Komputasi Numerik

PERTEMUAN 9

Interpolasi Newton dan Lagrange

2024/2025

Komnum Week 8

Apa Yang Akan Kita Pelajari?

01 Interpolasi Newton

02 Interpolasi Lagrange

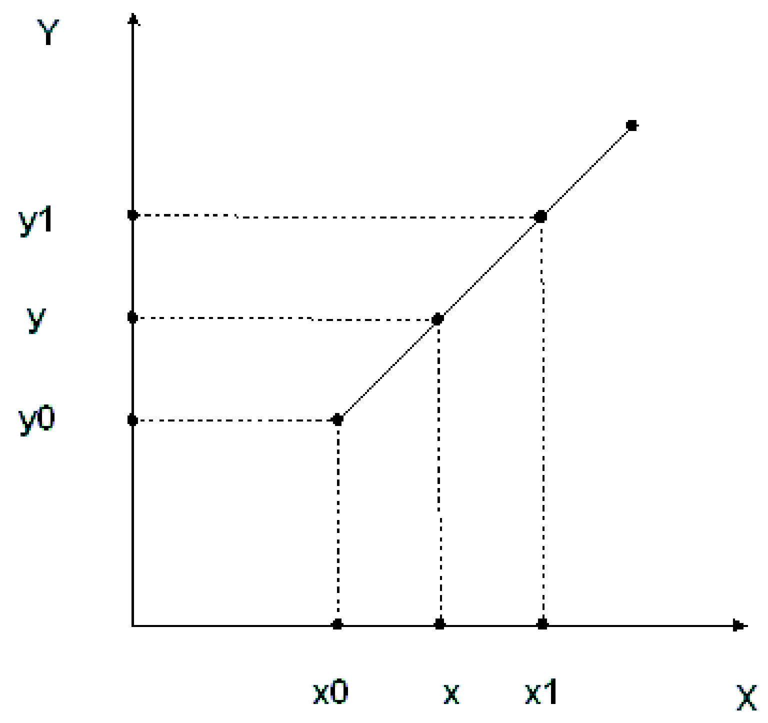
Komnum Week 8

Dalam pertemuan ini kita akan mempelajari metode-metode untuk mencari Interpolasi.

Interpolasi

Jika pada materi pencocokan kurva sebelumnya kita diminta menaksir bentuk fungsi melalui sederetan data, maka sekarang kita diminta untuk mengestimasi nilai fungsi f(x) di antara beberapa nilai fungsi yang diketahui (tanpa mengetahui bentuk fungsi yang menghasilkannya).

Contoh



Polinomial Newton

Bentuk Umum Polinomial Interpolasi Newton:

$$
\begin{aligned}
f_n(x) = &b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \\
&+ b_3(x - x_0)(x - x_1)(x - x_2) + \cdots + \\
&b_n(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{x-1})
\end{aligned}
$$

Contoh Untuk n = 3:

$$
\begin{aligned}
f(x) = &b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \\
&+ b_3(x - x_0)(x - x_1)(x - x_2)
\end{aligned}
$$

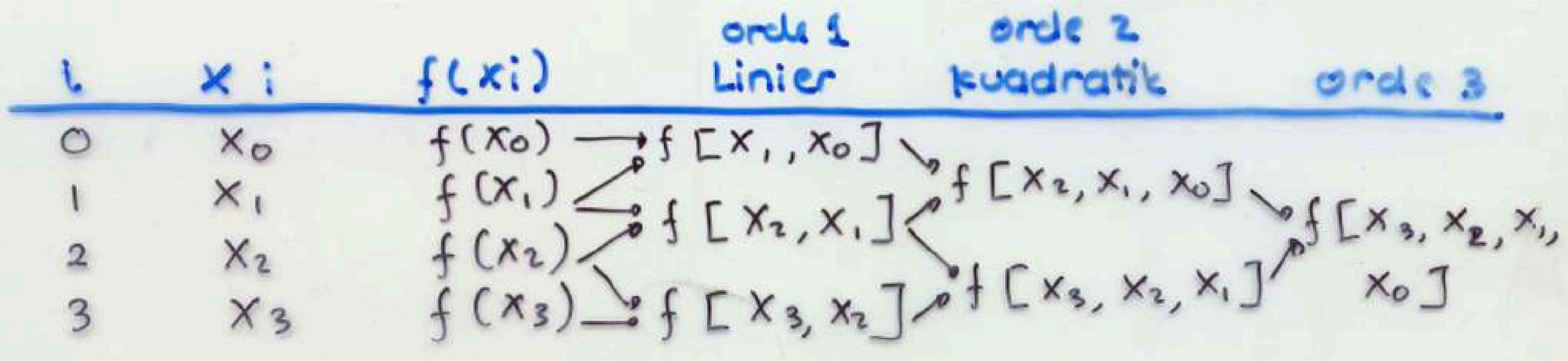
Polinomial Newton

$$b_0 = f(x_0)$$

$$b_1 = f\left[x_1, x_0\right] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$b_2 = f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0}$$



Interpolasi Linear

Menghubungkan 2 titik dengan sebuah garis lurus

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Interpolasi Linear

Contoh Soal 1

Taksirlah nilai ln 2 menggunakan Interpolasi Linear → x = 2 → ln 2 yang nilai sebenarnya ln 2 = 0.64314718

Diketahui:

- ln 1 = 0
- ln 6 = 1.7917595

Rumus

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Jawaban Contoh 1

$x_0 = 1 \rightarrow f(x_0) = 0$
$x_1 = 6 \rightarrow f(x_1) = 1,7917595$

u $x = 2$, mk :

$$f_1(2) = 0 + \frac{1,7917595 - 0}{6-1}(2-1)$$

$$= 0,3583519 \rightarrow E_t = 48,3\ \%$$

kedua diket : $\ln 1 = 0$
$\ln 4 = 1,3862944$

$x_0 = 1 \rightarrow f(x_0) = 0$
$x_1 = 4 \rightarrow f(x_1) = 1,3862944$

u $x = 2$, mk :

$$f_1(2) = 0 + \frac{1,3862944 - 0}{4-1}(2-1)$$

$$= 0,4620981 3 \rightarrow E_t = 33,3\ \%$$

Interpolasi Kuadratik

Terkadang jika suatu kurva didekatkan oleh persamaan garis, terjadi kesalahan, maka untuk mendekatkan gunakan parabola atau polinom orde ke-2 atau interpolasi kuadratik

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Interpolasi Kuadratik

Contoh Soal 1

Cocokkan polinomial orde ke-2 terhadap 3 titik yang digunakan dalam contoh:

$$x_0 = 1 \rightarrow f(x_0) = 0$$
$$x_1 = 4 \rightarrow f(x_1) = 1{,}3862944$$
$$x_2 = 6 \rightarrow f(x_2) = 1{,}7917595$$

Gunakan polinomial untuk mengevaluasi ln 2

Rumus

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Jawaban Contoh 1

$b_0 = 0$

$$b_1 = \frac{1{,}3862944 - 0}{4-1} = 0{,}4620913$$

$$b_2 = \frac{\dfrac{1{,}79\ldots - 1{,}38\ldots}{6-4} - 0{,}4620913}{6-1}$$

$$= -0{,}051\ldots$$

maka persamaan kuadratiknya adalah :

$$f_2(x) = 0 + 0{,}46\ldots(x-1) - 0{,}051\ldots(x-1)(x-4)$$

kmd masukkan $\rightarrow x = 2$

$f_2(2) = 0{,}5658\ldots \Rightarrow E_t = 18{,}4\%$ $\rightarrow$ lebih rendah drpd interpolasi Linier

Interpolasi Orde 3

Contoh Soal 1

Tafsirkan ln 2 denga Polinomial Interpolasi terbagi hingga Newton orde ke-3

$x_0 = 1 \longrightarrow f(x_0) = 0$

$x_1 = 4 \longrightarrow f(x_1) = 1,3862\ldots$

$x_2 = 6 \longrightarrow f(x_2) = 1,7917\ldots$

$x_3 = 5 \longrightarrow f(x_3) = 1,6094\ldots$

Persamaan yang akan dibentuk:

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$b_0 = f(x_0) = 0$$

Jawaban Contoh 1

$$b_1 = f[x_1, x_0] = \frac{1,3862\ldots - 0}{4-1} = 0,4620\ldots$$

$$f[x_2, x_1] = \frac{1,7917\ldots - 1,3862\ldots}{6-4} = 0,2027\ldots$$

$$f[x_3, x_2] = \frac{1,6094\ldots - 1,7917\ldots}{5-6} = -0,1823\ldots$$

$$b_2 = f[x_2, x_1, x_0] = \frac{0,2027\ldots - 0,4620}{6-1} = -0,0518\ldots$$

$$f[x_3, x_2, x_1] = \frac{0,1823\ldots - 0,2027}{5-1} = -0,0204\ldots$$

$$b_3 = f[x_3, x_2, x_1, x_0] = \frac{-0,0204\ldots - (-0,0518)}{5-1}$$

$$= 0,0078\ldots$$

Jawaban Contoh 1

∴ pers. orde 3 :

$$f_3(x) = 0 + 0,46\ldots(x-1) - 0,051\ldots(x-1)(x-4) + 0,00786\ldots(x-1)(x-4)(x-6)$$

mk u $x = 2 \Rightarrow f_3(2) = 0,628\ldots$ $E_t = 9,3\%$

lebih baik dari kuadrat

Interpolasi Orde 3

Contoh Soal 2

Taksirlah ketika $x = 7$ dengan polinomial interpolasi terbagi hingga Newton orde ke-3

$$x_0 = 2 \rightarrow f(x_0) = 31$$

$$x_1 = 5 \rightarrow f(x_1) = 382$$

$$x_2 = 8 \rightarrow f(x_2) = 1543$$

$$x_3 = 11 \rightarrow f(x_3) = 4000$$

Jawaban Contoh 2

pers yang akan dibentuk :

$$f(x) = b_0 + b_1 (x - x_0) +$$
$$b_2 (x - x_0)(x - x_1) +$$
$$b_3 (x - x_0)(x - x_1)(x - x_2)$$

$$b_0 = f(x_0) = 31 \quad 2$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{382 - 31}{5 - 2} = 117 \quad 2$$

Jawaban Contoh 2

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1543 - 382}{8 - 5} = 387 \quad (2)$$

$$f[x_3, x_2] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{4000 - 1543}{11 - 8} = 819 \quad (2)$$

Jawaban Contoh 2

$$b_2 = f[X_2, X_1, X_0] = \frac{f[X_2, X_1] - f[X_1, X_0]}{X_2 - X_0}$$

$$= \frac{387 - 117}{8 - 2} = 45 \quad (2)$$

$$f[X_3, X_2, X_1] = \frac{f[X_3, X_2] - f[X_2, X_1]}{X_3 - X_1}$$

$$= \frac{819 - 387}{11 - 5} = 72 \quad (2)$$

Jawaban Contoh 2

$$b_3 = f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0}$$

$$= \frac{72 - 45}{11 - 2} = 3 \quad 2$$

$\therefore$ pers orde 3 :

$$f(x) = 31 + 117(x-2) +$$

$$2 \quad 45(x-2)(x-5) +$$

$$3(x-2)(x-5)(x-8)$$

Jawaban Contoh 2

maka : $x = 7 \Rightarrow f(7) = 31 + 117(7-2) + 45(7-2)(7-5) + 3(7-2)(7-5)(7-8)$

2

$$= 31 + 117(5) + 45(5)(2) + 3(5)(2)(-1)$$

$$f(7) = 1036$$

2

Interpolasi Orde 3

Contoh Soal 3

Diketahui:

- X = 11
- f(6) = 234
- f(9) = 960
- f(12) = 2280
- f(15) = 4356

Ditanya:

- Carilah hasil dari fungsi berikut
  - i. $f[X1, X0]$
  - ii. $f[X2, X1]$
  - iii. $f[X3, X2]$
  - iv. $f[X2, X1, X0]$
  - v. $f[X3, X2, X1]$
  - vi. $f[X3, X2, X1, X0]$
- Carilah nilai f(11) dengan Interpolasi Newton Orde 3!

Jawaban Contoh 3a

Bentuk umum *Polinomial Interpolasi Newton Orde 3*:

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

=> Mencari b_0, b_1, b_2, b_3

- $b_0 = f[X0]$ = 234
- $b_1 = f\left[X1, X0\right] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = \dfrac{f[960] - f[234]}{9-6} = 242$
- $f[X2, X1] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1} = \dfrac{2280 - 960}{12-9} = 440$
- $f[X3, X2] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2} = \dfrac{4356 - 2280}{14-12} = 692$
- $b_2 = f\left[X2, X1, X0\right] = \dfrac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \dfrac{440 - 242}{12-6} = 33$
- $f[X3, X2, X1,] = \dfrac{f[x_3, x_2] - f[x_1, x_0]}{x_3 - x_0} = \dfrac{692 - 440}{15-9} = 42$
- $b_3 = f[X3, X2, X1, X0] = \dfrac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0} = \dfrac{42 - 33}{15-6} = 1$

Jawaban Contoh 3b

Mencari f(11):

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$= 234 + 242(11 - 6) + 33(11 - 6)(11 - 9) + (11 - 6)(11 - 9)(11 - 12)$$

$$= 1764$$

Jadi hasil dari f(11) *adalah* 1764

Interpolasi Orde 3

Contoh Soal 4

Diketahui:

a) X = 11
b) Xo = 8; f(Xo) = 660
c) X1 = 10 f(X1) = 1326
d) X2 = 12 f(X2) = 2280
e) X3 = 14 f(X3) = 3570

Ditanya:

a) Carilah hasil fungsi berikut berikut:
   i. $f[X1, X0]$
   ii. $f[X2, X1]$
   iii. $f[X3, X2]$
   iv. $f[X2, X1, X0]$
   v. $f[X3, X2, X1]$
   vi. $f[X3, X2, X1, X0]$
b) Carilah nilai f(11) dengan **Interpolasi Newton Orde 3**!

Jawaban Contoh 4a

Bentuk umum *Polinomial Interpolasi Newton Orde 3*:

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

=> Mencari b_0, b_1, b_2, b_3

- $b_0 = f[X0]$ = 660
- $b_1 = f\left[X1, X0\right] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = \dfrac{f[1326] - f[660]}{10-8} = 333$
- $f[X2, X1] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1} = \dfrac{2280 - 1326}{12-10} = 477$
- $f[X3, X2] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2} = \dfrac{3570 - 2280}{14-12} = 645$
- $b_2 = f\left[X2, X1, X0\right] = \dfrac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \dfrac{477 - 333}{12-8} = 36$
- $f[X3, X2, X1,] = \dfrac{f[x_3, x_2] - f[x_1, x_0]}{x_3 - x_0} = \dfrac{645 - 477}{14-10} = 42$
- $b_3 = f[X3, X2, X1, X0] = \dfrac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0} = \dfrac{42 - 36}{14 - 8} = 1$

Jawaban Contoh 4b

Mencari f(11):

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$= 660 + 333(11 - 8) + 36(11 - 8)(11 - 10) + (11 - 8)(11 - 10)(11 - 12)$$

$$= 1764$$

Jadi hasil dari f(11) *adalah* 1764

Polinomial Interpolasi Lagrange

- Modifikasi Newton
- Mencegah komputasi diferensiasi terbagi

Contoh Untuk Orde ke-1:

$$f_1(x) = f(x_0) + \underbrace{f[x_1, x_0]} (x - x_0)$$

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$\Downarrow$ dipisah

$$= \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1}$$

$\Downarrow$ dimasukkan kembali

$$f_1(x) = f(x_0) + \left(\frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \right) (x - x_0)$$

$$= f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) + \frac{(x - x_0)}{(x_0 - x_1)} f(x_0)$$

Polinomial Interpolasi Lagrange

$\forall\, f(x_0)$ dikumpulkan dengan $f(x_0) \Rightarrow$ sejenis

$$f_1(x) = \left(\frac{(x_0 - x_1)}{(x_0 - x_1)} + \frac{(x - x_0)}{(x_0 - x_1)} \right) \cdot f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} \cdot f(x_1)$$

$$f_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} \cdot f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} \cdot f(x_1)$$

$\Rightarrow$ U orde kedua :

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \cdot f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \cdot f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \cdot f(x_2)$$

Interpolasi Lagrange

$$
\begin{aligned}
f(x_s) = &\frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)} f_0 \\
&+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} f_1 \\
&+ \cdots \\
&+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} f_n
\end{aligned}
$$

Interpolasi Lagrange : facts and figures

- Lagrange tidak memerlukan tabel beda
- Aplikatif untuk kasus equispaced (h konstan) maupun non-equispaced (h tidak konstan)
- Aplikatif untuk kasus interpolasi dan invers interpolation
- Efisien untuk mencari nilai fungsi di dekat titik awal, tengah, maupun akhir

Interpolasi Lagrange

Contoh Soal 1

carilah nilai log 656, jika diketahui nilai-nilai log 654 = 2,8156, log 658 = 2,8182, log 659 = 2,8189, log 661 = 2,8202

n	Log	Nilai
0	654	2,8156
1	658	2,8182
3	659	2,8189
3	661	2,8202

$$\text{Log } 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \cdot (2{,}8156)$$
$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \cdot (2{,}8182)$$
$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \cdot (2{,}8189)$$
$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-654)(661-654)} \cdot (2{,}8202)$$
$$= 2{,}8168$$

Interpolasi Lagrange

Contoh Soal 2

Gunakan Interpolasi lagrange orde ke-1 dan ke-2 untuk mengevaluasi ln 2

$x_0 = 1 \qquad f(x_0) = 0$

$x_1 = 4 \qquad f(x_1) = 1,3862944$

$x_2 = 6 \qquad f(x_2) = 1,7917595$

Jawaban Contoh 2

→ u orde pertama :

$$f_1(x) = \frac{2-4}{1-4} \cdot 0 + \frac{2-1}{4-1} \cdot 1,386\ldots = 0,462\ldots$$

→ u orde kedua :

$$f_2(x) = \frac{(2-4)(2-6)}{(1-4)(1-6)} \cdot 0 + \frac{(2-1)(2-6)}{(4-1)(4-6)} \cdot 1,386\ldots$$

$$+ \frac{(2-1)(2-4)}{(6-1)(6-4)} \cdot 1,791\ldots$$

$$= 0,56584$$

∴ terbukti hasil newton = lagrange

Interpolasi Lagrange

Contoh Soal 3

Diketahui:

- X = 11
- Xo = 6 f(Xo) = 234
- X1 = 9 f(X1) = 960
- X2 =12 f(X2) = 2280
- X3 =15 f(X3) = 4356

Ditanya:

a. Selesaikan persamaan berikut!

- $\frac{(x- x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0)$
- $\frac{(x- x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$
- $\frac{(x- x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$
- $\frac{(x- x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$

b. Carilah nilai f(11) menggunakan **Interpolasi Langrange Orde 3**!

Jawaban Contoh 3a

Masukkan nilai variabel ke dalam persamaan

$$-\ \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,f(x_0)=\frac{(11-9)(11-12)(11-15)}{(6-9)(6-12)(6-15)}\,f(6)=$$

$-11{,}56$

$$-\ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,f(x_1)=\frac{(11-6)(11-12)(11-15)}{(9-6)(9-12)(9-15)}\,f(9)=$$

$355{,}36$

$$-\ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\,f(x_2)=\frac{(11-6)(11-9)(11-15)}{(12-6)(12-9)(12-15)}\,f(12)=$$

$1688{,}89$

$$-\ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,f(x_3)=\frac{(11-6)(11-9)(11-12)}{(15-6)(15-9)(15-12)}\,f(15)=$$

$-268{,}89$

Jawaban Contoh 3b

Masukkan nilai variabel ke dalam persamaan

- $f(x) = \frac{(x - x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x - x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x - x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x - x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$
- $f(11) = -11{,}56 + 355{,}36 + 1688{,}89 - 268{,}89 = 1764$

Jadi hasil dari f(11) adalah 1764

Interpolasi Lagrange

Contoh Soal 4

Diketahui:

X_0 =	3	$f(X_0)$ =	11
X_1 =	5	$f(X_1)$ =	-5
X_2 =	7	$f(X_2)$ =	-37
X_3 =	9	$f(X_3)$ =	-85

Ditanya :

- (Nilai 18) Taksirlah ketika $x = 6$ dengan menggunakan interpolasi polynomial newton orde ketiga
- (Nilai 15) Taksirlah ketika $x = 6$ dengan menggunakan interpolasi lagrange orde ketiga

https://its.id/m/komnum25

Komnum Week 8

Tugas Kelompok

1. Buatlah contoh soal sendiri, boleh mengarang atau mengambil dari internet:
   a. Polinomial Newton = 10 kelompok
      i. Orde 1, Error = ...?
      ii. Orde 2, Error = ...?
      iii. Orde 3, Error = ...?
   b. Polinomial Lagrange = 10 kelompok
      i. Orde 1, Error = ...?
      ii. Orde 2, Error = ...?
      iii. Orde 3, Error = ...?
2. Bentuk file PPT + nama kelompok dan anggota
3. Berikan contoh implementasi di dunia nyata dari metode yang digunakan

Komnum Week 8

TERIMA KASIH

Sampai Bertemu Kembali