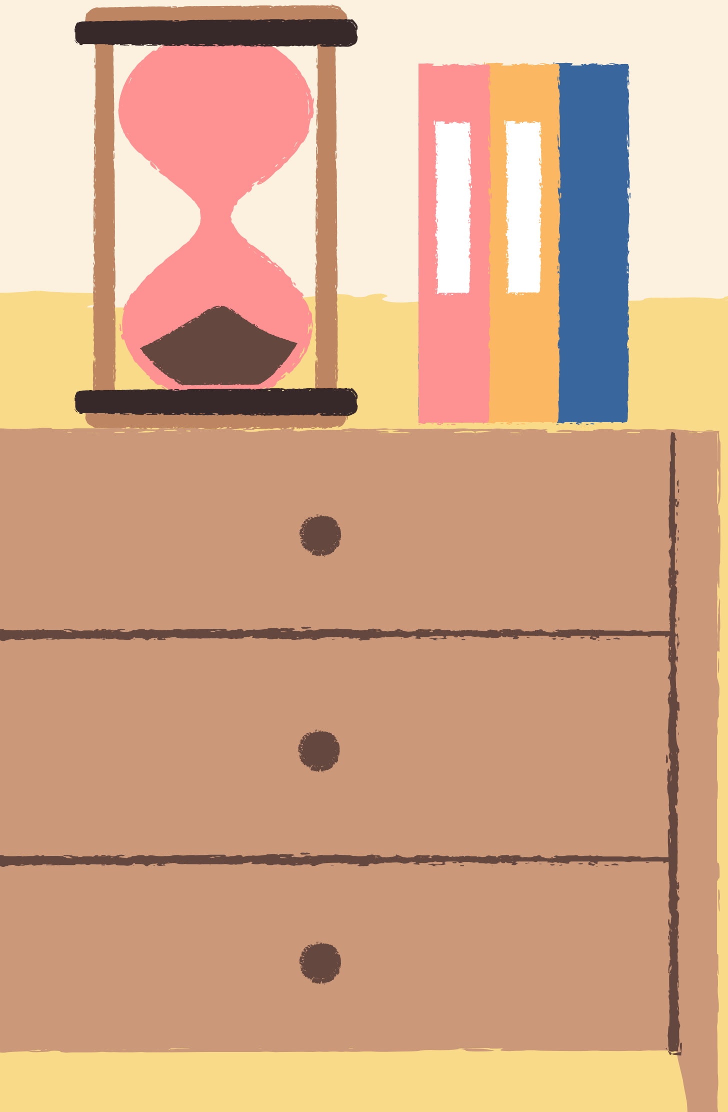
PPT 14

ALJABAR LINEAR

Lesson Outline

- INNER PRODUCT
- BASIS ORTHONORMAL
- GRAM SCHMIT
- KOORDINAT BASIS BARU




INNER
PRODUCT

INNER PRODUCT??

Hasil kali dalam pada ruang vektor nyata V adalah suatu fungsi $\langle \cdot,\cdot \rangle$: $V \times V \rightarrow R$ yang mengaitkan suatu bilangan real $\langle u,v \rangle$ pada setiap pasangan vektor u dan v dalam V dan memenuhi aksioma-aksioma berikut untuk semua vektor $u,v,w \in V$ dan semua skalar $k \in R$

1. $\langle u, v \rangle = \langle v, u \rangle$ [aksiom simetris]
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ [aksiom penjumlahan]
3. $\langle \boldsymbol{k}u, v \rangle = \boldsymbol{k}\langle u, v \rangle$ [aksiom homogenitas]
4. $\langle v, v \rangle \geq 0 \; dan \; \langle v, v \rangle = 0 \; jika \; dan \; hanya \; jika \; v = 0$ [aksiom positif]

$$\langle \bar{u}, \bar{v} \rangle = \bar{u} \cdot \bar{v}$$
$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

karena aksioma <u,v> memenuhi perkalian dot dari 2 vektor, maka:

<u,v> = u.v

= U1.V1+U2.V2+...+Un.Vn

Namun, dalam beberapa aplikasi, kita mungkin perlu memodifikasi hasil kali dalam ini dengan memberi BOBOT berbeda pada setiap komponen. Misalnya, mungkin ada kasus di mana beberapa elemen dari vektor memiliki pengaruh lebih besar atau lebih kecil dibandingkan yang lain.

Oleh karena itu, kita memperkenalkan bobot wl,w2,...,wn

misal $w_1\ w_2\ \ldots\ w_n$ → riil (+)
→ dinamakan bobot

mk:

$$\langle \bar{u}, \bar{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

↳ dinamakan → hasil kali dalam Euclidis yg dibobotti dgn bobot $w_1\ w_2 \ldots w_n$

contoh 4.7 : → lebih fleksibel dibanding $\bar{u} \cdot \bar{v}$
→ $\langle \bar{u}, \bar{v} \rangle$ → bisa $\bar{u}$ bisa matrix, pers. linier

$\bar{u} = (u_1, u_2)$
$\bar{v} = (v_1, v_2)$ } → vektor² pd R^2, buktikan ! bahwa hasil kali dalam Euclidis yg dibobot :

$$\langle \bar{u}, \bar{v} \rangle = 3 u_1 v_1 + 2 u_2 v_2$$

memenuhi 4 aksiom pd hal 175

→ polanya sama

jwb:

① ruas kiri ⇒ $\langle \bar{u}, \bar{v} \rangle = 3 u_1 v_1 + 2 u_2 v_2$
ruas kanan ⇒ $\langle \bar{v}, \bar{u} \rangle = 3 u_1 v_1 + 2 u_2 v_2$
⇒ ruas kiri = ruas kanan , mk aksioma 1 terpenuhi

② $\langle \bar{u} + \bar{v}, \bar{w} \rangle = \langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$

kiri ⇒ $\langle \bar{u} + \bar{v}, \bar{w} \rangle = 3 (u_1 + v_1) w_1 + 2 (u_2 + v_2)$

$= 3 u_1 w_1 + 3 v_1 w_1 + 2 u_2 w_2 + 2 v_2 w_2$
$= 3 u_1 w_1 + 2 u_2 w_2 + 3 v_1 w_1 + 2 v_2 w_2$

kanan ⇒ $\langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$
$= 3 u_1 w_1 + 2 u_2 w_2 + 3 v_1 w_1 + 2 v_2 w_2$

kiri = kanan ⇒ mk aksioma 2 terpenuhi

③ $\langle k\bar{u}, \bar{w} \rangle = k \langle \bar{u}, \bar{w} \rangle$

kiri ⇒ $\langle k\bar{u}, \bar{w} \rangle = 3 k \bar{u}_1 w_1 + 2 k u_2 w_2$

kanan ⇒ $k \langle \bar{u}, \bar{w} \rangle = k (3 u_1 w_1 + 2 u_2 w_2)$
$= 3 k u_1 w_1 + 2 k u_2 w_2$

kiri = kanan ⇒ mk aksioma 3 terpenuhi

→ kiri = kanan ⇝ 4.22 (b)

(4) $\langle \bar{v}, \bar{v} \rangle = 3 v_1 v_1 + 2 v_2 v_2$

$= 3 v_1^2 + 2 v_2^2$

jelas bahwa $3 v_1^2 + 2 v_2^2 \geq 0$

selanjutnya $\langle \bar{v}, \bar{v} \rangle = 3 v_1^2 + 2 v_2^2 = 0$ iff $v_1 = v_2 = 0$

contoh 49 : ⇝ lihat buku

$$\bar{u} = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

rumus berikut mendefinisikan hasil kali dalam pd M_{22}

$$\langle \bar{u}, \bar{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

$$\| \bar{u} \| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

misal :

$$\bar{u} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

mk:

$$\langle \bar{u}, \bar{v} \rangle = 1(-1) + 2(0) + 3(3) + 4(2)$$
$$= 16$$

contoh soal:

$$p = a_0 + a_1 x + a_2 x^2$$
$$q = b_0 + b_1 x + b_2 x^2$$

mk hasil kali dalam pd P_2 adalah :

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

$$\| p \| = \sqrt{a_0^2 + a_1^2 + a_2^2}$$

- $\|\bar{u}\| = \langle \bar{u}, \bar{u} \rangle^{1/2}$ → if ada bobot

  $= (\bar{u}, \bar{u})^{1/2}$

  $= \sqrt{\bar{u} \cdot \bar{u}}$

  $= \sqrt{u_1^2 + u_2^2}$ } → if ≠ bobot (biasa)

- jarak antar 2 vektor $\bar{u}$ & $\bar{v}$ ⇒ $d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$

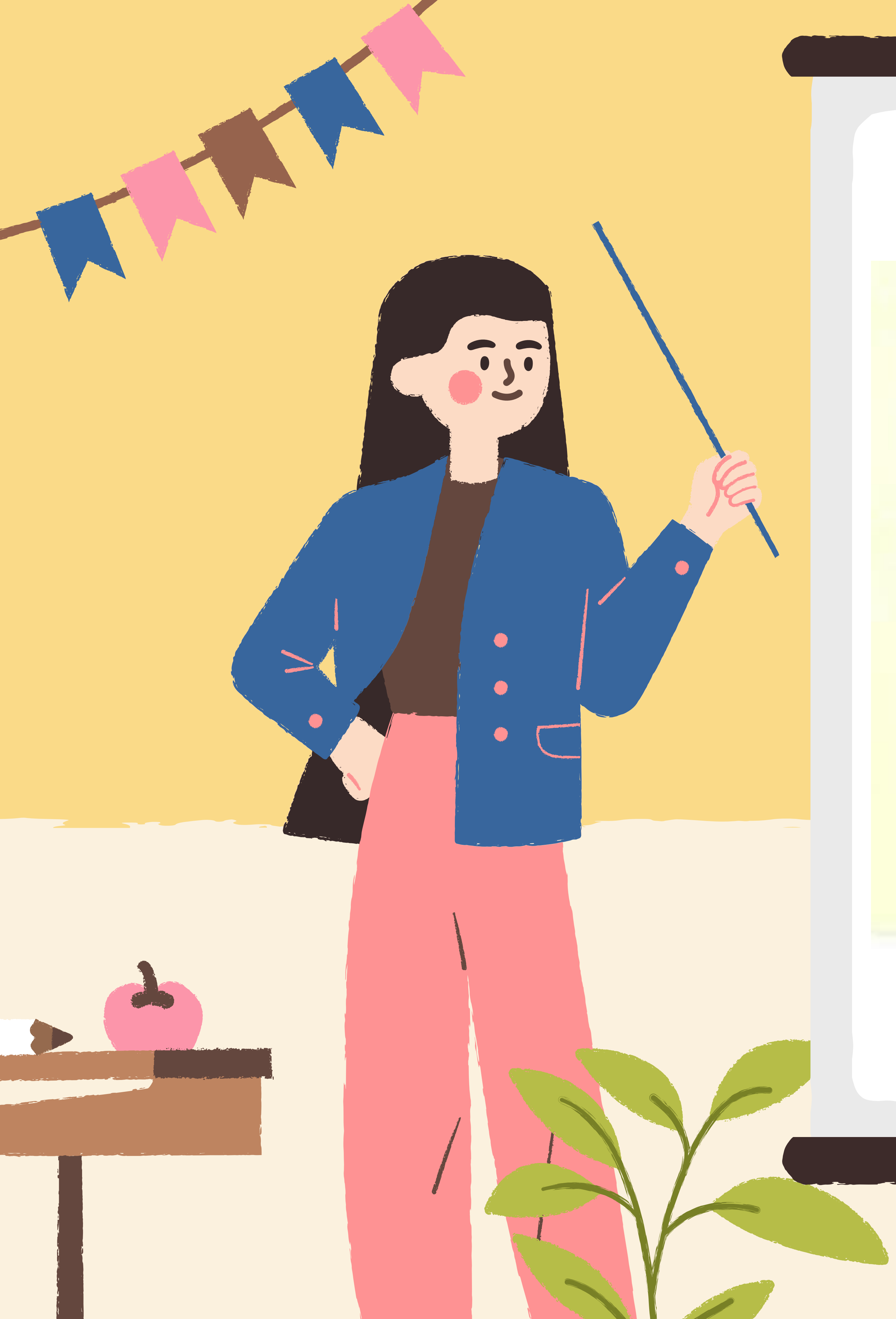
  ada bobot → $= \langle \bar{u} - \bar{v}, \bar{u} - \bar{v} \rangle^{1/2}$

  ≠ bobot → $= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$

- aturan ketaksamaan Cauchy - Schwarz

$$\langle \bar{u}, \bar{v} \rangle^2 \le \langle \bar{u}, \bar{u} \rangle . \langle \bar{v}, \bar{v} \rangle$$

sifat dasar panjang	sifat dasar jarak
$\|\bar{u}\| \geq 0$	$d(\bar{u}, \bar{v}) \geq 0$
$\|\bar{u}\| = 0$ if $\bar{u} = 0$	$d(\bar{u}, \bar{v}) = 0$ if $\bar{u} = \bar{v}$
$\|k\bar{u}\| = \|k\| \, \|\bar{u}\|$	$d(\bar{u}, \bar{v}) = d(\bar{v}, \bar{u})$
$\|\bar{u} + \bar{v}\| \le \|\bar{u}\| + \|\bar{v}\|$	$d(\bar{u}, \bar{v}) \le d(\bar{u}, \bar{w}) + d(\bar{w}, \bar{v})$

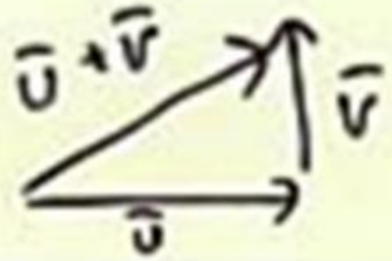


- Sudut antara $\bar{u}$ & $\bar{v}$

$$\cos\theta = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| . \|\bar{v}\|}$$

- $\bar{u}$ & $\bar{v}$ saling $\perp$ if $\Rightarrow \langle \bar{u}, \bar{v} \rangle = 0$
- if $\bar{u} \perp \bar{v}$, mk :

$$\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$$



Contoh 55 :

$$\bar{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

pola $\langle \bar{u}, \bar{v} \rangle = 3 u_1 v_1 + 2 u_2 v_2$

Contoh 55 :

$$\bar{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

pola $\quad \langle \bar{u}, \bar{v} \rangle = 3\, u_1 v_1 + 2\, u_2 v_2$

$$\begin{aligned} \|\bar{u}\| &= \langle \bar{u}, \bar{u} \rangle^{1/2} \\ &= (3 . u_1 u_1 + 2 . u_2 u_2)^{1/2} \\ &= (3 . 1 . 1 + 2 . 0 . 0)^{1/2} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} d(\bar{u}, \bar{v}) &= \| \bar{u} - \bar{v} \| \\ &= \langle \overset{\bar{u}}{\bar{u} - \bar{v}}, \overset{\bar{u}}{\bar{u} - \bar{v}} \rangle^{1/2} \\ &= (3 . u_1 u_1 + 2 . u_2 u_2)^{1/2} \\ &= (3 . 1 . 1 + 2 . -1 . -1)^{1/2} \\ &= (3 + 2)^{1/2} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} u_1 &= u_1 - v_1 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} u_2 &= u_2 - v_2 \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

Example 2 Let R^4 have the Euclidean inner product. Find the cosine of the angle θ between the vectors $\mathbf{u} = (4, 3, 1, -2)$ and $\mathbf{v} = (-2, 1, 2, 3)$.

Solution. We leave it for the reader to verify that

$$\|\mathbf{u}\| = \sqrt{30}, \qquad \|\mathbf{v}\| = \sqrt{18}, \qquad \text{and} \qquad \langle \mathbf{u}, \mathbf{v} \rangle = -9$$

so that

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\frac{9}{\sqrt{30}\sqrt{18}} = -\frac{3}{2\sqrt{15}}$$

INNER PRODUCT dikatakan ORTHOGONAL, jika:

<u,v> = 0

Example 3 If M_{22} has the inner product of Example 7 in the preceding section, then the matrices

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are orthogonal, since

$$\langle U, V \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$$

Definition. Let W be a subspace of an inner product space V. A vector $\mathbf{u}$ in V is said to be *orthogonal to W* if it is orthogonal to every vector in W, and the set of all vectors in V that are orthogonal to W is called the *orthogonal complement of W*.

Theorem 6.2.6. *If A is an $m \times n$ matrix, then:*

(a) *The nullspace of A and the row space of A are orthogonal complements in R^n with respect to the Euclidean inner product.*

(b) *The nullspace of A^T and the column space of A are orthogonal complements in R^m with respect to the Euclidean inner product.*

Example 6 Let W be the subspace of R^5 spanned by the vectors

$$\mathbf{w}_1 = (2, 2, -1, 0, 1), \qquad \mathbf{w}_2 = (-1, -1, 2, -3, 1),$$
$$\mathbf{w}_3 = (1, 1, -2, 0, -1), \qquad \mathbf{w}_4 = (0, 0, 1, 1, 1)$$

Find a basis for the orthogonal complement of W.

Example 6 Let W be the subspace of R^5 spanned by the vectors

$$\mathbf{w}_1 = (2, 2, -1, 0, 1), \quad \mathbf{w}_2 = (-1, -1, 2, -3, 1),$$
$$\mathbf{w}_3 = (1, 1, -2, 0, -1), \quad \mathbf{w}_4 = (0, 0, 1, 1, 1)$$

Find a basis for the orthogonal complement of W.

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

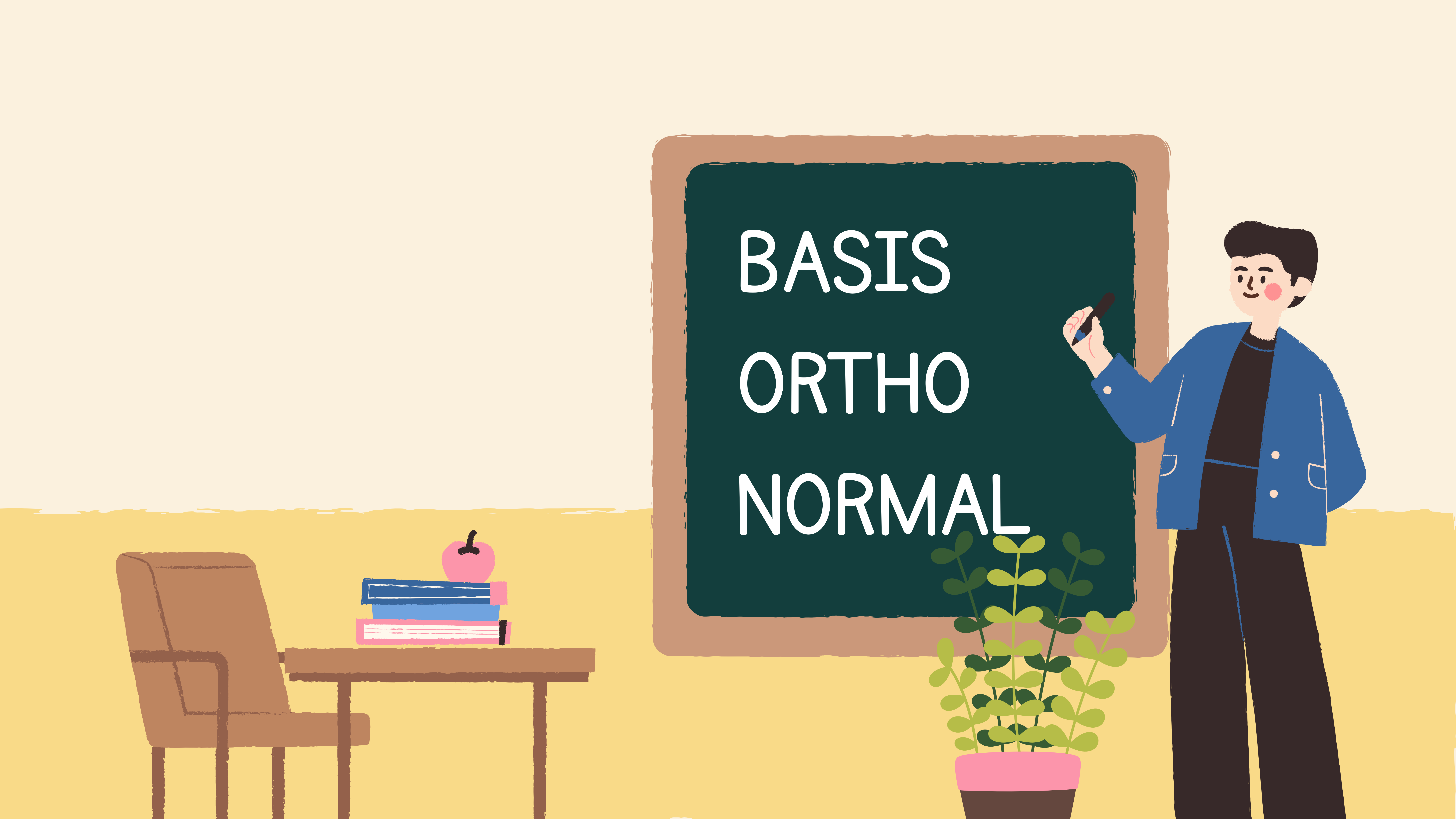
and by part (*a*) of Theorem 6.2.6 the nullspace of A is the orthogonal complement of W. In Example 4 of Section 5.5 we showed that

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for this nullspace. Expressing these vectors in the same notation as $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$, we conclude that the vectors

$$\mathbf{v}_1 = (-1, 1, 0, 0, 0) \quad \text{and} \quad \mathbf{v}_2 = (-1, 0, -1, 0, 1)$$

form a basis for the orthogonal complement of W. As a check, the reader may want to verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal to $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ by calculating the necessary dot products.


BASIS
ORTHO
NORMAL

BASIS ORTHONORMAL

- Cara menentukan basis
  - Cari basis sendiri ➔ basis baru
  - Dari vektor-vektor yang ada ➔ basis dari vektor-vektor lama
- Ortonormal
  - Ortogonal
  - Tiap vektor normanya(panjangnya) 1
- Normalisasi ➔ proses membagi V dengan panjangnya ||V|| agar normanya 1

Example 1 Let

$$\mathbf{u}_1 = (0, 1, 0), \qquad \mathbf{u}_2 = (1, 0, 1), \qquad \mathbf{u}_3 = (1, 0, -1)$$

and assume that R^3 has the Euclidean inner product. It follows that the set of vector $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$.

Example 1 Let

$$\mathbf{u}_1 = (0, 1, 0), \qquad \mathbf{u}_2 = (1, 0, 1), \qquad \mathbf{u}_3 = (1, 0, -1)$$

and assume that R^3 has the Euclidean inner product. It follows that the set of vector $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$.

If $\mathbf{v}$ is a nonzero vector in an inner product space, then by part (*c*) of Theorem 6.2.2 the vector

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has norm 1, since

$$\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$$

The process of multiplying a nonzero vector $\mathbf{v}$ by the reciprocal of its length to obtain a vector of norm 1 is called ***normalizing*** $\mathbf{v}$. An orthogonal set of *nonzero* vectors can always be converted to an orthonormal set by normalizing each of its vectors.

$$V_1 = \frac{u_1}{|u_1|} = \frac{(0,1,0)}{1} = (0,1,0)$$

$$V_2 = \frac{u_2}{|u_2|} = \frac{(1,0,1)}{\sqrt{2}} = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right)$$

$$V_3 = \frac{u_3}{|u_3|} = \frac{(1,0,-1)}{\sqrt{2}} = \left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right)$$

ortonormal

$\Uparrow$

$V_1 \quad V_2 \quad V_3$

$\rightarrow \perp$

- $\langle V_1, V_2 \rangle = \emptyset$
- $\langle V_1, V_3 \rangle = \emptyset$
- $\langle V_2, V_3 \rangle = \emptyset$

normal = 1

- $V_1 = 1$
- $V_2 = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right.}$

  $= 1$
- $V_3 = 1$

Example 2 The Euclidean norms of the vectors in Example 1 are

$$\|\mathbf{u}_1\| = 1, \qquad \|\mathbf{u}_2\| = \sqrt{2}, \qquad \|\mathbf{u}_3\| = \sqrt{2}$$

Consequently, normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = (0, 1, 0), \qquad \mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

We leave it for you to verify that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthonormal by showing

That <v1,v2> = <v1,v3> = <v2,v3> = 0

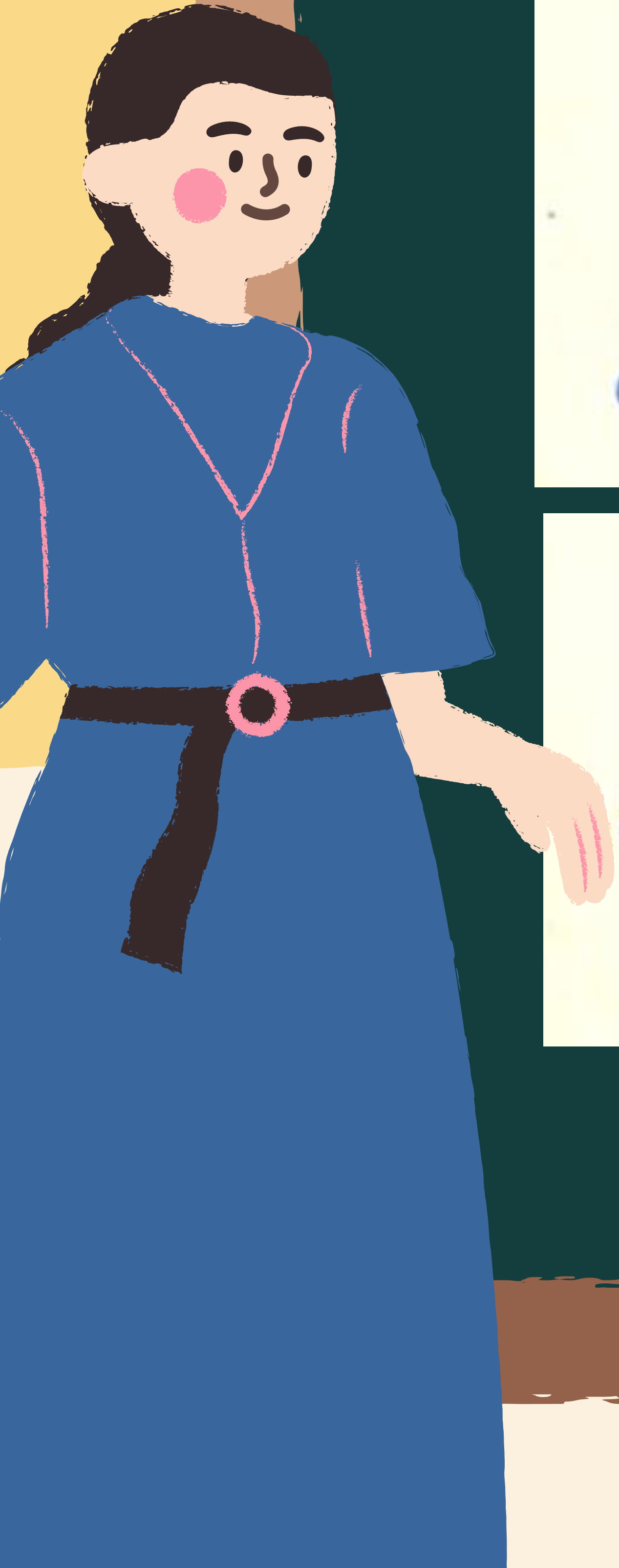
||v1|| = ||v2|| = ||v3|| = 1

Keuntungan Orthonormal

Theorem 6.3.1. *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and $\mathbf{u}$ is any vector in V, then*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Kita bisa langsung mencari k1, k2, k3, … , kn untuk kombinasi linier



teorema 23:

$\rightarrow S = \{\bar{v}_1, \bar{v}_2 \dots \bar{v}_n\}$ ⇝ basis ortonormal

nyatakan bahwa vektor $\bar{u}$ adalah kombinasi linier dr vektor² S, mk:

$$\Rightarrow \bar{u} = k_1 \bar{v}_1 + k_2 \bar{v}_2 + \dots + k_n \bar{v}_n$$

$$\langle \bar{u}, \bar{v}_1 \rangle \qquad \langle \bar{u}, \bar{v}_2 \rangle \qquad \langle \bar{u}, \bar{v}_n \rangle$$

$k_1, k_2 \dots k_n$ bisa dicari spt ini, krn $v_1\ v_2 \dots v_n$ → saling $\perp$ → norma = 1

contoh 62:

$\bar{v}_1 = (0, 1, 0) \qquad \bar{v}_2 = (-4/5, 0, 3/5) \qquad \bar{v}_3 = (3/5, 0, 4/5)$

$S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ ⇝ basis ortonormal → ortogonal → norma = 1

nyatakan $\bar{u} = (1, 1, 1)$ sbg kombinasi linier dr himp. S

$\overline{v_1} = (0, 1, 0)$ $\quad$ $\overline{v_2} = (-4/5, 0, 3/5)$ $\quad$ $\overline{v_3} = (3/5, 0, 4/5)$

$S = \{ \overline{v_1}, \overline{v_2}, \overline{v_3} \}$ → basis ortonormal → ortogonal

→ norma = 1

nyatakan $\overline{u} = (1, 1, 1)$ sbg kombinasi linier dr himp. S

$$\overline{u} = k_1 \overline{v_1} + k_2 \overline{v_2} + k_3 \overline{v_3}$$

$k_1 = \langle \overline{u}, \overline{v_1} \rangle = 1$; $k_2 = \langle \overline{u}, \overline{v_2} \rangle = -1/5$; $k_3 = \langle \overline{u}, \overline{v_3} \rangle = 7/5$

<u>mk:</u> $\overline{u} = \overline{v_1} - \frac{1}{5} \overline{v_2} + \frac{7}{5} \overline{v_3}$

•) basis ortonormal
- → merentang
- → bebas linier
- → saling $\perp$
- → norma = 1

contoh 63 :

$\bar{v}_1 = (0, 1, 0)$ $\quad \bar{v}_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})$ $\quad \bar{v}_3 = (1/\sqrt{2}, 0, -1/\sqrt{2})$

- → merentang → $\bar{u} = k_1 \bar{v}_1 + k_2 \bar{v}_2 + k_3 \bar{v}_3$
- → bebas linier → $0 = k_1 \bar{v}_1 + k_2 \bar{v}_2 + k_3 \bar{v}_3$

jwb:

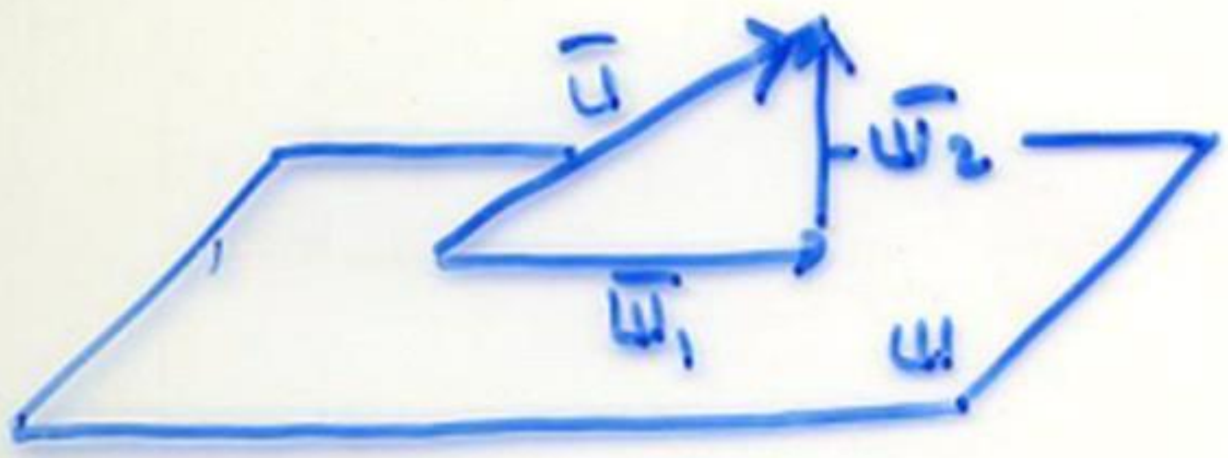
- → merentang
  - → mis : $\bar{u} = (1,2,3)$, mk
    - → $k_1 = \langle \bar{u}, \bar{v}_1 \rangle = 2$
    - → $k_2 = \langle \bar{u}, \bar{v}_2 \rangle = 4/\sqrt{2}$
    - → $k_3 = \langle \bar{u}, \bar{v}_3 \rangle = -2/\sqrt{2}$

    mk ⇒ $(1,2,3) = 2\bar{v}_1 + 4/\sqrt{2}\, \bar{v}_2 - 2/\sqrt{2}\, \bar{v}_3$
  - → mis : $\bar{u} = (2,2,2)$, mk
    - → $k_1 = 2$
    - → $k_2 = 4/\sqrt{2}$
    - → $k_3 = 0$

    mk ⇒ $(2,2,2) = 2\bar{v}_1 + 4/\sqrt{2}\, \bar{v}_2$
- → bebas linier → $0 = (0,0,0)$; mk
  - → $k_1 = 0$
  - → $k_2 = 0$
  - → $k_3 = 0$

  mk ⇒ $(0,0,0) = 0\bar{v}_1 + 0\bar{v}_2 + 0\bar{v}_3$

• proyeksi



$\bar{u} = \bar{w}_1 + \bar{w}_2$

$\bar{w}_1$ → proyeksi ortogonal $\bar{u}$ pd W

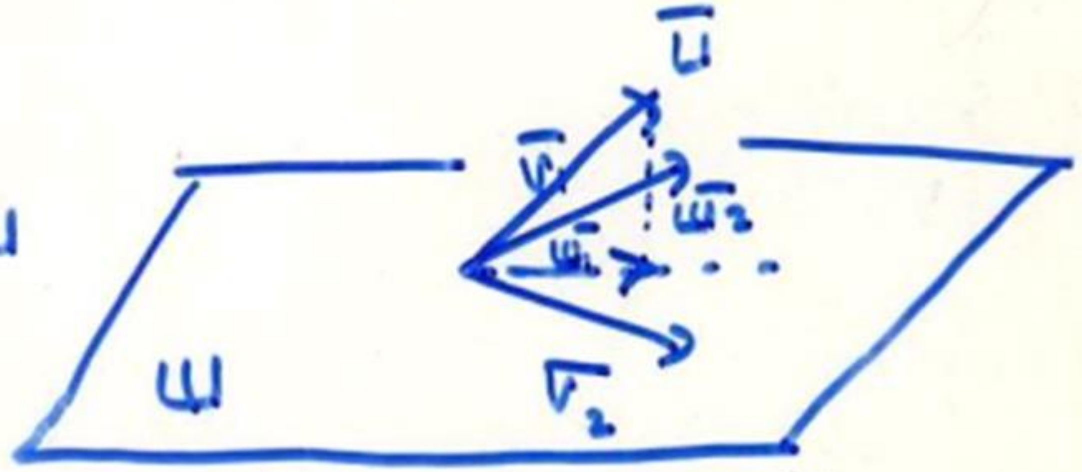
→ $\text{Proy}_W \, \bar{u} = \langle \bar{u}, \bar{v}_1 \rangle \bar{v}_1 + \langle \bar{u}, \bar{v}_2 \rangle \bar{v}_2 + \dots + \langle \bar{u}, \bar{v}_n \rangle \bar{v}_n$

$\bar{w}_2$ → komp. $\bar{u}$ yg ortogonal thp W

→ $\bar{u} - \text{Proy}_W \, \bar{u} = \bar{u} - \langle \bar{u}, \bar{v}_1 \rangle \bar{v}_1 - \langle \bar{u}, \bar{v}_2 \rangle \bar{v}_2 + \dots - \langle \bar{u}, \bar{v}_n \rangle \bar{v}_n$

contoh 69 :

$$\left.\begin{aligned} \bar{v}_1 &= \langle 0, 1, 0) \\ \bar{v}_2 &= (-4/5, 0, 3/5) \end{aligned}\right\} \text{pd } W$$



? Proyeksi ortogonal $\bar{u} = (1,1,1)$ pd W adalah :

$$\bar{w}_1 = \text{Proy}_W\, \bar{u} = \langle \bar{u}, \bar{v}_1 \rangle \bar{v}_1 + \langle \bar{u}, \bar{v}_2 \rangle \bar{v}_2$$

$$= \bar{v}_1 + -\tfrac{1}{5}\, \bar{v}_2$$

$$= \langle 0, 1, 0) + (4/25, 0, -3/25)$$

$$= (4/25, 1, -3/25)$$

4.27

(?) komp $\bar{u}$ yg ortogonal thp W adalah :

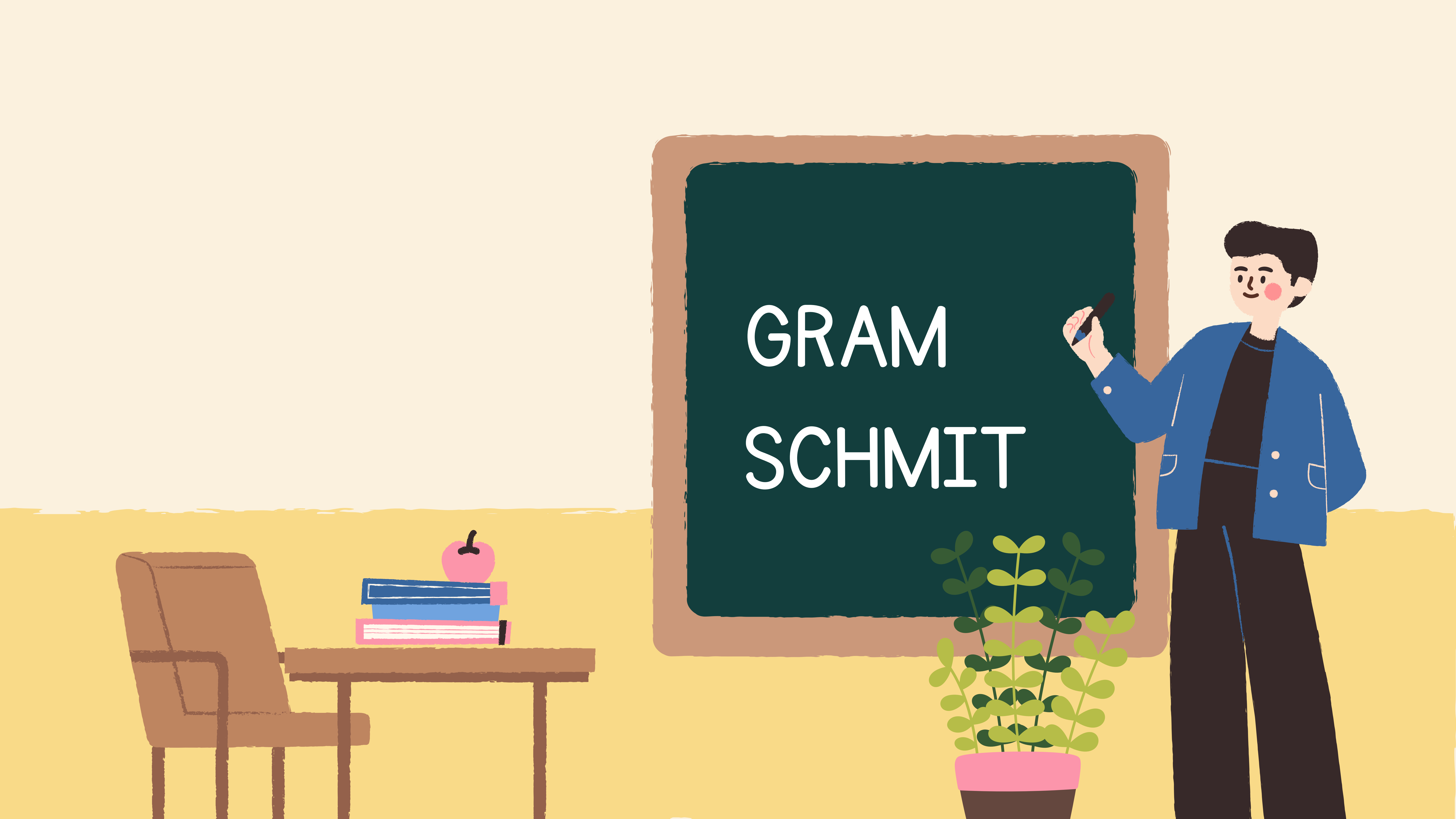
$$\bar{w}_2 = \bar{u} - \text{proy}_W\, \bar{u} = (1, 1, 1) - (4/25, 1, -3/25)$$
$$= (21/25, 0, 28/25)$$

$\bar{w}_2$ → ortogonal thp $\bar{v}_1$ & $\bar{v}_2$
→ $\perp$ thp setiap vektor pd W yg direntang o $\bar{v}_1$ & $\bar{v}_2$

Teorema 26 :

→ setiap ruang hasil kali dalam berdimensi berhingga taknol memp. sebuah basis ortonormal.

→ $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3 \ldots \bar{v}_n\}$ → basis biasa → diket
$R = \{\bar{v}_1, \bar{v}_2, \bar{v}_3 \ldots \bar{v}_n\}$ → basis ortonormal
→ yg dicari :


GRAM
SCHMIT

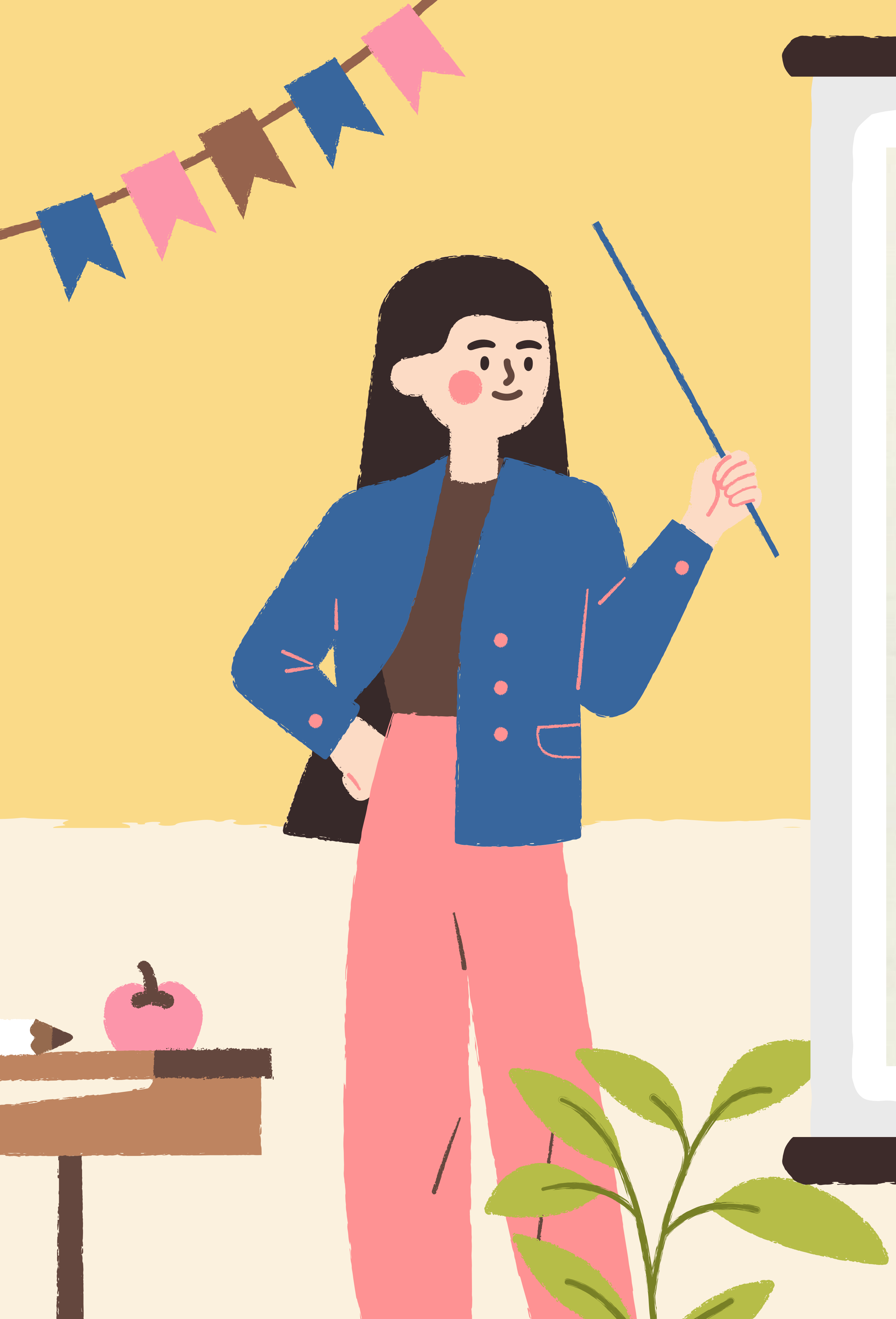
GRAM-SCHMIT

contoh 65 :

terapkan proses Gram Schmit u mentransfor-masikan basis

$$\left.\begin{aligned} \bar{v}_1 &= (1,1,1) \\ \bar{v}_2 &= (0,1,1) \\ \bar{v}_3 &= (0,0,1) \end{aligned}\right\} \text{ ke dlm basis ortonormal}$$

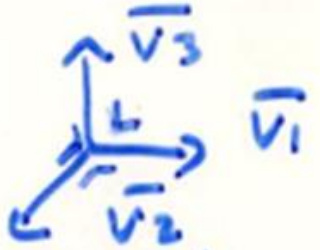
Metode ini digunakan untuk : Mencari basis orthonormal



contoh 65:

terapkan proses Gram Schmit u mentransformasikan basis

$$\left.\begin{aligned}\bar{u}_1 &= (1, 1, 1)\\ \bar{u}_2 &= (0, 1, 1)\\ \bar{u}_3 &= (0, 0, 1)\end{aligned}\right\}$$ ke dlm basis ortonormal



jwb:

$S = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\} \rightarrow$ basis biasa

$R = \{\bar{v}_1\ \bar{v}_2\ \bar{v}_3\} \rightarrow$ basis ortonormal yg akan dicari

langkah 1 $\rightarrow \bar{v}_1 = \dfrac{\bar{u}_1}{\|\bar{u}_1\|} = \dfrac{(1, 1, 1)}{\sqrt{3}} = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

langkah 2 $\rightarrow \bar{u}_2 - \text{Proy}_{w_1} \bar{u}_2 = \bar{u}_2 - \langle \bar{u}_2, \bar{v}_1 \rangle \bar{v}_1$

$= (0, 1, 1) - 2/\sqrt{3}\,(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$= (-2/3, 1/3, 1/3)$

mk $\rightarrow \bar{v}_2 = \dfrac{\bar{u}_2 - \text{Proy}_{w_1} \bar{u}_2}{\|\bar{u}_2 - \text{Proy}_{w_1} \bar{u}_2\|} = 3/\sqrt{6}\,(-2/3, 1/3, 1/3)$

$= (-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6})$

langkah 3 $\longrightarrow \bar{u}_3 - \text{proy}_{W_2}\bar{u}_3 = \bar{u}_3 - \langle \bar{u}_3, \bar{v}_1 \rangle \bar{v}_1 - \langle \bar{u}_3, \bar{v}_2 \rangle \bar{v}_2$

$$= (0, 0, 1) - 1/\sqrt{3}\,(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) - 1/\sqrt{6}\,(-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6})$$

$$= (0, -1/2, 1/2)$$

mk:

$$\bar{v}_3 = \frac{\bar{u}_3 - \text{proy}_{W_2}\bar{u}_3}{\|\bar{u}_3 - \text{proy}_{W_2}\bar{u}_3\|} = \sqrt{2}\,(0, -1/2, 1/2) = (0, -1/\sqrt{2}, 1/\sqrt{2})$$

$\therefore\ \bar{v}_1, \bar{v}_2, \bar{v}_3$ membentuk basis ortonormal u R^3

$$V_1 = \frac{u_1}{|u_1|} = \frac{(1, 1, 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$V_2 = \frac{u_2 - \langle u_2, V_1 \rangle V_1}{\left| u_2 - \langle u_2, V_1 \rangle . V_1 \right|}$$

$$u_2 - \langle u_2, V_1 \rangle . V_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$V_2 = \frac{(-2/3,\ 1/3,\ 1/3)}{\frac{\sqrt{6}}{3}} = \frac{3}{\sqrt{6}} (-2/3,\ 1/3,\ 1/3)$$

$$= (-2/\sqrt{6},\ 1/\sqrt{6},\ 1/\sqrt{6})$$

$$V_3 = \frac{u_3 - \langle u_3, V_1 \rangle V_1 - \langle u_3, V_2 \rangle V_2}{|\ldots\ldots\ldots|}$$

$$V_3 = \frac{(0,0,1) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}{|\ldots\ldots\ldots|}$$

$$V_3 = \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{|\ldots|}$$

$$V_3 = \sqrt{2}\left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$V_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

CONTOH SOAL

Terapkan proses Gram Schimt untuk mentransformasikan basis u1, u2, u3 ke dalam basis ortonormal.

u1	=	-4	8	2
u2	=	7	-3	6
u3	=	6	-3	7

CONTOH SOAL

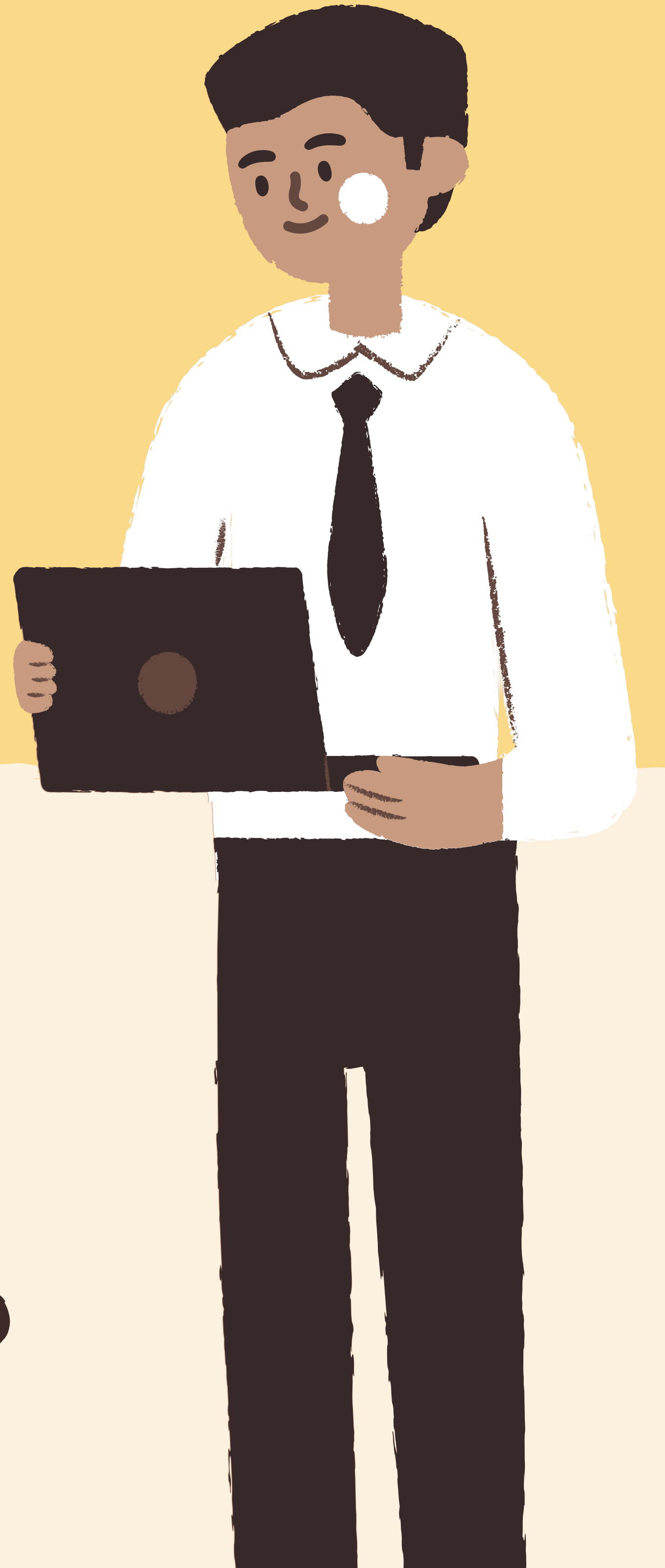
Terapkan proses Gram Schimt untuk mentransformasikan basis u1, u2, u3 ke dalam basis ortonormal.

u1	=	-4	8	2
u2	=	7	-3	6
u3	=	6	-3	7

v1 =	$u1 / \lvert u1 \rvert$			
	$\lvert u1 \rvert$	=	9.17	
v1 =	**-0.44**	**0.87**	**0.22**	

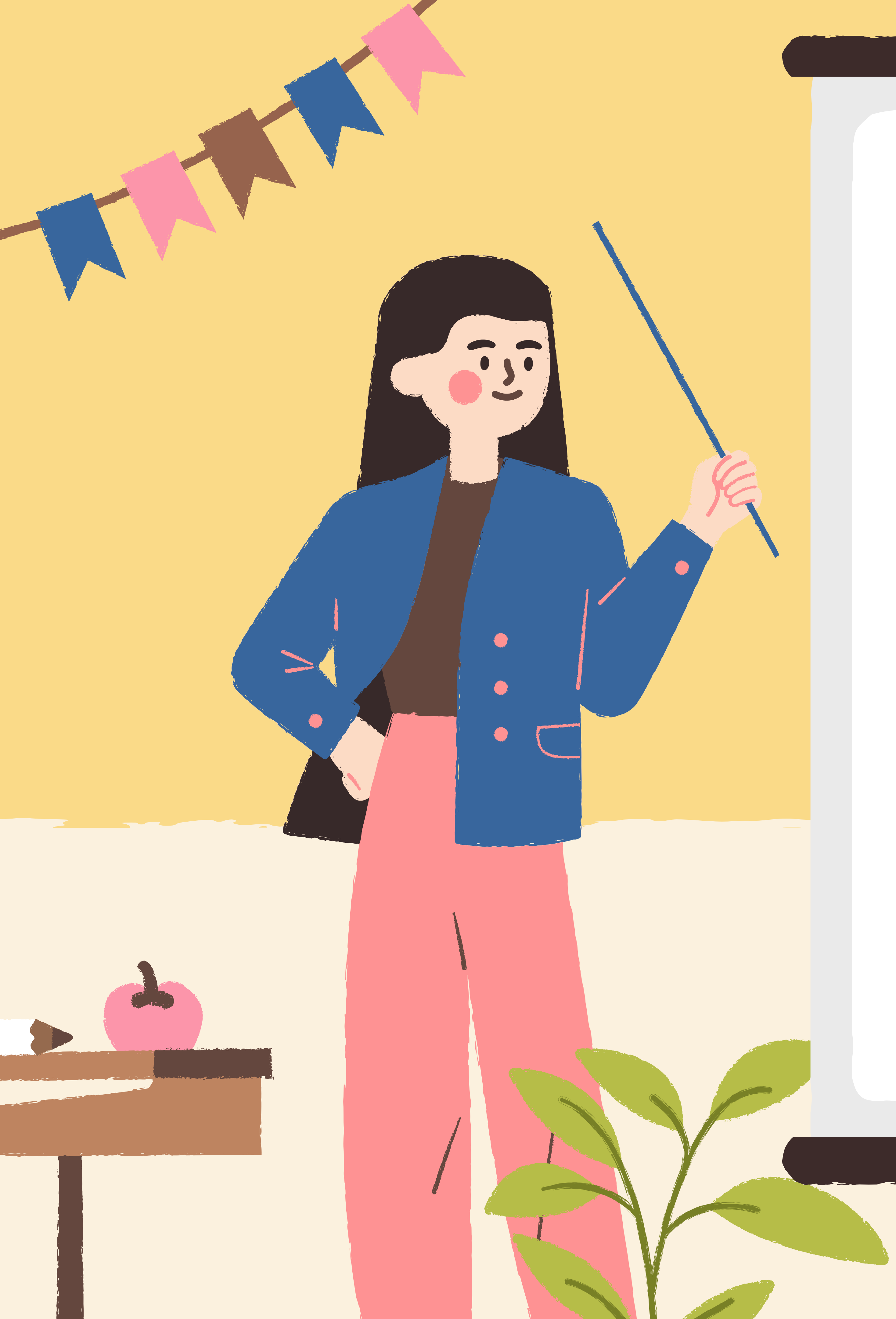
<table>
<tr><td>v2 =</td><td colspan="2">u2 - <u2,v1>.v1</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="2">|u2 - <u2,v1>.v1|</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td><u2,v1> =</td><td>-4.37</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="2">u2 - <u2,v1>.v1</td><td>=</td><td>5.08</td><td>0.80</td><td>6.96</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="2">|u2 - <u2,v1>.v1|</td><td>=</td><td>8.65</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>V2 =</td><td>0.59</td><td>0.09</td><td>0.80</td><td></td><td>|v2| =</td><td>0.998098</td></tr>
</table>

CONTOH SOAL



CONTOH SOAL

v3 =	u3 - <u3,v2> v2 - <u3,v1> v1						
	|u3 - <u3,v2> v2 - <u3,v1> v1|						
	<u3,v2> =	8.87					
	<u3,v2> V2 =		**5.23**	**0.80**	**7.10**		
	<u3,v1> =	-3.71					
	<u3,v1>V1 =		**1.63**	**-3.23**	**-0.82**		
	u3 - <u3,v2> v2 - <u3,v1> v1 =				-0.86	-0.57	0.72
	|u3 - <u3,v2> v2 - <u3,v1> v1|				**1.26**		
v3	**-0.68**	**-0.45**	**0.57**		|v3| =	0.994887	



6.5 Ortogonal matrix ; change of Basis

Ortogonal matrix :

$$A \cdot A^T = A^T \cdot A = I$$

$$A^{-1} = A^T$$

Example 1 The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal, since

$$A^TA = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

TEOREMA

Theorem 6.5.1. *The following are equivalent for an $n \times n$ matrix A.*

(a) *A is orthogonal.*

(b) *The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.*

(c) *The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.*

Contoh :

Show that

$$A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

is an orthogonal matrix by

(a) calculating $A^T A$

(b) using part (b) of theorem 6.5.1

(c) using part (c) of theorem 6.5.1

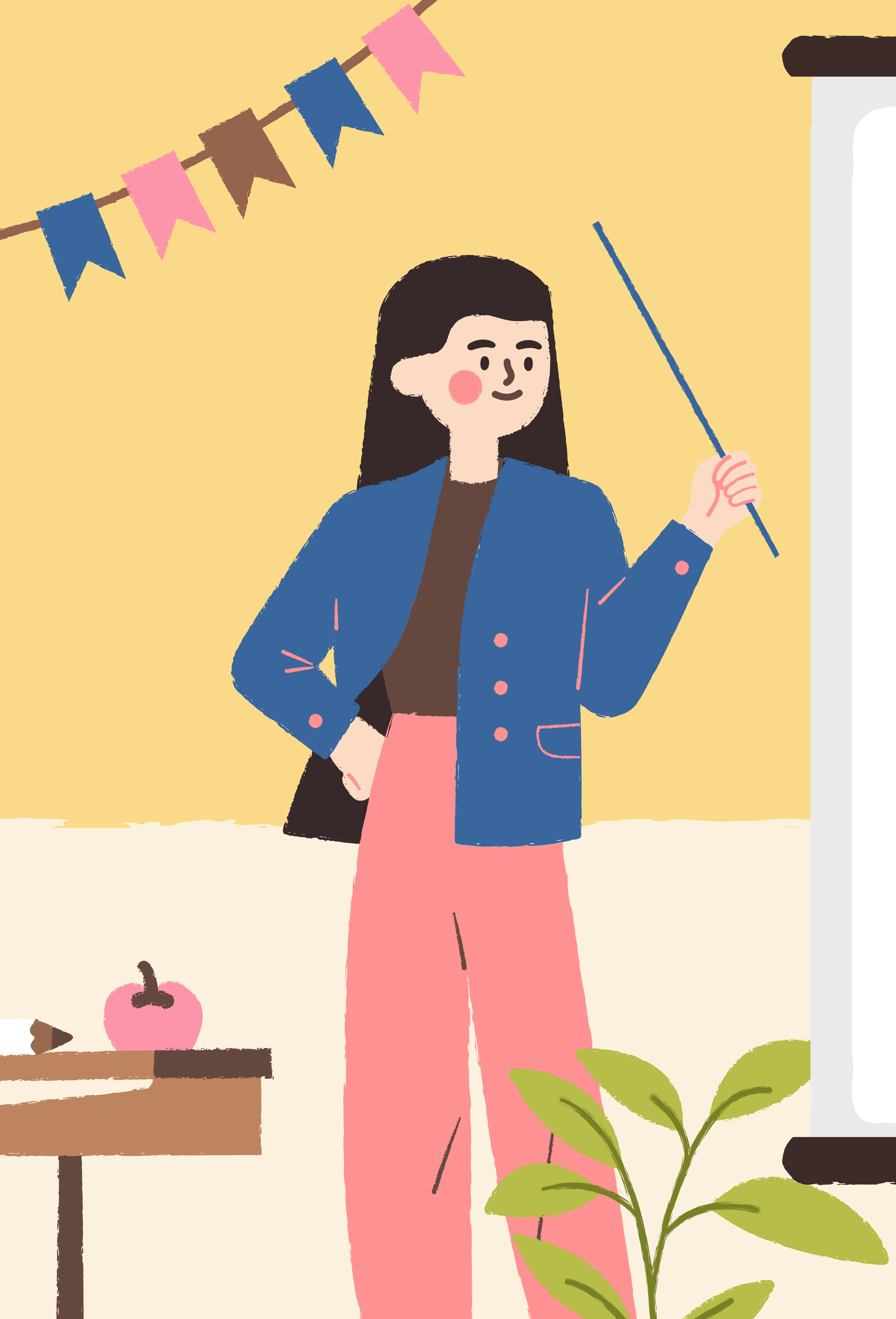
CONTOH SOAL

jawab :

(a) $A^T \cdot A$

$$= \begin{bmatrix} 4/5 & -9/25 & 12/25 \\ 0 & 4/5 & 3/5 \\ -3/5 & -12/25 & 16/25 \end{bmatrix} \cdot \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



(b) row vektor membentuk orthonormal

$r_1 = \{ 4/5 , 0, -3/5)$

$r_2 = (-9/5 , 4/5 , -12/25)$

$r_3 = (12/25 , 3/5 , 16/25)$

(c) colom vektor membentuk orthonormal

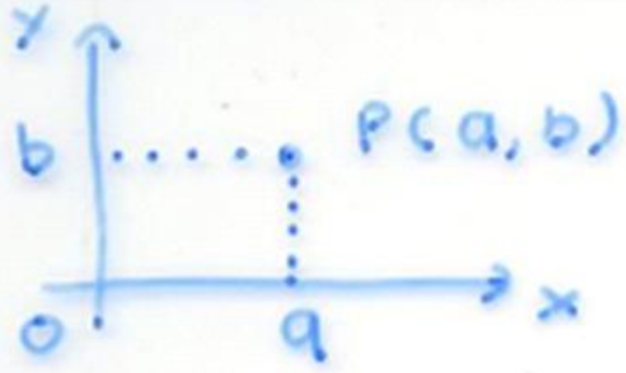
$$C_1 = \begin{bmatrix} 4/5 \\ -9/25 \\ 12/25 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 \\ 4/5 \\ 3/5 \end{bmatrix} \qquad C_3 = \begin{bmatrix} -3/5 \\ -12/25 \\ 16/25 \end{bmatrix}$$

KOORDINAT BASIS BARU

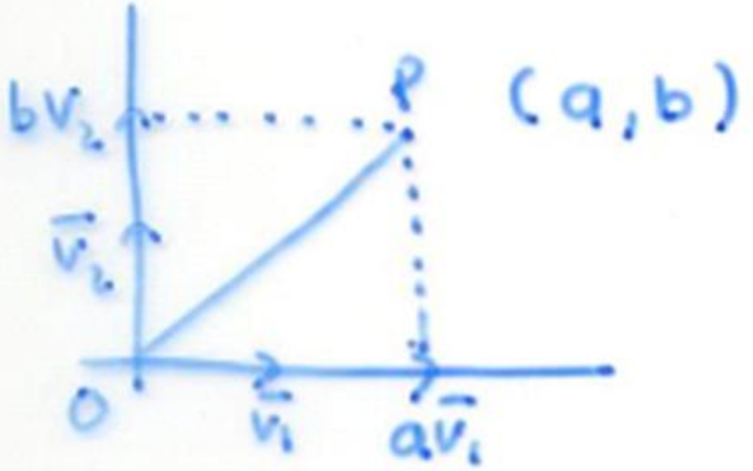
4.10 Koordinat, perubahan basis

→ basis berhubungan erat dgn koordinat

→ koordinat biasa → P(a,b) dlm koordinat basis (x,y)

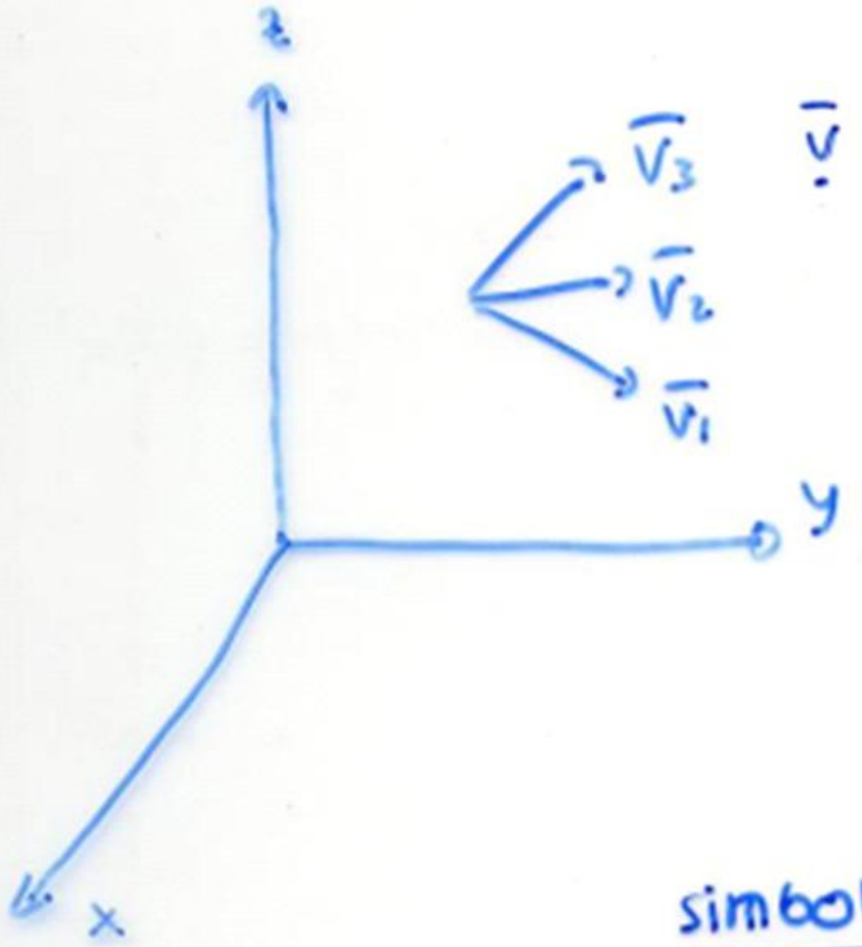


→ koordinat vektor → P(a,b) dlm koordinat vektor basis $S = \{\bar{v}_1, \bar{v}_2\}$



$\overline{OP} = a\bar{v}_1 + b\bar{v}_2$

titik (a,b) ⇒ koordinat P yg relatif thp basis $S = \{\bar{v}_1, \bar{v}_2\}$



(?) koordinat $\bar{v}$ thp basis baru $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ jk koordinat $\bar{v}$ thp basis lama $Q = \{x\ y\ z\}$ diket

(?) koordinat $\bar{v}$ thp basis lama $Q = \{x\ y\ z\}$ jk koordinat $\bar{v}$ thp basis baru $S = \{\bar{v}_1, \bar{v}_2, v_3\}$ diket

simbol :

$[B]_S$ = matrix transisi dr basis baru (S) ke basis lama (Q)
$= [\bar{v}_1\ \ \bar{v}_2\ \ \bar{v}_3]$

$[B]_Q$ = matrix transisi dr basis lama (Q) ke basis baru (S)
$= ([B]_S)^{-1}$

$[\bar{v}]_S$ = koordinat $\bar{v}$ thp basis baru $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

$[\bar{v}]_Q$ = koordinat $\bar{v}$ thp basis lama $Q = \{x\ y\ z\}$

umos:

$$\boxed{[B]_S\,[\vec{v}]_S = [\vec{v}]_Q}$$

$$[\vec{v}]_S = [B]_Q \cdot [\vec{v}]_Q$$

$$= ([B]_S)^{-1} \cdot [\vec{v}]_Q$$

contoh :

basis baru $S = \{ \bar{v}_1 \; \bar{v}_2 \; \bar{v}_3 \}$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$[B]_S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$

Pertanyaan :

a) Cari koordinat $[\bar{v}]_Q = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$ thp basis baru $S = \{ \bar{v}_1 \; \bar{v}_2 \; \bar{v}_3$

$\Rightarrow$ cari $[\bar{v}]_S$

b) Cari koordinat $[\bar{v}]_S = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ thp basis lama $Q = \{ x \; y \; z \}$

$\Rightarrow$ cari $[\bar{v}]_Q$

วิธี:

a)

$$[B]_S\,[\bar{v}]_S = [\bar{v}]_Q$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_S = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_S = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore\ [\bar{v}]_S = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

b) $[P]_S \, [\bar{v}]_S = [\bar{v}]_Q$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 31 \\ 7 \end{bmatrix}$$

$$\therefore \; [\bar{v}]_Q = \begin{bmatrix} 11 \\ 31 \\ 7 \end{bmatrix}$$

<u>contoh:</u>

basis baru $\quad S = \{ \bar{v}_1 , \bar{v}_2 \}$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$[P]_S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Pertanyaan :

a) Cari koordinat $[\vec{v}]_Q = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ terhadap basis baru $S = \{ \vec{v}_1 \; \vec{v}_2 \}$

$\Rightarrow$ cari $[\vec{v}]_S$

Jwb :

$$[B]_S \, [\vec{v}]_S = [\vec{v}]_Q$$

$$[\vec{v}]_S = \left([B]_S \right)^{-1} . [\vec{v}]_Q$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\therefore [\vec{v}]_S = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

b) Cari koordinat $[\bar{v}]_S = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ thp basis lama

$\Longrightarrow$ cari $[\bar{v}]_Q$

PR <u>4.10</u>
2.b
8
10

<u>jwb:</u>

$$[B]_S \, [\bar{v}]_S = [\bar{v}]_Q$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\therefore \; [\bar{v}]_Q = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

TUGAS!!!!

- Tugas Kelompok ➔
  - cari 2 soal dan jawaban di internet yang berhubungan dengan materi ppt ini
  - Tulis alamat internetnya
  - Di kirim ke elearning, terakhir ➔
    - Minggu depan
- Format ➔ subject ➔
  - Alin-B-melati
  - Bentuk ➔ ppt ➔ informasi nama kelompok + anggota

TERIMA KASIH!!!