DIMENSIONALITY REDUCTION

QUESTION 1

Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7,3/7,6/7], and another is [6/7, 2/7, -3/7]. Let the third column be [x,y,z]. Since the length of the vector [x,y,z] must be 1, there is a constraint that $x^2+y^2+z^2 = 1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

SOLUTION

Assignment #8

Question 1

For orthonormal column, their dot product is equal to 0

col(i) * col(j) == 0 for i != j

$c1 = [2/7, 3/7, 6/7]$ $\quad c2 = [6/7, 2/7, -3/7]$

$c3 = [x, y, z]$

$c1 * c2 = [2/7, 3/7, 6/7]\,[6/7, 2/7, -3/7]$

$= 12/49 + 6/49 + (-18)/49$

$= 0$

$c1 * c3 = [2/7, 3/7, 6/7]\,[x, y, z]$

$= \frac{2}{7}x + \frac{3}{7}y + \frac{6}{7}z$

$c2 * c3 = [6/7, 2/7, -3/7]\,[x, y, z]$

$= \frac{6}{7}x + \frac{2}{7}y - \frac{3}{7}z$

Equation ~~c3~~ C1 * C3 == C2 * C3

$\frac{2}{7}x + \frac{3}{7}y + \frac{6}{7}z\ 0$

$2x + 3y + 6z = 0$ (divide by 2)

$x + \frac{3}{2}y + 3z = 0$ (divide by 3) — A

$\frac{x}{3} + \frac{1}{2}y + z = 0$ — B

C2 × C3

$\frac{6}{7}x + \frac{2}{7}y + -\frac{3}{7}z = 0$

$6x + 2y - 3z = 0$ (divide by 6)

$x + \frac{1}{3}y - \frac{1}{2}z = 0$ (multiply by 2) — C

$2x + \frac{2}{3}y - z = 0$ —— D

Eqn A − eqn C

$(\cancel{x} + \frac{3}{2}y + 3z) - (\cancel{x} + \frac{1}{3}y - \frac{1}{2}z) = 0$

$(\frac{3}{2} - \frac{1}{3})y + (3 + \frac{1}{2})z = 0$

$\cancel{4} \; \frac{7}{6}y + \frac{7}{2}z = 0$

$\frac{7}{6}y = -\frac{7}{2}z$

$y = -\frac{\cancel{7}}{\cancel{2}} \times \frac{\cancel{6}^{3}}{\cancel{7}} z$

$\boxed{y = -3z}$ OR $\boxed{z = -\frac{1}{3}y}$

Eqn B + D

$(\frac{x}{3} + \frac{1}{2}y + \cancel{z}) + (2x + \frac{2}{3}y - \cancel{z}) = 0$

$(\frac{1}{3} + 2)x + (\frac{1}{2} + \frac{2}{3})y = 0$

$\frac{7}{3}x + \frac{7}{6}y = 0$

$\frac{14x + 7y}{6} = 0$

$14x + 7y = 0$

$7y = -14x$

$\boxed{y = -2x}$ OR $\boxed{x = -\frac{1}{2}y}$

x: y: z = -2: 1: -3

QUESTION 2

Find the eigenvalues and eigenvectors of the following matrix:

2	3
3	10

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.

SOLUTION

Assignment 8, question 2

let $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$

if λ is eigen value of A then let $\lambda I - A = 0$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda-2 & -3 \\ -3 & \lambda-10 \end{bmatrix} = 0$$

$$(\lambda-2)(\lambda-10)-9=0$$

$$\lambda^2-10\lambda-2\lambda+20-9=0$$

$$\lambda^2-12\lambda+11=0$$

$$\lambda^2-11\lambda-\lambda+11=0$$

$$\lambda(\lambda-1)-1(\lambda-11)=0$$

$$\lambda=1 \quad \text{or} \quad \lambda=11$$

When $\lambda = 11$

$$\begin{bmatrix} 11-2 & -3 \\ -3 & 11-10 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = B$$

$B.\bar{x} = 0 \quad \because \bar{x}$ is eigen vector

$$\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left[\begin{array}{cc|c} 9 & -3 & 0 \\ -3 & 1 & 0 \end{array}\right]$$

using row reduction $R_2 = 3R_2 + R_1$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\left[\begin{array}{cc|c} 9 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$9x_1 - 3x_2 = 0$$

$$\cancel{3}9x_1 = \cancel{3}x_2$$

$$x_2 = 3x_1$$

if $x_1 = 1 \quad x_2 = 3$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \text{eigen vectors.}$$

When $\lambda = 1$

$$\begin{bmatrix} \lambda-2 & -3 \\ -3 & \lambda-10 \end{bmatrix} = 0$$

$$B = \begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix} = 0 \quad \cancel{(B)}$$

$\therefore B.\bar{x} = 0$

$$\begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

using row reduction $R_2 = 3R_1 + R_2$

$$\begin{array}{cc} x_1 & x_2 \\ \end{array}$$
$$\begin{bmatrix} -1 & -3 \\ -6 & 0 \end{bmatrix}$$

$$-x_1 - 3x_2 = 0$$

$$-x_1 = 3x_2$$

if $x_1 = 1$ $\quad x_2 = -1/3$

$\therefore \begin{bmatrix} 1 \\ -1/3 \end{bmatrix}$ = eigen vectors.

QUESTION 3

Suppose [1,3,4,5,7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

SOLUTION

Question 3

Given eigen vector = $[1, 3, 4, 5, 7]$

To find out unit eigen vector in same direction,

$$D = \sqrt{1+9+16+25+49}$$
$$= \sqrt{100}$$
$$= 10$$

unit eigen vector = $[1/10, 3/10, 4/10, 5/10, 7/10]$

Distance to the unit eigen vector should always be 1.

So unit eigen vector of $[1, 3, 4, 5, 7]$ is

$[1/10, 3/10, 4/10, 5/10, 7/10]$

QUESTION 4

Suppose we have three points in a two-dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

SOLUTION

Question 4

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

$$M^T \times M$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 1+4+12 \\ 1+4+12 & 1+4+16 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 17 \\ 17 & 21 \end{bmatrix}$$

QUESTION 5

Consider the diagonal matrix M =

1	0	0
0	2	0
0	0	0

Compute its Moore-Penrose pseudoinverse.

SOLUTION

Question 5

Given diagonal matrix = $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

To compute the Moore penrose pseudo-inverse of above diagonal matrix,

Moore penrose pseudo-inverse of a diagonal matrix Σ i.e if the i^{th} diagonal element of Σ is $\sigma \neq 0$, then replace it by $1/\sigma$

pseudo inverse matrix = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

QUESTION 6

When we perform a CUR decomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

1	2	3
4	5	6
7	8	9
10	11	12

Calculate the probability distribution for the rows.

SOLUTION

Question 6

row probability $p_i = \sum_j \frac{M_{ij}^2}{f}$

$$f = \sum_{ij} M_{ij}^2$$

$$\text{matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Row sum values
14
77
194
365

$$f = 650$$

row probabilities are $\Rightarrow p_1 = \frac{14}{650} = 0.022$

$$p_2 = 77/650 = 0.118$$

$$p_3 = 194/650 = 0.298$$

$$p_4 = 365/650 = 0.562$$