

Title

Simulation and Modeling

Text

Simulation, Modeling And Analysis

By

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# **Simulation Principles and Methods**

## **1. Introduction**

The basic idea of Simulation is to build an experimental device (model) that act like (simulate) the real system in certain important aspects. The purpose is to understand or evaluate the behavior of a complex real-world system over extended period without risk to the real system performance.

## **2. Why using Simulation**

- a)** If the experimentation on the real-world system is not feasible or not possible:

- Too expensive,
- Too risky (atomic problems),
- Too complex,
- Not existing system,
- Intractable.

**b)** Simulation is the best (may be the only) technique available under conditions of uncertainty due to stochastic (random) variables, or nonlinearity. Mathematical treatment of such systems is frequently not possible.

**c)** Computer simulation gives control over time, which may be compressed or expanded, such as an aircraft; simulation may gather data on many months of operations in minutes.

### **3. Simulation Application**

Simulation can be used in Theoretical problems in basic science areas such as math., physics, and chemistry.

It can be useful in practical problems in all aspects of life such as industrial problems, business, economic, biomedical systems, social problems (Population dynamics), space researches, and War strategies.

### **4. Basic concepts and terminology**

**a) System :** is defined as an isolated collection of interacting components or objects with defined set of interactions among them. A Jet aircraft is an excellent example of a complex system consisting of mechanical, electronic, chemical and human components.

**b) State of a system:** It is the minimal collection of information with which the system future behavior can be predicted. For the Jet aircraft, the state of the system can be determined by Speed, Direction of travel, weather conditions, number of passengers, and amount of remaining fuel.

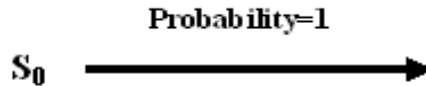


**c) An Activity:** It is the process or Event Which changes the system state.

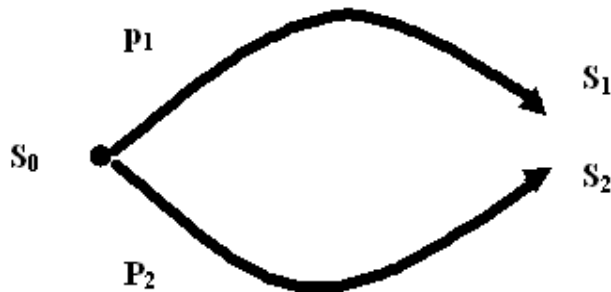


## d) System Classification:

- Natural and Man-Made,
- Deterministic



- Stochastic (Probabilistic)



- **Continuous:** A system whose changes in its state occurs continuously over time (liquid flow in a pipeline),
- **Discrete:** Growth of world population with respect to time.

## **5. System methodology**

When simulation is used to solve a problem, the following time-tested steps, or stages are applied:

- A. Observation of the system,
- B. Formulation of hypotheses or theories that account for the observed behavior,
- C. Prediction of the future behavior of the system based on the assumption that the hypotheses are correct,
- D. Comparison of the predicted behavior with the actual behavior.

The system being studied may impose constraints on certain steps of this scientific method. For example, consider the simulation of a system that does not yet exist. Obviously the observation of such system is not possible, but the simulation of such system may still be possible if the analysis is carefully conducted and if the ultimate requirements are known. The problem solving process is divided into four phases:

- . Planning**
- . Modeling**
- . Validation**
- . Application.**

**a) Planning** or premodeling includes the initial encounter with the system, the problem to be solved, and the factors pertaining to the system and its environment that are likely to affect the solution of the problem. This means the problem must be well defined. Obviously the more accurate and precise the problem statement is, the more smoothly the solution process can proceed .

Resources must be considered and estimated such as money, time, personal, and special equipment .If crucial resources are not available, solution of the problem can be judged infeasible before a significant amount of time or money is spent.

**b) Modeling** in this phase the analyst constructs the system model, which is a representation of the real system. The characteristics of this model should be Representative (not identical) of the characteristics of the real system. You must select some minimal set of the system's characteristics so that the model approximates the real system to be cost-effective and manageable. There are many types of simulation models:

1-Descriptive models (verbalizations of the system's composition)

2-Physical models (scaled facsimiles of the system: aircraft, cars, building,..)

3-Mathematical models (abstract expressions of the relationships among the system variables: Newton's laws of motion,..)

4-Flowcharts ( Logical interaction between system's components)

5-Schematics ( Logical interaction between system's components)

6-Computer programs : If the system is too complex then the reduction technique or subsystem modeling is used.

**c) Validation** is to check that the system or the model is a correct representation of the real system. But verification means that the logical of a computer program is ok. So, a verified computer program can represent invalid model . There are many techniques used to validate a computer program:

- Compare the results of simulation model with results historically produced by the real system operating under the same conditions.
- Use simulator to predict results, and then compare with the real data for some period.

Various statistical procedures can be used to determine whether the simulated output could have been produced by the real systems and vice versa. Two such methods are the Chi-square goodness-of-fit test and the kolmogrove-Smirnov test.

**d) Application** Once the model has been properly validated, it can be applied to solve the problem at hand. A simulator may work error-free for a long period until it encounters a new and perhaps unique combination of program parameters that generates the next error.

## **6. Advantages and disadvantages of simulation**

Phillips Ravindran and Solberg [OR principles and practice, New York John Wiley, 1976] stated that simulation is one of the easiest tools of management science to use but probably one of the hardest to apply properly and perhaps the most difficult one from which to draw accurate conclusions.

Adkins and Pooch list 5 advantages of simulation modeling:

- a) It permits controlled experimentation,
- b) It permits time compression,
- c) It permits sensitivity analysis by manipulation of input variables,
- d) It does not disturb the real system,
- e) It is an effective training tool.



They also list 4 disadvantages:

- a) A simulation model may become expensive in terms of manpower and computer time,
- b) Extensive development time may be encountered,
- c) Hidden critical assumption may cause the model to diverge from reality,
- d) Model parameters may be difficult to initialize. These may require extensive time in collection, analysis, and interpretation.

Thus although simulation has proved an effective approach to problem solving, it has its drawbacks. The researcher should be aware of these drawbacks before becoming committed to this approach.

# Examples

## Example1

The first example we are going to see is the simulation of a tossing of a fair coin. First step is analyzing the problem. The fair coin means that when tossing that coin the probability of head equal the probability of tail equal 50%. So, using a digital computer to simulate this phenomenon we are going to use a uniform Random number generated by the package you are using or you can write its code. Uniform random number means that you have a set of random number between 0 and 1 all with the same probability. But most languages generate uniform random number integer from a to b with equal probability.

## See the following program written in c++

```
#include <iostream>
#include <stdlib.h>
void main(void)
{
    int x,nuber_or_trials, head=0, tail=0;
    float phead,ptail,error_head,error_tail;
    cout<<"enter number of trials"<<endl;
    cin>>nuber_or_trials;
    for(int i=0;i<nuber_or_trials;i++)
    {
        x=random(2);
        if (x==1) head++;
        else ++tail;
    }
    phead=head*1.0/nuber_or_trials;
    ptail=tail*1.0/nuber_or_trials;
    error_head = abs(((0.5 - phead)/0.5)*100);
    error_tail  = abs(((0.5 - ptail)/0.5)*100);
    cout<<"probability of head= " <<phead<<"
with   error ="<<error_head<<"%"<<endl;
    cout<<"probability of tai = " <<ptail<<"   with
error ="<<error_tail<<"%"<<endl;
    cin>>x;
}
```

## The output results are:

Enter number of trials = 1

probability of head= 0 with error =100%

Probability of tail = 1 with error =100%

Enter number of trials =5

probability of head= 0 with error =100%

Probability of tail = 1 with error =100%

Enter number of trials = 32767

probability of head= 0.496292 with error =0%

Probability of tail = 0.503708 with error =0%

## Example2

Get the average daily demand for a small grocery store selling a fresh bread according to the following table:

Daily demand D	Probability Of demand P(D)	Device to generate chance outcomes Number and color of balls	
100	0.20	20	Red
110	0.50	50	Blue
120	0.30	30	Yellow

Note that the proportion of balls of a specific color corresponds exactly to the probability of a specific level of daily demand.

## To simulate the daily demand (5 days)

- Draw one ball at a time, notice its color and then place it back in the bowl. Then translate the outcomes into unique **values of demand**.

Sample Number	Color of ball	Day of the Week	Simulated Demand
1	Blue	Monday	110
2	Blue	Tuesday	110
3	Yellow	Wednesday	120
4	Yellow	Thursday	120
5	Red	Friday	100
Total			560

- Expected value of simulated demand  
=  $560/5=112$  units / day
- Analytical solution  
Expected daily demand  
=  $100(0.2) + 110(0.5) + 120(0.3) = 111$  units / day

### **Example3**

Use the one-digit Random Number (6, 3, 5, 0, 8)  
to generate random observations for :

- a) Throwing an unbiased coin,
- b) Throwing a die,
- c) The color of a traffic found by a randomly arriving car when it is green 40% of the time, yellow 10% of the time, and red 50% of the time.

## Solution

a) To simulate a coin using one digit R.N.(0-9), let 0 - 4 Represent a Head and 5 - 9 Represent a Tail; so the solution will be

$(6, 3, 5, 0, 8) \xrightarrow{\text{R.Observations}} (T, H, T, H, T)$

b) To simulate a die using number 0 - 9. Let 1 - 6 represents the faces of the die; then 0, 7, 8 and 9 are rejected. So the solution will be

$(6, 3, 5, 0, 8) \xrightarrow{\text{R.Observations}} (6, 3, 5, \text{reject}, \text{reject})$



c) Divide 0 – 9 into 3 classes according to the given probability:

- Green      40%      0, 1, 2, 3
- Yellow    10%      4
- Red       50%      5, 6, 7, 8, 9

So (6,3,5,0,8)  $\xrightarrow{\text{R.Observations}}$  (Red,Green,Red,Green,Red)

# Probability and Distribution Theory

**Definition1** A random variable is a quantity whose value is determined by the outcome of a Random Experiment ,i.e., the Random Variable  $X$  is a function whose domain is the event space (set of all possible outcomes of a random experiment) and whose range is some subset of the real numbers.

$$X : E \longrightarrow R$$

## Set Theory and compound Events

### Definition2

An event is some subset of the event space of a random experiment.

$\Omega$  : Event space or the collection of all the possible outcomes of the experiment.

### Example

Consider the experiment of tossing two dice and noting the sum of the faces. The event space is  $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  and some events are

$E_1 = \{2\}$   $E_2 = \{3, 4, 5\}$   $E_3 = \{2, 4, 6, 8, 10, 12\}$  (an even number appears).

### Definition3

The Complement of an event  $E$  is  $\bar{E}$  which contains the set of elements that are in  $\Omega$  but not in  $E$ .

Note that:  $E + \bar{E} = \Omega$  ,  $E * \bar{E} = \Phi$  (null set )

## Example

$$\bar{E}_1 = \{3,4,5,6,7,8,9,10,11,12\}$$

$$\bar{E}_2 = \{2,6,7,8,9,10,11,12\}$$

$$\bar{E}_3 = \{3,5,7,9,11\}$$

## Definition4

Let a random experiment has  $N$  equally likely outcomes. Let some event  $E$  is composed of  $n$  outcomes, so the probability of  $E$  is denoted by  $P(E) = n/N$ ,  $P(\bar{E}) = 1 - P(E) = (N - n)/N$   
Note :  $P(\Omega) = 1$ ,  $P(\Phi) = 0$ .

## Definition5

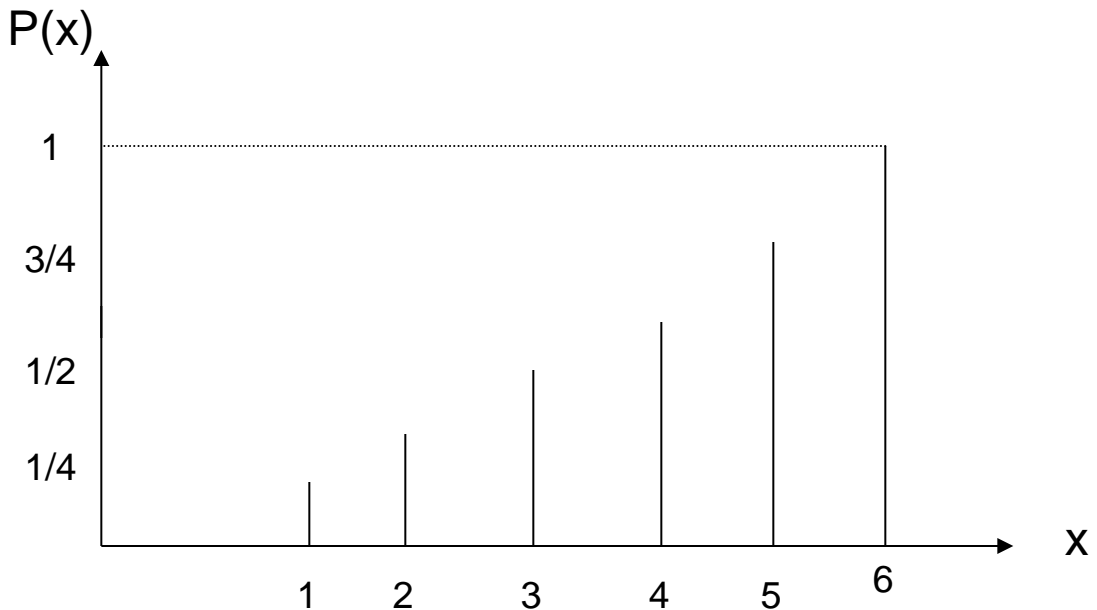
The Union of two events  $E_1$  and  $E_2$  is denoted by  $E_1 + E_2$  and it is defined as the outcomes in either  $E_1$  OR  $E_2$  OR both.

The intersection of two events  $E_1$  &  $E_2$  is denoted by  $E_1 \cdot E_2$  and is defined as the outcomes that the events have in common. Two events that have no outcomes in common are said to be mutually exclusive.

## Discrete Distribution

**Example** Tossing a single die Let  $X$  a random Variable that counts the spots of the side facing up. Then the values of  $X$  is 1, 2, 3, 4, 5, 6

$X$	1	2	3	4	5	6
$P(X = x)$	1/21	2/21	3/21	4/21	5/21	6/21

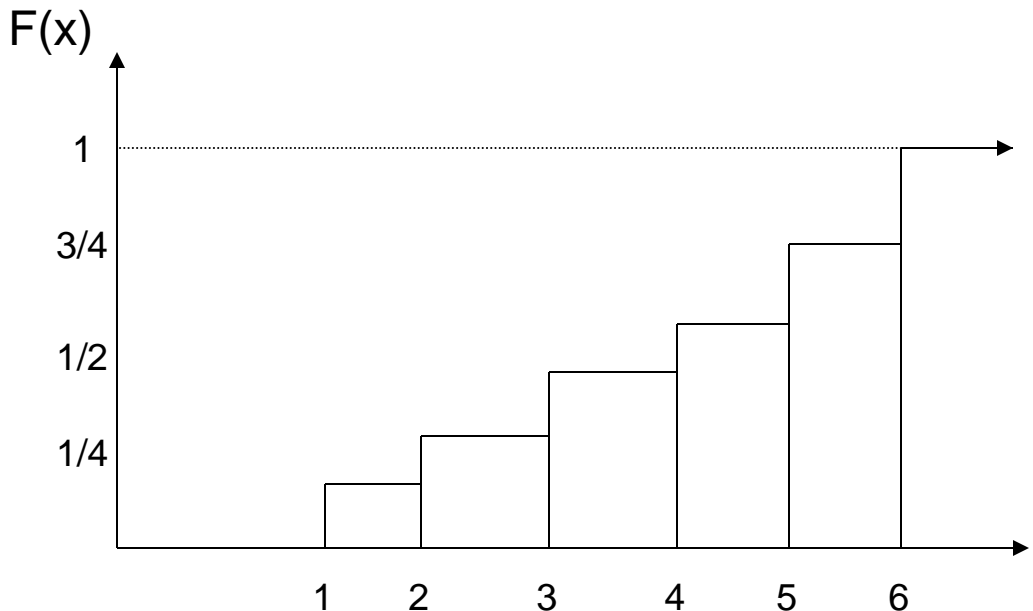


The properties that every discrete probability function must have :

1)  $P(x_i) \geq 0$  for all  $x$  It is called probability density function (p.d.f.)

$$2) \sum_{i=1}^{\infty} P(x_i) = 1$$

Another useful function is the Cumulative distribution  
 $\text{CDF} = F(x) = P(X \leq x)$  {probability that  $X \leq x$ }



The CDF has the following properties:

- 1)  $0 \leq F(X) \leq 1 \quad -\infty < X < \infty$
- 2) If  $X_1 \leq X_2 \longrightarrow F(X_1) \leq F(X_2)$  that is F is monotonically increasing.

- 3)  $\lim_{X \longrightarrow \infty} F(X) = F(\infty) = 1$

$$\lim_{X \longrightarrow -\infty} F(X) = F(-\infty) = 0$$

Expected value (Mean)

$$E(X) = \sum_{i=0}^{\infty} X_i P(X_i)$$

For the above example

$$E(X) = (1)(1/21) + (2)(2/21) + (3)(3/21) + (4)(4/21) + (5)(5/21) + (6)(6/21) = 91/21$$

## Variance

$$\begin{aligned} V(X) &= \sum_{i=0}^{\infty} [x_i - E(X)]^2 P(x_i) \\ &= E[(X - E(X))^2] \\ &= E(X^2) - (E(X))^2 \end{aligned}$$

For the above example  $V(X) = 2.222$

Standard Deviation of X is given by  $S(X) = V(X)^{1/2}$

For the above example

$$S(X) = [2.222]^{1/2} = 1.4907$$

# Continuous Distribution

- 1-  $f(x) = 0$  if  $x$  is not in the range of  $X$ ,
- 2-  $f(x) \geq 0$ ,
- 3- 
$$\int_{-\infty}^{\infty} f(x) dx = 1$$
- 4- 
$$F(x_1) = \int_{-\infty}^{x_1} f(x) dx$$
- 5- 
$$E(X) = \int_{-\infty}^{\infty} xf(x) dx$$
- 6- 
$$V(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx$$
- $$= \int_{-\infty}^{\infty} x^2 f(x) dx - (E(X))^2$$



## Function of a Random Variable

- Let  $X$  denote a R.V. Suppose the R.V.  $Y$  is related to  $X$  by  $Y = aX + b$  where  $a, b$  are arbitrary constant.
- $E(Y) = E(aX+b)$
- $$= \int_{-\infty}^{\infty} (ax + b) f(x) dx = a \int_{-\infty}^{\infty} xf(x) dx + b \int_{-\infty}^{\infty} f(x) dx$$
- So  $E(Y) = aE(X) + b$
- $V(X) = V(aX + b)$
- $$= \int_{-\infty}^{\infty} (ax + b)^2 f(x) dx - (E(aX+b))^2$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} a^2 x^2 f(x) dx + 2ab \int_{-\infty}^{\infty} x f(x) dx + b^2 \int_{-\infty}^{\infty} f(x) dx - (E(aX + b))^2 \\
&= \int_{-\infty}^{\infty} a^2 x^2 f(x) dx + 2ab E(X) + b^2 - (E(aX + b))^2 \\
&= \int_{-\infty}^{\infty} a^2 x^2 f(x) dx - a^2 (E(X))^2 \\
&= a^2 \left[ \int_{-\infty}^{\infty} x^2 f(x) dx - (E(X))^2 \right] \\
&= a^2 V(X)
\end{aligned}$$

# Some Common Distribution

## 1) Bernoulli Distribution

It is a discrete distribution having only two outcomes. Let  $X$  has been defined and it takes the value 0 with probability  $p$  and 1 with probability  $q=1-p$  i.e.,

$X$	$p(x)$
0	$p$
1	$q$

$$\text{CDF} = F(x) = 0$$

$$x < 0$$

$$p$$

$$0 \leq x < 1$$

$$1$$

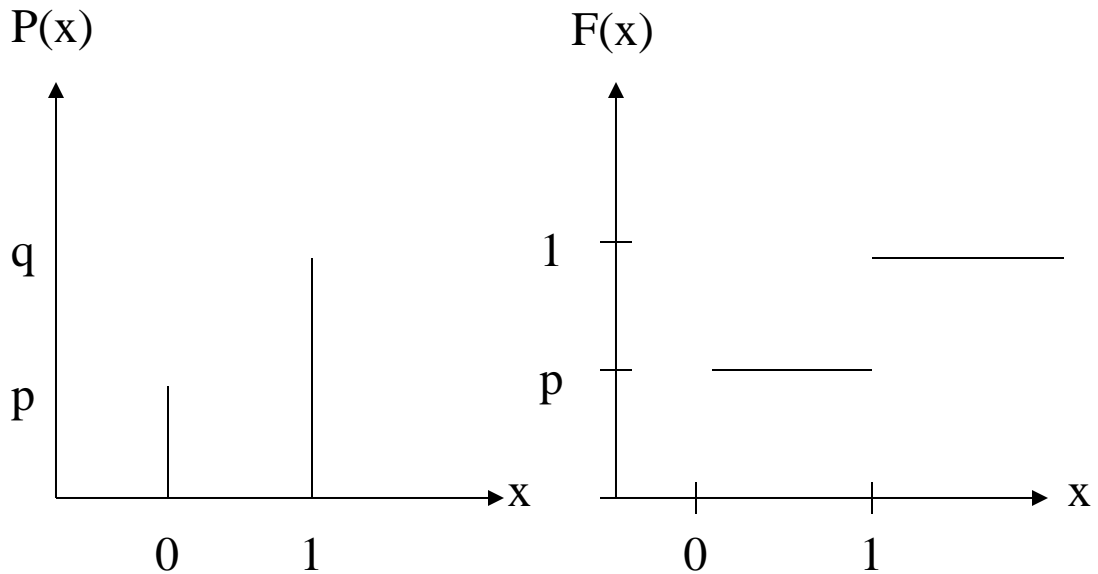
$$x \geq 1$$

$$\mu = E(X) = 0.p + 1.q = q$$

$$\sigma^2 = V(X) = E(X^2) - (E(X))^2$$

$$= 0.p + 1.q - q^2 = q(1-q) = pq$$

$$\sigma = (pq)^{1/2}$$



### Example

Consider the experiment of tossing a fair coin. Let  $X$  assume the value  $0$  if a head appears and  $1$  for tail. Then  $p = q = \frac{1}{2}$  and  $\mu(\text{mean}) = p = \frac{1}{2}$  &

$$\sigma^2(V(X)) = pq = 1/4$$

## Poisson Distribution

It is a discrete distribution.

$$P(X = k) = P_X(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

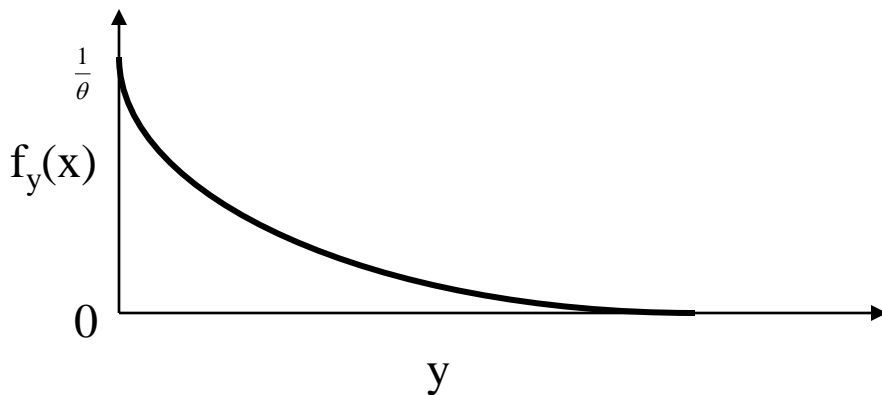
$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = 1$$

Mean  $\lambda$

Variance =  $\lambda$

# Exponential Distribution

$$f_x(y) = \begin{cases} \frac{1}{\theta} e^{-\frac{y}{\theta}} & y \geq 0 \\ 0 & y < 0 \end{cases}$$



$$\int_{-\infty}^{\infty} f_x(y) dy = \int_0^{\infty} \frac{1}{\theta} e^{\frac{-y}{\theta}} dy = -e^{\frac{-y}{\theta}} \Big|_0^{\infty} = 1$$

$$\mu = E(X) = \frac{1}{\theta} \quad , \quad \sigma^2 = V(X) = \frac{1}{\theta^2}$$

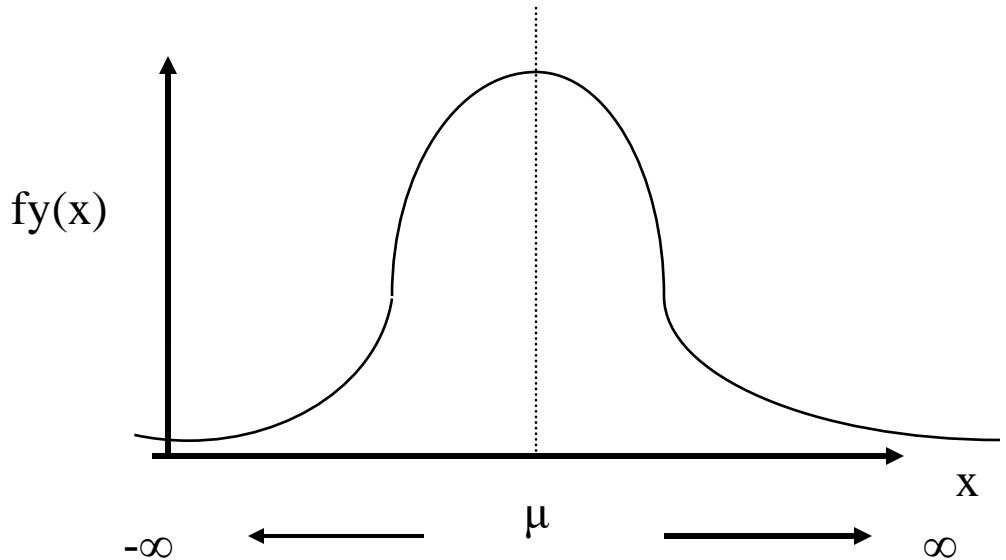
$$\begin{aligned} CDF = F_x(b) &= \int_{-\infty}^{\infty} f_x(y) dy \\ &= 0 \qquad b < 0 \end{aligned}$$

$$= \int_0^{\infty} \frac{1}{\theta} e^{\frac{-y}{\theta}} dy = 1 - e^{\frac{-b}{\theta}} \qquad b \geq 0$$

# Normal Distribution

$$f_x(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad -\infty < y < \infty$$

it is called a  $N \approx (\mu, \sigma)$  i., e., it has two parameters.





$$F_x(b) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

This integral does not exist in closed form. But there exist tables of values for the standard normal distribution with mean=0 and var.=1 which is given by:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad (\mu=0, \sigma=1)$$

*If  $X$  is  $N \approx (\mu, \sigma)$  then the random variable  $Z$  is*

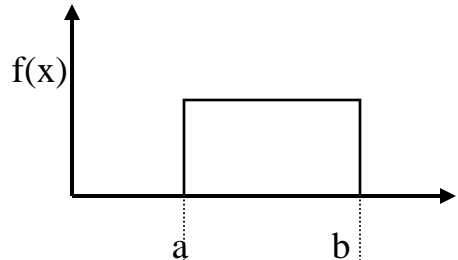
*$Z = (X - \mu) / \sigma$  is a  $N \approx (0,1)$*

*$\therefore X = \mu + Z\sigma$  i.e.,  $X$  is  $N \approx (\mu, \sigma)$*

# Uniform Distribution

$$f(x) = \frac{1}{(b-a)} \quad a \leq x \leq b$$

$$= 0 \quad \text{otherwise}$$



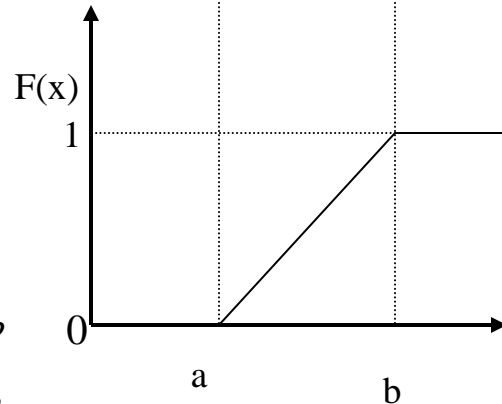
- Rectangular distribution
- The interval  $[a, b]$  is called the range of the distribution

- CDF is given by:

$$F(X) = \int_{-\infty}^x f(t) dt = 0 \quad x < a$$

$$= \frac{x-a}{b-a} \quad a \leq x \leq b$$

$$= 1 \quad x > b$$



$$Mean = \mu = \frac{a+b}{2} \quad Var. = \sigma^2 = \frac{(b-a)^2}{12}$$

Find the above mean using the form  $F(X) = \int_{-\infty}^{\infty} xf(x)dx$

# Variance Reduction Techniques

## 1 Increase sample size N where

$$V = E(x^2) - (E(X))^2 \propto \mathbf{V} \quad 1/N$$

### Examples

use  $f(x) = e^{-x}$  it is exponential distribution with mean  $\lambda=1$  i.e.,  $F(X) = 1 - e^{-x}$

i	R.N.	$X_i = -\ln(1 - r_i)$
1	0.495	0.684
2	0.335	0.408
3	0.791	1.568
4	0.469	0.633
5	0.279	0.326
6	0.698	1.199
7	0.013	0.014
8	0.761	1.433
9	0.290	0.343
10	0.693	1.183

Total = 7.793 so mean =  $7.793/10 = 0.779$   
with error =  $((1-0.779)/1)*100 = 22.1\%$

## 2-Use Complement Random Numbers for the above example $f(x) = e^{-x}$

i	$r_i$	$x_i = -\ln(1-r_i)$	$\hat{r}_i = 1-r_i$	$x_i = -\ln(1-\hat{r}_i)$
1	0.495	0.684	0.505	0.702
2	0.335	0.408	0.665	1.092
3	0.791	1.568	0.209	0.234
4	0.469	0.633	0.531	0.756
5	0.279	0.326	0.721	1.275
6	0.698	1.199	0.302	0.359
7	0.013	0.014	0.987	4.305
8	0.761	1.433	0.239	0.272
9	0.290	0.343	0.710	1.236
10	0.693	1.183	0.307	0.366
Total		= 7.793		=10.597

$$\begin{aligned}\text{Estimated mean} &= 0.5(0.7793 + 1.0597) \\ &= 0.92\end{aligned}$$

$$\text{With Error} = ((1 - 0.92) / 1) * 100 = 8.05\%$$

### Example:

determine the mean of the random variable with

$$F(X) = x^2 \quad 0 \leq x \leq 1$$

$$f(x) = 2x, \quad 0 \leq x \leq 1$$

$$\bar{x} = \int_0^1 x f(x) dx = \int_0^1 2x^2 dx = 2/3$$

$$F(X) = X^2 = r \quad \text{So } X_i = + r_i^{0.5}$$

$R_i$	$X_i = r_i^{0.5}$
0.628	0.792
0.207	0.455
0.662	0.814
0.774	0.880
0.360	0.600
0.951	0.975

TOTAL = 4.516   Estimated mean =  $4.516/6 = 0.753$  with error = 13%

## Using Complement Random Variables:

$r_i$	$\hat{r}=1-r_i$	$x_i=\sqrt{r_i}$	$\sqrt{(1-r_i)}$
0.628		0.792	
	0.372		0.61
0.207		0.455	
	0.793		0.891
0.662		0.814	
	0.338		0.581

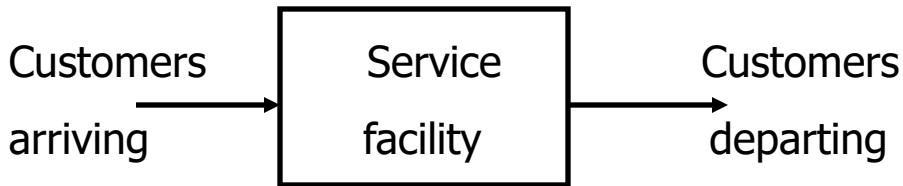
$$\text{Total} = 2.061 + 2.082 = 4.143$$

$$\text{So the mean} = 4.143/6 = 0.691$$

$$\text{with Error} = ((2/3 - 0.691)/(2/3)) * 100 = 4\%$$

# Introduction to Queuing Theory

A queuing system is a system in which customers arrive, wait if that service is not immediately available, receive the necessary service, and then depart.



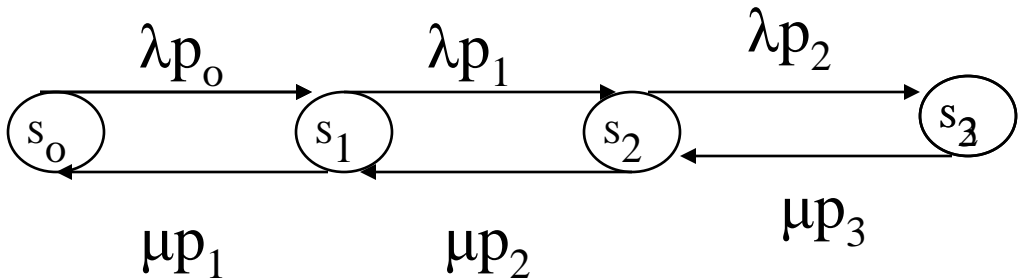
## Characteristics of any queuing system

- (A) The arrival pattern (distribution of arriving customers)
- (B) The service process (distribution of service time)
- (C) The number of parallel servers
- (D) The system capacity
- (E) The queuing discipline (FIFO, LIFO, priority)

So any queuing system can be described as A/B/C/D/E

## The items of interest concerning any queuing model:

- 1. Queuing length ( $L_q$ )
- 2. Time in the system ( $W$ )
- 3. Idle and busy time of a server
- **M/M/1/ $\infty$ /FIFO system**
- Derivation using stochastic balance



- Using balance
- at point  $S_0$   $\mu p_1 = \lambda p_0$   $p_1 = \lambda / \mu p_0$
- at point  $S_1$   $(\lambda + \mu) p_1 = \lambda p_0 + \mu p_2$   
 $p_2 = (\lambda / \mu)^2 p_0$
- and so on we get  $p_n = (\lambda / \mu)^n p_0$



$$\sum_{n=0}^{\infty} p_n = 1 = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n p_0$$

$$\therefore p_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{\sum_{n=0}^{\infty} \rho^n} =$$

$$\therefore p_0 = \frac{1}{\frac{1}{1-\rho}} = 1-\rho$$

$$\therefore p_n = (1-\rho)\rho^n \quad \text{for } n = 0, 1, 2, 3, \dots$$

$$L = \frac{\rho}{1-\rho} \quad \rho < 1,$$

$$L_q = \frac{\rho^2}{1-\rho} \quad \rho < 1$$

- Expected number in the system (L)
- $L = E(X)$

$$= \sum_{x=0}^{\infty} x p_n = \sum_{x=0}^{\infty} x(1-\rho)\rho^x$$

$$L = (1-\rho) \sum_{x=0}^{\infty} x \rho^x$$

$$= (1-\rho)\rho \sum_{x=0}^{\infty} x \rho^{x-1}$$

$$= (1-\rho)\rho \frac{d}{d\rho} \sum_{x=0}^{\infty} \rho^x$$

$$= (1-\rho)\rho \frac{d}{d\rho} \frac{1}{1-\rho}$$

$$= (1-\rho)\rho \frac{1}{(1-\rho)^2}$$

$$\therefore L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda} \quad \rho < 1 \text{ i.e., } \mu > \lambda$$

- Expected Number in the queue  $L_q$

$$L_q = \sum_{x=1}^{\infty} (x-1)p_x$$

$$= \sum_{x=1}^{\infty} xp_x - \sum_{x=1}^{\infty} p_x$$

$$= L - (1 - p_0)$$

$$= L - 1 + p_0$$

$$= \frac{\rho}{1-\rho} - 1 + 1 - \rho$$

$$\therefore L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$\rho < 1, \text{ i.e., } \mu > \lambda$$

## Using Little's formula

The Expected time in the system ( $W$ )

$$\begin{aligned} W &= \frac{L}{\lambda} \\ &= \frac{1}{\mu - \lambda} \end{aligned} \quad \mu > \lambda$$

The Expected time in the Queue ( $w_q$ )

$$\begin{aligned} W_q &= W - \frac{1}{\mu} \\ &= \frac{\lambda}{\mu(\mu - \lambda)} \end{aligned} \quad \mu > \lambda$$

## Random Number Generation (Pseudo R.N.)

\_There are many algorithms for random number generations, but we are going to emphasize on the most common one which is known as a Congruential method.

$$R_{i+1} = ( a R_i + c ) \text{ modulo } T$$

where

$R_{i+1}$  is the new uniform random number,

$R_i$  is the previous uniform random number

$a$ ,  $c$ , and  $T$  are arbitrary constants

## Example

For  $a = 1$ ,  $c = 4$ ,  $T = 5$ ,  $R_0 = 2(\text{seed})$ ; generate 10 random numbers.

## Solution

$$R_0 = \textcircled{2}$$

$$R_1 = \text{Rem}[\{(1)(2) + 4\}/5] = 1$$

$$R_2 = \text{Rem}[\{(1)(1) + 4\}/5] = 0$$

$$R_3 = \text{Rem}[\{(1)(0) + 4\}/5] = 4$$

$$R_4 = \text{Rem}[\{(1)(4) + 4\}/5] = 3$$

$$R_5 = \text{Rem}[\{(1)(3) + 4\}/5] = \textcircled{2}$$

$$R_6 = \text{Rem}[\{(1)(2) + 4\}/5] = 1$$

$$R_7 = \text{Rem}[\{(1)(1) + 4\}/5] = 0$$

$$R_8 = \text{Rem}[\{(1)(0) + 4\}/5] = 4$$

$$R_9 = \text{Rem}[\{(1)(4) + 4\}/5] = 3$$

$$R_{10} = \text{Rem}[\{(1)(3) + 4\}/5] = \textcircled{2}$$

# Generation of a Random Observation Following a certain distribution

1- For a Simple Discrete Distribution we use the allocation method, see the previous examples.

2- For continuous Distribution we use the inverse method if and only if the distribution is reversible, see the following examples.

## Example 1

Generate a random observation from the uniform distribution between  $a$  and  $b$ , i.e.,  $U(a, b)$

$$f(x) = \frac{1}{b - a}$$

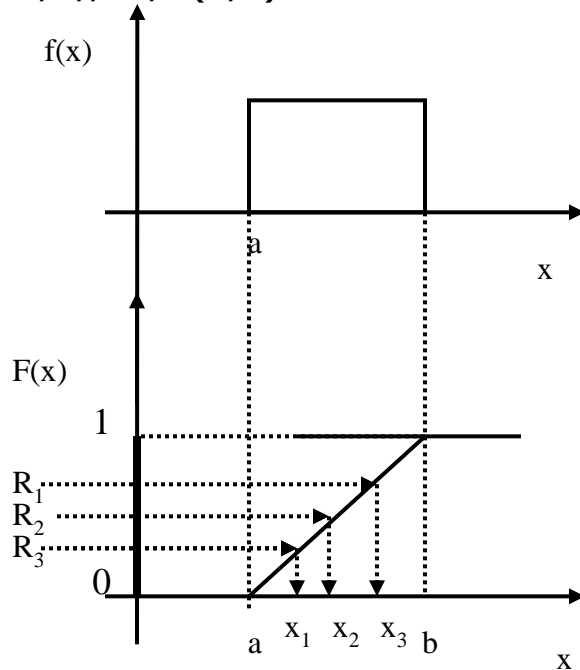
$$F(x) = \frac{x - a}{b - a}$$

$$U(0,1) \longrightarrow U(a,b)$$

$$R = F(x) = \frac{x - a}{b - a}$$

$$\therefore x = a + R(b - a)$$

$$\text{i.e., } x = F^{-1}(R)$$



## Most important Example:

Generate Random Observations from Exponential distribution with mean  $\mu$

$$f(x) = \mu e^{-\mu x}$$

$$F(x) = 1 - e^{-\mu x}$$

$$R = F(x) = 1 - e^{-\mu x}$$

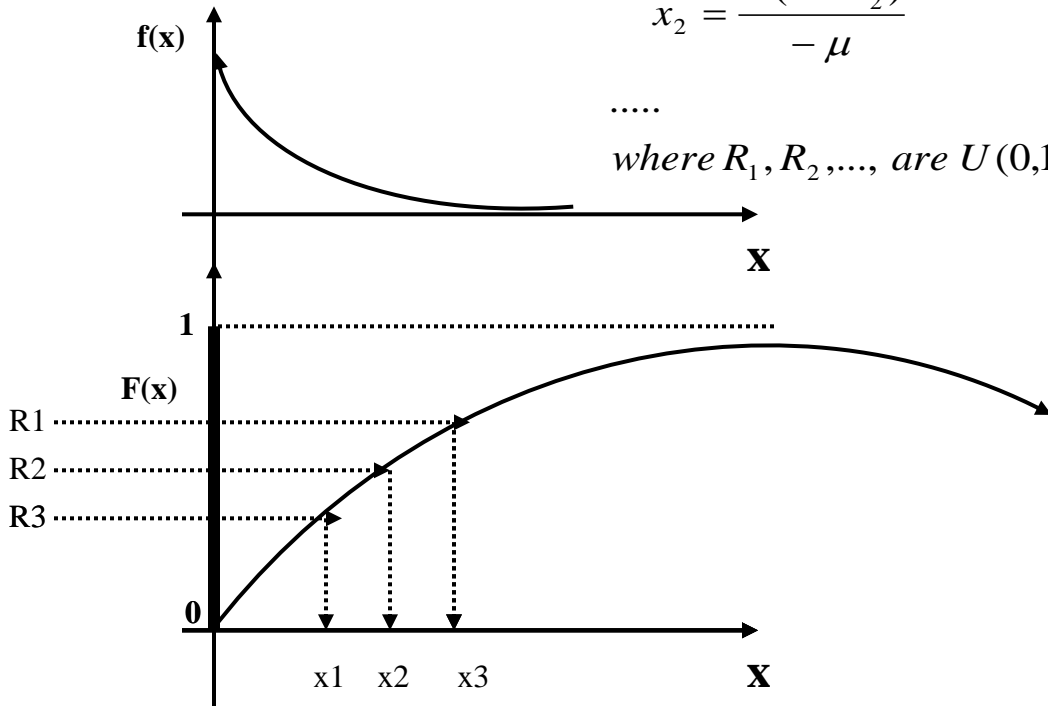
$$\therefore x = \frac{\ln(1 - R)}{-\mu}$$

$$\text{so } x_1 = \frac{\ln(1 - R_1)}{-\mu}$$

$$x_2 = \frac{\ln(1 - R_2)}{-\mu}$$

.....

where  $R_1, R_2, \dots$ , are  $U(0,1)$





## Illustrative Example:

### Sales of life insurance problem:

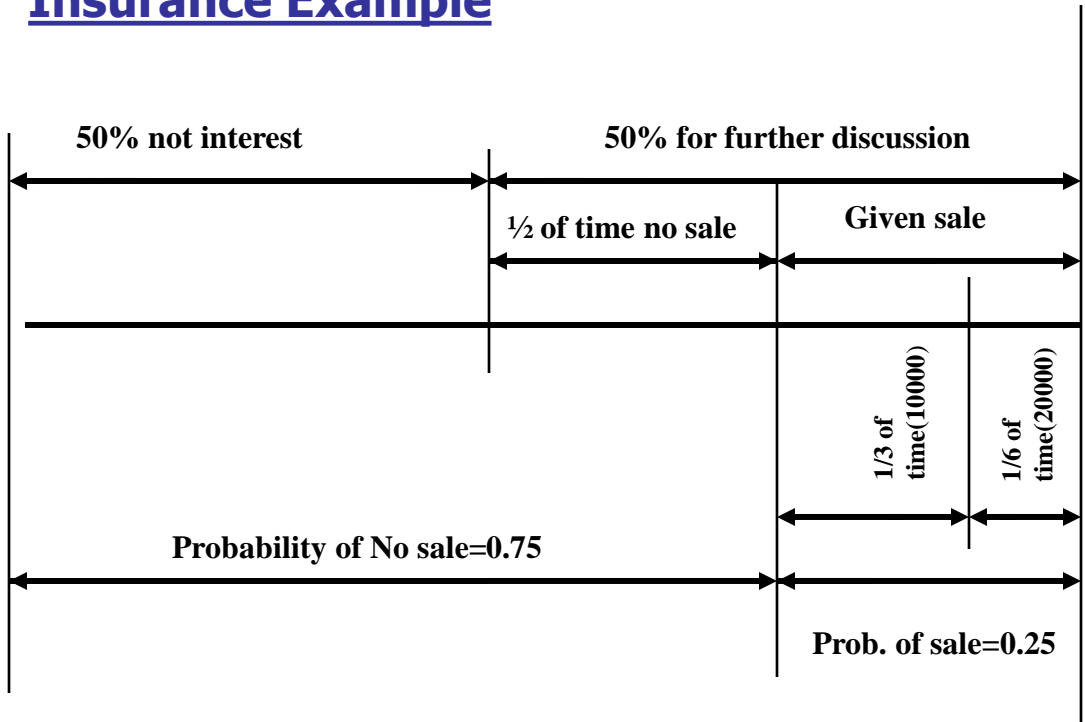
From the previous sales history, each time a house call is made:

50% not interested in purchasing,

50% willing for further discussions.

After discussions  $\frac{1}{2}$  of time the visit will result in "No Sale",  $\frac{1}{3}$  of time will result in sale of 10,000 L.E. policy,  $\frac{1}{6}$  of time will result in sale of 20,000 L.E. policy. Using a simulated sample of 20 visits, determine the probability that a sale is made and the expected policy value for each policy, assuming a sale is made.

## Insurance Example



### Theoretical Solution:

Probability of No Sale = 0.75

Probability of Sale = 0.25

Pro(10000 policy | sale) =  $\frac{2}{3}$  = 0.667

Pro(20000 policy | sale) =  $\frac{1}{3}$  = 0.333

Expected value =  $0.667 \times 10000 + 0.333 \times 20000$   
= 13333 L.E.

## Simulation of 20 insurance calls

[illegible]

From the table, 13 of the visits will result in a “No sale”.

$$P(\text{No sale}) = 13/20 = 0.65$$

Given Sale, we have 4x10,000 and 3x20,000 policies soled so

$$P(10000) = 4/7 = 0.5714$$

$$P(20000) = 3/7 = 0.4286$$

$$\begin{aligned}\text{The Expected Vale} &= 0.5714 \times 10000 + 0.4286 \times 20000 \\ &= 14,286 \text{ L.E.}\end{aligned}$$

### **Comparison between Theoretical & Simulation (Validation):**

	Prob.of No Sale	Prob.of Sale	Prob.of 10,000	Prob.of 20,000	Expected Value
Simulation	0.65	0.35	0.5714	0.4286	14,285
Theoretical	0.75	0.25	0.667	0.3333	13,333

## Time Management Methods

There are two time references used in simulation:

1) Simulation Time: The period of time simulated by the model – 30min., 5years, 100 years whatever interval the researcher is interested in. This time is reset to zero at the beginning of the simulation run and acts as a counter to the number of simulation time units.

2) Run time: The time that the computer takes to simulate the period of interest.

### We have two ways for time changes (periodic &Event Scan):

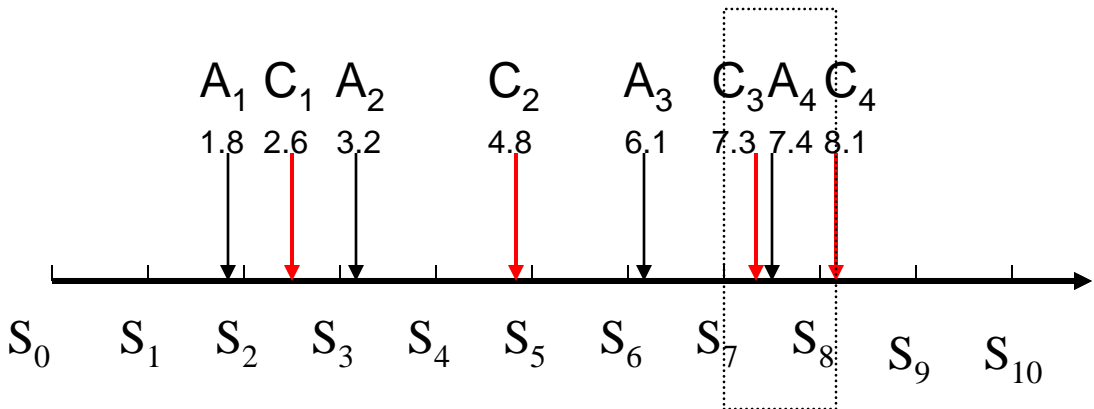
- i) The Periodic Scan or Fixed time increment technique: In this technique the simulation clock is adjusted by one predetermined uniform unit and then examines the system to determine whether any events occurred during that interval. If any occurred, the event or events are simulated; otherwise no action is taken.

The simulation clock is then advanced another unit, and the process is repeated. In this technique, the exact time of the occurrence of particular event is largely ignored. All events that occur during a given interval are treated as if these events occurred at the end of that interval.

### **Example :**

Consider the simulation of a single-bay-service station for a period of ten minutes. Assume that the system is empty at the beginning and at the end of the period. Assume also that four customers are serviced, their arrivals occur at simulation times 1.8, 3.2, 6.1 and 7.4, and that service completions occur at simulation times 2.6, 4.8, 7.3 and 8.1.

If a time interval of one minute is used, arrival events are denoted by  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$ . While completion events are denoted by  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  (see the following figure.).



$C_3$  and  $A_4$  are both considered to have occurred at clock time 8.0; although  $C_3$  actually occurred before  $A_4$ .

The problem of periodic scan is that events separated in time appear to occur simultaneously. In this example at time period  $S_7 - S_8$  the system appears to be idle while actually it is not.

- The big problem is to determine the length of the interval to be used.
- The time increment used in the periodic scan approach should be as small as possible to minimize the possibility of lost information. But the smaller the time unit, the longer the number of calculation necessary to complete the simulation.

Thus the actual run time of the simulation and hence the cost of simulation is increased. So there is a trade-off between the precision and maximum run time.

## ii ) The Event Scan

In this technique the clock is advanced by the amount necessary to trigger the occurrence of the next event. Thus the time advance intervals are of variable lengths.

While some information can be lost in the periodic scan, the event scan avoids this problem; because the clock is advanced only to the next occurrence time.

In periodic scan, the exact occurrence time of events is not known, because each event is treated as if occurred at the end of the time interval in which it occurred.. This problem is overcome in Event Scan approach.



### Example

A bank of machines in a manufacturing shop breaks down according to the following inter arrival time distribution. The time it takes one repair person to complete the repair of a machine is given in the service time distribution.

Inter arrival time (hours)	P(X)	Mapping of R.N.	Service Time (hours)	P(X)	Mapping of R.N.
0.5	0.30	00 – 29	0.5	0.25	00 – 24
1.0	0.22	30 – 51	1.0	0.20	25 – 44
1.5	0.16	52 – 67	2.0	0.25	45 – 69
2.0	0.10	68 – 77	3.0	0.15	70 – 84
3.0	0.14	78 – 91	4.0	0.10	85 – 94
4.0	0.08	92 – 99	5.0	0.05	95 – 99
	1.00			1.00	

Simulate the breakdown of 5 machines. Calculate the average machine downtime using two repair persons and the following random numbers sequence. Both repair persons cannot work on the same machine.

R.N.: 30, 81, 02, 91, 51, 08, 28, 44, 86, 84, 29, 08, 37, 34, 99.

**Solution**

RN	Interarrival Time	Time of Arrival	RN	Service Time	Repair Man1		Repair Man2		Waiting Time	Down Time
					On	Off	On	Off		
30	1.0	1.0	81	3	1	4			0	3
02	0.5	1.5	91	4			1.5	5.5	0	4
51	1.0	2.5	08	0.5	4	4.5			1.5	2
28	0.5	3.0	44	1	4.5	5.5			1.5	2.5
86	3.0	6.0	84	3	6	9			0	3
Total Down Time										14.5

Average Down Time = 14.5 / 5 = 2.9 hours.

## Example for a continuous distribution:

Simulate a system M/M/1 with arrival rate  $\lambda = 20/\text{h}$  and service rate  $\mu = 25/\text{h}$ . Assuming system empty at time zero.

- Using time scan (periodic scan)
- Event scan

### Solution:

#### 1. Using Time Scan :

$$\begin{aligned} P(\text{arrival in one minute period}) &= 1 - e^{-\lambda t} \\ &= 1 - e^{-1/3(1)} \\ &= 0.281 \end{aligned}$$

$$\begin{aligned} P(\text{departure in one minute period}) &= 1 - e^{-5/12(1)} \\ &= 0.341 \end{aligned}$$

So let  $0 \leq \text{R.N.} \leq 0.281$   $\Longrightarrow$   
Arrival,

$0 \leq \text{R.N.} \leq 0.341$   $\Longrightarrow$  Departure.

Time in min.	Customer In system	Random number		Arrival Yes/No	Departure Yes/No
		Arrival	Departure		
0	0	0.096	-	Yes	-
1	1	0.569	0.665	No	No
2	1	0.764	0.842	No	No
3	1	0.492	0.224	No	Yes
4	0	0.950	-	No	-
5	0	0.610	-	No	-
6	0	0.145	-	Yes	-
7	1	0.484	0.552	No	No
8	1	0.350	0.59	No	No
9	1	0.450	0.410	No	Yes
10	0	0.802	-	No	-

## 2. Using Event Scan (Event Driven):

The time until next arrival is given by

$$X_{arr.} = \frac{-\ln(1-R)}{\lambda} \quad \lambda = \frac{1}{3} \text{ min.}$$
$$= -3\ln(1-R)$$

The time until next departure is given by

$$X_{dep.} = \frac{-12\ln(1-R)}{5} \quad \mu = \frac{5}{12} \text{ min.}$$

Time in min.	Event	Customer In system	Random Number		Min. until next		Next
			Arrival	Departure	Arr.	Dep.	Event
0		0	0.096	-	0.303	-	A
0.303	A	1	0.569	0.665	2.525	2.625	A
2.828	A	2	0.764	-	4.332	0.1	C
2.928	C	1	-	0.842	4.232	4.428	A
7.160	A	2	0.492	-	2.032	0.196	C
7.356	C	1	-	0.224	1.836	0.609	C

## Example for discrete distribution:

Two-stage assembly line are producing a large product. There is no room in between the two line ; so the output of the first line is fed to the second line directly else the first line must wait until second line to be free. The service distribution for both line are given below:

Service seconds	Time Freq. for 1 <sup>st</sup> line	R.N. Allocation	Time Freq. for 2nd line	R.N Allocation
10	4	00 – 03	4	00 - 07
20	6	04 - 09	5	08 - 17
30	10	10 – 19	6	18 - 29
40	20	20 – 39	7	30 - 43
50	40	40 – 79	10	44 - 63
60	11	80 – 90	8	64 - 79
70	5	91 – 95	6	80 - 91
80	4	96 - 99	4	92 - 99
	100		50	

Item	First Line				Waiting Time	Second Line			
	No.	R.N.	Start Time	Service Time	Finish Time	R.N.	Start Time	Service Time	Finish Time
1	56	00	50	50	20	83	50	70	120
2	55	50	50	100		47	120	50	170
3	84	120	60	180		08	180	20	200
4	36	180	40	220		05	220	10	230
5	26	220	40	260		42	260	40	300
6	95	260	70	330	30	95	330	80	410
7	66	330	50	380		17	410	20	430
8	03	410	10	420	10	21	430	30	460
9	57	430	50	480		31	480	40	520
10	69	480	50	530		90	530	70	600

470

60

- The output rate average = 600/10 = 60 seconds
- Utilization of 1<sup>st</sup> line = 470/530 = 88.7%
- Utilization of 1nd line = 430/550 = 78.2%