J. Opl Res. Soc. Vol. 39, No. 9, pp. 823–831, 1988
Printed in Great Britain.

0160-5682/88 $3.00 + 0.00

A Deterministic Inventory System with an Inventory-Level-Dependent Demand Rate

R. C. BAKER[1] and TIMOTHY L. URBAN[2]

[1]Department of Information Systems and Management Sciences, University of Texas at Arlington, USA and [2]Department of Quantitative Methods and Management Information Systems, University of Tulsa, USA

This analysis is concerned with the continuous, deterministic case of an inventory system in which the demand rate of an item is of a polynomial functional form, dependent on the inventory level. Differential and integral calculus are used to find the inventory function with respect to time. From this, the objective function (to maximize average profit per unit time) is developed. For the continuous, multiperiod situation, a non-linear programming algorithm—separable programming—is utilized to determine the optimal order level (the quantity to order up to) and the order point (the quantity at which an order is placed). A numeric example and a sensitivity analysis are also presented.

Key words: dependent demand rate, inventory control

INTRODUCTION

The purpose of this paper is to evaluate an inventory system in which the demand rate of the product is a function of the inventory level (quantity on-hand). With this type of product, the probability of making a sale would increase as the amount of the product in inventory increases.

The inventory models currently available do not recognize a demand pattern of this type. The classical models assume that the demand rate is constant. Deterministic inventory models have been developed in which the demand rate is not required to be constant; these generally fall into three major categories. First, dynamic lot-size models, such as the dynamic programming approach by Wagner and Whitin[1] and the heuristics developed by Silver and Meal,[2,3] are those in which the planning horizon is divided into given time periods with a known demand for each. Second, time-varying models, many of which are based on the work by Donaldson,[4] are those in which the demand rate is a known function of time. Finally, there are inventory models incorporating economic and marketing policies such as the effect of price, advertising, etc. on demand rates; these would include the work done by Whitin[5] and the recent work of Lee and Rosenblatt.[6]

There is also a class of marketing models related to, but not directly associated with, inventory theory. These models, such as the geometric programming approach by Corstjens and Doyle,[7] are space-allocation models in which the demand rate is a known function of shelf space. However, no explicit consideration is given to inventory-related costs (holding, shortage or ordering costs) in these models; for example, in the Corstjens and Doyle model, an operating-cost elasticity factor is included which assumes the costs are proportional to the sales of the product.

A substantial amount of research has been conducted concerning the impact of shelf-space allocation on retail-product demand, suggesting the existence of products experiencing this type of demand. According to Levin *et al.*,[8] one of the functions of inventories is that of a motivator; they indicated:

> 'At times, the presence of inventory has a motivational effect on the people around it. It is a common belief that large piles of goods displayed in a supermarket will lead the customer to buy more.'

Silver and Peterson[9] (p. 412) have also noted that sales at the retail level tend to be proportional to inventory displayed. In fact, Silver[10] has enumerated several potential inventory-management research problems whose solution would have a major beneficial impact on the practice of inventory management. One of the important research problems noted in this article is the allocation of shelf space in which sales are directly affected by the allocation. However, no inventory model has been found in the literature that addresses an inventory-level-dependent demand pattern.

MODEL DEVELOPMENT

The model presented is the continuous, deterministic case of an inventory system in which the demand rate is dependent on the inventory level. The assumptions of this model are as follows:

1. Replenishments are instantaneous with a known, constant lead time.
2. The selling price, p, and the unit cost of the item, c, are known and constant (no price discounts and no effects of inflation).
3. The procurement cost, C_0, and the holding cost, C_h, are known and constant.
4. No backorders are allowed.
5. Demand rate is deterministic and is a known function of the level of inventory.
6. The time horizon of the inventory system is infinite.
7. The inventory system involves only one item.
8. The inventory system involves only one stocking point.

This analysis will concentrate on the situation in which the demand function is of a polynomial functional form. That is, the demand rate will take the form

$$D(i) = \alpha i^{\beta} \quad \alpha > 0, \quad 0 < \beta < 1, \tag{1}$$

where $D(i)$ is the demand rate of the product, i is the inventory level, α is the scale parameter, and β is the shape parameter. For this functional form, as the inventory level increases, so does the demand rate. Reasons why this particular functional form was chosen for analysis are shown in Appendix A.

Given this demand-rate function, the inventory level as a function of time will decrease rapidly initially, since the quantity demanded will be greater at a high level of inventory. As the inventory is depleted, the quantity demanded will decrease, resulting in the inventory level decreasing more slowly. To determine what the mathematical expression of the inventory function over time is, we can interpret the slope of the curve at any point (the rate of change of inventory level per unit time) as minus the demand rate, and solve the resulting differential equation as shown in Appendix B. This yields

$$i = \begin{cases} (Q^{1-\beta} - [\alpha(1-\beta)t])^{1/(1-\beta)} & \text{for} \quad t \leqslant \dfrac{Q^{1-\beta}}{\alpha(1-\beta)} \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where

Q = the order level (the quantity to order up to);
t = time from the start of a cycle.

It has been shown that in the case of deterministic, constant demand rate and instantaneous replenishment, it is optimal to let the inventory level reach zero before reordering (see Silver and Peterson,[9] p. 176). This is not necessarily so in the inventory-level-dependent demand-rate situation, since there are essentially lost sales (opportunity costs) as the inventory level decreases. With this type of demand-rate pattern, the quantity demanded decreases as the inventory level decreases, resulting in lower sales and potentially lower profits. To keep sales higher, the inventory level would need to remain higher. Of course, this would also result in higher holding or procurement costs.

The objective of this model will be to maximize the average net profit. The function would be as follows:

$$\text{Maximize} \quad \begin{matrix}\text{Average}\\ \text{net}\\ \text{profit}\end{matrix} = \begin{matrix}\text{Average}\\ \text{gross}\\ \text{profit}\end{matrix} - \begin{matrix}\text{Average}\\ \text{procurement}\\ \text{costs}\end{matrix} - \begin{matrix}\text{Average}\\ \text{holding}\\ \text{costs}\end{matrix}$$

$$\pi = \frac{(s-c)(Q-i_T)}{T} - \frac{C_0}{T} - C_h \bar{i}, \tag{3}$$

where

s is the selling price of the product;
c is the cost of the product (variable cost);
C_0 is the fixed procurement cost;
C_h is the holding cost;
T is the order interval (cycle length);
i_T is the order point;
$\bar{i}$ is the average inventory level; and
π is the average net profit.

Only one cycle (from zero to T) will need to be evaluated since all remaining cycles will be identical. Also, shortage situations will not need to be evaluated explicitly since there is no demand with the demand-rate patern under consideration when the inventory level reaches zero; therefore, there will be no backorders. While not allowing backorders may seem like an unrealistic restriction, the products that have this type of demand-rate pattern generally are not those that consumers would tend to order. There will be, however, opportunity costs (or lost sales) resulting as the inventory level decreases.

The average inventory level can be evaluated by integrating the inventory function over one cycle, as shown in Appendix C, resulting in

$$\bar{i} = \frac{1}{\alpha(2-\beta)T}\left[Q^{2-\beta} - (Q^{1-\beta} - [\alpha(1-\beta)T])^{(2-\beta)/(1-\beta)}\right].$$

By replacing i_T and $\bar{i}$ with appropriate values, the objective function can be shown to equal

$$\pi = \frac{(s-c)}{T}[Q - (Q^{1-\beta} - [\alpha(1-\beta)T])^{1/(1-\beta)}] - \frac{C_0}{T}$$
$$- \frac{C_\mathrm{h}}{\alpha(2-\beta)T}[Q^{2-\beta} - (Q^{1-\beta} - [\alpha(1-\beta)T])^{(2-\beta)/(1-\beta)}]. \tag{4}$$

There would be two constraints associated with this objective function. First, as indicated earlier (equation 2), the inventory function is valid only when t is less than or equal to $Q^{1-\beta}/\alpha(1-\beta)$. In other words, subsequent replenishments would occur at or before the time the current order was depleted, and no time would expire in which there is zero inventory. Also, there is likely to be a maximum amount of on-hand inventory due to storage limitations. Situations in which the demand rate is dependent on the inventory level will tend to be those products in which the items are displayed to the customer, and will likely have storage-space limitations. Therefore, the constraints corresponding to the objective function would be

$$Q^{1-\beta} - \alpha(1-\beta)T \geqq 0$$

$$Q \leqq Q_{\max}.$$

It can also be shown that this model reduces to the classical economic order quantity when the shape parameter, β, is set equal to zero (constant demand rate) and the inventory level reaches zero before reordering.

SOLUTION METHODOLOGY

While the objective function can be differentiated, the resulting equation is mathematically intractable; that is, it is difficult to solve the first derivative (set equal to zero) for Q and T (or i_T). Therefore, a solution to the general model will not be obtained in explicit form. The objective function is twice differentiable, so gradient search techniques available in standard scientific libraries might be used; however, this function does not satisfy the usual quasi-concavity condition.

Fortunately, the model can be solved using separable programming. Separable programming is a non-linear programming algorithm which provides an approximate solution by utilizing a piecewise linear function in place of the original non-linear function. A function is considered 'separable' if it can be expressed as the sum of functions, each involving only one variable. This type of problem can then be solved using mixed-integer programming.[11] Although the objective function is not quasi-concave for all values of the parameters, the mixed-integer programme will yield the global optimum to the approximate problem.

While the non-linear objective function (equation 4) is not directly separable, it can be made so by utilizing various substitutions. This can be done in the following manner. As previously indicated, the inventory level at time T is

$$i_T = (Q^{1-\beta} - [\alpha(1-\beta)T])^{1/(1-\beta)},$$

so

$$i_T^{1-\beta} = Q^{1-\beta} - \alpha(1-\beta)T. \tag{5}$$

Hence

$$\pi = \frac{(s-c)}{T}(Q - i_T) - \frac{C_0}{T} - \frac{C_\mathrm{h}}{\alpha(2-\beta)T}(Q^{2-\beta} - i_T^{2-\beta}).$$

Collecting like terms,

$$\pi = \frac{1}{T}\left[-C_0 + \left((s-c)Q - \frac{C_\mathrm{h}}{\alpha(2-\beta)}Q^{2-\beta}\right) - \left((s-c)i_T - \frac{C_\mathrm{h}}{a(2-\beta)T}i_T^{2-\beta}\right)\right].$$

The profit in one order cycle can be expressed as

$$\pi_T = -C_0 + \left((s-c)Q - \frac{C_\mathrm{h}}{\alpha(2-\beta)}Q^{2-\beta}\right) - \left((s-c)i_T - \frac{C_\mathrm{h}}{\alpha(2-\beta)}i_T^{2-\beta}\right). \tag{6}$$

So

$$\pi = \pi_T/T. \tag{7}$$

Maximizing the logarithm of π will result in maximizing π, so this can be restated as

$$\ln \pi = \ln \pi_T - \ln T.$$

Therefore, the separated problem would be:

maximize

$$\ln \pi = \ln \pi_T - \ln T$$

subject to

$$i_T^{1-\beta} = Q^{1-\beta} - \alpha(1-\beta)T$$

$$\pi_T = -C_0 + \left((s-c)Q - \frac{C_\mathrm{h}}{\alpha(2-\beta)}Q^{2-\beta}\right) - \left((s-c)i_T - \frac{C_\mathrm{h}}{\alpha(2-\beta)}i_T^{2-\beta}\right).$$

Neither of the two constraints previously mentioned (replenishing at or before reaching zero inventory and having a maximum inventory storage limitation) needs to be included explicitly in this problem since a non-negative i_T will satisfy the first constraint and an upper break point in the separable programming methodology equal to the maximum storage will satisfy the second. If there is no obvious maximum value, an arbitrary amount can be used as an upper bound. If the order-level solution is determined to be equal to this upper bound, the upper bound can be respecified and the programme re-evaluated.

In order to solve the separable programme, the range of values that each variable can take on must be determined. The order level, Q, must be greater than or equal to zero and will range to the storage capacity, $Q_{\max}$. The order point, i_T, must also be non-negative and will not be greater than the order level; therefore, it will range from zero to $Q_{\max}$. The order interval, T, must be greater than zero (since the objective function contains this variable in the denominator and since the separated problem is evaluating the logarithm of this variable) and will range to $Q^{1-\beta}/\alpha(1-\beta)$, as indicated in the development of the inventory function. To ensure that the order interval does not equal zero, a small constant can be used as a lower break point for T in the separable programme.

The profit in one order cycle, π_T, can range from negative to positive values. The minimum value that will need to be considered for this variable will be zero since it is necessary to maintain a positive profit in one order cycle to maintain a positive overall profit. As with the order interval, this variable would need to be slightly greater than zero since the objective function of the separable programme contains the logarithm of this variable, so a small, positive constant must be used as a lower break point. If no solution provides a positive profit, there will be no feasible solution to the separable programme; if the optimal solution to the problem is needed in this situation, the problem can be reformulated to minimize the net loss $(-\pi)$.

To determine the maximum value that the profit in one order cycle can realize, each of the components of the separated function must be evaluated. As shown earlier, this function consists of three parts: a constant, a function of Q and a function of i_T. The maximum value of the profit in one order cycle would occur when the separated function of Q is at a maximum and the separated function of i_T is at zero. The separated function of Q will reach a maximum at

$$f(Q) = (s-c)Q - \frac{C_h}{\alpha(2-\beta)} Q^{2-\beta}$$

$$df/dQ = (s-c) - \frac{C_h}{\alpha} Q^{1-\beta} = 0$$

$$Q^* = [\alpha(s-c)/C_h]^{1/(1-\beta)}.$$

This point is a maximum since the second derivative of the function $(d^2f/dQ^2 = -C_h(1-\beta)Q^{-\beta}/\alpha)$ is negative for all possible values of the parameters. Therefore, the maximum value of π_T will be at $Q = Q_{\max}$ when $Q_{\max} \leqslant Q^*$ or at $Q = Q^*$ when $Q_{\max} > Q^*$.

NUMERIC EXAMPLE

As an example, suppose the parameters of the inventory function are as follows:

$\alpha = 0.5$ units per time period;
$\beta = 0.4$;
$C_0 = \$10$ per order;
$C_h = \$0.50$ per unit per time period;
$s = \$20$ per unit;
$c = \$10$ per unit.

In order to solve this problem, a computer program written in SAS was developed. The input required for this program includes the demand (shape and scale) parameters, the inventory (holding and procurement cost) parameters, the selling price and cost of the item, the maximum inventory level and the number of break points.

With a maximum order level of 40 and using nine break points, the separable programme provides a solution to this problem of:

order level, $Q = 22.2$ U;
order point, $i_T = 5$ U.

The time between orders, T, and the profit in one order cycle, π_T, can also be determined from the output; however, since separable programming performs a piecewise approximation of the non-linear objective and constraint functions, these values may not correspond exactly with the order level and order point calculated. Therefore, to maintain consistency between the values of all of the variables, the time between orders, the profit in one order cycle and the average profit can be calculated from equations 5, 6 and 7, respectively:

$$\begin{aligned} \text{time between orders,} \quad & T = 12.7 \text{ time periods;} \\ \text{profit in one order cycle,} \quad & \pi_T = \$81.2; \\ \text{average profit,} \quad & \pi = \$6.40/\text{time period.} \end{aligned}$$

(Note that the values of T and π_T obtained from the separable programme are 12.6 and 80.7, respectively.)

The values obtained from the separable programme are very close to the actual optimal solution. Using a numerical search procedure, the optimal solution to this example is:

$$\begin{aligned} \text{order level,} \quad & Q = 20.7 \text{ U;} \\ \text{order point,} \quad & i_T = 3.4 \text{ U;} \\ \text{time between orders,} \quad & T = 13.6 \text{ time periods.} \end{aligned}$$

This optimal solution would result in an average profit of \$6.46 per time period, which is less than 1% higher than that obtained from the results of the separable programme.

SENSITIVITY ANALYSIS

Using the numerical example above, three types of sensitivity analyses are performed. First, the effects that various changes in either of the decision variables (the order level or the order point) will have on the average profit while holding the other decision variable at its optimal value are analysed. Next, the effects that various changes in any of the model parameters will have while holding the other parameters constant are analysed. Finally, the effect that the number of break points used in the separable programming methodology has on the resulting solution is investigated.

As shown in Table 1, this model is very insensitive to changes in either the order level or order point, particularly to changes in the order point. For example, if the optimal value of the order level is used (20.7 U), a 25% change from the optimal value of the order point would result in a reduction in profit (penalty) of approximately one-fourth of 1%. A 25% positive change of the order level (while holding the order point at its optimal level) will result in a 1.33% penalty; a 25% negative change will result in a 2% penalty. Only as the order level approaches the order point will substantial penalties be realized.

As shown in Table 2, this model is generally insensitive to errors made in estimating the model parameters. A 25% error in measuring the ordering cost parameter will incur a penalty of less than 0.2%. Only as the estimated value of the ordering cost approaches zero (−100% error) does the penalty become substantial; this is due to frequent ordering, resulting in large procurement costs. The holding-cost parameter is somewhat more sensitive than that of the ordering cost; however, it is also shown to be fairly insensitive to errors, particularly positive errors. A positive error of 25% error will incur a penalty of less than 3%; a negative error of 25% will result in a penalty

TABLE 1. *Effect of changes in decision variables on the average profit*

Percentage change in variable	Percentage change of average profit	
	Order point	Order level
−50	1.20	11.7
−25	0.26	2.0
−10	0.04	0.3
+10	0.04	0.2
+25	0.21	1.3
+50	0.80	4.6

TABLE 2. *Effects of changes in model parameters on the average profit*

Percentage change in parameter	Percentage change of average profit			
	Ordering cost	Holding cost	Demand scale	Demand shape
−50	0.85	56.3	19.7	23.8
−25	0.17	7.0	4.5	9.0
−10	0.03	0.8	0.7	1.9
+10	0.02	0.6	0.7	3.1
+25	0.11	2.9	4.0	30.2
+50	0.34	7.9	15.1	336.6

of 7%. But as the estimated holding cost gets very small (large negative errors), the penalty increases rapidly. An error of −80% will result in a penalty of nearly 700%.

The effects of using a value for the scale and shape parameters of the demand-rate function other than the true value are also shown in Table 2. The model is fairly insensitive to changes in the scale parameter; a 25% error will incur a 4–4.5% penalty. However, this model is quite sensitive to large, positive errors made in estimating the shape parameter of the demand-rate function while being fairly insensitive to negative or small, positive errors. A negative error of 25% will incur a penalty of 9%. However, a positive 25% error will incur a 30% penalty, and an error of +50% will incur a penalty of 337%.

Since the separable programming methodology utilizes a piecewise linear estimate of the non-linear objective and constraint functions, the solution will be dependent on the number of break points used in the programme. It would be expected that the accuracy of the problem would increase as the number of break points increases. However, the size of problem (the number of variables and constraints) will also increase as the number of break points increases. Each additional break point results in eight additional variables and four additional constraints. The number of iterations required to solve this problem is small for few break points (13 iterations for four break points) but increases rapidly as the number of break points increases (over 1000 iterations for 15 break points). On the other hand, the accuracy of this problem quickly approaches the optimal; the solution is within 1% of the optimal for as few as nine break points.

CONCLUSION

In conclusion, no inventory model has been found in the literature that addresses an inventory-level-dependent demand-rate pattern, despite the apparent applications of such a model. This type of problem can be solved fairly simply, even for the general case in which an integer programming procedure is all that is required. While the model generally appears to be insensitive to changes in the model parameters, it is somewhat sensitive to changes in the shape parameter (which is what primarily distinguishes this model from the classical model). Therefore, it may be misleading to ignore this type of demand-rate pattern and to use the EOQ model.

APPENDIX A

The advantages of the polynomial functional form for the demand rate

$$D(i) = \alpha i^{\beta},$$

where $D(i)$ is the demand rate of the product, i is the inventory level, α is the scale parameter, and β is the shape parameter, are characterized by:

(1) Diminishing returns. By constraining the shape parameter, β, to remain between zero and one, the marginal increase in the demand rate will decrease for larger values of the inventory level. Although there is no maximum limit on the level of demand with this functional form (that is, the curve does not approach a constant limiting value as the inventory level approaches infinity), it can be shown that the slope of the curve approaches zero. We would not realistically expect a situation in which the demand rate continued to increase with an 'infinitely' large level of inventory.

(2) Inventory-level elasticity. The shape parameter of the demand-rate function represents the ratio of the percentage change in the quantity demanded to the percentage change in the inventory

level. It is a measure of the responsiveness of the demand rate to changes in the level of inventory, all other things being equal.

(3) Richness. Unless the actual distribution is quite irregular, we can expect this function (with varying values of the shape and scale parameters) to provide a good approximation for many situations. As β approaches zero, the function will approach a horizontal line such that the demand rate will equal a constant, α. On the other extreme, as β approaches one, the demand-rate function will approach a straight line with an intercept of zero and a positive slope (αi).

(4) Intrinsic linearity. Another reason this function is desirable is its simplicity and ease of use. This function is intrinsically linear; a logarithmic transformation can be used to transform it to a straight line. Simple linear regression can then be used to estimate the parameters if the error terms are also assumed to be multiplicative.

APPENDIX B

The mathematical expression of the inventory function over time can be determined by equating the rate of change of inventory level per unit time with minus the demand rate, and solving the resulting differential equation as follows:

$$\frac{di}{dt} = -D(i)$$

$$\frac{di}{dt} = -\alpha i^{\beta} \quad \alpha > 0, \quad 0 < \beta < 1$$

$$\int i^{-\beta}\, di = \int -\alpha dt$$

$$\frac{i^{1-\beta}}{1-\beta} = -\alpha t + \kappa$$

$$i^{1-\beta} = -[\alpha(1-\beta)t] + \kappa,$$

when $t = 0$, $i = Q$, so $\kappa = Q^{1-\beta}$ and $i^{1-\beta} = -[\alpha(1-\beta)t] + Q^{1-\beta}$. Solving for i yields

$$i = \begin{cases} (Q^{1-\beta} - [\alpha(1-\beta)t])^{1/(1-\beta)} & \text{for} \quad t \leqslant \dfrac{Q^{1-\beta}}{\alpha(1-\beta)} \\ 0 & \text{otherwise.} \end{cases}$$

APPENDIX C

The average inventory level can be evaluated by integrating the inventory function over one cycle as follows:

$$\bar{i} = \frac{1}{T}\int_0^T i(t)dt$$

$$= \frac{1}{T}\int_0^T (Q^{1-\beta} - [\alpha(1-\beta)t])^{1/(1-\beta)}\, dt$$

$$= \frac{1}{T}\left|\frac{(Q^{1-\beta} - [\alpha(1-\beta)t])^{(1/1-\beta)+1}}{[(1/1-\beta)+1][-\alpha(1-\beta)]}\right|_0^T$$

$$= \frac{1}{T}\left[\frac{(Q^{1-\beta} - [\alpha(1-\beta)T])^{(2-\beta)/(1-\beta)} - Q^{2-\beta}}{-\alpha(2-\beta)}\right]$$

$$\bar{i} = \frac{1}{\alpha(2-\beta)T}[Q^{2-\beta} - (Q^{1-\beta} - [\alpha(1-\beta)T])^{(2-\beta)/(1-\beta)}].$$

REFERENCES

1. H. M. Wagner and T. M. Whitin (1958) Dynamic version of the economic lot size model. *Mgmt Sci.* **5,** 89–96.
2. E. A. Silver and H. C. Meal (1969) A simple modification of the EOQ for the case of a varying demand rate. *Prod. Invent. Mgmt* **10,** 52–65.
3. E. A. Silver and H. C. Meal (1973) A heuristic for selecting lot size quantities for the case of a deterministic time-varying demand rate and discrete opportunities for replenishment. *Prod. Invent. Mgmt* **14,** 64–74.
4. W. A. Donaldson (1977) Inventory replenishment policy for a linear trend in demand—an analytical solution. *Opl Res. Q.* **28,** 663–670.
5. T. M. Whitin (1955) Inventory control and price theory. *Mgmt Sci.* **2,** 61–68.
6. H. L. Lee and M. J. Rosenblatt (1986) The effects of varying marketing policies and conditions on the economic ordering quantity. *Int. J. Prod. Res.* **24,** 593–598.
7. M. Corstjens and P. Doyle (1981) A model for optimizing retail space allocations. *Mgmt Sci.* **27,** 822–833.
8. R. I. Levin, C. P. McLaughlin, R. P. Lamone and J. F. Kottas (1972) *Production/Operations Management: Contemporary Policy for Managing Operating Systems*, p. 373. McGraw–Hill, New York.
9. E. A. Silver and R. Peterson (1985) *Decision Systems for Inventory Management and Production Planning*, 2nd edn. Wiley, New York.
10. E. A. Silver (1981) Operations research in inventory management. *Opns Res.* **29,** 628–645.
11. H. A. Taha (1982) *Operations Research*, 3rd edn, p. 771. Macmillan, New York.