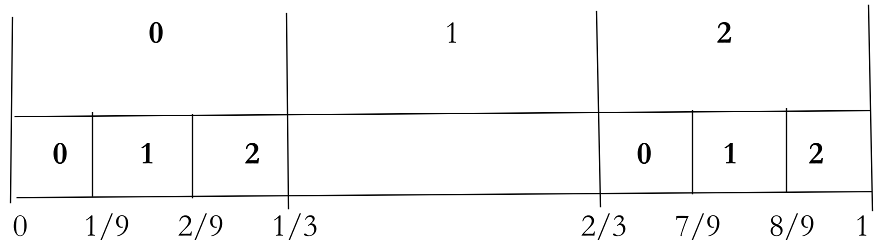


Figure 19 – To create the Cantor Set, we divide the interval (0,1) into thirds and then continue dividing each subinterval into thirds. We associate the number 0 with the first third, 1 with the second third, and 2 with the final third. We can characterize the set as the sequence of such divisions with all 1s removed from the set.

We can make see an analogy between the middle-third Cantor Set and the idea of itineraries that we previously explored. Rather than using the labels *L* and *R* to designate the lower and upper half of the original interval, we can use the labels *0, 1,* and, *2* to designate the original interval (0,1) and any subsequent disjoint subinterval. *0* is associated with the first third, *1* is associated with the middle third, and *2* is associated with the final third. In this manner, any number in (0,1) can be represented as a sequence of 0s, 1s, and 2s. As the number *1* is associated with middle thirds, the Cantor Set consists of all sequences of 0s and 2s. Since the number of such sequences is uncountably infinite, the Cantor Set has uncountably many elements. (Dangello and Seyfried, 2000).

The Cantor Set is an example of a mathematical structure that behaves and appears the same at any scale. Even if we were to zoom in the number line to view an extremely small subinterval, we would see the removal of a middle-third subinterval that looks the same graphically. This self-similarity is one of the characteristics that defines fractals.

2. The Sierpinski gasket

   The Cantor Set is most easily understood graphically on a one-dimensional number line, but there is an easy way to extend the concept to the two-dimensional case. Begin with an equilateral triangle. **The Sierpinski gasket** consists of the points in a plane that remain after the sequential removal of middle triangles as depicted in figure 21.

Figure 20 – The Sierpinski gasket consists of the points that remain after the successive removals of middle triangles. This is the gasket of a triangle of side length 1. (MATLAB code created by Paulo Silva, https://www.mathworks.com/matlabcentral/profile/authors/2021205-paulo-silva).

While the principles of the gasket are largely similar to those that construct the Cantor Set, some of the characteristics are different. Whereas the Cantor Set is the union of 2^n intervals of length $1/3^n$, the Sierpinski gasket is the union of 3^n intervals of triangles of side length ½.

3. The Sierpinski carpet

   The Sierpinski carpet demonstrates that fractals can be constructed from the repeated removal of any fraction of its components. Here, instead of removing a middle triangle, we remove a central square from a square.

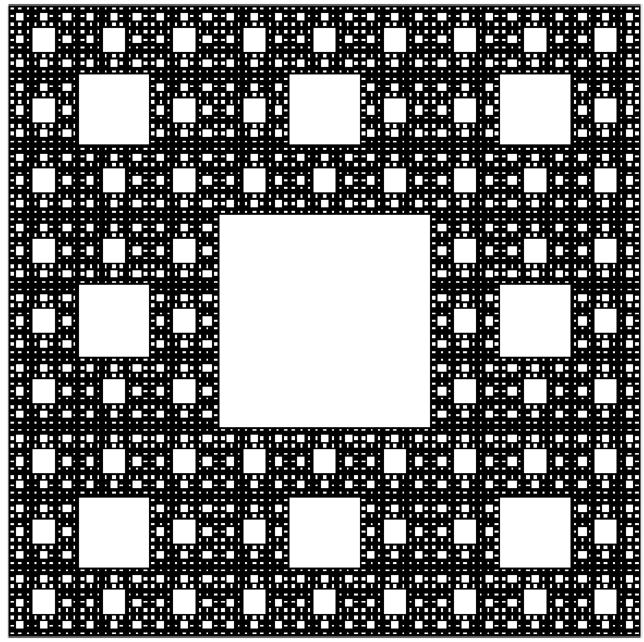


Figure 21 – The Sierpinski carpet consists of the points that remain after the successive removals of middle squares. (MATLAB code created by Jochen Schurr, http://m2matlabdb.ma.tum.de/download.jsp?MC_ID=5&SC_ID=13&MP_ID=275).

The Sierpinski carpet is the union of 8^n intervals of side length $1/3^n$.

4. Fractal dimension

In the above examples—the middle-third Cantor Set, the Sierpinski gasket, the Sierpinski carpet—and many others, there is a scaling that occurs as $n \to \infty$. Focusing momentarily on the Sierpinski carpet, if our original square has side lengths of 1, then it has an area of 1 x 1 = 1. After the first removal of a middle square, we are left with a shape of smaller area. $1 - (1/3) \cdot (1/3) = 8/9$. With each subsequent removal, the total area of the resulting shape is further reduced by a factor of (8/9). As $n \to \infty$, $(8/9)^n = 0$. Since the Sierpinski carpet has an area of 0, we can see that it is not a two-dimensional object. However, just as points and lines are geometric objects without area, the Sierpinski carpet has a **fractal dimension** that can be analyzed. The scaling relationship is central to any such analysis.

Let $N(\varepsilon)$ be the number of resulting disjoint subintervals and let ε be the length of the subintervals. We say that if $N(\varepsilon) = C(1/\varepsilon)^d$, then the set is d-dimensional.

Note that $N(\varepsilon) = C(1/\varepsilon)^d \Rightarrow d = \dfrac{\ln N(\varepsilon) - \ln C}{\ln(1/\varepsilon)}$

As ε gets very small, ln C will become negligible, so we proceed using the formula $d = \lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$

Example

Recall from our discussion of the middle-third Cantor set that the set consists of 2^n subintervals of length $(1/3)^n$. $\varepsilon = 1/3^n$, so $\varepsilon\to 0$ as $n\to\infty$. We can calculate d as:

$$\lim_{n\to\infty} \frac{\ln 2^n}{\ln(1/(1/3)^n)} = \lim_{n\to\infty} \frac{n \ln 2}{n \ln 3} = 0.6309$$

We can general for a middle-*k*th Cantor set if we let:

$$\varepsilon = \frac{1 - (1/k)}{2}$$

$$d = \lim_{n\to\infty} \frac{\ln 2^n}{\ln(2/1 - (1/k))^n)} = \frac{\ln 2}{\ln(2k/(k-1))}$$

For cases in two or three dimensions, the process is the same except that *N* represents the number of two-or three-dimensional segments remaining and ε represents the length of one side.

Example

For the Sierpinski carpet, we have $N(\varepsilon) = 8$ and $\varepsilon = 1/3$.

$$d = \lim_{n\to\infty} \frac{\ln 8^n}{\ln(1/(1/3)^n)} = \lim_{n\to\infty} \frac{n \ln 8}{n \ln 3} = 1.8928$$

VII. Chaos in Two Dimensions

A. Key Concepts: Chaos in *m* dimensions; Lyapunov dimension; Markov partitions; Backward and forward itineraries; Forward Limit Sets; Chaotic attractors

1. Chaos in *m* dimensions

We saw in section V-A-5 a system for denoting the *m* Lyapunov numbers associated orbits of *m*-dimensional maps. The Lyapunov number in the direction with the greatest average stretching is L_1, the number in direction with the second greatest average stretching is L_2, and so on. We rely in part on this notation to define chaos in *m* dimensions.

An orbit of an *m*-dimensional map is chaotic if it is not asymptotically periodic, if no Lyapunov number equals 1, and if $L_1 > 1$.

2. Lyapunov dimension
   An orbit's **Lyapunov dimension** indicates the connection between Lyapunov exponents and fractal dimension and serves as a means for understanding the characteristics of overall stretching in *m*-dimensional maps.

   Let f: $R^m \rightarrow R^m$ and consider an orbit $\{f(x), f^2(x), \ldots f^n(x), \ldots\}$. Then there are *m* Lyapunov numbers denoted L_m such that $L_1 \geq L_2 \geq L_3 \ldots L_m$. Denote as *j* the largest integer such that $L_1 + L_2 + \ldots + L_j \geq 0$, if such a *j* exists.
   The Lyapunov dimension of the orbit is $j + \dfrac{L_1 + L_2 + \ldots + L_j}{|L_{j+1}|}$.

   Recall that the procedure for calculating fractal dimension that we explored in section VI relied upon counting $N(\varepsilon)$, the number of subintervals resulting from an iteration of a map. The Lyapunov dimension provides a mechanism for calculating fractal dimension of maps for which such geometric analysis is not as easy.

3. Markov partitions
   To help characterize the orbits of two-dimensional maps, it is helpful to revisit the idea of itineraries that we explored in section IV-A-1. In plotting numbers along the one-dimensional number line, we found that we can describe orbits as sequences of *Ls* and *R*s according whether particular elements of the orbit fell in the subinterval (0, ½) or (1/2, 1). We are not restricted to using two partitions *L* and *R*, but for our purposes this works well.

   We can create a similar scheme in two dimensions by describing as *L* the two dimensional space including all points (x,y) such that $0 \leq x \leq ½$ and $0 \leq y \leq 1$ and describing as *R* all points (x,y) such that $½ \leq x \leq 1$ and $0 \leq y \leq 1$.

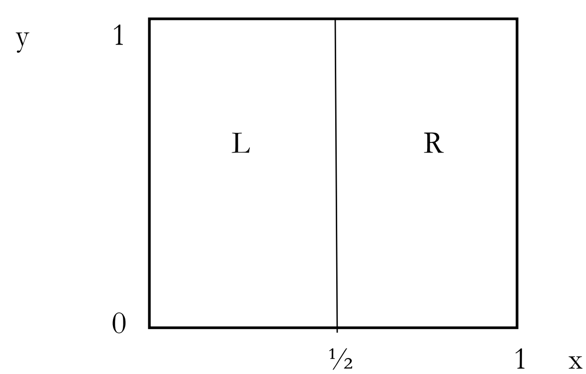


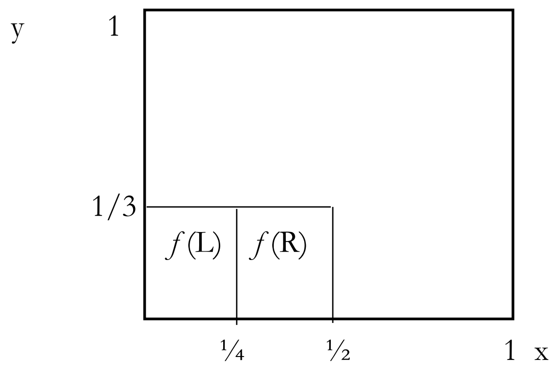
   Figure 22 – A basis of a Markov partition in two dimensions.

   We can use these labels *L* and *R* to denote what happens to values in the interval $0 \leq x \leq 1$ and $0 \leq y \leq 1$ under different maps.
   Example

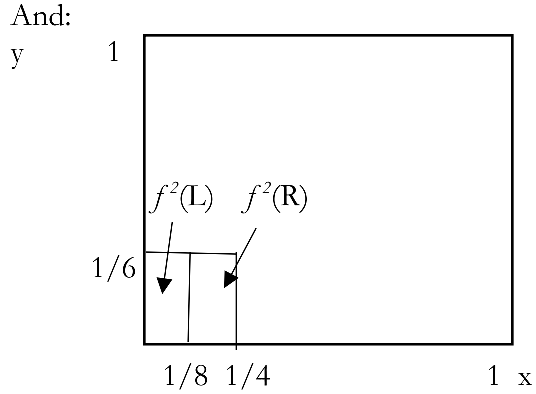
Let f: $R^2 \rightarrow R^2$ such that

$$f(x,y) = (x/2,\ y/3) \qquad 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1$$

We can draw partitions as:



And:



We can see graphically in this case that there exists a fixed point for the map at (0,0). This demonstrates the two-dimensional fixed-point theorem, which tells us that a bound, continuous map of a two-dimensional map to itself has a fixed point.

4. Backward and forward itineraries
   In inverse two-dimensional Markov partitions, we find bi-infinite itineraries. Rather than itineraries limited to the form $S_1S_2\ldots S_n\ldots$, we have itineraries of the form $\ldots S_{-n}\ldots S_{-2}S_{-1}S_0.S_1S_2\ldots S_n\ldots$

   The **forward itineraries** consist of the portions $S_1S_2\ldots S_n\ldots$The **backward itineraries** consist of the portions $\ldots S_{-n}\ldots S_{-2}S_{-1}S_0$. To construct backward itineraries, it is necessary to find the inverse of the maps in question.

   Just as the itineraries of one-dimensional maps give an indication of the periodicity or lack thereof for a given map, two-dimensional itineraries can represent periodicity. If an itinerary consists of a repeating sequence of k symbols or at least ends with such a repeating sequence, then we know that the orbit is periodic or asymptotically periodic. If, on the contrary, an itinerary is not periodic at its right end, then the orbit is not asymptotically periodic. Therefore, if the largest Lyapunov number of the orbit is greater than 1, and none of the Lyapunov numbers is exactly 1, then the orbit is chaotic.

5. Chaotic attractors
   To this point, we have characterized chaotic orbit as resulting from the repulsion of a given orbit. Using concepts like sensitive dependence on initial conditions and Lyapunov numbers and exponents, we have shown that chaos occurs due to the stretching that orbits undergo at their points. This characterization is helpful in predicting the behavior of dynamical systems, but it obscures the truth that if chaotic orbits are observed then they must have attracted some initial conditions. Even though we regard these orbits as unstable, we can view them as having a certain kind of order due to the fact that the chaotic orbit will restrict itself to a specific **chaotic attractor**.

   Reconsider the Hénon map. Instead of setting b = -0.5 as in figure 18, we let b = 0.5.

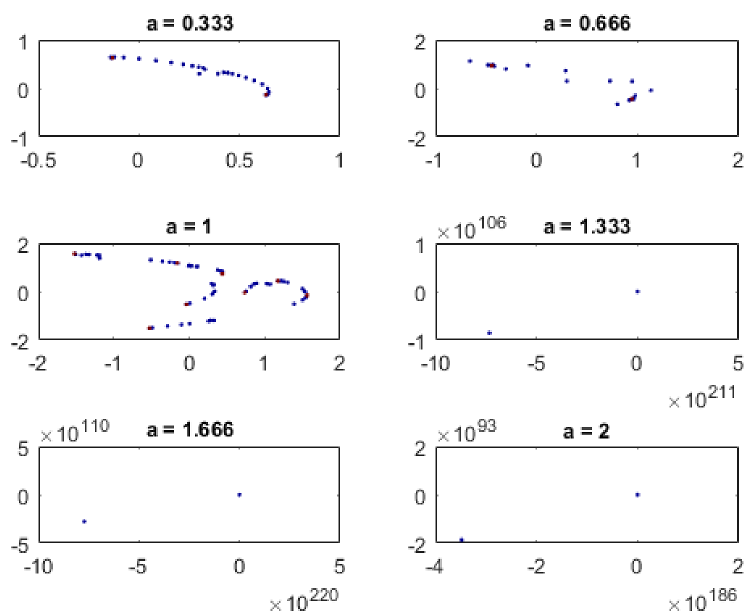


Figure 23 – Plots of the Hénon map at initial point (0.3,0.3), b = 0.5. 3000 iterations.

Observe that at values of a between 0 and 1, points of a given orbit are attracted to a particular attractor. Compare these plots to figure 20, which plots the Hénon map at b = 0.3 with the same values of a.

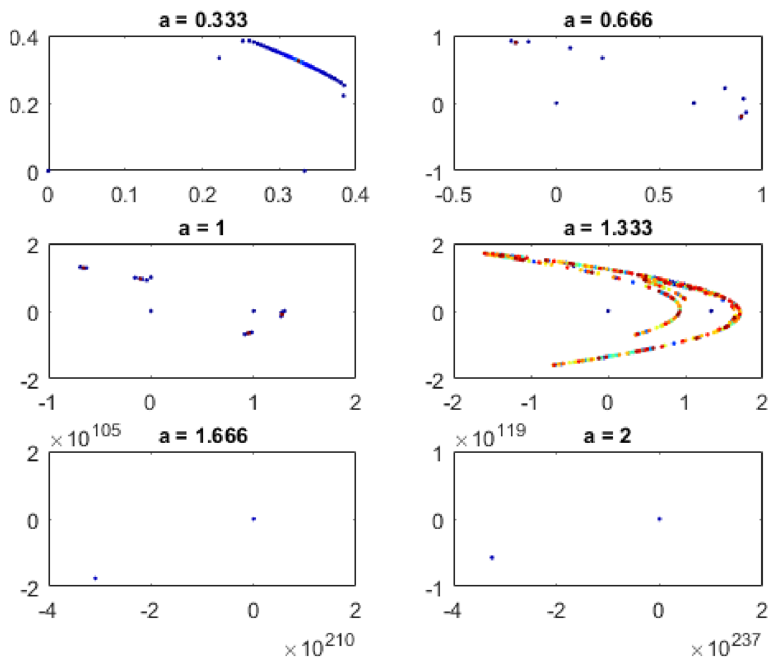


Figure 24 – Plots of the Hénon map at initial point (0.3,0.3), b = 0.3. 3000 iterations.

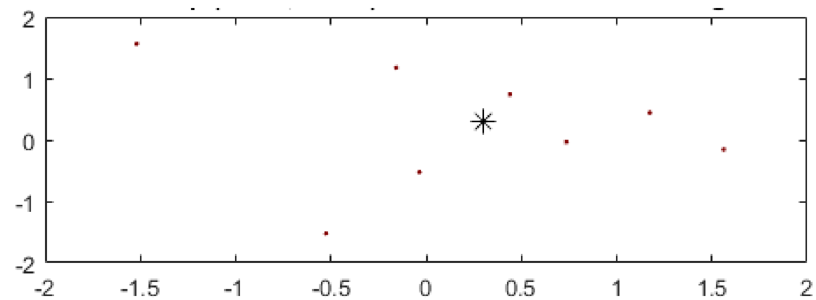
Once again, we see shapes that indicate attractors. As we will see in the section on forward limit sets, these different attractors have important distinctions. In the case of $b = 0.3$, $a = 4/3$, we see chaotic attractors whereas in the plots for $b = 0.5$, we do not.

The two characteristics that define an orbit as a chaotic attractor are (1) that the orbit is indeed chaotic, as demonstrated by Lyapunov numbers greater than 1 and itineraries that are not eventually periodic and (2) that the orbit attracts points from some portion of the plane.

6. Forward Limit Sets

We can provide a more precise definition of a chaotic attractor through the concept of a **forward limit set**, or ω-limit set. Let f be a map and x_0 be the initial condition for an orbit. The forward limit set of an orbit is the set of all points x such that for all N and for all $\varepsilon > 0$, $\exists n > N$, $|f^n(x_0) - x| < \varepsilon$. (Alligood, Sauer, and Yorke, 1996). This is the set of points to which the orbit always returns arbitrarily closely. It may or may not include points that are actually on the obit.

The ω-limit set can be helpful in understanding the attractors that affect and orbit. If we ignore the first elements of the orbit, say the first 1000 or 5000, we can focus on the eventual behavior of the orbit. In some cases, this may show a fixed or periodic sink. In other cases, it could reveal a chaotic attractor. Revisit the plots of figures 23 and 24. The plots showed orbits taking the shape of attractors near $a = 1.0$, $b = 0.5$ and $a = 1.333$, $b = 0.3$. Now, let's compare the plots of $a = 1.0$, $b = 0.5$ and $a = 1.4$, $b = 0.3$, but now seeing only iterations 1001 through 2000.



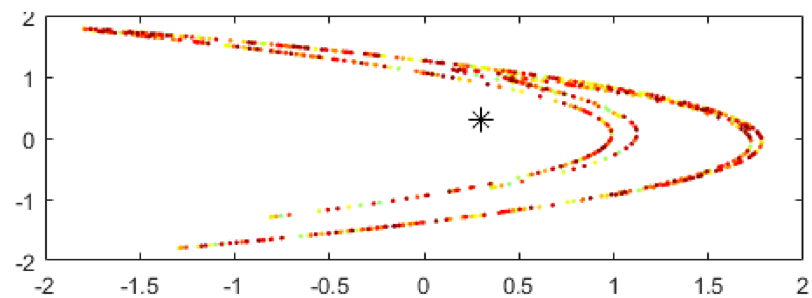


Figure 24 – Plots of the Hénon map at initial point (0.3,0.3), $a = 1.0$, $b = 0.5$ (above) and $a = 1.3$, $b = 0.4$ (below). Only iterations 1001 through 2000 are shown.

We see that only the orbit of $a = 1.3$, $b = 0.3$ continues to show chaotic behavior. The set of points plotted in red and yellow represent the orbit's ω-limit set. Because the set is a chaotic set, the orbit is a chaotic orbit.

B. Maps

1. Skinny baker map: $B(x, y): \begin{cases} ((1/3)\bullet x, 2y) & \text{if } 0 \leq y \leq ½ \\ (1/3)\bullet x + 2/3, 2y - 1) & \text{if } ½ \leq y \leq 1 \end{cases}$

The skinny baker map presents a simple example of a two-dimensional map with shrinking and stretching. We can begin to explore the dynamics of the map by first looking at its associated Markov partitions.

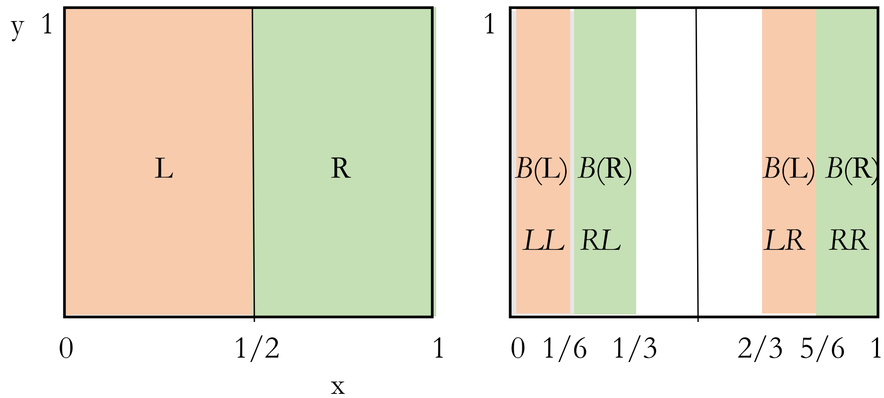


Figure 25 – The unit square and the partitions of the skinny baker map.

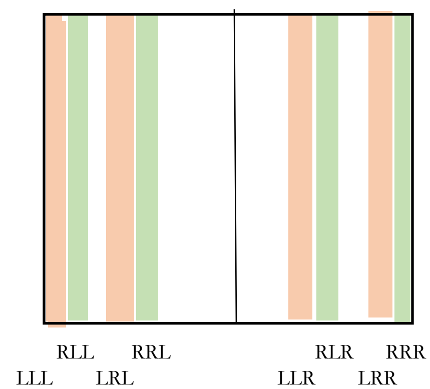
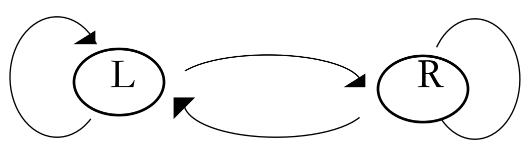


Figure 26 – After a second iteration of the skinny baker map, with red indicating the plots of points that began in the subinterval *L* and green indicating the plots of points that began in *R*.

Plots of the partitions indicate a few characteristics of the map that will be helpful in our exploration of its dynamics.

First. Points in *L* can map to *L* and *R*. and points in *R* can map to *L* and *R*. Therefore, the map has the following transition, which we know to be conducive to chaotic orbits.



Second, the map bears similarities to the middle-third Cantor set. In As opposed to the one-dimensional case, the skinny baker map removes the middle third section of a two-dimensional plane. The result, instead of an uncountably infinite number of endpoints comprising the Cantor set, is an uncountably infinite number of vertical lines of length 1 associated with the endpoints of the Cantor set.

To more clearly understand the stability of the map, we can look at its Lyapunov numbers. Regardless of the piecewise nature of the map, the Jacobian matrix is straightforward.

$$\mathbf{DB}(\mathbf{v}) = \begin{pmatrix} 1/3 & 0 \\ 0 & 2 \end{pmatrix}$$

Therefore, the Lyapunov numbers are 2 and 1/3. As the largest Lyapunov number is greater than 1 and no Lyapunov number is equal to 1, we have met two of the conditions necessary for chaos in two dimensions.

We can rewrite the Lyapunov numbers as Lyapunov exponents -ln3 and ln2. With these Lyapunov exponents in hand, we can calculate the maps' Lyapunov dimension.

$$D_L = \ln 2 + \frac{\ln 2 - \ln 3}{-\ln 3} = 0.693 + 0.599 = 1.292$$

VIII. Differential Equations

A. Key Concepts: Equilibrium; Phase portraits; Slope field; Phase planes and phase spaces; Lotka-Volterra models; Poincaré-Bendixson Theorem

1. Equilibrium

Whereas all of the previous maps presented the state of a system in terms of the system's previous state, differential equations offer a method for presenting the current state in terms of its own current rate of change. Analogous to the concept of *sinks* in one-and two-dimensional maps is the idea of **equilibria** in differential equations. An equilibrium is the solution of differential equation or system of differential equations that takes the form of $f\,'(x) = f(x)$. Because $f\,'(x)$ is the rate of change, the map is unchanging, then necessarily $x' = f(x) = 0$.

2. Phase portraits

Many of the nonlinear differential equations that are relevant in the study of dynamical systems are impossible to solve analytically, so we rely on graphical tools for understanding their behavior. The simplest tool is the **phase portrait**, which uses arrows along a number line to indicate where equilibria will be found.

Example

Let x' = *a*x, $-1 \leq x \leq 1$. For different values of *a*, we will find that the map behaves differently. We can use a series of phase portraits to illustrate the alternatives.

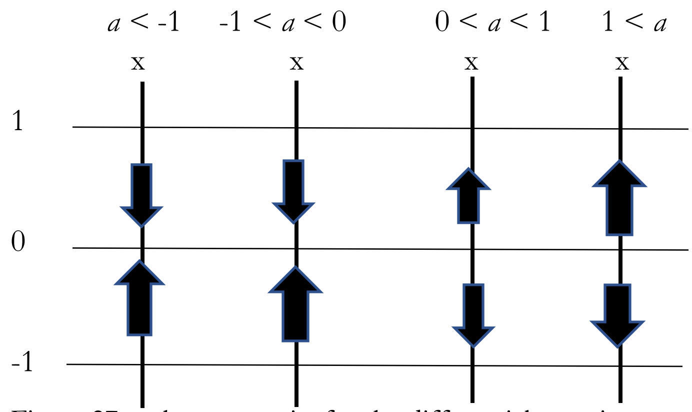


Figure 27 – phase portraits for the differential equation x' = *a*x at various values of *a*. At *a* < 0, the arrows converge at x =0, indicating that orbits reach an equilibrium at that point.

3. Slope field

   A second tool for analyzing the behavior of differential equations is the **slope field**, which shows the slopes of an equation at difference points in the plane.

   Example

   x' = x – y

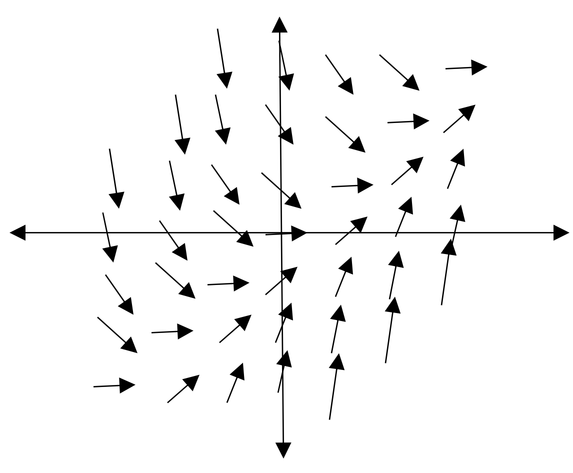


Figure 28 – Slope field for x' = x – y.

The slopes of figure 28 appear to points towards a line running along y = x – 1. Indeed, we can draw a solution curve along this line.

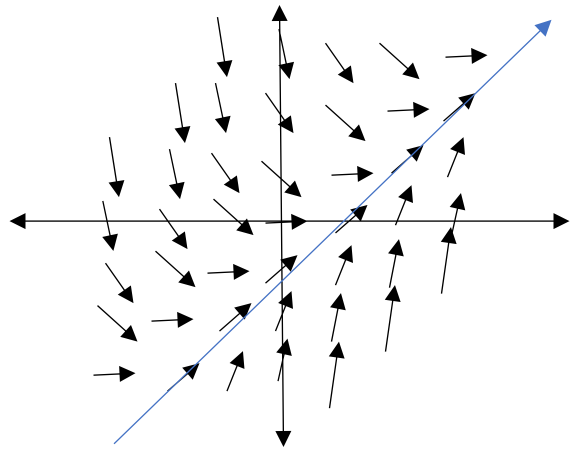


Figure 29 – Slope field for x' = x – y with solution curve.

B. Maps:

1. Lotka-Volterra models

   Differential equations are particularly helpful in studying population dynamics, where the rate of change in a population is typically dependent on the size of the population at any given point. **Lotka-Volterra** equations model the interactions of different populations.

   Lotka-Volterra equations fall into two categories: competing species models and predator-prey models. In the competing species case, we model two groups of species, denoted as x and y, competing over limited resources. We can denote the per-individual reproduction rates as:

$$x' = ax \bullet (1 - x) - bxy$$
$$y' = cy \bullet (1 - y) - dxy, \qquad a, b, c, d > 0$$

   The product xy subtracted from both equations represents the competition for resources. With any increase in either population, the growth rate of either will be limited. On the same note, if either population has an initial condition equal to 0, then the growth of the other population will follow a logistic model.

   To illustrate the behavior of the system, we draw nullclines—lines x' = 0 and where y' = 0—on the x-y plane.

   Example

   Let $a = 1$, $b = 0.5$, $c = 3$, $d = 2$.

   Then we have $x' = ax \bullet (1 - x) - bxy \quad = x - x^2 - 0.5xy = 0$

   $\Longrightarrow x^2 + (0.5y - 1)x = 0$

And x' = *c*y•(1 – y) – *d*xy = 3y – 3y^2 – 2xy = 0
==> 3y^2 + (2x – 3)y = 0

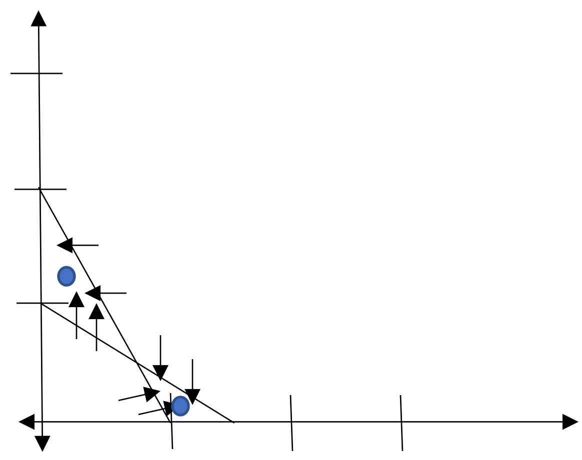


Figure 30 – Finding solution for competing species with *a* =1, *b* = 0.5, *c* = 3, *d* = 2 using the method of nullclines. The regions denoted by blue dots are trapping regions. The model has an equilibrium where the lines x' = 0 and y' = 0 intersect.

We can construct a similar graphic to model predator-prey relationships using the equations

x' = *a*x – *b*xy
y' = -*c*y+ *d*xy, *a, b, c, d* > 0

Example

Let *a* = 1, *b* = 4, *c* = 3, *d* = 2
Then x' = *a*x – *b*xy = x – 4xy = 0
And y' = -*c*y+ *d*xy = -3y + 2xy = 0

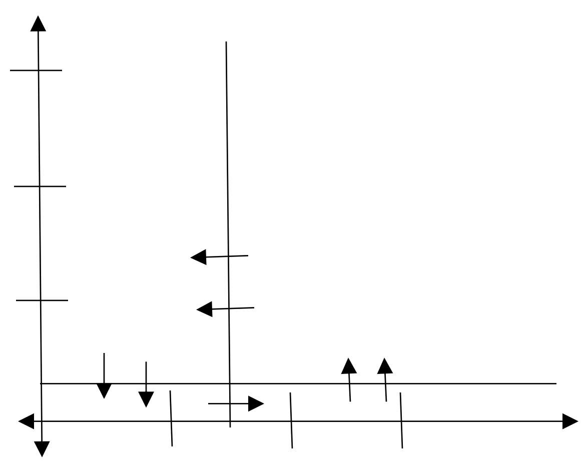


Figure 31 – Finding solution for predator-prey with *a* =1, *b* = 4, *c* = 3, *d* = 2 using the method of nullclines. The populations are at equilibrium at the intersection of the two nullclines.

IX. Chaos in Differential Equations

A. Maps: Lorenz equations; Rössler Attractor

1. Lorenz equations

$$x' = -\sigma x + \sigma y$$
$$y' = -xz + rx - y$$
$$z' = xy - bz \qquad \text{where } \sigma, r, b \text{ are constants}$$

The Lorenz equations model atmospheric convection and they represent changes in three quantities with respect to time. The variable x represents convection, y is the horizontal variable, and z is vertical.

The constants σ, r, and b are assumed to be positive and we most frequently let $\sigma = 10$ and $b = 8/3$, and then we watch what occurs at various values of r.

We can find equilibria by finding the points x, y, and z where $x' = y' = z' = 0$. We can look at stability via the Jacobian matrix:

$$\mathbf{Df} = \begin{pmatrix} -\sigma & \sigma & 0 \\ -z+r & -1 & -x \\ y & x & -b \end{pmatrix}$$

After some complicated linear algebra, we find that $\lambda_1 = -b$ and

$$\lambda_{2,3} = \frac{-(r+1) \pm \sqrt{(r+1)^2 + 4r(r-1)}}{2}$$

Because the parameters are all positive, we know that $\lambda_1 < 0$. The values of $\lambda_{2,3}$ are determined by the value of r, and thus, the overall stability of the system depends on this value. Figures 32 through 36 show plots of the system for different values of r.

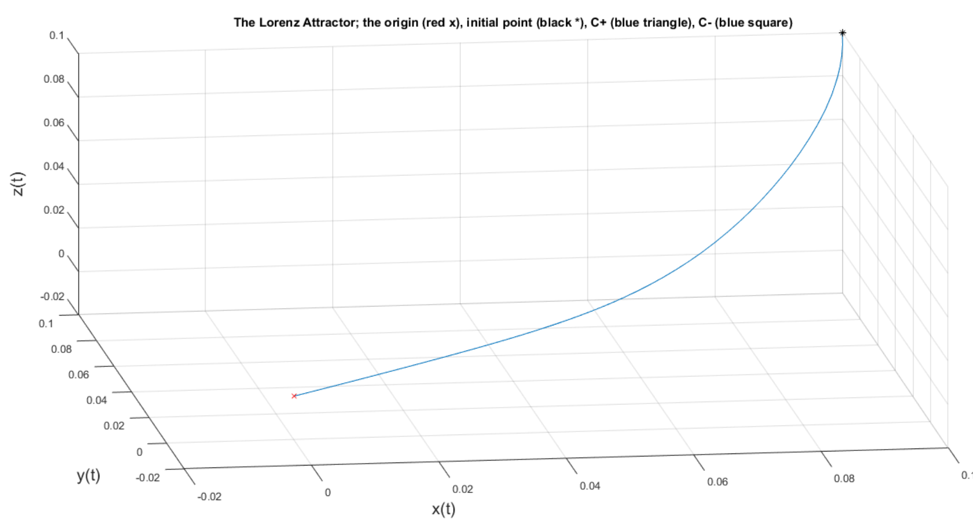


Figure 32 – The Lorenz equations at r = 0.5. The orbit moves towards the stable fixed point at (0,0,0).

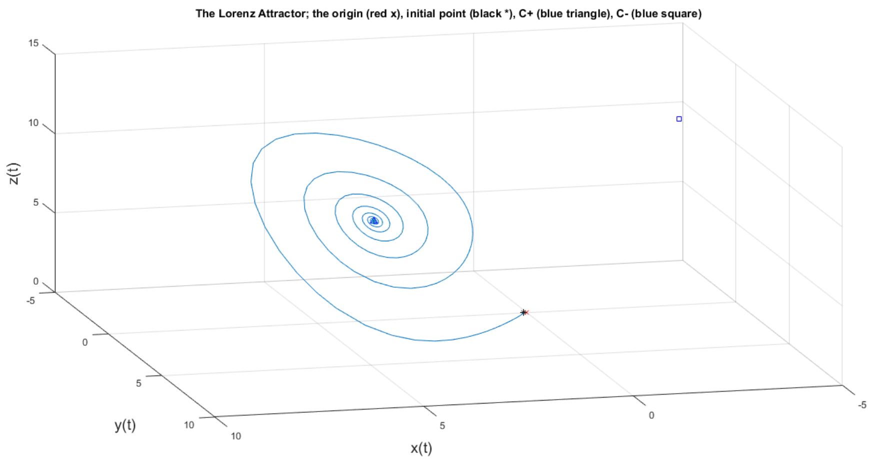


Figure 33 – The Lorenz equations at r = 10. The orbit moves towards a stable fixed point.

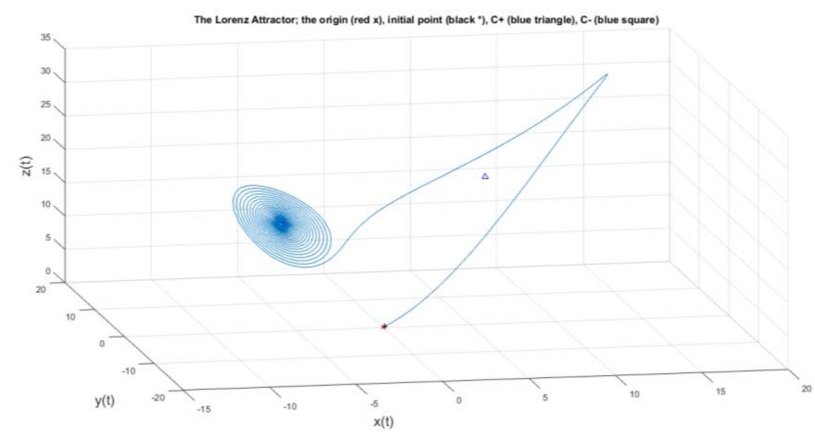


Figure 34 – The Lorenz equations at r = 20. The orbit still moves towards a stable fixed point. Other initial conditions at this value of r may land on an isolated chaotic orbit.

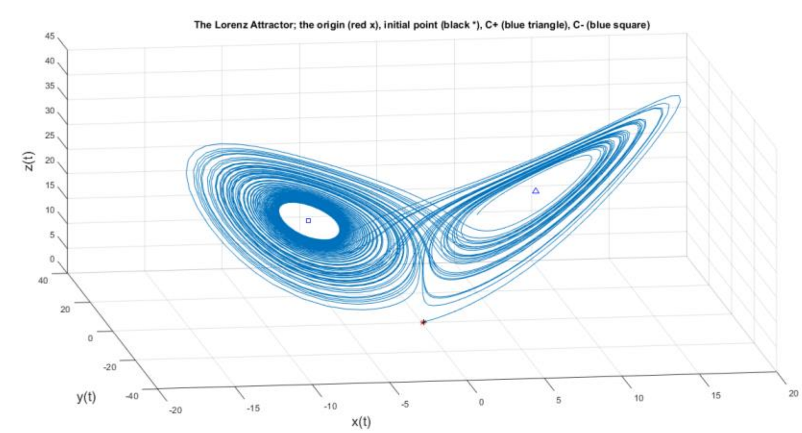


Figure 35 – The Lorenz equations at r = 24.5. The orbit still moves towards a chaotic attractor, which exists alongside the stable fixed point.

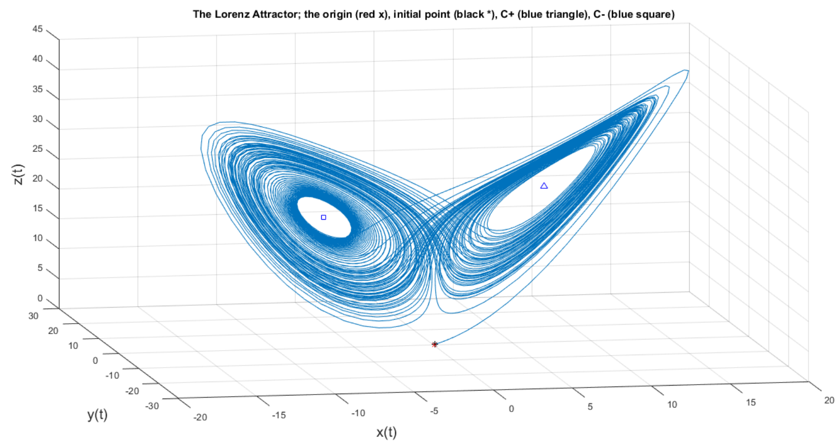


Figure 36 – The Lorenz equations at r = 25. The orbit still moves towards a chaotic attractor.

Through the Lorenz system of equations, we see that chaotic attractors, like the one demonstrated in figure 36, can be described by differential equations.

X. <u>Conclusion</u>

The most fundamental idea gleaned from this semester-long study of dynamical systems and chaos amounts to a more precise definition of the word *chaos*. Colloquially, the term is used somewhat interchangeably with *random*. From afar, the association of chaos with random events creates an impression that dynamical systems theory and chaos theory are studies in probabilistic processes. An introduction to dynamical systems and chaos demonstrates that this widespread supposition is not only inaccurate, but antithetical to the mathematical definition of chaos. This statement is meant as any conceit, but instead to introduce the idea that the discovery that chaos emerges as a result of purely deterministic processes is surprising. Those without backgrounds in math might even find these results to be counter-intuitive.

One of my goals after completing this course is to be able to communicate the principles of chaos in a way that is logically thorough, even if not mathematically rigorous. The models explored here have great relevance in demonstrating a valuable truth about deterministic behavior: the distinction between being able to understand and being able to predict. A common though perhaps poorly scrutinized assumption is that in any given situation requiring thoughtful planning, limitations of control are due only to an inability to construct an accurate model of circumstances. This model would include a precise identification of initial conditions and the set of rules that guides the interaction of different variables. Some people may grant that such a model is impossible to construct due to limitations in collecting and analyzing data, but fewer would contend that such a perfect model would still leave its administrator incapable of predicting the long-term future. But this is essentially what we learn from studying chaos theory. We can identify the initial conditions, parameters, and rules that govern a system, and still have no way of accounting for tiny, inevitable perturbations in predicting long-term behavior. It appears that this truth is not a result of a shortfall of computing power, but instead an innate feature of complex systems.

The study of the behavior of logistic maps provides an introduction to how relatively simple systems show chaotic behavior. During the course, we looked closely at the behavior of the map between $a = 3$ and $a = 4$, where the rate of bifurcations accelerates rapidly. At the left side of this interval, the parameter a induces a period-two orbit. By the right side of the interval, after a huge number of bifurcations, a induces chaotic orbits. All of this behavior occurs as a result of a function simple enough to be studied in a high school algebra course.

If such a simple map exhibits chaotic behavior, we have reason to suspect that natural systems exhibit structures that are at least as complicated. This semester's coursework introduced students to the relevance of dynamical systems in the fields of meteorology, biology, health sciences, and physics, among other topics. Anywhere we find systems with large numbers of variables interacting under sets of rules, we can use principles examined in this course to study them. In continuing studies of dynamical systems, I am interested in exploring how different types of orbits—fixed, periodic, chaotic, near-chaotic—perform within these different kinds of systems. Do the orbits that we observe within various systems represent attractors, chaotic or otherwise? Is there reason to suspect that the orbits are the result of self-organization, or that the orbits exist because they somehow "out-perform" other orbits in manner analogous to natural selection in biology?

XI. References

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