

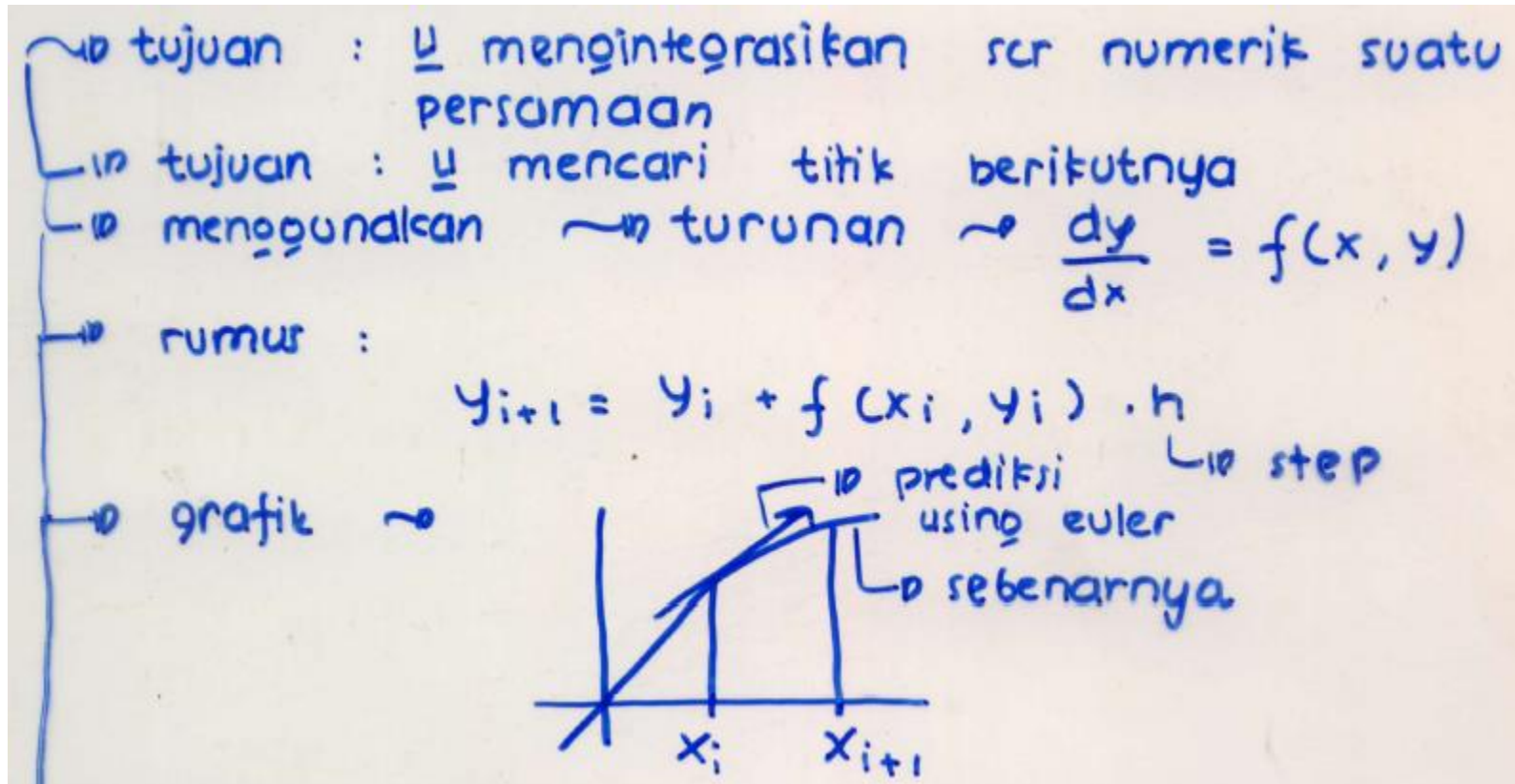
Pertemuan 15

# Euler, Heun, Range-Kutta

Bilqis

# Metoda Euler

Tujuan → mencari nilai  $y(x)$  pada titik  $x$ , jika diketahui deferensiasi  $f(x,y)$



Contoh 16.1 :

gunakan metode euler u mengintegrasikan  
scr numerik pers :

$$\frac{dy}{dx} = f(x, y) = -2x^3 + 12x^2 - 20x + 8,5$$

dari  $x=0$  sampai  $x=4$  dgn ukuran step  $h=0,5$   
pers. sebenarnya :

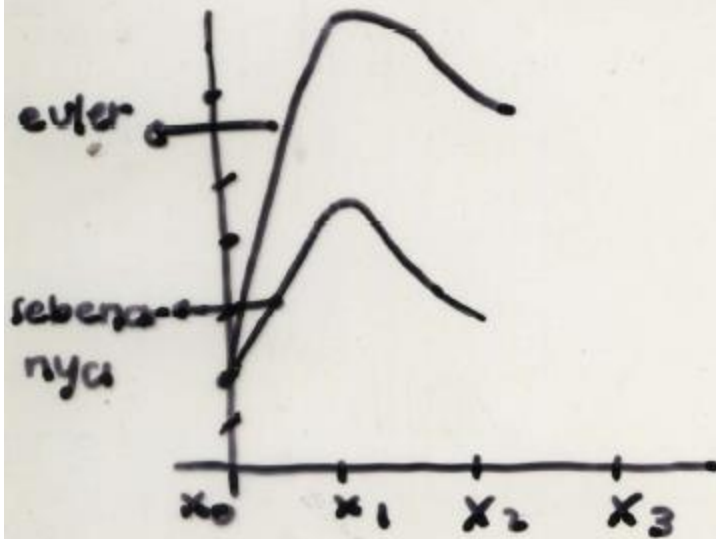
$$y = -0,5x^4 + 4x^3 - 10x^2 + 8,5x + 1$$

Jwb:

step 1:

sebenarnya

$$\begin{array}{l} \rightarrow x_0 = 0 \rightarrow y_0 = 1 \\ \rightarrow x_1 = 0,5 \rightarrow y_1 = 3,21875 \end{array}$$



euler

$$\begin{aligned} \rightarrow x_0 &= 0 \rightarrow y_0 = 1 \rightarrow \underline{y} \text{ pertama kali} \\ &= \text{sebenarnya} \\ \rightarrow x_1 &= 0,5 \rightarrow y_1 = y_0 + f(x_0, y_0) \cdot h \\ &= 1 + 8,5 \cdot (0,5) \\ &= 5,25 \end{aligned}$$

$$\begin{aligned} f(0, 1) &= -2(0)^3 + 12(0)^2 - 20(0) + 8,5 \\ &= 8,5 \end{aligned}$$

$$E_t = \left| \frac{3,21875 - 5,25}{3,21875} \right| \cdot 100\% = 63,1\%$$

Step 2:

Sebenarnya  $\begin{cases} X_1 = 0.5 \\ X_2 = 1 \end{cases} \rightarrow \begin{cases} y_1 = 3.21875 \\ y_2 = 3 \end{cases}$

Euler  $\begin{cases} X_1 = 0.5 \\ X_2 = 1 \end{cases} \rightarrow \begin{cases} y_1 = 5.25 \\ y_2 = y_1 + f(x_1, y_1) \cdot h \\ = 5.25 + 1.25 \cdot (0.5) \\ = 5.875 \end{cases}$

$$f(0.5, 5.25) = -2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5 \\ = 1.25$$

$$E_t = \left| \frac{3 - 5.875}{3} \right| \cdot 100 \% = 95.0 \%$$



Step 3:

sebenarnya  $\begin{cases} \rightarrow X_2 = 1 \rightarrow Y_2 = 3 \\ \rightarrow X_3 = 1,5 \rightarrow Y_3 = 2,21875 \end{cases}$

euler

$$\begin{cases} \rightarrow X_2 = 1 \rightarrow Y_2 = 5,875 \\ \rightarrow X_3 = 1,5 \rightarrow Y_3 = Y_2 + f(X_2, Y_2) \cdot h \\ \quad = 5,875 + (-1,5) \cdot (0,5) \\ \quad = 5,125 \end{cases}$$

$$\begin{aligned} f(1, 5,875) &= -2(1)^3 + 12(1)^2 - 20(1) + 8,5 \\ &= -1,5 \end{aligned}$$

$$B_t = \left| \frac{2,21875 - 5,125}{2,21875} \right| \cdot 100\% = 121\%$$

teruskan hingga  $X = 4$

1. Integrasikan persamaan  $f(x,y) = 4x^4 - 12x^2$  dari  $X_0=2$  sampai  $x_3=11$  dengan ukuran step  $h = 3$ . dengan menggunakan :

Nilai sebenarnya :

i	$X_i$	$f(x_i)$
0	2	-6.4
1	5	2.000
2	8	24.166
3	11	123.517

- a) (nilai 18) metoda Euler + error

Jawab :

step 1									
$X_0 =$	2	$Y_0 =$	(6)	nilai awal					
$X_1 =$	5	$Y_1 =$	-6,4	+	16	*	3		
		$y_1 =$	41,60						
		Error =	2.000	-	41,60	*	100		
			2.000						
		Error =	97,92						



step 2								
x1 =	5		y 1=	41,60				
x2 =	8		y 2=	41,6	+	2.200	*	3
			y 2=	6.642				
			Error =	24.166	-	6.642	*	100
				24.166				
			Error =	72,52				
step 3								
x2 =	8		y2 =	6.642				
x3 =	11		y3 =	6641,6	+	15.616	*	3
			y3 =	53.490				
			Error =	123.517	-	53.490	*	100
				123.517				
			Error =	56,69				

# Metoda Heun

kesalahan dr metode euler  $\sim$  turunan awal interval dianggap akan diterapkan sepanjang keseluruhan interval

uping metode Heun  
 $\sim$  2 titik direkte - rata

Rumus:

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} \cdot h$$

Contoh soal ibem dgn 9.1

jawab :

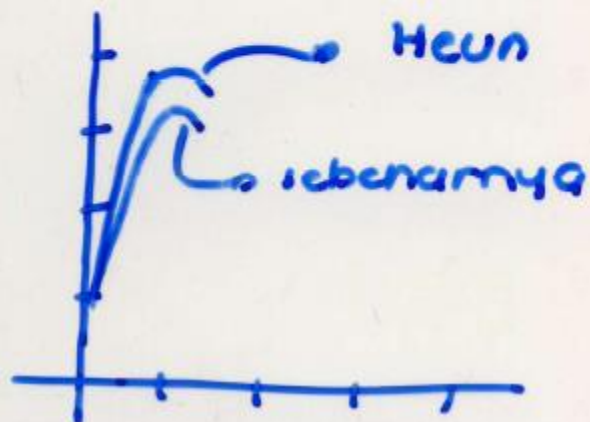
step 1

sebenarnya

$$\begin{cases} x_0 = 0 \\ x_1 = 0.5 \end{cases}$$

$$\rightarrow y_0 = 1$$

$$\rightarrow y_1 = 3,21875$$



Heun  $\begin{cases} \rightarrow x_0 = 0 \rightarrow y_0 = 1 \rightarrow \text{pertama kali} \\ \rightarrow x_1 = 0.5 \rightarrow y_1 = y_0 + \frac{f(0, y_0) + f(0.5, y_1)}{2} \cdot h \end{cases}$

$$= 1 + \frac{8.5 + 1.25}{2} \cdot 0.5$$

$$= 3.4375$$

$$Et = \left| \frac{3.21875 - 3.4375}{3.21875} \right| \cdot 100\% = 6.8\%$$

Step 2:

Sebenarnya

$$\begin{aligned} \rightarrow X_1 &= 0.5 & \rightarrow y_1 &= 3.21875 \\ \rightarrow X_2 &= 1 & \rightarrow y_2 &= 3 \end{aligned}$$

Itu en

$$\begin{aligned} \rightarrow X_1 &= 0.5 & \rightarrow y_1 &= 3.4375 \\ \rightarrow X_2 &= 1 & \rightarrow y_2 &= y_1 + \frac{f(0.5, y_1) + f(1, y_2)}{2} \cdot h \end{aligned}$$

$$= 3.4375 + \frac{1.25 + (-1.5)}{2} \cdot 0.5$$

$$= 3.375$$

$$Et = \left| \frac{3 - 3.375}{3} \right| \cdot 100 \% = 12.5 \%$$

b) (nilai 18) metoda Heun + error

Jawab :

step 1										
Xo =	2	Yo =	(6)	nilai awal = nilai sebenarnya						
X1 =	5	Y1 =	-6,4	+	$\frac{16}{2}$	+	$\frac{2.200}{2}$	*	3	
		y 1 =	3.318							
		Error =	$\frac{2.000 - 3.318}{2.000}$							
		Error =	65,88							
step 2										
x1 =	5	y 1 =	3.318							
x2 =	8	y 2 =	3317,6	+	$\frac{2.200}{2}$	+	$\frac{15.616}{2}$	*	3	
		y 2 =	30.042							
		Error =	$\frac{24.166 - 30.042}{24.166}$							
		Error =	24,31							

step 3										
x2 =	8		y2 =	30.042						
x3 =	11		y3 =	$30041,6 + \frac{15.616 + 57.112}{2} * 3$						
			y3 =	139.134						
			Error =	$\frac{123.517 - 139.134}{123.517} * 100$						
			Error =	12,64						



## Metode Runge - Kutta orde kedua :

Rumus  $\leadsto$   
dimana  $\leadsto$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) \cdot h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$a_1 + a_2 = 1 \quad \leadsto \quad a_1 = 1 - a_2$$

$$a_2 p_1 = \frac{1}{2} \quad \leadsto \quad p_1 = \frac{1}{2} a_2$$

$$a_2 q_{11} = \frac{1}{2} \quad \leadsto \quad q_{11} = \frac{1}{2} \cdot a_2$$

$$\left. \begin{array}{l} a_2 p_1 = \frac{1}{2} \\ a_2 q_{11} = \frac{1}{2} \end{array} \right\} p_1 = q_{11} = \frac{1}{2 \cdot a_2}$$

$\therefore$  kita bisa memberi sembarang nilai  $a_2$

1) misal kita masukkan pd metoda Heun dgn sebuah korektor tunggal  $\rightarrow a_2 = \frac{1}{2}$ , mk

$$\begin{aligned} a_1 &= 1 - a_2 \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \\ p_1 &= q_{11} = \frac{1}{2 \cdot \frac{1}{2}} = 1 \end{aligned}$$

pers

$$\begin{cases} y_{i+1} = y_i + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \\ k_1 = f(x_i, y_i) \\ k_2 = f(x_i + h, y_i + k_1 h) \end{cases}$$

2) misal kita memasukkan pd metode poligon yg diperbaiki

$$\sim \Delta z = 1, \text{ mk } \begin{cases} a_1 = 0 \\ p_1 = q_{11} = \frac{1}{2} \end{cases}$$

$$\begin{aligned} \text{pers} \rightarrow y_{i+1} &= y_i + k_2 h \\ k_1 &= f(x_i, y_i) \\ k_2 &= f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \end{aligned}$$

3) misal kita masukkan pd metode Ralston

$$\sim a_2 = \frac{2}{3}, \text{ maka } \begin{cases} a_1 = \frac{1}{3} \\ p_1 = q_{11} = \frac{1}{2 \cdot \frac{2}{3}} = \frac{3}{4} \end{cases}$$

pers :

$$\begin{cases} y_{i+1} = y_i + \left( \frac{1}{3} k_1 + \frac{2}{3} k_2 \right) \cdot h \\ k_1 = f(x_i, y_i) \\ k_2 = f\left(x_i + \frac{3}{4} \cdot h, y_i + \frac{3}{4} k_1 h\right) \end{cases}$$

### contoh 16.6

9.6

~ gunakan metode Runge kutta . Orde kedua yg kita masukkan pd metoda Polygon yg diperbaiki

~  $a_2 = 1$  , mk  $\begin{cases} a_1 = 0 \\ p_1 = q_{11} = \frac{1}{2} \end{cases}$

pers

$$y_{i+1} = y_i + k_2 h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

u mengintegrasikan scr numerik pers :

$$\frac{dy}{dx} = f(x, y) = -2x^3 + 12x^2 - 20x + 8,3$$

dari  $x=0$  sampai  $x=4$  dgn ukuran step = 0,5

jawab:

step 1:

pers. sebenarnya

$$\begin{cases} y = -0,5x^4 + 4x^3 - 10x^2 + 8,5x + 1 \\ x_0 = 0 \rightarrow y_0 = 1 \\ x_1 = 0,5 \rightarrow y_1 = 3,21875 \end{cases}$$



Polygon

$$\begin{aligned} k_1 &= f(x_0, y_0) = f(0, 1) \\ &= -2(0)^3 + 12(0)^2 - 20(0) + 8,5 \\ &= 8,5 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1h\right) \\ &= f\left(0 + \frac{1}{2} \cdot 0,5, 1 + \frac{1}{2}(8,5)(0,5)\right) \\ &= f(0,25, 3,125) \\ &= -2(0,25)^3 + 12(0,25)^2 - 20(0,25) + 8,5 \\ &= 4,21875 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + k_2 \cdot h \\ y(0,5) &= 1 + 4,21875 \cdot 0,5 \\ &= 3,109375 \end{aligned}$$

$$\text{Error} = \frac{3,21875 - 3,109375}{3,21875} \cdot 100 \% = 3,4 \%$$



step 2:

sebenarnya

$$\begin{cases} \rightarrow x_1 = 0,5 & \leadsto y_1 = 3,21875 \\ \rightarrow x_2 = 1 & \leadsto y_2 = 3 \end{cases}$$

Polygon

$$\begin{cases} \rightarrow x_1 = 0,5 & \leadsto y_1 = 3,109375 \\ \rightarrow x_2 = 1 & \leadsto y_2 = \dots ? \\ \rightarrow k_1 = f(x_1, y_1) \end{cases}$$

$$= f(0,5, 3,109375)$$

$$= -2(0,5)^3 + 12(0,5)^2 - 20(0,5) + 8,5$$

$$= 1,25$$

$$\begin{aligned}
 \rightarrow k_2 &= f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1h\right) \\
 &= f\left(0,5 + \frac{1}{2} \cdot 0,5, 3,109371 + \frac{1}{2}(1,25) \cdot (0,5)\right) \\
 &= f(0,75, 3,421871) \\
 &= -2(0,75)^3 + 12(0,75)^2 - 20(0,75) + 8,3 \\
 &= -0,59375
 \end{aligned}$$

$$\begin{aligned}
 \rightarrow y_{\text{true}}^2 &= y_1 + k_2 \cdot h \\
 y_2 &= 3,109371 + (-0,59375) \cdot (0,5) \\
 &= 2,81250
 \end{aligned}$$

$$\text{Error} = \frac{3 - 2,81250}{3} \cdot 100 \% = 6,25 \%$$

c) (nilai 18) metoda Runge-Kutta orde kedua jika diketahui  $a_2 = \frac{1}{2} + \text{error}$

Jawab :

$$y_{i+1} = y_i + \left( \frac{1}{2} \cdot k_1 + \frac{1}{2} \cdot k_2 \right) \cdot h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 \cdot h)$$

step 1									
Xo =	2	Yo =	(6)	nilai awal					
X1 =	5	Y1 =	....						
	k 1 =	16							
	k 2 =	2.200							
		Y1 =	$-6,4 + (0,50 \cdot 16 + 0,50 \cdot 2.200) \cdot 3$						
		Y1 =	3.317,60						
		Error =	$\frac{2.000 - 3.318}{2.000} \cdot 100$						
		Error =	65,88						

step 2					
X 1 =	5		Y 1 =	3.317,60	
X 2 =	8		Y 2 =	....	
	k 1 =	2.200			
	k 2 =	15.616			
			Y 2 =	$3317,6 + (0,50 * 2.200 + 0,50 * 15.616) * 3$	
			Y 2 =	30.041,60	
			Error =	$\frac{24.166 - 30.042}{24.166} * 100$	
			Error =	24,31	
step 3					
X 2 =	8		Y 1 =	30.041,60	
X 3 =	11		Y 2 =	....	
	k 1 =	15.616			
	k 2 =	57.112			
			Y 2 =	$30041,6 + (0,50 * 15.616 + 0,50 * 57.112) * 3$	
			Y 2 =	139.133,60	
			Error =	$\frac{123.517 - 139.134}{123.517} * 100$	
			Error =	12,64	



step 2					
X 1 =	5		Y 1 =	2.186,13	
X 2 =	8		Y 2 =	....	
	k 1 =	2.200			
	k 2 =	10.421			
			Y 2 =	$2186,13125 + (0,33 * 2.200 + 0,67 * 10.421) * 3$	
			Y 2 =	25.227,16	
			Error =	$\frac{24.166 - 25.227}{24.166} * 100$	
			Error =	4,39	
step 3					
X 2 =	8		Y 1 =	25.227,16	
X 3 =	11		Y 2 =	....	
	k 1 =	15.616			
	k 2 =	42.892			
			Y 2 =	$25227,16 + (0,33 * 15.616 + 0,67 * 42.892) * 3$	
			Y 2 =	126.626,69	
			Error =	$\frac{123.517 - 126.627}{123.517} * 100$	
			Error =	2,52	