MTH 101 LECTURE NOTE

ELEMENTARY MATHEMATICS 1
DEPARTMENT OF MATHEMATICS
FACULTY OF NATURAL SCIENCES
UNIVERSITY OF JOS
NIGERIA
mathslist@ unijos.edu.ng

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Contents

Chapter 1

Elementary Set Theory and Number System

1.1 Introduction

The theory of sets is an important tool in modern Mathematics. The study of sets which began in the later part of the nineteenth century by George Cantor (1845-1918) has assumed a central role in every branch of Mathematics today. For example, algebra is concerned with sets of numbers and operations on these sets whereas analysis deals mainly with sets of functions.

Definition 1.1 (*set*)

A set can be defined as a collection of well-defined objects.

Every object that belongs to the set is called a member or an element of the set. For example, the collection of all traders in Jos forms a set and any trader in Jos would be a member of this set. A teacher in Jos would however not be a member of this set. Another example of a set is the collection of all students in Nigerian Universities. This clearly shows that any student who is not in any of the Nigerian Universities would not be a member of this set. A teacher in Jos is also not a member of this set. We use upper cases such as $A, B, C, ...$ to denote sets whereas the lower cases such as $a, b, c, ...$ are used for members of the set.

The statement $x \in A$ means "x is a member of or an element of or x belongs to the set A" while $y \notin A$ means that y is not a member of or is not an element of A.

1.1.1 Methods of Describing Sets

(a) **Roster method:** The most elementary method of describing a set is by listing its members. This is conventionally done by enclosing them in a pair of braces, $\{\}$. In this convention, each element's name is separated by a comma. For example the set X, whose elements are the months of the year that begin with the letter A is $X = \{April, August\}$

The set of integers whose squares is less than 20 can be written as $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

(b) **The Set-Builder Notation:** The set-builder notation (or the rule method) is a method of specifying a set in which, rather than listing all the elements of a set, specifies a set by writing within braces a formal description that describes or identifies the members of the set. As part of this formal description, we use a variable (such as x) to represent any one of the members of the set. The following convention is used in a set-builder notation.

Name of Set = $\{x|x$ is word description of the set $\}$ or

Name of Set = $\{x{:}x$ is a word description of the set$\}$

Note that the slash (|) or the colon (:) is pronounced "such that". Now considering the example we mentioned earlier, that is the set X whose elements are the months of the year that begin with the letter A, we can describe the set using the set-builder notation as follows:

$$X = \{x : x \text{ is a month of the year that begins with the letter } A\}.$$

The statement is read: X is the set of all x such that x is a month of the year that begins with the letter A. Consider the set whose elements are the days in a week. Such a set can be described using the set-builder notation as:

$A = \{x|x$ is a day of the week.$\}$

If we use the roster notation discussed earlier, the set can be written as:

$A = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\}$.

Exercise 1.1

1) Using the roster method, describe the following sets:

(i) $A = \{x : x$ is a month of the year$\}$. (ii) $B = \{x : x^2 - 3x + 2 = 0\}$

(iii) $C = \{x : x$ is a continent$\}$ (iv) Set of integers between 1 and 50 divisible by 9

2) Use the set-builder notation to describe the following sets.

(a) $A = \{a, e, i, o, u\}$ (b) $B = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

1.1.2 Types of Sets

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(1) **Finite Set**: Any set that contains a definite number of elements is called a finite set. The following are examples of finite sets,

$A = \{1, 2, 3, 4, 5, 6\}$
$B = \{-6, -5, -4, ..., 4, 5, 6\}$
$C = \{a, b, x, y, z\}$

(2) **Infinite Sets**: These are sets that have unlimited number of elements. The following are examples of infinite sets.

$X = \{1, 2, 3, ...\}$ $Y = \{... - 4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$

The three dots indicate that the established pattern will continue indefinitely.

(3) **Empty Sets (or Null Sets)**. Any set that has no element in it is called an empty or a null set. We denote such sets by the symbol $\emptyset$ or $\{\}$. We must exercise caution not to write $\{\emptyset\}$ to indicate the null set. This notation does not represent the null set because there is an object contained within the braces. The symbol $\{\emptyset\}$ represents the set with the element $\emptyset$. An obvious example of the null set is the set of traders who live on the moon. This is an empty set because no trader lives on the moon.

(4) **The Unit Set (singleton set)**. Any set that has exactly one element is called a unit set or a singleton set. Consider the set of all months with less than 30 days. This set has only one element, namely, February. The following are also singleton sets.(i) $A = \{\text{gari}\}$,(ii) $B = \{2\}$

Definition 1.2(Subsets)

A set X is said to be a subset of set Y if every element of set X is an element of set Y. We use the notation $X \subseteq Y$ to describe this, so that the symbol $\subseteq$ should be read as " is a subset of '. We also say that X is contained in Y. The set X is said to be a proper subset of Y if every element of set X is an element of set Y, and also set Y has at least one other element that is not in X. We denote this by $X \subset Y$. This symbol is read as "X is a proper subset of Y" sometimes we use $Y \supset X$ to mean the same thing but that should be read as Y contains X. For instance if the set $X = \{-3, -2, -1, 0, 1, 2, 3\}$ and $Y = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ then the set $X \subset Y$ since every element of the set X is in Y and in addition the set Y has some elements that are not in X.

Definition 1.3(Equality of Sets)

Two sets are said to be equal if they have exactly the same elements. In other words equality between two sets A and B imply that $A \subseteq B$ and $B \subseteq A$. For example if $X = \{a, b, f, g\}$ and $Y = \{f, a, b, g\}$ then the set X is equal to set Y and it is denoted as $X = Y$. The order in which the elements of a set are written is not important. The set $A = \{\text{all the vowels in English Alphabet}\}$ and the set $B = \{e, o, a, u, i\}$ are also equal since both sets have the same elements. To prove that $A = B$ it suffices to show that $A \subset B$ and $B \subset A$.

Example

Let $A = \{a, b, c, d\}$ and $B = \{a, b, f, d, e, c, g, h\}$. Every element in set A, i.e., a, b, c and d is also contained in set B. Notice further that the two sets are not equal; so $A \subseteq B$. The set $X = \{\text{All basketball players in Jos }\}$ is a subset of the set $Y = \{\text{all athletes in Jos.}\}$

Remark

Every set is a subset of itself and the empty is a subset of any given set; and these two sets are the improper subsets of any given set. This calls for a clear distinction between a subset and a proper subset. The notations used to represent improper subset is $\subseteq$ and proper subset is $\subset$ should serve as an aid in distinguishing between the two concepts.

Definition 1.4: Sets whose members are sets are referred to as class or family.

Definition 1.5 (Power Set)

Given a set A, the power set A denoted $\rho(A)$ is the set of all subsets of A. For instance the power set of the set $\{x, y\}$ is $\rho(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Generally for all integers $n \geq 0$ if a set X has n elements, the the power set, $\rho(X)$ has 2^n elements.

Definition 1.6 (Cardinality of a Set)

The cardinal number of any set A is defined as the number of elements in set A. This is denoted by the symbol $n(A)$, read as "the number of elements in set A."

Example (i) The set $X = \{1, 2, 4\}$ has cardinal number 3. That is $n(X) = 3$

(ii) $A = \{x, y, z, Moses, Yam\}$ has cardinal number $n\,(A) = 5$.

(iii) The set $Y = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ has cardinal number $n(Y) = 9$.

Remark Repetition of elements in a given set does not alter the set nor its cardinality. For example, the set $A = \{x, y, z\}$ is the same as the set $B = \{x, x, z, y, z\}$ .Their cardinality is the same.

Definition 1.7 (Equivalent Sets)

Set A and B are said to be equivalent if they can be put into one-one correspondence. Consider the two sets $A = \{2, 4, 6\}$ and $B = \{a, b, c\}$. These two sets are not equal, but they have exactly the same number of elements. Each set contains three elements. Note that in every day English "equivalent" means the same thing. However in the discussion of sets, equality and equivalence are not the same thing. If two sets are equal, they are also equivalent, but if two sets are equivalent they are not necessarily equal.

Definition 1.8 (The Universal set)

Whenever we are discussing problems involving sets, we normally employ the use of a general set called the universal set. This contains the elements of the other sets being considered in a given problem. We may wish to define the universal set as follows: If all sets in a given discussion are to be subsets of a fixed overall set U then the set U is called the Universal set. For example if we wish to discuss the set of negative integers and the set of positive integers, then a suitable universal set is the set of integers. Note that the universal set may vary from discussion to discussion.

1.3 SET OPERATIONS:

If A and B are sets, it is sometimes natural to wish to unite their elements into a comprehensive set. One way of describing such a comprehensive set is to require it to contain all the elements that belong to at least one of the two members of the pair (A, B). The comprehensive set that consist of elements from A and B is called the union of the two sets.

Definition 1.9 (Union of Sets) The union of two sets A or B denoted $A \cup B$ (read as " A union B ") means the set of all elements that are either in A or B or both. Using mathematical notation, we write

$A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Example If $A = \{1, 2, , 5, 6\}$ and $B = \{0, 1, 2, 3, 5\}$ then $A \cup B = \{1, 2, , 5, 6\} \cup \{0, 1, 2, 3, 5\} = \{0, 1, 2, 3, 5, 6\}$

Example If $A = \{$Orange, Apple, John$\}$ and $B = \{$Orange,Yam,Goat,Rice$\}$ then

$A \cup B = \{$Orange, Apple, John, Yam, Goat, Rice.$\}$

More generally if $A_1, A_2, ..., A_n$ are sets, then their union is the set of all objects which belong to at least one of them and it is denoted by $A_1 \cup A_2 \cup ... \cup A_n = \bigcup\limits_{i=1}^{n} A_i$.The union of infinite sets, $A_1, A_2, ..., A_n...$ is given by $A_1 \cup A_2 \cup ... \cup A_n \cup ... = \bigcup\limits_{i=1}^{\infty} A_i$.

Here are some basic properties about the union of sets.

(i) $A \cup \phi = A$

(ii) $A \cup B = B \cup A$ Commutativity of union

(iii) $A \cup (B \cup C) = (A \cup B) \cup C$ Associativity of union

(iv) $A \cup A = A$ Idempotent

(v) $A \cup U = U$ Universal union

(vi) $A \subset B$ if and only if $A \cup B = B$

The proofs are based on the corresponding elementary properties of the logic operator " or".

Definition 1.10 (Intersection of Sets)

The intersection of two sets A and B, denoted as $A \cap B$ (read as "A intersects B") means the set of all elements that are in both sets A and B at the same time. In another notation we write the intersection of A and B as

$A \cap B = \{x : x \in A \text{ and } x \in B.\}$

For instance, if $A = \{-1, 0, 2, 3, 4\}$ and $B = \{0, 2, 4, 6, 8\}$ then $A \cap B = \{0, 2, 4\}$.

The intersection of n sets $A_1, A_2, ..A_n$ is the set of all objects which belong to every one of them, and is denoted by $A_1 \cap A_2 \cap ... \cap A_n = \bigcap\limits_{i=1}^{n} A_i$. If the sets are infinite, that is, $A_1, A_2, ...A_n...$ then we write $A_1 \cap A_2 \cap ... \cap A_n \cap ... = \bigcap\limits_{i=1}^{\infty} A_i$.

The basic facts about intersection, as well as their proofs are similar to the basic facts about unions.

(i) $A \cap \phi = \phi$ Null intersection

(ii) $A \cap B = B \cap A$ Commutativity of intersection

(iii) $A \cap (B \cap C) = (A \cap B) \cap C$ Associativity of intersection

(iv) $A \cap U = A$

(v) $A \cap A = A$ Idempotent

(iv) $A \subset B$ if and only if $A \cap B = A$

Two useful facts about unions and intersection involve both the operations at the same time. For any subsets A, B,and C of the set U,

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

These identities are called the distributive laws.

We shall attempt proving the first identity. The proof involves showing that

$[A \cap (B \cup C)] \subseteq [(A \cap B) \cup (A \cap C)]$ and conversely, $[(A \cap B) \cup (A \cap C)] \subseteq [A \cap (B \cup C)]$.

For $[A \cap (B \cup C)] \subseteq [(A \cap B) \cup (A \cap C)]$

Let $x \in A \cap (B \cup C)$ (and show that $x \in (A \cap B) \cup (A \cap C)$)

$\Rightarrow x \in A$ and $x \in (B \cup C)$. definition of intersection

$\Rightarrow x \in A$ and $x \in$ B or $x \in C$ definition of union

$\Rightarrow x \in A$ and $x \in$ B or $x \in A$ and $x \in C$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$ definition of intersection

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$ definition of union

Thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ (1)

Conversely

Let $x \in (A \cap B) \cup (A \cap C)$ (and show that $x \in A \cap (B \cup C)$).
$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$ definition of union
$\Rightarrow x \in A$ and $x \in$ B or $x \in A$ and $x \in C$ definition of intersection
$\Rightarrow x \in A$ and $x \in$ B or $x \in C$ definition of union
$\Rightarrow x \in A$ and $x \in (B \cup C)$.
$\Rightarrow x \in A \cap (B \cup C)$
Thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ (2)
From (1) and (2) the result follows.

The general distributive laws are:
(i) $A \cup (B_1 \cap B_2 \cap ... \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap ... \cap (A \cup B_n)$.
(ii) $A \cap (B_1 \cup B_2 \cup ... \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup ... \cup (A \cap B_n)$.
These can be proved by the principle of mathematical induction (which will be discussed later on).

Definition 1.11 (Disjoint Sets)

Two sets are called disjoint(or mutually exclusive) if, and only if, they have no elements in common. Symbolically, A and B are disjoint if and only if $A \cap B = \phi$.

Example. Let $A = \{3, 7, 9\}$ and $B = \{4, 8, 10\}$. A and B are disjoint sets since they have no element in common.

In general sets $A_1, A_2, ..., A_n$ are mutually disjoint (pairwise disjoint or non over lapping) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, ..., n$,

$A_i \cap A_j = \emptyset$ where $i \neq j$.

Example: Let $A_1 = \{2, 4, 7, 8\}$, $A_2 = \{1, 3, 6\}$ and $A_3 = \{5, 9\}$. A_1, A_2 and A_3 are mutually disjoint since A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.

Definition 1.12(Partition of Sets)

A collection of non empty sets $\{A_1, A_2, ..., A_n\}$ is a partition of set A if, and only if,

(i) $A = A_1 \cup A_2 \cup ... \cup A_n$ and (ii) $A_1, A_2, ..., A_n$ are mutually disjoint.

Example: Let $A = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$, $A_1 = \{-3, -2, -1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{3, 4\}$ and $A_4 = \{5, 6, 7\}$. Is $\{A_1, A_2, A_3, A_4\}$ a partition of A?

Solution. Yes. By inspection. $A = A_1 \cup A_2 \cup A_3 \cup A_4$ and sets A_1, A_2, A_3, A_4 are mutually disjoint.

Definition 1.13 (Complementation)

The complement of set A with respect to the universal set U, denoted by A' or A^c(read as "A complement") is the set of all elements in the universal set that are not in A. For instance if the universal set $U = \{1, 2, 4, 8, 9, 10, 11\}$ and $A = \{2, 4, 8, 9\}$ then $A^c = \{1, 10, 11\}$.

If the universal set $U = \{\text{all men}\}$ and $Y = \{\text{all men who are married}\}$, then

$Y^c = \{\text{all men who are not married}\}$

The following are some of the basic facts about complementation. If A and B are subsets of the universal set U then

i) $(A')' = A$

ii) $\varnothing' = U$ and $U' = \varnothing$ (where U is the universal set)

iii) $A \cap A' = \varnothing$ and $A \cup A' = U$

iv) $A \subset B$ if and only if $B' \subset A'$; and

v) $(A \cap B)' = A' \cup B'$. Its dual is $(A \cup B)' = A' \cap B'$.

Statement (v) is the so-called De-Morgan's law. These laws hold also for larger collection of sets and not just two sets.

We shall prove the De Morgan's law for any two pair of sets; that is, if A and B are any two sets,

$(A \cup B)' = A' \cap B'$.

Proof:

Let $y \in (A \cup B)'$

$\Rightarrow y \notin (A \cup B)$ definition of complement

$\Rightarrow y \notin A$ or $y \notin B$ definition of union

$\Rightarrow y \in A'$ and $y \in B'$ definition of intersection

$\Rightarrow y \in (A' \cap B')$

Thus $(A \cup B)' \subseteq (A' \cap B')$

Conversely

Let $y \in (A' \cap B')$

$\Rightarrow y \in A'$and $y \in B'$ definition of intersection

$\Rightarrow y \notin A$ or $y \notin B$ definition of union

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

Thus $(A' \cap B') \subseteq (A \cup B)'$ and the result follows.

Generally if $A_1, A_2, ..., A_n$ is a collection of finite sets, then $(A_1 \cap A_2 \cap ... \cap A_n)' = A_1' \cup A_2' \cup ... \cup A_n'$ and $(A_1 \cup A_2 \cup ... \cup A_n)' = A_1' \cap A_2' \cap ... \cap A_n'$.

Definition 1.14 (Difference of Sets)

The difference between sets A and B denoted as $A - B$ (read as "A minus B") means the set of all elements that belong to set A, but not to set B. Symbolically,

$A - B = \{x : x \in A \text{ but } x \notin B\}$.

Example (i) If $A = \{2, 4, 5, 8\}$ and $B = \{2, 5\}$, then $A - B = \{4, 8\}$. (ii) Similarly, consider the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If $A =$

$\{1,2,4,5,8,9,10\}$ and $B = \{3,5,6,7,8\}$, then the set $A-B = \{1,2,4,9,10\}$ and $A \cap B' = \{1,2,4,9,10\}$ since $B' = \{1,2,4,9,10\}$. Note that $A - B = A \cap B'$ and observe that this is an illustration and not a proof.

Definition 1.15 (Symmetric difference of Sets)

If A and B are subsets of the universal set U, then the symmetric difference (or Boolean sum) of A and B is the set $A + B$ (or $A\Delta B$) defined by

$A\Delta B = (A - B) \cup (B - A)$.

An alternative way of defining symmetric difference is

$A\Delta B = \{x : x \in A$ or $x \in B$, but not both$\}$

For example if $A = \{1,2,4,5,8,9,10\}$ and $B = \{3,5,6,7,8\}$ then the symmetric difference between the two sets is $A + B = (A - B) \cup (B - A) = \{1,2,4,9,10\} \cup \{3,6,7\} = \{1,2,3,4,6,9,10\}$.

Another way of stating De Morgan's laws for any sets A , B and C using the concept of difference and symmetric difference is

i) $A - (B \cup C) = (A - B) \cap (A - C)$.

ii) $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof

If $x \in A-(B\cup C)$,then $x \in A$ but $x \notin (B\cup C)$. Consequently $x \notin B$ or C, so that $x \in A-B$ and $x \in A-C$. But this implies that $x \in (A-B)\cap(A-C)$; hence $A - (B \cup C) \subseteq (A - B) \cap (A - C)$. Similarly, if $x \in (A - B) \cap (A - C)$ then $x \in A - B$ and $x \in A - C$ which implies that $x \in A$ but not B and $x \in A$ but not C. It follows that $x \in A$ but $x \notin B$ or C. Therefore, $(A - B) \cap (A - C) \subseteq A - (B \cup C)$. This completes the proof of the De Morgan's law. The second is obtained by similar reasoning.

Definition 1.16 (Cartesian product of sets)

The Cartesian product (also called the cross product) of two sets A and B, denoted $A \times B$, is defined as

$A \times B = \{(x, y) : x \in A,\ y \in B\}$.

Example: Suppose that $A = \{Musa, Pam, Neshi\}$ and $B = \{a, b, c\}$ then

$$A \times B = \big\{(musa, a), (musa, b), (musa, c), (Pam, a), (Pam, b),\\ (Pam, c), (Neshi, a), (Neshi, b), (Neshi, c)\big\}.$$

Example: If $A = \{1, 2, \}$ and $B = \{2, 4, 5\}$ .Find (i) $A \times B$ (ii) $B \times A$

Solution: (i)$A \times B = \{(1, 2), (1, 4), (1, 5), (2, 2), (2, 4), (2, 5)\}$

(ii) $B \times A = \{(2, 1), (2, 2), (4, 1), (4, 2), (5, 1), (5, 2)\}$

Definition 1.17 (Relation)

A relation is any subset of the Cartesian product $A \times B$ of any two sets A and B.

If $\Re$ is a relation in $A \times B$ (i.e. $\Re \subset A \times B$) we say $\Re$ is a relation from A to B. If $A = B$, we simply say $\Re$ is a relation in A. To give an example, let A be an arbitrary set and consider the subset $\Re$ of $A \times A$ consisting of all pairs (x, y) such that $x = y$. Here the relation $\Re$ is the relation of equality between elements of A.

If $\Re$ is a relation, then $(x, y) \in \Re$ will sometimes be written as $x\Re y$.

The domain of a relation $\Re$ is the set $Dom(\Re) = \{x : (x, y) \in \Re \; for \; some \; y\}$ and the range of $\Re$ is the set $ran(\Re) = \{y : (x, y) \in \Re \; for \; some \; x\}$. The inverse of $\Re$ is the set $\Re^{-1} = \{(x, y) : (y, x) \in \Re\}$. Thus $\Re^{-1}$ is obtained by reversing each pair in $\Re$.

Definition 1.18 (Reflexive, symmetric and transitive)

Let A be a set. A relation $\Re$ in A is said to be

(i) reflexive if $x\Re x$ for all $x \in A$;

(ii) symmetric if $x\Re y$ implies that $y\Re x$ for all $x, y \in A$;

(iii) transitive if $x\Re y$ and $y\Re z$ implies that $x\Re z$ for all $x, y, z \in A$.

For any set A, an equivalence relation $\Re$ on A is a relation $\Re \subset A \times A$ satisfying (i), (ii) and (iii) above.

Definition 1.19 (Image)

If $\Re$ is a relation and E is a set, we define the image of E under $\Re$ to be the set $\Re(E) = \{y : (x, y) \in \Re \; for \; some \; x \in E\}$

Definition 1.20 (Single-valued Relation)

A relation $\Re$ is single-valued if $(x, y) \in \Re$ and $(y, z) \in \Re$ imply that $y = z$; that is, if no two distinct members of $\Re$ can have the same first element.

If $\Re$ and $\Re^{-1}$ are singled-valued, we say that $\Re$ is a one-to-one relation.

Definition 1.21 (Function)

Let A and B be sets. A function from A to B is a single valued relation $\Re$ such that $Dom(\Re) \subset A$ and $Ran(\Re) \subset B$.

Remark

From now on, we shall adopt the well-established convention of using small letters f, g, and so on for functions in the place of $\Re$ but it is important to bear in mind all the time that a function is a subset of the Cartesian product of two sets.

Definition 1.22 Let f be a function having A as its domain and having its range contained in B. Let $x \in A$; then there is a unique $y \in B$ such that $x \; f \; y$. It is convenient to denote this unique element by the symbol $f(x)$ and call it the image of x under f or the value of f at x.

If $Dom(f) = A$ and $Ran(f) \subset B$, we shall say that f is a function on A and B and write $f : A \to B$. If $Ran(f) = B$, we say that f is a function from A onto B.

Examples

1. Suppose that the set $U = \{-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10\}$ is a universal set, and the sets A, B and C given as $A = \{-5,-4,0,1,2,6,9,10\}$, $B = \{-3,0,2,4,5,6,7,8,\}$ and $C = \{-5,-4,-3,-2,-1,0,1,2,3\}$ are subsets of the universal set U. Find the followings:

(a) $A \cup B$ (b) $A \cup C$ (c) $(A \cup B) \cap C$ (d) $A - B$ (e) $(A - B) \cap C$ (f) A^c (g) B^c (h) $A \cap B^c$

(i) $(A \cup B) \cap C$

2. If $X = \{2,3\}$ and $Y = \{3,5,6\}$. Find (i) $X \times Y$ (ii) $Y \times X$

3. If $A = \{1,2,3,4\}$. Find (i) $\rho(A)$. (ii) Cardinality of A.

4. Prove each of the following set identities.

i) $(A \cup B) - C = (A - C) \cup (B - C)$ ii) $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

5. Simplify each set- theoretic expression as far as possible, naming the set laws used

(a) $(A \cap B')' \cup B$ (b) $A \cap B \cap C \cup Y' \cup A' \cap C \cup B' \cap C \cup C \cap Y$

Solutions

1. Given that $U = \{-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10\}$ and that $A = \{-5,-4,0,1,2,6,9,10\}$, $B = \{-3,0,2,4,5,6,7,8,\}$ and $C = \{-5,-4,-3,-2,-1,0,1,2,3\}$. Then

(a)

$$\begin{aligned} A \cup B &= \{-5,-4,0,1,2,6,9,10\} \cup \{-3,0,2,4,5,6,7,8,\} \\ &= \{-5,-4,-3,0,1,2,4,5,6,7,8,9,10\} \end{aligned}$$

(b) $$\begin{aligned} A \cup C &= \{-5,-4,0,1,2,6,9,10\} \cup \{-5,-4,-3,-2,-1,0,1,2,3\} \\ &= \{-5,-4,-3,-2,-1,0,1,2,3,6,9,10\}. \end{aligned}$$

(c)

$$\begin{aligned} (A \cup B) \cap C &= (\{-5,-4,0,1,2,6,9,10\} \cup \{-3,0,2,4,5,6,7,8,\}) \\ &\quad \cap \{-5,-4,-3,-2,-1,0,1,2,3\} \\ &= \{-5,-4,-3,0,1,2,4,5,6,7,8,9,10\} \\ &\quad \cap \{-5,-4,-3,-2,-1,0,1,2,3\} \\ &= \{-5,-4,-3,0,1,2\}. \end{aligned}$$

(d)

$$\begin{aligned} A - B &= \{-5,-4,0,1,2,6,9,10\} - \{-3,0,2,4,5,6,7,8,\} \\ &= \{-5,-4,1,9,10\} \end{aligned}$$

(e) $(A-B)\cap C = \{-5,-4,1,9,10\} \cap \{-5,-4,-3,-2,-1,0,1,2,3\} = \{-5,-4,1\}$.

(f)

$$\begin{aligned} A^c = U - A = &\{-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10\} \\ &-\{-5,-4,0,1,2,6,9,10\} \\ &= \{-3,-2,-1,3,4,5,7,8\}. \end{aligned}$$

(g) $B^c = \{-5,-4,-2,-1,1,3,9,10\}$

(h)

$$\begin{aligned} A\cap B^c &= \{-5,-4,0,1,2,6,9,10\} \cap \{-5,-4,-2,-1,1,3,9,10\} \\ &= \{-5,-4,1,9,10\}. \end{aligned}$$

(i) $(A\cup B)^c \cap A^c = \{-2,-1,3\} \cap \{-3,-2,-1,3,4,5,7,8\} = \{-2,-1,3\}$.

2. Given that $X = \{2,3\}$ and $Y = \{3,5,6\}$. Then
(i) $X\times Y = \{(2,3),(2,5),(2,6),(3,3),(3,5),(3,6)\}$
(ii) $Y\times X = \{(3,2),(3,3),(5,2),(5,3),(6,2),(6,3)\}$

3. Given that $A = \{1,2,3,4\}$. Then

(i)

$$\rho(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \\ , \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\Big\}.$$

(ii) Cardinality of A is given as $n(A) = 4$.

4. i) Let sets A, B and C be given. Then
$(A\cup B) - C = (A\cup B)\cap C'$ By alternate representation of set difference.
$= C'\cap (A\cup B)$ By commutative law for $\cap$
$= (C'\cap A)\cup (C'\cap B)$ By distributive law.
$= (A\cap C')\cup (B\cap C')$ By Commutative law for $\cap$
$= (A-C)\cup (B-C)$ By alternate representation of set difference.

ii) $A\cup B = (A\cup B)\cap U$ By definition of $\cap$ (Where U is the universal set).

$= (A \cup B) \cap [(A' \cup B') \cup (A \cap B)]$ By definition of universal set

$= [(A \cup B) \cap U] \cap [((A' \cup B') \cap U) \cup (A \cap B)]$ By the definition of intersection.

$= [(A \cup B) \cap (B \cup B'] \cap [((A' \cap B') \cap (A \cup A')) \cup (A \cap B)]$ definition of universal set

$= [(A \cap B') \cup B] \cap [((A \cap B^{'}) \cup A') \cup (A \cap B)]$ By the distributive law.

$= [(A \cap B') \cup B] \cap [((A \cap B^{'}) \cup A')] \cup (A \cap B)$

$= [(A \cap B') \cup (B \cap A')] \cup (A \cap B)]$

$= (A - B) \cup (B - A) \cup (A \cap B)$

5(a) $(A \cap B')' \cup B = A' \cup (B')' \cup B$ De Morgan's law.

$= A' \cup B \cup B$ Double complement

$= A \cup B$ Idempotence

(b) $A \cap B \cap C \cup Y' \cup A' \cap C \cup B' \cap C \cup C \cap Y$

$= A \cap B \cap C \cup Y' \cup (A' \cup B' \cup Y) \cap C$ Distributivity of $\cap$ over $\cup$

$= (A \cap B \cap Y' \cup (A' \cup B' \cup Y)) \cap C$ Distributivity of $\cap$ over $\cup$

$= (A \cap B \cap Y' \cup (A' \cup B' \cup Y'')) \cap C$ Double complement

$= (A \cap B \cap Y' \cup (A \cap B \cap Y')') \cap C$ De Morgan's law

$= U \cap C$ Union of complements

$= C$ $\cap$ identity element.

Exercise 1.3

1. Let $U = \{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ be a universal set and let $A = \{x \in Z : -2 < x \leq 3\}$, $B = \{x \in Z : -8 \leq x \leq 5\}$, and $C = A = \{x \in Z : 0 < x \leq 12\}$ where Z is an integer. Find if possible (a) $A \cup B$ (b) $B \cup C$ (c) A^c (d) B^c (e) C^c (f) $(A \cup B) - C$
(g) $(A - B) \cap (A - C)$ (h) $(A \cup B)^c \cap (A - B)$ (i) $(A^c \cup B^c)^c \cap C^c$. (j) $\rho(A)$ (k) $\rho(B)$

2. Let $X = \{John, Mary\}$ *and* $Y = \{2, 4, 6\}$. Find (i) $X \times Y$ (ii) $Y \times Y$ (iii) $Y \times X$

3. Suppose that A, B, C and D are subsets of the universal set U. Prove that

i) $(A - B) \cap B = \emptyset$ ii) $B - A = B \cap A'$

iii) $(A - B) \cap (C - D) = A \cap C - B \cup D$ iv) $A - (B - C) = (A - B) \cup (A \cap C)$

v) $A - (A \cap B) = A - B$ vi) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$

4. Simplify each set- theoretic expression as far as possible, naming the set laws used.

(i) $[(A \cap B)' \cup (A' \cup B)] \cup [(A \cup B)' \cup (A' \cap B)]$

(ii) $[(A \cap B') \cup (A' \cup B)'] \cup [(A \cap B) \cap (A \cap B')']$.

5. If A and B are subsets of the universal set U, show that (i) $A \cup (A \cap B) = A$

(ii) $A \cap (A \cup B) = A$.

6 Show that the following statements about sets A and B are equivalent

(a) $A \subseteq B$ (b) $A \cap B = A$ (c) $A \cup B = B$

1.4 THE NUMBER SYSTEM

The number system as we know it today is a result of gradual development as indicated in the following list.

1.4.1 Natural numbers

The natural numbers are numbers that occur commonly and obviously in nature. As such, they are whole, non-negative numbers. This set of numbers, which were first used in counting, can be defined by: $IN = \{1, 2, 3, ...\}$, the three dots leave much to the imagination. The symbols varied with the times, for example,the Romans used I,II,III,IV,...

In mathematical expressions, lowercase, italicized letters represent unknown or unspecified numbers and the most common for the natural numbers is n.

Addition

If a and b are natural numbers, then their sum $a + b$ is also a natural number, for example $5 + 3 = 8$. For this reason the set of natural numbers is said to be closed under the operation of addition.

Multiplication

Given any two natural numbers a and b, their product, $a \times b, a.b$ or ab is also a natural number, for instance $2 \times 3 = 6$. This therefore means that the natural numbers are closed with respect to multiplication.

It should be observed that the order of the products just like the summands has no influence on the result; for example $5 \times 4 = 4 \times 5 = 20$ and $6 \times 8 = 8 \times 6 = 48$. Since this property of the summands or products holds for all natural numbers, one can write that $a+b = b+a$ or $ab = ba$ and this is known as the commutative law of addition and multiplication, respectively.

Remark

It should be noted that the difference, $a - b$ between any two natural numbers a and b, is not always a natural number, e.g. $3 - 7 = -4$. In otherwords, the closure property on subtraction $-$, of the natural numbers is not satisfied.

The natural numbers can be **ordered** in the sense that given any two natural numbers n_1 and n_2, exactly one of the following relations holds: $n_1 < n_2$ i.e n_1 is less than n_2, for example $10 < 35$, or $n_1 = n_2$ i.e n_1 is equal to n_2 for example $48 = 48$ or $n_1 > n_2$ i.e n_1 is greater than n_2, for example $95 > 40$.

These order relations have a property called transitivity; for example if $n_1 < n_2$ and $n_2 < n_3$ then $n_1 < n_3$.Also if $n_4 > n_5$ and $n_5 > n_6$ then $n_4 > n_{6.}$ This relationship orders the natural numbers linearly. The illustration of this linear order is the number ray/ line. In it, the natural numbers are represented by a set of isolated (discrete) points. The fact that n_1 is less than n_2 means that the point on the number ray belonging to n_1 lies to the left of the point n_2.

The set IN, is a *denumerable* (countable) set. *Denumerability* refers to the fact that, even though there might be an infinite number of elements in the set, those elements can be orderly arranged in a list that implies the identity of every element in the set. Every natural number a has a successor denoted by $a + 1$

1.4.2 Integers

Integers arose to permit solutions of equations such as $x + b = a$,where a and b are any natural numbers. The integers therefore consist of the numbers $\{0, 1, 2, 3...\}$and their negatives $\{-1, -2, -3, ...\}$. The set of all integers is usually denoted by Z which stands for *Zahlen* (German name for "number").

Integers can be added, subtracted, multiplied, and compared(**ordered**). The ordering is given by

$... < -2 < -1 < 0 < 1 < 2 < ...$

Introducing the negative integers makes it possible to solve all equations of the form $a + x = b$ (where a and b are constant natural numbers) for the unknown x; if x is constrained to the natural numbers, only some of these equations are solvable.

We call an integer positive if it is greater than zero; zero itself is neither considered to be positive nor negative.

The order is compatible with the algebraic operations in the following way:

1. if $a < b$ and $c < d$, then $a + c < b + d$
2. if $a < b$ and $\dot{c} > 0$, then $ac < bc$

1.4.3 Rational numbers

In mathematics, a rational number (or informally fraction) is a ratio of two integers, usually written as a/b, where b is not zero.

The operations of rational numbers are as follows:

(i) $a/b + c/d = (ad + bc)/bd$, and (ii) $a/b \times c/d = ac/bd$

Two rational numbers a/b and c/d are equal if and only if $ad = bc$.

The set of rational numbers is denoted by Q.

Rational numbers such as $3/4, -7/2, 1/5, ...$arose from the need to find solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$.This leads to the operation of division or inverse of multiplication, and we write $x = \frac{a}{b}$ or $a \div b$ (called the quotient of a and b) where a is the numerator and b is the denominator.

Each rational number can be written in many forms, for example $3/6 = 2/4 = 1/2$. The simplest form is when a and b are coprime i.e.they have no common factors, and every rational number has a simplest form of this type. The decimal expansion of a rational number is either finite or eventually periodic, and this property characterises rational numbers.

It should be noted that the set of integers is a subset of the set of rational numbers, since the integers correspond to rational numbers $\frac{a}{b}$ where $b = 1$.The set of rational numbers is closed under the operations of addition, multiplication, subtraction and division, so long as division by zero is excluded.

If r and t are rational numbers such that $r < t$, then there exists a (infinitely many) rational number s such that $r < s < t$. This is true no matter how small the difference between r and t, as long as the two are not equal.

1.4.4 Irrational numbers

An irrational number is any number that is not a rational number, i.e.a number that cannot be expressed as a/b with a and b integers and b not zero. The irrational numbers are precisely those numbers whose decimal expansion never ends and never enters a periodic pattern.

Some irrational numbers are algebraic numbers such as $\sqrt{2}$ (the square root of two) and $\sqrt[3]{3}$ (the cube root of 3).

Examples of some irrational numbers

Perhaps the numbers most easily proved to be irrational are logarithms like $\log_2 3$ and $\sqrt{2}$.

Proof

(I) $\log_2 3$

The argument by contradiction (reductio ad absurdum) is as follows:
Suppose $\log_2 3$ is rational. Then for some positive integers m, n we have
$\log_2 3 = m/n$. Consequently $2^{m/n} = 3$. So $2^m = 3^n$.But 2^m is even
(because at least one of its prime factors is 2) while 3^n is odd(because
none of its prime factors are 2 (they're all 3)) so that is impossible.
This therefore implies that $\log_2 3$ is not rational but irrational.

(II)$\sqrt{2}$

The discovery of irrational number is usually attributed to Pythagoras or one of his followers, who produced a (most likely geometrical) proof of the irrationality of the square root of 2.

One proof of this irrationality is the following reductio ad absurdum. The proposition is proved by assuming the opposite and showing that, it is false,which in mathematics means that the proposition must be true.

1. Assume that $\sqrt{2}$ is a rational number. Meaning that there exists integers a and b so that $a/b = \sqrt{2}$.

2. Then $\sqrt{2}$ can be written as an irreducible fraction a/b(the fraction is simplified as much as possible) such that a and b are coprime (i.e.a and b have no common factors other than1) .Squaring both sides implies that$(a/b)^2 = 2$.

3. It follows that $a^2/b^2 = 2$ or $a^2 = 2b^2$.

4. Therefore a^2 is even because it is equal to $2b^2$ which is obviously even.

5. It follows that a must be even.

(Odd numbers have odd squares and even numbers have even squares.)

6. Because a is even, there exists an integer k such that $a = 2k$.

7. We insert $a = 2k$ in the last equation of (3) giving $2b^2 = (2k)^2$ which is equivalent to $2b^2 = 4k^2$ or $b^2 = 2k^2$.

8. Because $2k^2$ is even it follows that b^2 is also even which means that b is even because only even numbers have even squares, i.e.$b = 2l$ for some l.

9. By (5) and (8) a and b are both even, which contradicts that a/b is irreducible as stated in (2).Since we have found a contradiction to the assumption (1) that $\sqrt{2}$ is a rational number. Our supposition is therefore false. The opposite is proven, that is $\sqrt{2}$ is irrational. This proof can be generalized to show that any root of any natural number is either a rational or irrational number.

1.4.5 Real Numbers

The real numbers IR is made up of the rational and irrational numbers, i.e.$IR = Q \cup Q^c$.

If x and z are real numbers such that $x < z$, then there is always a real number y such that $x < y < z$.

In working with real numbers, it is often convenient to have a geometrical representation before us or in our minds. This representation of numbers by points takes the form of a line called the **number line** or **ideal ruler**. To do this, we draw a line L which is assumed to extend indefinitely in both directions. A reference point is chosen arbitrarily on L and labelled 0. The point 0 divides the line L into two equal half-lines,the half to the right of 0 is designated positive and the half to the left of 0 is designated negative. The point 0 is called the origin. Suppose now that P is any point on L distinct from 0 and suppose that the distance from 0 to P equals the real number x (which must be positive since distance is always nonnegative), then we take P to represent the number x or the number $-x$ according as P lies on the negative or positive half of L. In either case,given the point P on the line L, the real number x or $-x$ is uniquely determined.

Conversely, given a real number $x > 0$,we shall assume (atleast from intuition) that there is one and only one point P lying to the right of 0 such that the distance of P from 0 is x. If $x < 0$, then P is assumed to lie to the left of 0. In this way we obtain a one to one correspondence between the set of all points on L and the set of all real numbers(the point 0 corresponding to the real number zero).In view of this correspondence, no distinction need be made between the line L and the real number system IR and for this reason we often use point and real number interchangeably.

We shall denote the set of real numbers by IR. There are some certain sets of real numbers called intervals. An interval is the set of all points between given points a and b. If an interval contains the points a and b then such an interval is called a closed interval denoted by $[a, b]$. More formally, if $a \in IR$ and $b \in IR$, with $a < b$, then

$[a, b] = \{x \in IR : a \leq x \leq b\}$

is called a closed interval. This interval can be represented geometrically as:

xx aa bb

The shaded part constitute the closed interval $[a, b]$. For example, the set $[1, 2] = \{x \in IR : 1 \leq x \leq 2\}$ is a closed interval. Its graphical representation is

xx a1 b2

On the other hand if the interval does not contain its end points , then such an interval is called an open interval. We write

$(a, b) = \{x \in IR : a < x < b\}$

to indicate this open interval. Graphically an open interval of the form (a, b) can be represented as follows:

xx aa bb

An example of an open interval is the set $(-2, 1) = \{x \in IR : -2 < x < 1\}$. This interval consist of the set of real numbers lying between -2 and 1. Its graphical representation is

xx -2-2 11

Intervals that include only one of the end points are called half-open intervals(sometimes referred to as half-closed). We write $[a, b) = \{x \in IR : a \leq x < b\}$ and $(a, b] = \{x \in IR : a < x \leq b\}$ to denote these intervals respectively. A half open interval of the form $(a, b]$ can be represented as follows:

The lower end of the interval is open whereas the upper end is closed.

Sets of the form

i) $[a, \infty) = \{x \in IR : x \geq a\}$ are referred to as unbounded intervals.

Other unbounded intervals are

ii) $(a, \infty) = \{x \in IR : x > a\}$

iii) $(-\infty, b] = \{x \in IR : x \leq b\}$

iv) $(-\infty, b) = \{x \in IR : x < b\}$ and

v) $(-\infty, \infty) = \{x \in IR : -\infty < x < \infty\}$.

1.5 ELEMENTARY OPERATIONS ON SET OF REAL NUMBERS

Elementary set operations such as the union of sets, intersection of sets, complements and a host of others can be performed on the set of real numbers also.

For instance if,

$A = \{x \in IR : -2 \leq x \leq 20\}$

$B = \{x \in IR : 0 \leq x \leq 10\}$

and $C = \{x \in IR : -15 < x \leq 5\}$

Then $A \cup B = \{x \in IR : -2 \leq x \leq 20\} \cup \{x \in IR : 0 \leq x \leq 10\}$

$= \{x \in IR : -2 \leq x \leq 20\}$

and

$A \cap B = \{x \in IR : 0 \leq x \leq 10\}$

In a similar way, the complement of the set C denoted as $C^{'}$ or C^c is given by

$C^{'} = (-\infty, -15] \cup (5, \infty)$.

We can equally determine $(A \cup B)' \cap C$ and $A \cap C^{'}$ as follows:

Since $A \cup B = \{x \in IR : -2 \leq x \leq 20\}$ it follows that

$(A \cup B)' = \{x \in IR : -\infty < x < -2\} \cap \{x \in IR : 20 < x < \infty\}$

or

$(A \cup B)' = (-\infty, -2) \cup (20, \infty)$

Therefore $(A \cup B)' \cap C = \{(-\infty, -2) \cup (20, \infty)\} \cap (-15, 50]$

$(A \cup B)' \cap C = \{(-\infty, 0) \cap (-15, 5)\} \cup \{(20, \infty) \cap (-15, 5]\}$

by distributive laws

$(A \cup B)' \cap C = (-15, 2)$

This can be written in another form as

$(A \cup B)' \cap C = \{x \in IR : -15 < x < 0\}$.

Now to determine $A \cap C'$, recall that $C' = (-\infty, -15] \cup (5, \infty)$

Therefore,

$A \cap C' = [-2, 20] \cap \{(-\infty, -15] \cup (15, \infty)\}$

$= \{[-2, 20] \cup (-\infty, 15)\} \cap \{[-2, 20] \cup (15, \infty)\}$
$= (-\infty, 20] \cap [-2, \infty)$
$= [-2, 20]$

Exercise 1.5

1) If $A = \{x \in IR : -12 \leq x < 30\}$, $B = \{x \in IR : -5 \leq x \leq 35\}$ and $C' = \{x \in IR : -5 < x < 13\}$.

Determine if possible

(i) A' (ii)B' (iii)C' (iv) $(A \cap B)'$ (v) $(A \cup B)'$ (vi) $(C \cup B) \cap A'$
(vii) $(A \cap B) \cup C'$ (viii) $A' \cap (B' \cup C)$

2) Suppose IN is the set of natural numbers and that the sets E, F are defined as

$E = \{x \in IN \cup \{0\} : 0 \leq x \leq 35\}$
$F = \{x \in IN : 2 \leq x \leq 100\}$

Determine if possible

i) $(E \cap G) \cup F$ ii) $(E' \cup F') \cap E$ iii) $E \backslash \{0\}$ iv) $(E - G) \cup (E \triangle G)$

3) Suppose that Z, the set of integers is the university set, and the sets X, Y and W are defined as

$X = \{x \in Z : -15 \leq x \leq 70\}$
$Y = \{x \leq Z : -10 < x < 10\}$ and $W = \{x \leq Z : -17 < x < 23\}$

Find if possible

(i)X', (ii) Y' (iii) W' (iv) $(X \cup Y)'$ (v) $(X \cup W) \cap Y'$ (vi) $X - Y$ (vii) $(X \triangle Y) - W$

Chapter 2

Mathematical Induction and Binomial Expansion

2.1 Mathematical Induction

It is a common knowledge that whereas experimental scientists search for evidence to support a theory, mathematicians search for a proof to validate a statement. The mathematical induction is one of the famous ways of establishing mathematical proofs.

Mathematical induction is a technique for proving statements (or propositions) about the natural numbers. It also provides a uniform framework for studying the natural numbers. We will now formally state the principle of mathematical induction.

2.1 The Principle of Mathematical Induction

Suppose that for each positive integer n we have a statement $P(n)$ that is either true or false. Suppose that

$P(1)$ is true; (2.1).

If $P(k)$ is true, for all $k < k+1$ (2.2)

$\Rightarrow P(k+1)$ is true, then $P(n)$ is true for every positive integer n. The first step (i.e. condition (2.1)) is usually called the basis step and the second step (2.2) is called the inductive step. What the inductive step does is to assert that if the process has reached some stage, then it will go one stage further.

Example

Use the principle of Mathematical induction to show that

$1+2+3+...+n = \frac{n(n+1)}{2}$.

Proof: (By Mathematical induction).

Let P(n) be the assertion that

$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$.

Basis step: Since $1 = \frac{1(1+1)}{2}$, it means that $P(1)$ is true.

Inductive step: Assume that the kth case is true and proceed to prove that the $(k+1)$th case is true. By the inductive hypothesis

$1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$.

Now consider the case $n = k + 1$, we have

$$\begin{aligned} 1 + 2 + 3 + .. + k + (k+1) &= \frac{k(k+1)}{2} + k + 1 \\ &= (k+1)[\tfrac{k}{2} + 1] \\ &= \tfrac{1}{2}(k+1)(k+2). \end{aligned}$$

This is the same when we substitute $n = k + 1$ in the assertion. That is,

$1 + 2 + 3 + ... + k + 1 = \frac{(k+1)(k+1+1)}{2} = \frac{1}{2}(k+1)(k+2)$.

Since the basis step and the inductive step have been verified, the principle of mathematical induction tells us that $P(n)$ is true for all positive integers n.

Example

Establish the following identities using induction

(a) $1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$

(b) $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Proof:(By Mathematical Induction).

(a) Let $P(n)$ be the proposition that

$1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Basis Step: Note that $1.2 = \frac{1(1+1)(1+2)}{3} = 2$ and therefore the assertion is true for the case $n = 1$.

Inductive Step: Assume that the proposition is true for the case $n = k$ and proceed to show that it is also true for the case $n = k+1$. By hypothesis,

$1.2 + 2.3 + 3.4 + ... + k(k+1) = \frac{k(k+1)(k+2)}{3}$,

therefore

$$\begin{aligned} 1.2+2.3+3.4+...+k(k+1)+(k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)(k+2)[\tfrac{k}{3} + 1] \\ &= \tfrac{1}{3}(k+1)(k+2)(k+3). \end{aligned}$$

Since the inductive step is also true, then the proposition $P(n)$ must be true for all positive integers n.

(b) Given that $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Basis Step: When $n = 1$, we see that

$1^3 = \left[\frac{1(1+1)}{2}\right] = 1$.

Therefore the statement is true for the case $n = 1$.

Inductive Step: Assume that

$1^3+2^3+3^3+...+k^3=\left[\frac{k(k+1)}{2}\right]^2$ is true.

Next we show that the statement is true for $n=k+1$. That is,

$$\begin{aligned}1^3+2^3+3^3+...+k^3+(k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3\\ &= (k+1)^2\left[\left(\frac{k}{2}\right)^2+(k+1)\right]\\ &= (k+1)^2\left(\frac{k^2+4k+4}{4}\right)\\ &= \left[\frac{(k+1)(k+2)}{2}\right]^2.\end{aligned}$$

This shows that the statement is true for the $(k+1)th$ case and therefore by the principle of mathematical induction, the statement must be true for all positive integers n.

Example

Use induction to show that 4^n-1 is divisible by 3 for $n=1,2,3,...$

Proof:

Basis Step: If $n=1, 4^n-1=4^1-1=3$ which is divisible by 3.

Inductive Step: Assume that the statement is true for $n=k$. That is 4^k-1 is divisible by 3. We now show that $4^{k+1}-1$ is divisible by 3. To relate the the $(k+1)$th case to the kth case, we write

$4^{k+1}-1=4.4^k-1=4^k-1+3.4^k$.

By assumption, 4^k-1 is divisible by 3 and since 3.4^k is divisible by 3, the sum

$\left(4^k-1\right)+3.4^k=4^{k+1}-1$, is divisible by 3.

Since the basis step and the inductive step have been verified, the principle of mathematical induction tells us that 4^n-1 is divisible by 3 for $n=1,2,3,...$

Exercise 2.1

Establish the following identities by induction:

(i) $\frac{1}{1\times3}+\frac{1}{3\times5}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$.

(ii) $\frac{1^2}{1\times3}+\frac{2^2}{3\times5}+...+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$

(iii) $a+ar+...+ar^{n-1}=\frac{a(1-r^n)}{1-r}$

(iv) $\frac{1}{1.2}+\frac{1}{2.3}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$

(v) $\frac{1}{1.3}+\frac{1}{2.4}+...+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$ for $n\geq1$

2.2 BINOMIAL THEOREM

Expansions of expressions of the form $(a+x)^n$ where $n=1,2,3,...$ can easily be obtained. However, when n becomes very large,negative or fractional, such expansions cannot easily be obtained. Pascal (1623-1662) in search of a way to expand such expressions with large and fractional n produced what is known as the pascal triangle. This he derived by observing the pattern formed in the expansion

$(1+x)^n, n>0$ (2.3)

Binomial	**expansion**				**coefficients**			
$(1+x)^0$	1				1			
$(1+x)^1$	1	x			1	1		
$(1+x)^2$	1	$2x$	x^2		1	2	1	
$(1+x)^3$	1	$3x$	$3x^2$	x^3	1	3	3	1

It is interesting to note that in the coefficients of the various expansions which form the Pascal triangle,the next line is generated from the previous. In all cases,the first and the last terms is 1.

2.2.1 Binomial theorem for n a positive number

To carry out expansions using the Pascal triangle for large n will not be easy. Thus the need for a more general form for such expansion. To achieve this,we desire a formula for the coefficients of x^r which is denoted by ${}^n C_r$.

The expansion $(1+x)^n$ is then given by:

$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + ... + {}^nC_rx^r + ... + x^n$ (2.4)

The coefficients in the expansion of $(1+x)^n$ give the pattern

where $\frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...\ 3.2.1} = {}^n C_r$ (2.5) and $n! = n(n-1)(n-2)...(n-r+1)...3.2.1$ is called the n factorial.

One notices that in the numerator of (2.5) factors begin at n and decrease to $(n-r+1)$ while in the denominator they begin at r and reduce to1.Equations(2.4) and (2.5) together form the binomial theorem.

Then binomial theorem gives the general expansion of the binomial $(a+x)^n$ where n is a positive,negative or fractional power and is expressed as follows:

$(a+x)^n = a^n + {}^nC_1xa^{n-1} + {}^nC_2x^2a^{n-2} + ... + {}^nC_rx^ra^{n-r} + ... + {}^nC_nx^n$ (2.6)

where ${}^nC_r$ is as in (2.5) and can be expressed as ${}^nC_r = \frac{n!}{r!(n-r)!}$

Proof of the Binomial theorem for n a positive number

The method (2.4) and (2.5) suggest that the result is true for all positive integer values of n. Thus we shall prove by mathematical induction.

For $n = 1, 2, 3...$ (2.4) and (2.5) are true.

We assume true for a value of $n = k$ then prove for $n = k+1$

$(1+x)^k = 1 + {}^k C_1 + {}^k C_2x^2 + ... + {}^k C_rx^r + ... + {}^k C_{k-1}x^{k-1} + x^k$

and

$(1+x)^{k+1} = (1+x)(1+x)^k = (1+x)(\ 1 +^k C_1 +^k C_2 x^2 + ... +^k C_r x^r + ... +^k C_{k-1} x^{k-1}\ + x^k)$

$= 1 + (1 +^k C_1)x + (^kC_2 +^k C_1)x^2 + ... + (^kC_r +^k C_{r-1})x^r + x^{k+1}$

but $^kC_1 + 1 = k + 1 =^{k+1} C_1$

and

$^{k+1}C_r =^k C_r +^k C_{k-1}$

Hence our assumption leads to

$(1+x)^{k+1} = 1 + \ ^{k+1}C_1 x +^{k+1} C_2 x^2 + ... +^{k+1} C_r x^r + ... +^{k+1} C_k x^k + x^{k+1}$

Since (2.2) and (2.3) are true for a positive number $n = k$, and true for the next value of n i.e $n = k+1$,then it is true for all positive integer values of n.

Example Expand $(x - 2y)^5$ by the binomial theorem.

Solution.

$(x-2y)^5 = x^5 +^5 C_1 x^4(-2y) +^5 C_2 x^3(-2y)^2 +^5 C_3 x^2(-2y)^3 +^5 C_4 x(-2y)^4 +^5 C_5(-2y)^5$

$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$

Example

Find the coefficient of x^6 in the expansion of $(\frac{1}{x^2} - x)^{18}$.

Solution.

Given the expansion $(x+y)^n$,the coefficient of x^r is given by $^nC_r x^{n-r} y^r$

Thus using (2.7), in our problem we have

$^nC_r(\frac{1}{x^2})^{n-r}(x)^r$, $n = 18$

$^{18}C_r(\frac{1}{x^2})^{18-r}(x)^r =^{18} C_r x^{-2(18-r)} x^r$

$=^{18} C_r (x^r)^{-36+2r+r}$

$=^{18} C_r x^{3r-36}$

but $3r - 36 = 6$

$\therefore\ r = 14$.

thus $^{18}C_r(x)^{3r-36}\ =\ ^{18}C_{14} x^{42-36}$

$=^{18} C_{14} x^6\ =\ 3060$.

Example

Find the term which is independent of y in the expansion of $(\frac{y^3}{x^2} + \frac{x}{3y})^8$

Solution

$^nC_r x^{n-r} y^r =\ ^nC_r(\frac{y^3}{x^2})^{n-r}(\frac{x}{3y})^r$

$=^8 C_r(\frac{y^3}{x^2})^{8-r}(\frac{x}{3y})^r$

$=\frac{^8C_r\ x^{-16+3r} y^{24-3r}}{3^r}$

For the term independent of y ,it means that $24 - 4r = 0$

$\therefore\ r\ = 6$.

For $r = 6$

$=\frac{^8C_r\ x^{-16+3r} y^{24-3r}}{3^r}$

$\therefore \quad {}^8C_6 \frac{x^2}{3^6} \quad = \frac{8!}{6!2!}\frac{x^2}{36} = \frac{28x^2}{729}$

Example

Find the value of k if the coefficients of y^2 in the expansion of $(1+ky)^4(2-y)^3$.

Solution

$(1+ky)^4 = 1 + 4(ky) + 6(ky)^2 + 4(ky)^3 + (ky)^4$

$(2-y)^3 = 2^3 + 3\times 2^2(-y) + 3\times 2(-y)^2 + (-y)^3$

$\quad = 8 - 12y + 6y^2 - y^3$

thus $\quad (1+ky)^4(2-y)^3. = (1+4ky+6k^2y^2+4k^3y^3+k^4y^4)(8-12y+6y^2-y^3)$

The coefficient of y^2 is $6 - 48k + 48k^2 = 6$

$k^2 - k = 0$

$k(k-1) = 0$

$\therefore \quad k = 0,\ k = 1$

Example

Expand $(x+3y)^5$.Hence evaluate $(1.03)^5$ correct to 3 decimal places.

Solution

$(x+3y)^5 = x^5 + 5x^4(3y) + 10x^3(3y)^2 + 10x^2(3y)^3 + 5x(3y)^4 + (3y)^5$

$\quad = x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

To evaluate $(1.03)^5$, it implies that $x = 1, y = 0.01$; thus

$(1.05)^5 = 1 + 15(0.01) + 90(0.01)^2 + 270(0.01)^3 + 405(0.01)^4 + 243(0.01)^5$

$\quad = (1 + 0.15 + 0.009 + 0.00027) = 1.159\,27$

2.2.2 Binomial expansion for n negative and n a fraction

It can be verified that for $-1 < \frac{n}{2} < 1$,

$(1+x)^{\frac{n}{2}} \ = \ 1 + \frac{n}{2}x + \frac{\frac{n}{2}(\frac{n}{2}-1)}{2!}x^2 + \frac{\frac{n}{2}(\frac{n}{2}-1)(\frac{n}{2}-2)}{3!}x^3 + ...$which is also a binomial theorem.

Example

If y is so small that its fourth and higher powers can be neglected.show that

$\sqrt{(\frac{1+2y}{1-2y})^n} \ = \ 1 + 2ny + 2n^2y^2 + \frac{4}{3}n(n^2+2)y^3$

By letting $y = \frac{1}{7}$ and $n = 1$,prove that $\sqrt{5} = \frac{2295}{1029}$

Solution

$\sqrt{(\frac{1+2y}{1-2y})^n} \ = (1+2y)^{\frac{n}{2}}(1-2y)^{-\frac{n}{2}}$

Expanding by the binomial theorem

$$(1+2y)^{\frac{n}{2}} \ = \ 1 + \frac{n}{2}(2y) + \frac{\frac{n}{2}(\frac{n}{2}-1)}{2!}(2y)^2 + \frac{\frac{n}{2}(\frac{n}{2}-1)(\frac{n}{2}-2)}{3!}(2y)^3 + ...$$

$$= 1 + ny + \frac{4n(n-2)}{8}y^2 + \frac{8n(n-2)(n-4)}{48}y^3 + ...$$

$$= 1 + ny + \frac{1}{2}(n^2 - 2n)y^2 + \frac{1}{6}(n^3 - 6n^2 + 8n)y^3 + ... \qquad (2.8)$$

$$(1-2y)^{-\frac{n}{2}} \ = \ 1 - \frac{n}{2}(-2y) + \frac{(-\frac{n}{2})(-\frac{n}{2}-1)}{2!}(-2y)^2 + \frac{(-\frac{n}{2})(-\frac{n}{2}-1)(-\frac{n}{2}-2)}{3!}(-2y)^3 + ...$$

$$= 1 + ny + \frac{n}{2}\frac{(\frac{n}{2}+2)}{2!}4y^2 + \frac{n}{2}\frac{(\frac{n}{2}+2)(\frac{n}{2}+4)}{3!}8y^3 + ...$$

$$= 1 + ny + (n^2 + 2n)y^2 + \tfrac{1}{6}(n^3 + 6n^2 + 8n)y^3 + ... \qquad (2.9)$$

$$(1+2y)^{\frac{n}{2}}(1-2y)^{-\frac{n}{2}} = (1 + ny + \tfrac{1}{2}(n^2 - 2n)y^2 + \tfrac{1}{6}(n^3 - 6n^2 + 8n)y^3 + ...)$$
$$\times(1 + ny + (n^2 + 2n)y^2 + \tfrac{1}{6}(n^3 + 6n^2 + 8n)y^3 + ...)$$

expanding yields

$$= 1 + 2ny + 2n^2y^2 + \tfrac{4}{3}n(n^2 + 2)y^3 + ...$$

for $n = 1, y = \frac{1}{7}$

$$\sqrt{\frac{(1+\frac{2}{7})}{(1-\frac{2}{7})}} = 1 + \frac{2}{7} + \frac{2}{49} + \frac{4}{343}$$

$$\sqrt{\frac{\frac{9}{7}}{\frac{5}{7}}} = \frac{343+98+14+4}{343}$$

$$\sqrt{\frac{9}{5}} = \frac{459}{343}$$

$\frac{3}{\sqrt{5}} = \frac{459}{343}$, $\frac{3\sqrt{5}}{5} = \frac{459}{343}$, $\sqrt{5} = \frac{459}{343} \times \frac{5}{3}$

$\therefore \sqrt{5} = \frac{2295}{1029}$

Example

Expand $\sqrt{1-3x}$ and hence evaluate $\sqrt{0.7}$correct to four decimal places.

Solution

$$(1-3x)^n = 1 + n(-3x) + \frac{n(n-1)}{2!}(-3x)^2 + \frac{n(n-1)(n-2)}{3!}(-3x)^3 + ...$$

then

$$\sqrt{(1-3x)} = (1-3x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}(3x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}9x^2 - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}27x^3 + ...$$
$$= 1 - \tfrac{3}{2}x - \tfrac{9}{8}x^2 - \tfrac{81}{48}x^3 + ...$$

if $\sqrt{0.7} = (0.7)^{\frac{1}{2}} = (1-3(0.1))^{\frac{1}{2}}$,thus $x = 0.1$

substituting $x = 0.1$ into the expansion of $\sqrt{1-3x}$ gives the value of $\sqrt{0.7} = (0.7)^{\frac{1}{2}} = (1-3(0.1))^{\frac{1}{2}}$ as

$$\sqrt{0.7} = 1 - \tfrac{3}{2}(0.1) - \tfrac{9}{8}(0.1)^2 - \tfrac{81}{48}(0.1)^3 + ...$$
$$= 1 - 0.15 - 0.01125$$
$$= 0.8388$$

Exercise 2.2

(1) Expand $(1+3x)^{\frac{1}{3}}$ hence evaluate $3\sqrt{1.06}$ correct to 4 decimal places.

(2) Obtain the first 4 terms of the expansion $(1 + 2x)^{12}$ in ascending powers of x.Hence find the value of $(0.998)^{12}$correct to four decimal places.

(3) Show that $\sqrt{(9 + y^2)} \approx 3 + \frac{1}{6}y^2 - \frac{1}{316}y^4$.For what values of y is the expansion valid.?

(4) Find the term independent of x in the expansion of $\left(\frac{x^5}{y^4} + \frac{y^3}{2x^3}\right)^{10}$

(5) Expand $\sqrt{\frac{1+2x}{1-2x}}$ in ascending powers of x up to and including the term in x^2.By substituting $x = \frac{1}{100}$, show that $\sqrt{5} \approx 7\frac{7}{50}$.

(6) If x is so small that its fourth and higher powers can be negleted. Show that $\sqrt{(\frac{1+x}{1-x})^3} = 1 + 3x + \frac{9}{2}x^2 + 38x^3$.

(7) Find the coefficient of y^9 in the expansion of $\left(\frac{1}{y^3}-y\right)^{18}$.

(8) Find the expansion of $\left(\frac{1+\frac{1}{2}y}{1-\frac{1}{2}y}\right)^{\frac{3}{2}}$ in ascending powers of y as far as the term in y^3.

(9) If x is so small that terms in $x^n, n \geqslant 3$ can be neglected and $\left(\frac{3+ax}{3+bx}\right) = (1-x)^{\frac{1}{3}}$. Find the values of a and b. Hence evaluate $(0.96)^{\frac{1}{3}}$.

(10) Find b if the coefficients of x in the expansion of $(1+bx)^8\ (1+3x)^4 - (1+x)^3\ (1+2x)^4$ is zero. What is the coefficient of x^3.

Chapter 3

Real Sequences and Series

3.1 Introduction:

In Mathematics, many sets of numbers are written in a definite order, and there is always a simple rule by which the terms are obtained.

Definition 3.1(Sequence)

A sequence $\{u_n\}_{n=1}^{\infty}$ or $\{u_n\}_1^{\infty}$is a function whose domain is the set of natural numbers or positive integers. In otherwords it is a set of numbers or quantities $u_1, u_2, ..., u_n$, each of which is obtained by a prescribed rule, that is $u_n = f(n)$. Individual terms can be obtained by substituting $n = 1, 2, 3...$, in the general term u_n.

For example:

(i) $1, 3, 5, 7, ...$, is a sequence in which $u_n = 1 + 2(n - 1)$. That is the fifth term would be $u_5 = 1 + 2(4) = 9$.

(ii) $2, 8, 32, ...$, is a sequence in which $u_n = 2 \times 4^{n-1}$. That is $u_5 = 2 \times 4^4 = 2 \times 256 = 512$.

(iii) $1^2, -2^2, 3^2, -4^2, ...$, is a sequence in which $u_n = (-1)^{n-1}n^2$.

The patterns in which the terms of a sequence can be obtained are not so easily recognised: thus the form of the general term u_n is more difficult to establish. For example, $-1, 7, -15, 25, ...$, is a sequence in which $u_n = (-1)^n[n(n + 3) - 3]$.

Definition 3.2 (Finite Sequence)

A sequence that contains only a finite number of terms is a finite sequence.The house numbers in a street form a finite sequence.

Definition 3.3 (Infinite Sequence)

A sequence is said to be infinite if the terms of the sequence are not finite.The natural numbers 1,2,3,... form an infinite sequence.

Definition 3.4 (Series)

A series is the sum of the terms of a sequence. Thus $1 + 3 + 5 + 7 + ...$ is a series. As with the terms of a sequence, u_n represents the nth term of a given sequence. The sum of the first n terms of a sequence (that is the nth term of a series) is denoted by S_n and it is a partial sum of the series.

3.1 ARITHMETIC PROGRESSION (AP)

We define an Arithmetic Progression (AP) as a sequence of terms in which each term other than the first is obtained from the preceding one by adding a constant number. We call the constant number the common difference and it is usually denoted by d. The first term of an Arithmetic Progression is usually denoted by a, while the nth term is denoted by l. In order to find out whether a given sequence is an AP, we find the difference between consecutive terms. If the difference obtained is the same, then we say the sequence is an AP. A typical AP is thus given by

$a, a + d, a + 2d, a + 3d, ..., a + (n - 1)d$. Hence $l = a + (n - 1)d$ is the nth term of the AP.

Example

The third term of an Arithmetic Progression is 10 and the seventh term is 34. Find the first term and the common difference of the AP.

Solution:

Let a be the first term and d the common difference of the AP. Then the third and seventh terms are $a + 2d$ and $a + 6d$ respectively.

Thus $a + 2d = 10$

$a + 6d = 34$

Solving for a and d we obtain $a = -2$ and $d = 6$.

Example

The 9th term of an AP is three times the 5th term. (i) Find a relation between a and d, and (ii) Prove that the 8th term is five times the 4th term.

Solution

(i) Let a be the first term and d the common difference of the AP. Then the 9th and 5th terms are $a + 8d$ and $a + 4d$ respectively.

Thus $a + 8d = 3(a + 4d)$

$\Rightarrow a + 8d = 3a + 12d$

$\Rightarrow -2a = 4d$

$\Rightarrow a = -2d$

Thus the relation between a and d is $a = -2d$.

(ii) To prove that the 8th term is five times the 4th term, we have that the 8th and the 4th terms are $a + 7d$ and $a + 3d$ respectively. But $a = -2d$, thus the 8th term is $5d$ and the 4th term is d which shows that the 8th term is five times the 4th term.

Definition 3.5 (Arithmetic Series)

The Arithmetic series is defined as the sum of the terms of an AP.

(i) Consider the sum of the first n positive integers, that is
$\Sigma_{r=1}^{n} r = 1 + 2 + 3 + ... + n$.

Writing the sum as S we have that

$S = 1 + 2 + 3 + ... + n.$ (3.1)

and

$S = n + (n - 1) + (n - 2) + ... + 1.$ (3.2)

Adding equations (3.1) and (3.2) termwise we have

$2S = (n + 1) + (n + 1) + ... + (n + 1)$ (n times)
$= n(n + 1)$

Thus $S = \frac{n}{2}(n + 1)$

(ii) Consider the sum of the first n odd positive integers. Writing this sum as S, we have

$S = 1 + 3 + 5 + ... + (2n - 1) = \sum_{r=1}^{n}(2r - 1)$ (3.3)

Also $S = (2n - 1) + (2n - 3) + (2n - 5) + ... + 1$ (3.4)

Adding the two equations together, we have

$2S = 2n + 2n + 2n + ... + 2n$ (n times)
$= n(2n)$
$= 2n^2$

Thus $S = n^2$.

(iii) In general, let S_n denotes the sum of n terms of an Arithmetic Progression. Then

$S_n = a+(a+d)+(a+2d)+...+[a+(n-2)d]+[a+(n-1)d]$ (3.5)

Also $S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + ... + (a + d) + a$ (3.6)

Adding the two equations together, we have

$2S_n = [2a+(n-1)d]+[2a+(n-1)d]+[2a+(n-1)d]+...+[2a+(n-1)d]$ (n times)
$= n[2a + (n - 1)d]$

Thus $S_n = \frac{n}{2}[2a + (n - 1)d]$.

Denoting the nth term of an AP by l we have that

$S_n = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}(a + l)$ where $l = a + (n - 1)d$.

Example

The 10th term of a certain AP is -29 and the 20th term is -69. Find the sum of the first 20 terms of the AP.

Solution:

The 10th and the 20th terms are $a + 9d$ and $a + 19d$ respectively. Thus

$a + 9d = -29$

$a + 19d = -69$

Solving for a and d, we obtain $a = 7$ and $d = -4$.

Now $S_{20} = \frac{20}{2}[2(7) + (20 - 1)(-4)]$

$= 10[14 - 76]$
$= -620$

Example

The 9th term of an AP is 24 and the sum of the first 9 terms is 126. Find an expression for the (i) nth term, (ii) sum to n terms.

Solution:

The 9th term is $a + 8d$. Thus
$a + 8d = 24 \qquad (3.7)$
and the sum of the first 9 terms is
$S_9 = \frac{9}{2}[2a + (9 - 1)d] = 126$
$\Rightarrow \frac{9}{2}[a + a + 8d] = 126 \qquad (3.8)$
Substituting (3.7) into (3.8) we have
$9(a + 24) = 252$
$\Rightarrow 9a + 216 = 252$
$\Rightarrow a = 4$
Substituting for a in (3.7) we have
$d = \frac{5}{2}$
(i) Thus the expression for the nth term is
$l = a + (n - 1)d = 4 + (n - 1)\frac{5}{2} = \frac{5n+3}{2}$, and
(ii) the expression for sum to n terms is
$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[8 + \frac{5n-5}{2}] = \frac{5n^2+11n}{4}$

Definition 3.6 (The Arithmetic Mean (AM)

Let x, y and z be three consecutive terms of an AP. We call y the Arithmetic mean between x and z. Now the common difference of the AP is given by $d = y - x = z - y$. Therefore $2y = x + z$ Hence $y = \frac{x+z}{2}$.

Also, in a finite Arithmetic Progression, $u_1 = a, u_2 = a + d, u_3 = a + 2d, ..., u_{n-1} = a + (n - 2)d, u_n = a + (n - 1)d$, the terms in between the first term u_1 and the last u_n (i.e $u_2, u_3, .., u_{n-1}$) are called the Arithmetic means between u_1, and u_n.

For example, in the finite AP $2, 7, 12, 17, 22, 27, 32$, the numbers $7, 12, 17, 22, 27$ are the five Arithmetic means between 2 and 32.

Example

Insert three Arithmetic means between 25 and 9.

In this case, we have $u_1 = 25, u_2, u_3, u_4, u_5 = 9$. Since $u_5 = a + 4d$, we have
$9 = 25 + 4d$
$\Rightarrow d = -4$
Thus $u_2 = a + d = 25 - 4 = 21$
$u_3 = a + 2d = 25 - 8 = 17$
$u_4 = a + 3d = 25 - 12 = 13$
Therefore, the AP is $25, 21, 17, 13, 9$.

Example

p, q, r are three consecutive terms of an AP whose sum is 21. The ratio $p : r = 6 : 1$. Find p, q and r.

Solution:

By assumption, the sum of the three terms is 21. That is $p + q + r = 21$ (3.9)

and $p : r = 6 : 1 \Rightarrow \frac{p}{r} = \frac{6}{1}$ or $p = 6r$ (3.10)

But $q = \frac{p+r}{2}$ (3.11)

Substitute (3.10) in (3.11) to have

$q = \frac{6r+r}{2} = \frac{7r}{2}$ (3.12)

Substitute (3.10) and (3.11) in (3.9) to get

$6r + \frac{7r}{2} + r = 21$

$\Rightarrow 12r + 7r + 2r = 42$

$\Rightarrow r = 2$

$\therefore p = 6r = 6.2 = 12$

and $q = \frac{7r}{2} = \frac{7.2}{2} = 7$

$\therefore (p, q, r) = (12, 7, 2)$.

3.2 GEOMETRIC PROGRESSION (GP)

A geometric Progression (GP) is a sequence of numbers in which each term other than the first is obtained from the preceding one by the multiplication of a non-zero constant number called the common ratio. In otherwords it is a progression in which the ratio between any two consecutive terms is constant (i.e the same). The common ratio is usually denoted by r. The first term of a GP is usually denoted by a. Hence a typical GP takes the following form:

$a, ar, ar^2, ..., ar^{n-1}$.

Consequently the nth term of a GP is given by $l = ar^{n-1}$.

Example

The third term of a Geometric Progression is 10, and the sixth term is 80. Find the common ratio and the first term.

Solution:

Let a and r denote the first term and the common ratio respectively. Then

$ar^2 = 10$ (3.13)

$ar^5 = 80$ (3.14)

Dividing equation (3.14) by (3.13) we have

$\frac{ar^5}{ar^2} = \frac{80}{10}$

$\Rightarrow r^3 = 8$

$\Rightarrow r = 2$.

Hence $4a = 10$

$\Rightarrow a = \frac{10}{4} = 2\frac{1}{2}$

Definition 3.7 (Geomtric Series)

A Geometric series is defined as the sum of the terms of a Geometric Progression. The sum of n terms of a GP is easily obtained by the following argument. A typical Geometric series is given by

$S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}$ (3.15)

Multiplying equation (3.15) throughout by r we get

$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$ (3.16)

Subtracting equation (3.16) from equation (3.15) we have

$S_n - rS_n = a - ar^n$

$\Rightarrow (1-r)S_n = a(1-r^n)$

$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}$.

Hence $S_n = \frac{a(1-r^n)}{1-r}$, for $-1 < r < 1$

or $S_n = \frac{a(r^n-1)}{r-1}$, for $\mid r \mid > 1$.

Or simply $S_n = \frac{a(1-r^n)}{1-r}$, if $r < 1$ (3.17)

and $S_n = \frac{a(r^n-1)}{r-1}$, if $r > 1$.

Example

Find the sum of the first eight terms of the GP $5 + 15 + 45 + ...$

Solution

In this case $a = 5$, and $r = 3$

Hence $S_n = \frac{a(r^n-1)}{r-1}$, $r > 1$

That is $S_8 = \frac{5(3^8-1)}{3-1} = \frac{5(6561-1)}{2} = 16400$.

Example

In a GP, the sum of the second and third terms is 9; and the seventh term is eight times the fourth term. Find the first term, the common ratio and the fifth term.

Solution

Let a and r denote the first term and the common ratio respectively. Then

$ar + ar^2 = 9 \Rightarrow ar(1+r) = 9$ (3.18)

and $ar^6 = 8ar^3 \Rightarrow r^3 = 8 \Rightarrow r = 2$.

Substitute $r = 2$ in equation (3.18) we have

$6a = 9 \Rightarrow a = \frac{3}{2}$. (3.19)

Thus the first term is $\frac{3}{2}$ and the common ratio is 2. Therefore the fifth term is ar^4.

That is $\frac{3}{2}(2^4) = 24$.

Definition 3.8 (Geometric Mean (GM))

Let x, y and z be three consecutive terms of a GP with common ratio r. We call y the geometric mean (GM) between x and z. Thus $r = \frac{y}{x} = \frac{z}{y}$. Hence $y^2 = xz$ and so $y = \sqrt{xz}$

Also in a finite Geometric Progreession, $u_1 = a, u_2 = ar, u_3 = ar^2 ..., u_{n-1} = ar^{n-2}, u_n = ar^{n-1}$, the terms in between u_1 and u_n (i.e $u_2, u_3, ..., u_{n-1}$) are called the Geometric means between u_1 and u_n. For instance, in the finite GP $2, 6, 18, 54, 162$, the numbers $6, 18, 54$ are the three Geometric means between 2 and162.

Example

Insert 4 Geometric means between -2 and $\frac{1}{16384}$.

Solution

In this case, we have $u_1 = -2, u_2, u_3, u_4, u_5, u_6 = \frac{1}{16384}$.

Since $u_6 = ar^5$, we have that

$\frac{1}{16384} = -2r^5 \Rightarrow r^5 = -\frac{1}{2(16384)}$ i.e $r^5 = -\frac{1}{2^{15}} \Rightarrow r = -\frac{1}{2^3} = -\frac{1}{8}$.

Hence $u_2 = ar = -2(-\frac{1}{8}) = \frac{1}{4}$

$$u_3 = ar^2 = -2\left(-\frac{1}{8}\right)^2 = -\frac{1}{32}$$

$$u_4 = ar^3 = -2\left(-\frac{1}{8}\right)^3 = \frac{1}{256}$$

$$u = ar^4 = -2\left(-\frac{1}{8}\right)^4 = -\frac{1}{2048}$$

Thus the 4 geometric means are $\frac{1}{4}, -\frac{1}{32}, \frac{1}{256}, -\frac{1}{2048}$.

Hence the GP is $-2, \frac{1}{4}, -\frac{1}{32}, \frac{1}{256}, -\frac{1}{2048}, \frac{1}{16384}$.

Example

The third term of a geometric progression is 36 and the sixth term is $\frac{243}{2}$. Find the first term, the common ratio and the sum of the first eight terms.

Solution

Let a be the first term and r the common ratio of the GP. Then we have that

$u_3 = ar^2 = 36$ (3.18)

and

$u_6 = ar^5 = \frac{243}{2}$ (3.19)

Solving for a and r, we have $a = 16$ and $r = \frac{3}{2}$. Thus the sum of the first eight terms is

$$S_8 = \frac{16\left[\left(\frac{3}{2}\right)^8 - 1\right]}{\frac{3}{2} - 1} = 788\frac{1}{8}.$$

Example

The fifth, nineth and the sixteenth terms of an AP are consecutive terms of a GP. The sum of the first twelve terms of the AP is 492. Find the first term of the AP and the common ratio of the GP.

Solution:

Let a be the first term and d the common difference of the AP. Then we have that

$u_5 = a + 4d, u_9 = a + 8d$ and $u_{16} = a + 15d$.

Since these form consecutive terms of a GP, the common ratio r is given by

$r = \frac{a+8d}{a+4d} = \frac{a+15d}{a+8d}$ (3.20)
$\Rightarrow \frac{a+8d}{a+4d} = \frac{a+15d}{a+8d}$
$\Rightarrow (a + 8d)^2 = (a + 4d)(a + 15d)$
$\Rightarrow a^2 + 16ad + 64d^2 = a^2 + 19ad + 60d^2$
$\Rightarrow 64d^2 - 60d^2 = 19ad - 16ad$
$\Rightarrow 4d^2 = 3ad$
$\Rightarrow d = \frac{3a}{4}$ (3.21)
And the sum of the first Twelve terms is 492. That is
$S_{12} = \frac{12}{2}[2a + 11d] = 492$
$\Rightarrow 6[2a + 11d] = 492$
$\Rightarrow 2a + 11d = 82$ (3.22)
Substitute equation (3.21) in equation (3.22) to get
$2a + 11\left(\frac{3a}{4}\right) = 82$
$\Rightarrow 8a + 33a = 328$
$\Rightarrow 41a = 328 \Rightarrow a = 8.$
Substitute for a in equation (3.22), to get
$d = \frac{3\times8}{4} = 6.$
Substituting for a and d in any of the equations in (3.20), we have
$r = \frac{8+8(6)}{8+4(6)} = \frac{8+48}{8+24} = \frac{56}{32} = \frac{7}{4}.$

Thus the first term of the AP is 8 and the common ratio of the GP is $\frac{7}{4}$. Alternatively, we can get the common ratio r of the GP by substituting equation (3.21) directly into any of the equation (3.20).

Exercise 3.1

(1) If $x+1, x+3$ and $x+7$ are consecutive terms of a GP, find the value(s) of x.

(2) The third term of a GP is five and the eighth term is 160. Find the first term and the common ratio of the GP.

(3) The first term of a GP is 18 and the sum to infinity is 20. Find the common ratio and the sum of the first 6 terms.

(4) Prove that the sum to n terms of the GP $1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + ... + \frac{1}{a^{n-1}}$ is equal to $\left(\frac{a^n-1}{a-1}\right)a^{1-n}$.

(5) The first three tems of a GP are also the first, nineth and eleventh terms respectively of an AP. Given that the terms of the GP are all different, find the common ratio r, if the sum to infinity of the GP is 8. Find also the first term and the common difference of the AP.

(6) The coefficients of y^n, y^{n+1}, y^{n+2} in the binomial expansion of $(1 + y)^{14}$ are in AP. Find the possible values of n.

(7) The sum of the first q terms of an AP is Q and the sum of the first r terms of the AP is R. Show that the common difference of the AP is $\frac{2(rQ-qR)}{qr(q-r)}$.

(8) The pth, qth and rth terms of an AP are consecutive terms of a GP. Prove that the ratio of the first term of the AP to its common difference is $1+\frac{q^2-pr}{r+p-2q}$. Hence given that $3(q^2-pr)=r+p-2q$ and that the 12th term of the AP is 74, find the common difference of the AP.

(9) Show that the sum to n terms of the series $\log a+\log ax+\log ax^2+...$ is $n\log a+\frac{1}{2}n(n-1)\log x$.

(10) If the sum of n terms of the series $1+8+15+22+...$ is 5500, then find n.

(11) The sum of the second and fourth terms of an AP is 15, and the sum of the fifth and sixth terms is 25. Find the first term and the common difference of the AP.

(12) The sum of the first two terms of a GP is 3, and the sum of the second and third terms is -6. Find the first term and the common ratio.

3.2.1. INFINITE SERIES

In the work so far on Arithmetic Progressions and Geometric Progressions, we have been concerned with a limited number of terms in the series, that is we have found S_n where n is a finite number. But in some situations, the series goes on for ever and the problem then arises to finding the sum S_n as n tends to infinity.

With an AP, $S_n=\frac{n}{2}\left[2a+(n-1)d\right]$. Therefore with finite values for a and d, as n increases, so does the value of S_n. Thus if n tends to infinity, then S_n tends to infinity in a positive or negative sense depending on the series.

Consider the sum of n terms of a GP given by $S_n=\frac{a(1-r^n)}{1-r}$ for $|r|<1$. Recall from equation (3.17) that for $|r|<1, r^n$ gets smaller and smaller as n gets larger and larger. In the limit, therefore, when n approaches infinity, r^n tends to zero. Then the sum of the GP becomes

$S_\infty=\frac{a}{1-r}$.

Chapter 4

Quadratic Equations and Remainder Theorem

4.1 Quadratic Equations

Definition 4.1(Quadratic Equations)

Any equation of the form $\mathrm{ax}^2 + bx + c = 0$ *where* $a \neq 0$, b *and* c are constants is called a quadratic equation. Examples include $3\mathrm{x}^2 + 2x + 4 = 0, x^2 - 6x = 0$ and $\mathrm{x}^2 - 7 = 0$

Definition 4.2 (Solutions to Quadratic Equations)

A number p is said to be the solution or root of a quadratic equation if by replacing x by p the LHS and the RHS of the Quadratic equation are equal. For example, x =3, is the solution or a root of $x^2 - 5x + 6 = 0$ because when 3 is substituted in the LHS, we have $3^2 - 5(3) + 6 = 9 - 15 + 6 = 0 = RHS$

Given any quadratic equation $ax^2 + bx + c = 0$, the quantity $b^2 - 4ac$ is called the discriminant and it is often needed to calculate or determine the nature of roots of the Quadratic Equation.

When the discriminant $b^2 - 4ac$ is a perfect square, we can factorise the Quadratic Equation. Consider $2x^2 - 7x + 3 = 0$;the discriminant is $b^2 - 4ac = 7^2{-}4(2)(3) = 25$ and 25 is a perfect square so we can go ahead to factorise $2x^2 - 7x + 3$. We take the following steps to factorise the given expression.

1. Multiply the first term by the third term.

2. Find the factors of the product in 1, which sums up to give the second term.

3. Replace the second term by the sum of these factors in 2

4 Pair the terms in 3 and factorise completely.

Example

Factorise $2x^2 - 7x + 3 = 0$ and solve the equation $2x^2 - 7x + 3 = 0$.

Solution

Step1 : $2x^2 \times 3 = 6x^2$

Step 2 : -6x and -x are the factors of $6x^2$sum up to $-7x$ i.e $-7x = -6x - x$
Step3 : $2x^2 - 7x + 3 = 2x^2 - 6x - x + 3$
Step4:$2x^2 - 6x - x + 3 = (2x^2 - 6x) + (-x + 3) = 2x(x-3) - 1(x-3)$
$= (x-3)(2x-1)$ $\therefore 2x^2 - 7x + 3 = (x-3)(2x-1)$
$\Rightarrow 2x^2 - 7x + 3 = 0 \Rightarrow (x-3)(2x-1) = 0 \Rightarrow (x-3) = 0$ or $(2x-1) = 0$
$\Rightarrow x = 3$ or $x = \frac{1}{2}$

The factorisation of the quadratic equation $ax^2 + bx + c = 0$ can be simple if it is of a special case where $b = 0$ or $c = 0$.

If $c = 0$, we have $ax^2 + bx = 0 \Rightarrow x(ax+b) = 0 \Rightarrow x = 0$ or $ax + b = 0$.
$\therefore$The solution is $x = 0$ or $x = \frac{-b}{a}$
If $b = 0$, we have:;
$ax^2 + c = 0$
$\Rightarrow x^2 + \frac{c}{a} = 0$
$\Rightarrow x^2 = \pm\sqrt{\frac{-c}{a}}$
i.e $x = -\sqrt{\frac{-c}{a}}$ or $x = \sqrt{\frac{-c}{a}}$

Example

Solve the following equations:
(i) $2x^2 - 14x = 0$
(ii)$3x^2 - 9 = 0$

Solution (i)
$2x^2 - 14x = 2x(x-7)$
$\therefore 2x^2 - 14x = 0 \Rightarrow 2x(x-7) = 0 \Rightarrow 2x = 0$ or $x - 7 = 0$
$\Rightarrow x = 0$ or $x = 7$
(ii) $3x^2 - 9 = 0 \Rightarrow 3x^2 = 9 \Rightarrow x = \pm\sqrt{3} \Rightarrow x = -\sqrt{3}$ or $x = +\sqrt{3}$

Example

Now consider $t(t-3) = t^2 - 4$ and the following solution.

Solution

$t(t-3) = t^2 - 4$
$\Rightarrow t^2 - 3t = t^2 - 4$
$\Rightarrow 4 - 3t = 0$
$\Rightarrow t = \frac{4}{3}$

Exercise 4.1

Solve the following quadratic equations
1.$x^2 - 5x - 6 = 0$
2.$3x^2 - 7x = 0$
3.$4x^2 - 8x + 4 = 0$
4.$x(x-3) = x^2 - 6$
5.$3 - x - 2x^2 = 0$
6 $x^2 - 2ax + a^2 = 0$

7.$\frac{1}{x^2} - 1 = \frac{1}{x} - 1$
8.$x(1-x) = x(2x-1)$
9 $(x-2)(x+3) = (x-2)(4-x)$
10.$\frac{1}{x+1} + \frac{2}{x+2} = 1$

4.2 SOLUTION OF QUADRATIC EQUATIONS THAT DO NOT FACTORIZE

When a quadratic equation $ax^2 + bx + c = 0$ does not factorize (and this we know when the discriminant $b^2 - 4ac$ is not a perfect square), we use the method of completing the square to obtain the solution.

For example, consider $2x^2 - 5x + 1 = 0$
Dividing through by 2 we obtain $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$
$\Rightarrow x^2 - \frac{5}{2}x = -\frac{1}{2}$
Add $(\frac{1}{2} \times \frac{5}{2})^2$ to both sides to make LHS a perfect square, we have:
$x^2 - \frac{5}{2}x^2 + \left(\frac{5}{4}\right)^2 = -\frac{1}{2} + \left(\frac{5}{4}\right)^2$
$\Rightarrow \left(x - \frac{5}{4}\right)^2 = \frac{17}{16}$
$\Rightarrow x - \frac{5}{4} = \pm\sqrt{\frac{17}{16}}$
$\Rightarrow x = \frac{5}{4} \pm \sqrt{\frac{17}{16}}$
$\Rightarrow x = \frac{5}{4} + \frac{\sqrt{17}}{4}$or $x = \frac{5}{4} - \frac{\sqrt{17}}{4}$
$\therefore x = 2.28$ or $x = 0.22$.
Generally, for $ax^2+bx+c = 0$, we follow the same procedure above. Thus
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
$\Rightarrow x^2 + \frac{b}{a}x = \frac{-c}{a}$
$\Rightarrow x^2 + \frac{b}{a}x + \left(\frac{1}{2} * \frac{b}{a}\right)^2 = \frac{-c}{a} + \left(\frac{1}{2} * \frac{b}{a}\right)^2$
$\Rightarrow \left(x + \frac{b}{a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$
$\Rightarrow \left(x + \frac{b}{a}\right)^2 = \frac{-4ac+b^2}{4a^2} = \frac{b^2-4ac}{4a^2}$
i.e$\left(x + \frac{b}{a}\right)^2 = \frac{b^2-4ac}{4a^2}$
$\Rightarrow \left(x + \frac{b}{a}\right) = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \frac{\sqrt{b^2-4ac}}{2a}$
$\Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$
$\therefore$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is called the quadratic formula.

4.3 NATURE OF THE ROOTS OF QUADRATIC EQUATIONS

Using the quadratic formula to solve the equation $ax^2+bx+c = 0$,we see that either $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Therefore in general, a quadratic equation has two solutions (called the roots).

If $b^2 - 4ac$ is positive, then$\sqrt{b^2 - 4ac}$ can be evaluated and the equation will have two real and distinct(different) roots.

If $b^2 - 4ac$ is zero, then the equation can be satisfied by $x = \frac{-b}{2a}$ and we say it has repeated roots or equal roots

If $b^2 - 4ac$ is negative,$\sqrt{b^2 - 4ac}$ has no real value and we say the equation has complex roots

In summary, the equation $ax^2 + bx + c = 0$

(i) has two real and distinct roots if $b^2 - 4ac > 0$

(ii) has equal roots if $b^2 - 4ac = 0$

(iii) has complex roots if $b^2 - 4ac < 0$

Example

Determine the nature of roots of the equations

(a) $4x^2 - 7x + 3 = 0$ (b) $x^2 + ax + a^2 = 0$ (c) $x^2 - px - q^2 = 0$

Solution

(a)$4x^2 - 7x + 3 = 0 \therefore b^2 - 4ac = (-7)^2 - 4(4)(3) = 1$ i.e $b^2 - 4ac$ $\{>\}0$, So the equation has two distinct roots. More over, 1 is a perfect square so the equation also has rational roots

(b) $x^2 + ax + a^2 = 0 \therefore b^2 - 4ac = a^2 - 4(1)(a^2) = -3a^2$ since a^2 is positive for all real and non zero a, $\therefore -3a^2 < 0$

i.e $b^2 - 4ac < 0$ so the equation has no real roots for any real and non zero value of a

(c) x$^2 - px^2 - q^2 = 0$

b$^2 - 4ac = (-p)^2 - 4(1)(-q^2) = p^2 + 4q^2$ As p^2 and q^2 are both positive $b^2 - 4ac > 0$ therefore, the equation has two real and distinct roots

Example

Find the value of k if $2x^2 - kx + 8 = 0$ has equal roots.

Solution

For the root of $2x^2 - kx + 8 = 0$ to be equal, the discriminant must be zero.

i.e $b^2 - 4ac = 0$

$\Rightarrow (-k) - 4(2)(8) = 0$

$\Rightarrow k^2 - 64 = 0$

$\Rightarrow k = \pm 8$

Exercise 4.2

Solve the following quadratic equations by completing the squares

(1) $2x^2 - 6x + 4 = 0$ (2) $x^2 + 4x - 8 = 0$ (3) $2x^2 + 7x + 3 = 0$

(4) $x^2 - 2x + a = 0$ (5) x^2 -2$ax + b = 0$ (6) $ax^2 - bx + c = 0$

Determine the nature of roots of the following equations and hence solve the equations.

(7) $x^2 - 6x + 9 = 0$ (8) $2x^2 - 5x + 3 = 0$ (9) $x^2 - 6x + 10 = 0$

(10) $3x^2 + 4x + 2 = 0$ (11) $4x^2 - 12x + 9 = 0$

(12) $4x^2 - 12x - 9 = 0$.

(13) For what value of k is $9x^2 + kx + 16$ a perfect square?

(14) The roots of $3x^2 + kx + 12 = 0$ are equal find k.

(15) Prove that $kx^2 - 2x - (k-2) = 0$ has real roots for any value of k.

(16) Show that the roots of $ax^2 + (a+b)x + b = 0$ are real for any values of a and b .

(17) Find the relationship between p and q if the roots of $px^2 + qx + 1 = 0$ are equal.

(18) If $f = \frac{x^2+1}{2x-1}$is real and $p = 3\left(\frac{x^2+1}{2x-1}\right)$, prove that $p^2 = -3\,(p+3) \geqslant 0$.

4.4 RELATIONSHIP BETWEEN THE ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

Let α and β be the roots of the quadratic equation

$ax^2 + bx + c = 0$

i.e $(x-\alpha)(x-\beta) = 0$ (4.1)

$ax^2 + bx + c = 0$ (4.2)

(4.1) and (4.2) must have the same solution

$\therefore x^2 - (\ \alpha + \beta\ )x + \alpha\beta = 0$ (4.3) and

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ (4.4)

must have the same solution. As the LHS and the RHS of (4.3) and (4.4) have the same coefficients of x^2, it follows that the coefficeints of x and the constant terms are equal

i.e $x^2 - (\ \alpha + \beta\ )x + \alpha\beta \equiv x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\therefore \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ and the equation may be written as

$x^2 -$ (sum of roots) x + product of roots $= 0$

So if $2x^2 - 3x + 6 = 0$ then the sum $\alpha + \beta = -(\frac{-3}{2}) = \frac{3}{2}$ and the product $\alpha\beta = \frac{6}{2} = 3$.

Also if a quadratic equation has roots whose sum is 7 and whose product is 10, then the equation can be written as $x^2 - 7x + 10 = 0$

Definition 4.3 (Symmetry of the roots)

An expression involving the roots of α, β of a quadratic equation is said to be symmetric if when α is subtituted for β and β for α, the expression either remains the same or the expression becomes exactly negetive of the original expression.

For example,$\frac{1}{\alpha} + \frac{1}{\beta}$ becomes$\frac{1}{\beta} + \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta}$ when we replace α by β and β by α in the expression. Another example is $\alpha^2 - \beta^2$ which becomes $\beta^2 - \alpha^2 = -\left(\alpha^2 - \beta^2\right)$.

But consider $\alpha^2 - \beta$, this becomes $\beta^2 -\ \alpha$ which is neither the same as the original expression nor negetive of the original expression

Example

If α and β are the roots of the quadratic equation $2x^2 - 7x + 4 = 0$, calculate the values of (i) $\alpha^2 + \beta^2$ (ii)$\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (iii) $\alpha - \beta$ (iv)$\alpha^3 + \beta^3$

Solution

From the given equation,

The sum of the roots, $\alpha + \beta = -\left(\frac{-7}{2}\right) = \frac{7}{2}$, and the product of the roots, $\alpha\beta = \frac{4}{2} = 2$. Thus

(i) $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\,\beta \Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\,\beta$

$\therefore \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\,\beta$ but $\alpha + \beta = \frac{7}{2}$ and $\alpha\,\beta = 2$

$\therefore \alpha^2 + \beta^2 = \left(\frac{7}{2}\right)^2 - 2(2) = \frac{49}{4} - 4 = \frac{33}{4}$

(ii)$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{\frac{33}{4}}{2} = \frac{33}{8}$

(iii) Observe that $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\,\beta = (\alpha + \beta)^2 - 2\alpha\,\beta - 2\alpha\,\beta$

i.e $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\therefore \alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm\sqrt{\left(\frac{7}{2}\right)^2 - 4\,(2)} = \pm\sqrt{\frac{17}{4}} = \pm\frac{\sqrt{17}}{2}$

(iv) Consider$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = \alpha^3 + \beta^3 + 3\alpha\beta\,(\alpha + \beta)$

$\therefore (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta) = \alpha^3 + \beta^3$

or $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta) = (\alpha + \beta)\left[(\alpha + \beta)^2 - 3\alpha\beta\right]$

$= \frac{7}{2}\left[\left(\frac{7}{2}\right)^2 - 3\,(2)\right] = \frac{175}{4}$

Example

Write the equation of the quadratic function whose roots are α^2, β^2 where α, β are the roots of the quadratic equation $2x^2 + 7x + 4 = 0$

Solution

$\alpha + \beta = -\frac{7}{2}, \alpha\beta = \frac{4}{2} = 2$ and the equation may be written as

$x^2 - (\mathit{sum}$ of roots$)\,x + \mathit{product}$ of roots $= 0$ and since α^2, β^2 are the roots of the equation, sum of roots $= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{7}{2}\right)^2 - 2\,(2) = \frac{33}{4}$

product of roots $= \alpha^2\beta^2 = (\alpha\beta)^2 = (2)^2 = 4$

$\therefore$The equation is $x^2 - \frac{33}{4}x + 4 = 0$ or $4x^2 - 33x + 16 = 0$

Example If α, β are the roots of the quadratic equation $x^2 - 3x - 2 = 0$, Find the quadratic equation whose roots are α^3, β^3

Solution

Since α, β are the roots of $x^2 - 3x - 2 = 0, \alpha + \beta = -\left(\frac{-3}{1}\right) = 3$ and $\alpha\beta = \frac{-2}{1} = -2$

Now sum of roots of desired equation $= \alpha^3 + \beta^3 = (\alpha + \beta)\left[(\alpha + \beta)^2 - 3\alpha\beta\right] = 3\,(3^2 - 3\,(-2)) = 45$

Product of roots of desired equation $= \alpha^3\beta^3 = (\alpha\beta)^3 = (-2)^3 = -8$

$\therefore$ The equation whose roots are α^3, β^3 is $x^2 - 45x - 8 = 0$

Example (i) Find the range of values of k for which the equation $x^2 - 2x - k = 0$ has real roots (ii) If the roots of this equation differ by one, find the value of k.

Solution

If $x^2-2x-k=0$ has real roots, then $b^2-4ac\geq 0\Rightarrow (-2)^2-4\left(1\right)\left(-k\right)=0\Rightarrow 4+4k\geq 0\Rightarrow k\geq -1$

(ii)Let one root of the equation be α, then the other root is $\alpha+1$ Sum of roots $=\alpha+\alpha+1=2\alpha+1=-\left(-2\right)\Rightarrow 2\alpha+1=2\Rightarrow \alpha=\frac{1}{2}$

product of roots $=\alpha\left(\alpha+1\right)=-k$, since $\alpha=\frac{1}{2}$, then $k=\frac{-3}{4}$

Exercise 4.3

(1) Write down the sum and the product of the roots of the following equations

(a) $x^2-3x+2=0$ (b) $4x^2+7x-3=0$ (c) $x(x-3)=x+4$ (d)$\frac{x-1}{2}=\frac{3}{x+2}$ (e) $x^2-kx+k^2=0$

(f) $ax^2-x(a+x)-a=0$

(2)Write down the sums, the products and the equations whose roots are (a) 3,4 (b)$-2,\frac{1}{2}$ (c) $\frac{1}{3},\frac{-2}{5}$ (d) -$\frac{1}{4},0$ (e) a,a^2 (f) $-\left(k+1\right),k^2-3$ (g)-$\frac{b}{a},\frac{c^2}{b}$

(3)The roots of the equation $2x^2-4x+5=0$ are α and β. Find the value of

(a) $\frac{1}{\alpha}+\frac{1}{\beta}$ (b) $\left(\alpha+1\right)\left(\beta+1\right)$ (c) $\alpha^2+\beta^2$ (d) $\alpha^2\beta+\alpha\beta^2$ (e)$\left(\alpha-\beta\right)^2$ (f)$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (g)$\frac{1}{\alpha+1}+\frac{1}{\beta+1}$ (h) $\frac{1}{2\alpha+\beta}+\frac{1}{\alpha+2\beta}$ (i) $\frac{1}{\alpha^2+1}+\frac{1}{\beta^2+1}$ (j)$\alpha^2-\beta^2$ (k)$\alpha^3+\beta^3$ (l)$\alpha^3-\beta^3$

(4)The roots of $x^2-2x+3=0$ are α and β, find the equations whose roots are (a)$\alpha+2$,$\beta+2$ (b)$\frac{1}{\alpha},\frac{1}{\beta}$ (c) α^2,β^2 (d)$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (e) $\alpha-\beta,\beta-\alpha$

(5) Write down and simplify the equation whose roots are the reciprocal of of the roots of $3x^2+2x-1=0$ without solving the given equation

(6)Write down and simplify the equation whose roots are the double of those of $4x^2-5x-2=0$ without solving the equation.

(7)Write down and simplify the equation whose roots are *one* less than those of $5x^2+3x-2=0$

(8) Write down the equation whose roots are minus those of $2x^2-3x-1=0$

(9) find the value of k if the roots of $3x^2+5x-k$ differ by two.

(10) Find the value of p if one root of $x^2+px+8=0$ is the square of the other.

(11)If α and β are the roots of $ax^2+bx+c=0$, find the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

(12) Find the relationship between a and c if one root of $ax^2+bx+c=0$ is the reciprocal of the other.

(13) If one root of $ax^2+bx+c=0$ is the triple of the other, prove that $3b^2-16ac=0$

(14) If α and β are the roots of the equation $x^2+px+q=0$, find the roots of the equation $x^2+p^2x+p^2q=0$ in terms of α andβ.

4.5 CURVE OF A QUADRATIC FUNCTION

Definition 4.4 (Quadratic function)

Any function f whose general form is $f(x) = ax^2 + bx + c$ where a, b and c are constants and $a \neq 0$ is called a quadratic function. Completing the square of the RHS of the expression gives

$f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[x^2 + \frac{b}{a}x\right] + c = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right] + c$

$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right] + \frac{-b^2}{4a} + \frac{c}{a} = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right] + \frac{-b^2-4ac}{4a}$

$= a\left[x + \frac{b}{2a}\right]^2 + \frac{4ac-b^2}{4a}$

Now what ever value x takes $\frac{4ac-b^2}{4a}$ is a constant and equal to k say and $\left[x + \frac{b}{2a}\right]^2 \geq 0$ as it is a square quantity.

Therefore the function has a general form

$f(x) = a(zero\ or\ a + ve$ quantity) $+k$. So if a is positive, i.e $a \geq 0$, $f(x)$ is at least k i.e $f(x)$ has at least $\frac{4ac-b^2}{4a}$ occuring when $x = \frac{-b}{2a}$. When a is negetive, i.e $a < 0$ then $f(x)$ can never be greater than k. i.e $f(x)$ has the greatest value of $\frac{4ac-b^2}{4a}$ when $x = -\frac{b}{2a}$

Now, $x = -\frac{b}{2a}$ is the value of x corresponding to the greatest and least value of $f(x)$. Taking two values of x that are symmetric about $x = -\frac{b}{2a}$ i.e -$\frac{b}{2a} \pm k$ gives $f(-\frac{b}{2a} + k) = f(-\frac{b}{2a} - k) = ak^2 + \frac{4ac-b^2}{4a}$ i.e input of x is symmetric about $x = -\frac{b}{2a}$ give the same output of $f(x)$

From the analysis, we can deduce that when $f(x) = ax^2 + bx + c$, then

$f(x)$ has (i) a least value of $f(-\frac{b}{2a})$ if $a > 0$ (ii) a greatest value of $f(-\frac{b}{2a})$ if $a < 0$ and is symmetric in shape around the line $x = -\frac{b}{2a}$ which is called the axis of the curve.

The diagrams below represent the two alternative graphs of the quadratic function.

The curves representing any partiular quadratic function can now be sketched using this information. For example, to sketch the curve representing $f(x) = 2x^2 - 7x - 4$, we proceed as follows:

$a = 2, b = -7$ and $c = -4$

$\therefore f(-\frac{b}{2a}) = f(\frac{7}{4}) = -\frac{81}{8}$ i.e $f(x)$ has the least value of -$\frac{81}{8}$, when $x = \frac{7}{4}$. To locate the curve accurately on the axes, we need one more pair of x and $f(x)$. $f(0)$ is easy to find i.e $f(0) = -4$. Again we need the points at which f crosses the x axis. This is the solution of $2x^2 - 7x - 4 = 0$ which is $x = \frac{1}{2}$ and $x = 4$ The curve is shown below:

For any quadratic function $f(x) = ax^2 + bx + c$ the point at which the curve cuts the x-axis is the solution of $f(x) = 0$ i.e $ax^2 + bx + c = 0$. If $ax^2 + bx + c = 0$ has real and distinct roots, the curve cuts the X-axis at two points and the roots are the point at which the curve cuts the x-axis.If the equation has equal roots, then the curve only touches the x-axis at the point $x = -\frac{b}{2a}$ .When the roots are complex, the curve never touches the x-axis.

Below are the possible diagrams of the curve of $f(x) = ax^2 + bx + c$ with respect to the x-axis

4.6 QUADRATIC INEQUALITIES

Any inequality relationship that involves a quadratic function is called a Quadratic inequality. For example $(x-2)(2x+1) > 0$

The range of values of x that saisfies the quadratic inequality can be found graphically as follows.

Let $f(x) = (x-2)(2x+1)$. The diagram below shows a sketch of $f(x)$ from which we see that $f(x) > 0$ (i.e the portion of the curve above the y-axis) are for values x greater than 2 and x less than $-\frac{1}{2}$.

Therefore the range of values of x satisfying the inequality $(x-2)(2x+1) > 0$ is $\left\{x : x < -\frac{1}{2} \text{or } x > 2\right\}$

Alternatively, we can solve the inequality by analytic method.

consider $(x-2)(2x+1) > 0$,

$\Rightarrow (x-2)$ *and* $(2x+1)$ must have the same signs. Either both are negetive or both are positive

$\Rightarrow (x-2) > 0$ and $(2x+1) > 0$ or $(x-2) < 0$ and $(2x+1) < 0$

$\Rightarrow$ either $x > 2$ and $x > -\frac{1}{2}$ or $x < 2$ and $x < -\frac{1}{2}$

$\Rightarrow x > 2$ or $x < -\frac{1}{2}$

Example

Find the range of values which are satisfied by the inequality $x^2 - 8x + 15 < 0$

Solution

The LHS of the inequality can be factorised as $(x-3)(x-5)$

Graphical method:

The sketch of $f(x) = (x-3)(x-5)$ is given below

The portion of the curve satisfied by the inequality (portion below the x-axis) are for x between 3 and 5 . Therefore the range of values satisfied by the inequality is $3 < x < 5$

Analytic method

Again, this can be solved analytically by observing that if $(x-3)(x-5) < 0$ then $x-3$ and $x-5$ must have different signs

$I.e$ either $x-3 > 0$ and $x-5 < 0$ or $x-3 < 0$ and $x-5 > 0$

$\Rightarrow$either $x > 3$ and $x < 5$ or $x < 3$ and $x < 5$ but $x < 3$ and $x > 5$ has no solution because if x is greater than 3, it cannot be less than 5.

$\therefore$ The range is $x > 3$ and $x < 5 \Rightarrow x \in (3, 5)$ or $3 < x < 5$

Exercise 4.4

Find the range(or ranges) of x satisfying the following inequalities

(1) $(x+2)(x-2) > 0$ (2)$(x-1)(x-2) \geqslant 0$ (3) $(2x-1)(x+1) < 0$ (4) $x^2 - 4x > 5$ (5) $(x-3)(x+5) < 0$ (6) $5x^2 > 3x + 2$ (7) $(2-x)(x+4) < 0$ (8) $3x > x^2 + 2$ (9) $(3-2x)(x+5) > 0$ 10 $(x-1)(x+2) < x(4-x)$

FURTHER EXAMPLES

Example

Find the range of values of k for which the equation $x^2 - kx + k + 3 = 0$ has real roots.

Solution

For $x^2 - kx + k + 3 = 0$ to have real roots, then $b^2 - 4ac \geq 0$

$\Rightarrow (-k)^2 - 4(1)(k+3) \geq 0$

$\Rightarrow k^2 - 4k - 12 \geq 0$

From the sketch of $f(k) \equiv k^2 - 4k - 12 \Rightarrow f(k) \equiv (k+2)(k-6)$

From the diagram, we see that $k \leq -2$ and $k \geq 6$

(2) Find the set of values of p for which $f(x) \equiv x^2 - 3px + p > 0$ for all x.

Solution

$f(x) \equiv x^2 - 3px + p = x^2 + 3px + \left(\frac{3p}{2}\right)^2 - \left(\frac{3p}{2}\right)^2 + p$

$= \left(x + \frac{3p}{2}\right)^2 + p - \frac{9p^2}{4}$

$\therefore The$ least value of $f(x)$ is $p - \frac{9p^2}{4}$

So for $f(x)$ to be greater than zero for all x, we must have that $p - \frac{9p^2}{4} > 0$

$\Rightarrow 4p - 9p^2 > 0 \Rightarrow p(4-9p) > 0.$

The sketch of $f(p) = p(4-9p)$ is as below:

$\Rightarrow p > 0$ and $p < \frac{4}{9}$

Therefore, $f(x) > 0$ for all real x holds for the set of values of p such that $0 < p < \frac{4}{9}$.

Example

Find the range of values of x for which $2x-1<x^2-4<12$

Solution

There are two inequality relationships here. viz

(a) $2x-1<x^2-4$ and (b) $x^2-4<12$

For (a) $2x-1<x^2-4 \Rightarrow -x^2+2x+3<0 \Rightarrow x^2-2x-3>0 \Rightarrow (x-3)(x+1)>0$

Let $f(x) \equiv (x-3)(x+1)$. The graph of $f(x)$ is as drawn below

From the diagram, we can see that the region satisfied by (a)is $x<-1$ and $x>3$

For (b) $x^2-4<12 \Rightarrow x^2-16<0 \Rightarrow (x+4)(x-4)<0$

Let $g(x)=(x+4)(x-4)$ The graph of g is as sketched below:

From the graph of $g(x)$, the $g(x)<0$ if $-4<x<4$

The range of $f(x)$ and $g(x)$ can be represented on the same number line (see chapter one) to obtain the range of values that satisfy (a) and (b) simultaneously.From the above we see that the ranges of x that satisfy both

a and b (this is where the lines overlap) are $-4<x<-1$ and $3<x<4$.

Exercise 4.5

(1) Find the range of values of k for which $x^2+(k-3)x+k=0$ has (a) real and distinct roots (b) roots of the same sign

(2) If x is real and $x^2+(2-k)x+1-2k=0$, show that k cannot lie between certain limits and find these limits

(3) Find the limitations required on the values of real number c in order that the equation $x^2+2cx-c+2=0$ shall have real roots.

(4) Prove that if $x^2>k(x+1)$ for all real x then $-4<k<0$

(5) Find the condition that must be satisfied by k in order that the expression $2x^2+6x+1+k(x^2+2)$ may be positive for all real values of x

(6) Find the the range of values of x for which $x-4<x(x-4)\leq 5$

(7)(a) If $x=2$ is a root of the equation $\alpha^2x^2+2(2\alpha-5)x+8=0$, find the possible value(s) of α and the corresponding value(s) of x .

(b)Find the range(s) of possible values of the real number α if $\alpha^2x^2+2(2\alpha+5)x+8=0$ for all real values of x.

(8) Determine for each of the expressions $f(x)$ and $g(x)$ the range(s) of values of x for which it is positive. Give your answer correct to 2 decimal places. Explain briefly the reason for your answers

(a) $f(x) = x^2 + 4x - 6$ (b) $g(x) = -x^2 - 8x + 2$.

(9) (a) State the range of values of x for which $2x^2 + 5x - 12$ is negative.

(b) The value of a constant is such that the quadratic function $f(x) = x^2 - 4x + a + 3$ is never negative. Determine the nature of roots of the function $af(x) = (x^2 + 1)(a - 1)$. Deduce the values of a for which the equation has equal roots.

(10) By eliminating x and y from the equation$\frac{1}{x} + \frac{1}{y} = 1, x + y = a$ and $\frac{y}{x} = m$, where $a \neq 0$. Obtain a relationship between m and a given that a is real. Determine the ranges of values of a for which m is real.

(11) If $a > 0$, prove that the quadratic expression $ax^2 + bx + c$ is positive for all real values of x, when $b^2 < 4ac$. Hence find the range of values of p for which the quadratic function $f(x) \equiv 4x^2 + 4px - (3p^2 + 4p - 3)$ is positive for all real values of x Illustrate your result by making sketch of the graphs for each case $p = 0$, $p = 1$.

(12)(a) If a is a positive constant, find the set of values of x for which $a(x^2 + 2x - 8)$ is negative.Find the value of a if this fuction has a minimum value of 11.

(b) Find two quadratic functions of x which are 0 at $x = 1$, which take value 10 when $x = 0$ and has a maximum value of 18. Sketch the graph of these two functions.

(13) Find the set of values of k for which $f(x) \equiv 3x^2 - 5x - k$ is greater than unity for all values of x. Show that for all k, the minimum value of $f(x)$ occurs when $x = \frac{5}{6}$. Find k if this minimum value is 0.

(14) Prove that $3x^2 - 4x + 2 > 0$ for all real values of x.

(15) Find the value of $k (\neq 1)$ so that the quadratic function $k(x + 2)^2 - (x - 1)(x - 2)$ is equal to 0. Find also (a) the range of k for which the function possesses a minimum value, (b) The range of values of k for which the value of the function never exceeds 12.5. Sketch the graph of the funtion for $k = \frac{1}{2}$ and for $k = 2\frac{1}{2}$.

(16) Find the set of values of x for which $2x^2 + 3x + 2 > 0$

(17)The root of the equation $9x^2 + 6x + 1 = 4kx$ where k is a constant are denoted by α and β

(a) Show that the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $x^2 + 6x + 9 = 4kx$

(b) Find the of the set of values of k for which α and β are real.

(c) Find also the values of k for which α and β are real and positive

(18) Define the function f as $f : x \to x^2 - 3x - 4$, for $x \in IR$. Find the range of f and the values of x for which $f(x) = 0$.

4.7 REMAINDER AND FACTOR THEOREMS

The remainder theorem enables us to find the remainder when certain algebraic expressions are divided by a given factor without necessarily performing the actual division. An immediate consequence of the remainder theorem is the factor theorem.

4.7.1 Remainder theorem

If $P(x)$ is a polynomial in x, then the remainder when $P(x)$ is divided by $(x-b)$

is $P(b)$.

Proof: Let $P(x)$ be a polynomial in $x, Q(x)$ be the quotient when $P(x)$ is divided

by $(x-b)$ and R be the remainder. Thus $P(x) = (x-b)Q(x) + R$.

$\therefore P(b) = (b-b)Q(b) + R = 0Q(b) + R = R$.

4.7.2 Factor Theorem

From 4.7.1, if $P(x)$ vanishes when $x = a$, then $(x-a)$ is

a factor of $P(x)$. Thus, $P(a) = 0$ if $(x-a)$ is a factor of $P(x)$.

Example

Find the remainder when $x^3 + 5x^2 - 2x - 1$ is divided by $(x+3)$.

Solution: Let $f(x) = x^3 + 5x^2 - 2x - 1$.

Applying the remainder theorem we have:

$f(-3) = (-3)^3 + 5(-3)^2 - 2(-3) - 1 = 27 + 45 + 6 - 1 = 23$.

$\therefore$ the remainder is 23.

Example

Find the values of the constants g and h if $(x-1)$and $(3x-1)$are both factors of the polynomial $3x^4 + gx^3 - 6x^2 + hx - 1$.

Solution: Let $f(x) = 3x^4 + gx^3 - 6x^2 + hx - 1$;

if $(x-1)$ is a factor, then $f(1) = 0$,

thus $f(1) = 3 + g - 6 + h - 1 = 0$, i.e. $g + h = 4...(i)$

If $(3x-1)$ is a factor, then $f(\frac{1}{3}) = 0$,

thus $f(\frac{1}{3}) = 3(\frac{1}{3})^4 + g(\frac{1}{3})^3 - 6(\frac{1}{3})^2 + h\frac{1}{3} - 1 = 0$.

$\frac{1}{27} + \frac{g}{27} - \frac{2}{3} + \frac{h}{3} - 1 = 0$

i.e. $\frac{g}{27} + \frac{h}{3} = \frac{44}{27}$ or $g + 9h = 44...(ii)$.

From (i) and (ii) we have a system of simultaneous equations and solving we

have $g = -1$ and $h = 5$.

Example

Find the value of m if $(x-2)$ is a factor of $x^3 + mx^2 - 4x + 4$.

Find the other factors of the expression.

Solution: Let $f(x) = x^3 + mx^2 - 4x + 4$.

Then $f(2) = 2^3 + 4m - 8 + 4 = 0$

$$\Longrightarrow 4m = -4$$
$$m = -1.$$

the expression is $x^3 - x^2 - 4x + 4$.

Using factor theorem let's check whether $(x \pm 1)$ and $(x+2)$ are factors:

$f(-1) = -1 - 1 + 4 + 4 = 6$, thus $(x+1)$ is not a factor.

$f(1) = 1 - 1 - 4 + 4 = 0$, hence $(x-1)$ is a factor

$f(-2) = (-2)^3 - 4 - 4.(-2) + 4 = -8 - 4 + 8 + 4 = 0$, giving $(x+2)$ as a factor. Now,

$(x-2)(x-1)(x+2) = (x^2 - 3x + 2)(x+2) = x^3 - x^2 - 4x + 4$.

$\therefore$ the other factors are $(x-1)$and $(x+2)$.

Example

The equations $x^2 + 9x + 2 = 0$ and $x^2 + hx + 5 = 0$ have a common root. Find the quadratic equation giving the two possible values of h.

Solution: Let the common root be a. Then $(x-a)$ will be a factor of both

$x^2 + 9x + 2$ and $x^2 + hx + 5$. Thus $a^2 + 9a + 2 = 0$ and $a^2 + ha + 5 = 0$.

i.e. $a^2 + 9a + 2 = a^2 + ha + 5$

i.e. $(9-h)a = 3$

$\therefore a = \frac{3}{9-h}$, substituting this in $a^2 + 9a + 2 = 0$;

we get: $\frac{9}{(9-h)^2} + \frac{27}{9-h} + 2 = 0$.

i.e. $9 + 27(9-h) + 2(9-h)^2 = 0$

i.e. $9 + 27.9 - 27h + 2.(9^2 - 18h + h^2) = 0$

or $9 + 27.9 - 27h + 2.9^2 - 36h + 2h^2 = 0$

$2h^2 - 63h + 414 = 0$ is the required quadratic equation for h.

Exercise 4.7

1. Find the remainder when:-

(i) $x^4 - 6x^2 + 1$ is divided by $(x+2)$

(ii) $x^{23} - 1$ is divided by $(x+1)$.

2. Find the value of the constant k if the remainder is 22 when the expression $3x^3 + kx^2 - 4x + 6$ is divided by $(x-4)$.

3. Find the values of m and n if $(x-2)$ and $(x-3)$are both factors of

$x^3 + mx^2 + nx + 6$.

4. Find the value of the constant k for which the polynomial $x^3 + kx^2 + 5x - 10$ has $(x+2)$ as a factor.

5. The expression $gx^2 + hx - 2$ has $(x+2)$ as a factor. When the expression is divided by $(x-1)$, the remainder is 6. Find the values of g and h .

6. Whem $x^2 + mx + n$ is divided by $x - h$ the remainder is the same as when

the quadratic expression is divided by $x - 2h$. Find two possible values of h.

Chapter 5

Indices and Logarithms

5.1 Introduction

The concept of indices and loarithms have been very important in the development of Mathematics. We start by defining a^m, where a is any integer, m a natural number, and develop rules for writing down the product, quotient, e.t.c., of two such quantities. We thereafter discover meanings for a^m when m is negative, fractional or zero. We finally apply these rules in solving related problems. We next introduce the concept of logarithms whose basic theory follows naturally from that of indices.

5.1 INDICES

When we mutliply a quantity a by itself, say five times, the resulting product we have is a to power $5, i.e. a \times a \times a \times a \times a = a^5$. Thus, in general, given m a natural number, a^m denotes the product of m factors each equal to a, and the number m expressing the power is called the index (indices for plural) and a is

called the base. a^m itself is called an index number.

We have three fundamental laws of indices, and we state them as follows:

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$.

Where m and n are natural numbers and $m > n$ in (ii) . These formulae can

easily be proved from first principles as follows:

(i) From the definiton of a^m and a^n,

$a^m = a \times a \times a \times ...$ to m factors,

$a^n = a \times a \times a \times ...$ to n factors, so that

$a^m \times a^n = a \times a \times a \times ...$ to $(m + n)$ factors,

$= a^{m+n}$.

(ii) $a^m \div a^n = \frac{a \times a \times a \times ... \text{ to } m \text{ factors}}{a \times a \times a \times ... \text{ to } n \text{ factors}}$

With $m > n$, the n factors in the denominator will cancel with n of the m factors

in the numerator leaving $(m - n)$ factors, so that,

$a^m \div a^n = a \times a \times a \times ...\text{to}(m - n)$ factors,

$= a^{m-n}$.

$(a^m)^n = a^m \times a^m \times a^m \times ...\text{to n factors}$

$= a^{m+m+m+m... \text{ to n terms,}}$ by (i),

$= a^{mn}$.

The results above can be extended to cases where m is negative, fractional and

zero. And this we do as follows:

(iv) $(a^{p/q})^q = a^{\left(\frac{p}{q}\right)q} = a^p$ by (iii)).

(v) $\frac{a^m}{a^m} = a^{m-m} = a^o$ by (ii)).

But we know that $\frac{a^m}{a^m} = 1$.

$\therefore a^o = 1$. Thus any number (except 0) raised to the power of zero equals 1.

(vi) $\frac{1}{a^m} = \frac{a^o}{a^m} = a^{o-m} = a^{-m}$ (by (ii)&(v)).

Hence $\frac{1}{a^m} = a^{-m}$.

Thus a^{-m} is the reciprocal of a^m.

Example If $\{(a^3)^4 \times a^2\}/a^9 = a^h$, find h.

Solution: $\frac{(a^3)^4 \times a^2}{a^9} = \frac{a^{12} \times a^2}{a^9} = \frac{a^{12+2}}{a^9} = a^{14-9} = a^5$, thus $h = 5$.

Example Given that $g = 27^{\frac{2}{3}}$ and $h = 3^{-4}$, find (without using tables) the value of g^2h

Solution: $g = 27^{\frac{2}{3}} = \sqrt[3]{(27)^2} = 3^2$

Thus, $g^2h = (3^2)^2 \times 3^{-4} = 3^4 \times \frac{1}{3^4} = 1$.

Example Evaluate:

(i) $4^{\frac{1}{2}} \times 8^{\frac{1}{3}}$, and (ii) $(24)^{\frac{1}{2}} \times (32)^{\frac{2}{3}} \times 6^{\frac{1}{6}} \times 3^{\frac{1}{3}}$.

Solution: (i) $4^{\frac{1}{2}} \times 8^{\frac{1}{3}} = (2^2)^{\frac{1}{2}} \times (2^3)^{\frac{1}{3}} = 2 \times 2 = 4$.

(ii) $(24)^{\frac{1}{2}} \times (32)^{\frac{2}{3}} \times 6^{\frac{1}{6}} \times 3^{\frac{1}{3}} = (2^3 x 3)^{\frac{1}{2}} \times (2^5)^{\frac{2}{3}} \times (2 \times 3)^{\frac{1}{6}} \times 3^{\frac{1}{3}}$

$= 2^{\frac{3}{2}} \times 3^{\frac{1}{2}} \times 2^{\frac{10}{3}} \times 2^{\frac{1}{6}} \times 3^{\frac{1}{6}} \times 3^{\frac{1}{3}} = 2^{\left(\frac{3}{2}+\frac{10}{3}+\frac{1}{6}\right)} \times 3^{\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{3}\right)}$

$= 2^{\frac{30}{6}} \times 3^{\frac{6}{6}} = 2^5 \times 3^1 = 32 \times 3 = 96$.

Example: Solve the equations.

(i) $32^h = 8$, (ii) $4^x - 6.2^x + 2^3 = 0$.

Solution:

(i) $32 = 2^5$ and $8 = 2^3$.

Hence $(2^5)^h = 2^3$

Thus $2^{5h} = 2^3$

$\therefore 5h = 3$ and we have $h = \frac{3}{5}$.

(ii) Let $y = 2^x$. Then the equation becomes

$y^2 - 6y + 8 = 0$

$\therefore (y-2)(y-4) = 0$ and $y = 2$ or $y = 4$

If $y = 2$, then $2^x = 2^1 \Longrightarrow x = 1$.

If $y = 4$, then $2^x = 2^2 \Longrightarrow x = 2$.

{\} $\therefore x = 1$ or 2.

Exercise 5.1

1. Evaluate:

(i) $81^{\frac{3}{4}} \times (\frac{121}{144})^{-\frac{1}{2}}$, (ii) $27^{\frac{1}{3}} \times 3^{-\frac{1}{3}} \times 18^{\frac{1}{2}} \times 2^{\frac{1}{2}}$,

(iii) $16^{-\frac{1}{4}} \times 81^{\frac{1}{2}} \times 5^{\frac{1}{3}} \times 25^{\frac{1}{3}}$.

2. Simplify: (i) $\frac{1}{1-x^2} - \frac{1}{x^{-2}-1}$, (ii) $\frac{(a+b)}{a^{-1}+b^{-1}}$,

(iii) $\frac{x-x^{-1}}{x-1}$, (iv) $\frac{y-1}{y^{\frac{1}{2}}-1}$, (v) $\frac{1+y^{-1}}{1+y}$, (vi) $\frac{1-x^{-1}}{x-1}$

(vii) $(2a^{\frac{7}{3}} - 4a^{\frac{4}{3}} + 2a^{\frac{1}{3}}) \div (2a^{\frac{1}{3}})$.

3. Solve the equation(s):

(i) $8x^{-\frac{2}{3}} = \frac{2}{9}$, (ii) $64x^{-\frac{3}{4}} = 27$, (iii) $3^y = \frac{1}{9}$,

(iv) $9^x - 10.3^x + 3^2 = 0$, (v) $9^x - 4.3^{x+1} + 3^3 = 0$.

4. If $g^h = h^g$, show that $(g/h)^{g/h} = g^{(g-h)/h}$; if $h = 2g$, find the value of h.

5. If $x = 1.44$, find the values of $x^{\frac{1}{2}}, x^o, x^{-\frac{1}{2}}$ and $(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \div x^{-\frac{1}{2}}$.

5.2 Logarithms

The logarithm of a number, say y, to base b is the power to which b must be raised to give the number y, i.e. if $y = b^x$ then $x = \log_b y$.Where the base is not indicated, we assume it to be base 10.

We have three important formulae connecting loarithms which correspond to the three formulae of indices. The logarithm formulae are:

(i) $\log_b xy = \log_b x + \log_b y$

Proof: Let $u = \log_b x \ and \ v = log_b y$.

Then (by definition) $b^u = x$ and $b^v = y$

$\therefore xy = b^u.b^v = b^{u+v}$.

Thus from definition;

$\log_b xy = u + v = \log_b x + \log_b y$.

(ii) $\log_b \frac{x}{y} = \log_b x - \log_b y$.

Proof: Let $u = \log_b x$ and $v = \log_b y$
Then $b^u = x$ and $b^v = y$.
Thus $\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v}$.
Hence from definition, $\log_b \frac{x}{y} = u - v = \log_b x - \log_b y$

(iii) $\log_b x^n = n \log_b x$
Proof: Let $u = \log_b^x \Longrightarrow b^u = x$.
$\therefore x^n = (b^u)^n = b^{un}$
From definition, $\log_b x^n = un = nu = n \log_b x$.

We can also find the *log* of 1 to any base, that of the base itself; and change from one base to another as follows:
(i) $\log_b^1 = 0$, since $b^o = 1$. **(ii)** $\log_b^b = 1$ since $b^1 = b$
(iii) $\log_b x = \frac{\log_a x}{\log_a b}$.
Proof: Let $u = \log_b x$, so that $b^u = x$.
Taking log to base a, we have: $\log_a b^u = \log_a x$ or $u \log_a b = \log_a x$

$\therefore u = \frac{\log_a x}{\log_a b}$ i.e. $\log_b x = \frac{\log_a x}{\log_a b}$.

Example
If $\log_a q = 5 + \log_a b$ and $c = a^4$. prove that $q = abc$.
Solution: $log_a q = 5 + \log_a b$ can be written as $\log_a q - \log_a b = 5$, thus $\log_a \frac{q}{b} = 5$.
By definition of logarithm we have: $a^5 = \frac{q}{b} \Longrightarrow q = a^5 b = aba^4 = abc$.

Example
Simplify: (i) $a^{\log_a x}$, (ii) $a^{-2\log_a x}$
Solution:
(i) Let $u = a^{\log_a x}$, then $\log_a u = \log_a(a^{\log_a x})$
i.e. $\log_a u = \log_a x \times \log_a a = \log_a x$, since $\log_a a = 1$.
Thus, $\log_a u = \log_a x$
$\therefore u = x$, i.e. $a^{\log_a x} = x$.
(ii) Let $v = a^{-2\log_a x}$, so that $\log_a v = \log_a(a^{\log_a x^{-2}})$

i.e. $\log_a v = \log_a x^{-2} \times \log_a a = \log_a x^{-2}$

$\therefore v = x^{-2} = \frac{1}{x^2}$

i.e. $a^{-2\log_a x} = \frac{1}{x^2}$.

Example
If $\log 2 = 0.3010$ and $\log 3 = 0.4771$ find the values of (i) $\log 5$ (ii) $\log (0.3)^{\frac{1}{3}}$
Solution:

(i) $\log 5 = log(\frac{10}{2}) = \log 10 - \log 2 = 1 - \log 2 = 1 - 0.3010 = 0.6990.$

(ii) $\log(0.3)^{\frac{1}{3}} = \frac{1}{3}\log(0.3) = \frac{1}{3}\log\frac{3}{10} = \frac{1}{3}(\log 3 - \log 10) = \frac{1}{3}(0.4771 - 1)$
$= \frac{1}{3} \times -0.5229 = -0.1743.$

Example

If $\log 5 = 0.6990$, find the number of digits in the integral part of $(\sqrt{5})^{21}$.

Solution: $\log(\sqrt{5})^{21} = \frac{21}{2}\log 5 = \frac{21}{2} \times 0.6990 = 7.3395.$

Hence 10 has to be raised to more than the seventh power to give $(\sqrt{5})^{21}$ and this quantity will $\therefore$ contain 8 digits in its integral part.

Example

Without using tables, find the value of $\log_2(\frac{5}{3}) + \log_2(\frac{6}{7}) - \log_2(\frac{5}{28})$.

Solution: $\log_2\frac{5}{3} + \log_2\frac{6}{7} - \log_2\frac{5}{28} = \log_2(\frac{5}{3} \times \frac{6}{7}/\frac{5}{28}) = \log_2(\frac{5\times2}{7\times5} \times 28)$
$= \log_2 2 \times 4 = \log_2 2^3 = 3.$

Exercise 5.2

1. Prove that:

(i) $\log_q p . \log_r q . \log_p r = 1.$ **(ii)** $\log_a N . \log_b M = \log_b N \log_a M$

(iii) If $\log_b a = c$ and $\log_c b = a$ then $\log_c a = ac.$

2. If $\log_8 h = \frac{1}{2}k, \log_2 2h = m$ and $m - k = 4$, find h.

3. Find, without using tables, the values of x if **(i)** $\log x - \log 3 = 1$;
(ii) $logx^{\frac{1}{2}} = \frac{3}{2}$,**(iii)** $x = \log_6 216.$

4. Given that $\log 2 = 0.30103$ and $\log 3 = 0.47712$ find the value of
$\log(2^{-1} + 3^{-1})$ to four decimal places.

5. If $\log 3 = 0.4771$, find the number of digits in the decimal point of $(\sqrt{3})^{20}$
and find the value of x for which $3^x = 100$.

Chapter 6

Trigonometry

6.1 Introduction

Trigonometry is a branch of Mathematics that deals with the study of the relationships between the sides and the angles of a triangle. These relationships are defined in terms of **trigonometric ratios** such as sine(sin),cosine(cos), and tangent(tan). Relative to a right-angled triangle ABC,with the angle marked θ (theta) as the reference angle, the side AB is the *opposite*, BC is the *adjacent*,and the *hypotenuse* is AC, which is the side opposite the right angle. We define the **trigonometric ratios** as follows:

where a, b and c are the sides of a triangle.

$$\sin\theta = \frac{Opposite\ side\ to\ angle\ \theta}{Hypotenuse} = \frac{c}{b},\ \ cos\theta = \frac{Adjacent\ \ side\ to\ angle\ \theta}{Hypotenuse} = \frac{a}{b},$$

and $tan\theta = \frac{Opposite\ side\ to\ angle\theta}{Adjacent\ side\ to\ angle\theta} = \frac{a}{c}$

Relative to angle$A = 90^\circ - \theta$, we obtain

$\sin(90^\circ - \theta) = \frac{a}{b} = \cos\theta$

$\cos(90^\circ - \theta) = \frac{c}{b} = \sin\theta.$

Note that angles θ and $(90^\circ - \theta)$ are **complementary** (that is, they add up to 90^0) hence we conclude that **the** sine **of an angle is always equal to the** cosine **of its complement**.

Other trigonometric ratios such as secant,cosecant and cotangent can be defined in terms of the other ratios sine,cosine and tangent thus:

$\sec\theta = \frac{1}{\cos\theta},\ cosect\theta = \frac{1}{\sin\theta},\ \cot\theta = \frac{1}{\tan\theta}.$

We derive from the figure above the following trigonometric identities:

$\tan(90° - \theta) = \frac{a}{c} = \cot\theta$, $\frac{\sin\theta}{\cos\theta} = (\frac{c}{b}) \div (\frac{a}{b}) = \frac{c}{a} = \tan\theta$.
$\therefore\ \tan\theta = \frac{\sin\theta}{\cos\theta}$ (6.1)
Also, $(\sin\theta)^2 + (\cos\theta)^2 = (\frac{c}{b})^2 + (\frac{a}{b})^2 = \frac{a^2+c^2}{b^2} = \frac{b^2}{b^2} = 1$,
since by Pythagoras' theorem, $a^2 + b^2 = c^2$.

$$\therefore\ \sin^2\theta + \cos^2\theta = 1 \qquad (6.2)$$

Dividing through (6.2) by $sin^2\theta$ and then by $\cos^2\theta$, we obtain

$$1 + \cot^2\theta = \cos ec^2\theta \qquad (6.3)$$

and

$$1 + \tan^2\theta = \sec^2\theta \qquad (6.4)$$

The identities (6.2), (6.3) and (6.4) are sometimes referred to as $Pythagoras'\ Theorem$.

6.2 Trigonometric Ratios of Some Special Angles

$$\{0°, 30°, 45°, 60°, 90°\}$$

We consider an *equilateral* triangle ABC of side 2 *units*, with a perpendicular *bisector* AD of the base BC drawn from A to D which also bisects the vertical angle as follows:
From the triangle above,the height $h = \sqrt{2^2 - 1} = \sqrt{3}$.
$\therefore\ \sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, *and* $\tan 30° = \frac{\sqrt{3}}{3}$.
Also, $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$, *and* $\tan 60° = \sqrt{3}$.

Consider a square $PQRS$ of side 1*unit* and draw the diagonal QS. From the figure,
$d^2 = \sqrt{1^2 + 1^2} = \sqrt{2}$.
Hence, $\sin 45° = \frac{\sqrt{2}}{2}$, $\cos 45° = \frac{\sqrt{2}}{2}$, *and* $\tan 45° = 1$.

For trigonometric ratios of 0° and 90°, we consider the right-angled triangle PQR.
Then, $\sin\theta = \frac{PQ}{QR}$. As P approaches$Q$, θ approaches 0°,
i.e. as $P \to Q$, $\theta \to 0°$, hence $PR \to QR$.
$\therefore\ \sin 0° = \frac{0}{|QR|} = 0$, thus $cos 90° = 0$, since the *sine* of angle is always equal to the *cosine* of its *complement*.
Similarly, $cos 0° = \frac{QR}{QR} = 1$ (since $\theta \to 0°$implies $PR \to QR$).

6.3 Trigonometric Ratios of Angles Between 0°and 360°

Consider the circle of radius r below

In the first Quadrant,	Second Quadrant
$\sin\theta = \frac{y}{r}$ ($\{positive\}$)	$\sin\theta = \frac{y}{r}$ ($\{positive\}$)
$cos\theta = \frac{x}{r}$ ($\{positive\}$)	$\cos\theta = \frac{-x}{r}$ (negative)
$\tan\theta = \frac{y}{x}$ ($\{positive\}$)	$\tan\theta = \frac{y}{-x}$ (negative)

All the trigonometric ratios in first quadrant are positive.

Only sine has a positive value in the second quadrant

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Third Quadrant

$\sin\theta = \frac{-y}{r}$ (negative)
$\cos\theta = \frac{-x}{r}$ (negative)
$\tan\theta = \frac{-y}{-x} = \frac{y}{x}$ (positive)

In the third quadrant, only $\tan$ *gent has a positve value.*

Fourth Quadrant

$\sin\theta = \frac{-y}{r}$ (negative)
$\cos\theta = \frac{x}{r}$ (positive)
$\tan\theta = \frac{-y}{x}$ (negative)

In the fourth quadrant, only cosine has a positive value.

Some people use acronyms such as **CAST** or **ACTS** to enable them remember the signs of trigonometric ratios of angles of all magnitudes, see diagram below:

Starting from the fourth quadrant and moving anti-clockwise, we use **CAST**, while

starting from the first quadrant and moving in the clockwise direction, we use **ACTS**.

We note that in the first quadrant,

$$\sin\theta = +\sin\theta,\ \cos\theta = +\cos\theta,\ and\ \tan\theta = +\tan\theta$$

In the second quadrant,

$$\sin\theta = +\sin(180^\circ - \theta),\ \cos\theta = -\cos(180^\circ - \theta),\ and\ \tan\theta = -\tan(180^\circ - \theta)$$

The third quadrant has

$$\sin\theta = -\sin(\theta - 180^\circ),\ \cos\theta = -\cos(\theta - 180^\circ),\ and\ \tan\theta = +\tan(\theta - 180^\circ)$$

and the fourth quadrant has

$$\sin\theta = -\sin(360^\circ - \theta),\ \cos\theta = +\cos(360^\circ - \theta),\ and\ \tan\theta = -\tan(360^\circ - \theta)$$

Example

1. Express in terms of the trigonometric ratios of positive acute angles:
(a) $sin 160^\circ$ (b) $\cos(400^\circ)$ (c) $\sin(-200^\circ)$ (d) $\cos(-300^\circ)$ (e) $\tan(-900^\circ)$.

Solution

1 (a) $\sin 160^\circ = \sin(180^\circ - 160^\circ) = \sin 20^\circ$

(b) $\cos 400^\circ = \cos(400^\circ - 360^\circ) = \cos 40^\circ$

(c) $\sin(-200^\circ) = \sin(-200^\circ + 360^\circ) = \sin 160^\circ = \sin(180^\circ - 160^\circ) = \sin 20^\circ$, or $sin(-200°) = -\sin 200° = -(-\sin(200° - 180°)) = \sin 20°$.

(d) $\cos(-300°) = \cos(-300° + 360°) = \cos 60°$

(e) $\tan(-900°) = \tan(-900° + 360°) = \tan(-540°) = \tan(-540° + 360°)$
$= \tan(-180°) = \tan(-180° + 360°) = \tan(180°) = -\tan(180° - 180°) = \tan 0°$.

Example

2. Evaluate without using tables:

(a) $\cos 210°$ (b) $\cos^2 330°$ (c) $2\sin 120° \cos 120°$ (d) $\cos^2 300° - \sin^2 300°$

Solution

(a) $\cos 210° = -\cos(210° - 180°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

(b) $\cos^2 330° = \cos^2(360° - 330°) = \cos^2 30° = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$.

(c) $2\sin 120° \cos 120° = 2\sin(180° - 120°)(-\cos(180° - 120°))$
$= -2\sin 60° \cos 60° = -2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = -\frac{\sqrt{3}}{2}$.

(d) $\cos^2 300° - \sin^2 300° = \cos^2(360° - 300°) - (-\sin(360 - 300°))^2$
$= \cos^2 60° - \sin^2 60° = (\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2 = -\frac{1}{2}$.

Example

3. Find in its simpest form, the value of $\frac{\sin 3\theta + \tan\theta}{1+\cos 2\theta}$ if $\theta = 60°$.

Solution: When $\theta = 60°$, $\frac{\sin 3\theta+\tan\theta}{1+\cos 2\theta} = \frac{\sin 180°+\tan 60°}{1+\cos 120°} = \frac{0+\sqrt{3}}{1-\frac{1}{2}} = 2\sqrt{3}$.

Example

4. Given that $\sin A = \frac{5}{13}$ and A is an obtuse angle, find without using tables, the value of $\frac{2\sin A - \cos A}{\frac{1}{2} - \tan A}$.

Solution: Note that $x^2 + y^2 = r^2$. Therefore,
$x = \sqrt{13^2 - 5^2} = \sqrt{144} = 12\ units$.

$$\therefore\ \cos A = -\frac{12}{13},\ \tan A = -\frac{5}{12}.$$

$$\therefore\ \frac{2\sin A - \cos A}{\frac{1}{2} - \tan A} = \frac{2\times\frac{5}{13} - (-\frac{12}{13})}{\frac{1}{2} - (-\frac{5}{12})} = \frac{\frac{10}{13} + \frac{12}{13}}{\frac{1}{2} + \frac{5}{12}} = \frac{24}{13}.$$

5. Show that

$$\frac{\sin\alpha + \tan\alpha}{\cot\alpha + \cos ec\alpha} = \sin\alpha\tan\alpha.$$

Solution:

$LHS = \frac{\sin\alpha+\tan\alpha}{\frac{1}{\tan\alpha}+\frac{1}{\sin\alpha}} = \frac{\sin\alpha+\tan\alpha}{\frac{\sin\alpha+\tan\alpha}{\sin\alpha\tan\alpha}} = (\sin\alpha+\tan\alpha)\times\frac{\sin\alpha\tan\alpha}{(\sin\alpha+\tan\alpha)} = \sin\alpha\tan\alpha = RHS$.

Exercise 6.3

1. Express in terms of the trigonometric ratios of positive acute angles:

$(i)\ \sin 140°$ (ii) $\sin(-945°)$ (iii) $\cos 310°$ (iv) $\tan(-700°)$ (v) $\cos(-910°)$

2. Evaluate without using tables, giving your answer in simplified surd form
where necessary:

(i) $\sin 135°$ (ii) $\sin^2 330°$ (iii) $\cos^2 210° - \sin^2 210°$ (iv) $\frac{1+\cos 210°}{1-\cos 210°}$

(v) $\frac{\tan 330^\circ + \tan 240^\circ}{1 - \tan 330^\circ \tan 240^\circ}$

3. If $\sin\theta = p$ and θ is obtuse, find expressions for $\cos\theta$ and $\tan\theta$ in terms of p.

4. If $\tan A = \frac{4}{3}$, where A is an acute angle, find the value of $\frac{4\cos A - 3\sin A}{4\cos A + 3\sin A}$.

5. Show that $\frac{\cos\beta}{1+\sin\beta} = \sec\beta - \tan\beta$.

6. Show that $(1+\tan\theta)^2 + (1-\tan\theta)^2 = \frac{2}{\cos^2\theta}$.

6.4 DEGREE AND RADIAN MEASURE OF AN ANGLE

Two kinds of units commonly used in measuring angles are the *degree* and the *radian*. The *degree* is a more familiar and more commonly used unit of angular measure in elementary mathematics than the radian. A *degree* ($\circ$) is defined as $\frac{1}{360}$ of a complete rotation. It has sub-divisions such as the *minute* ($'$) which is $\frac{1}{60}$ of a *degree* and the *second* ($''$) is $\frac{1}{60}$ of a *minute*.

The *radian* measure is employed almost exclusively in advanced mathematics and in many branches of science and engineering. In this section, we introduce the concept of *radian* and examine the relationship between *degrees* and *radians*.

Definition 6.2 : *A radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.*

Let us consider a circle with centre O and radius r and the arc PQ be equal in length to r and θ is the angle subtended at the centre of the circle, and $\theta = 1rad$.

(iv) $144^\circ = \frac{144\times\pi}{180}rad = \frac{4\pi}{5}rad.$

Example

Express each angle in degrees: (i) $\frac{2\pi}{3}$ (ii) $\frac{\pi}{30}$ (iii) $-\frac{5\pi}{9}$ (iv) $\frac{3\pi}{8}$.

Solution

Note that $1rad = \frac{180^\circ}{\pi}$.

$\therefore$ (i) $\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180^\circ}{\pi} = 120^\circ$ (ii) $\frac{\pi}{30} = \frac{\pi}{30} \times \frac{180^\circ}{\pi} = 6^\circ$

(iii) $-\frac{5\pi}{9} = -\frac{5\pi}{9} \times \frac{180^\circ}{\pi} = -100^\circ$

(iv) $\frac{3\pi}{8} = \frac{3\pi}{8} \times \frac{180^\circ}{\pi} = 67\frac{1}{2}^\circ$.

Exercise 6. 2

1. Express the following in degrees:

(i) $\frac{\pi}{7}rad$ (ii) $\frac{4\pi}{7}rad$ (iii) $\frac{8\pi}{9}rad$ (iv) $0.45rad$ (v) $\frac{\pi}{9}rad$.

2. Express the following in radians:

(i) 36° (ii) $112^\circ 40'$ (iii) 135° (iv) $1'$ (v) 270° (vi) $394^\circ 32'$.

Even and Odd Functions:

Definition 6.3 A function f is *even* if $f(-x) = f(x)$, for all x in its domain of definition. The function is *odd* if $f(-x) = -f(x)$.

For example, $f(x) = x^2$, $f(\alpha) = \cos\alpha$, are even functions while $f(x) = x^3$, $f(\alpha) = \sin\alpha$ are odd functions.

Periodic Functions: A function f is said to be **periodic** if there exists $T > 0$

such that $f(x+T) = f(x)$. The function has **period** T, if it is the smallest value

for which the relation holds. A function involving trigonometric ratios is called a **trigonometric function**. All trigonometric functions are **periodic**.

For example, $f(x) = \sin x$, $g(x) = \cos x$ are periodic with period $T = 2\pi$.

Graphs of Trigonometric Functions

The graphs of $y = \sin\theta$ $y = \cos\theta$, for $0 \leq \theta \leq 2\pi.\cos\theta$,$0 \leq \theta \leq 2\pi$

$y = \sin\theta$

$y = \cos\theta$

6.5 Fundamental Trigonometric Identities

We study in this section , some more important identities involving trigonometric functions.These identities can be employed to simplify or change the form of trigonometric expressions.

6.5.1 Addition Formulae

The following identities are often referred to as **addition formulae** or **addition theorems** in trigonometry:

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Proofs:

Consider the triangle ABC and construct a perpendicular bisector AD of BC.

aB bA cC dD

Using the sine rule, we obtain

$\frac{\sin<BAC}{|BC|} = \frac{\sin<ABC}{|AC|}$

$\therefore\ \frac{\sin(\alpha+\beta)}{x+y} = \frac{\sin(90^\circ-\alpha)}{w} = \frac{\frac{h}{z}}{w} = \frac{h}{wz}$.

$\therefore\ \sin(\alpha+\beta) = \frac{h(x+y)}{wz} = \frac{hx}{wz} + \frac{hy}{wz} = (\frac{x}{z})(\frac{h}{w}) + (\frac{h}{z})(\frac{y}{w}) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Using cosine rule, we obtain in a similar manner

$cos < BAC = \frac{|AC|^2+|AB|^2-|BC|^2}{2|AC||AB|} = \frac{1}{2}\left[\frac{|AC|}{|AB|} + \frac{|AB|}{|AC|} - \frac{|BC|}{|AC|}.\frac{|BC|}{|AB|}\right]$

This implies $\cos(\alpha+\beta) = \frac{1}{2}\left[\frac{w}{z} + \frac{z}{w} - \frac{(x+y)}{w}.\frac{(x+y)}{z}\right] = \frac{1}{2}\left[\frac{w^2+z^2}{wz} - \frac{x^2+2xy+y^2}{wz}\right]$

$= \frac{1}{2}\left[\frac{w^2+z^2-x^2-2xy-y^2}{wz}\right] = \frac{1}{2}\left[\frac{2h^2-2xy}{wz}\right]$, since $w^2-y^2 = z^2-x^2 = h^2$.

$= \frac{h}{z}.\frac{h}{w} - \frac{x}{z}.\frac{y}{w} = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

Thus,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \;\; \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Putting $\beta = -\beta$ in the above formulae, we obtain

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta, \;\; \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Thus, $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} = \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

$$\therefore \; \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Similarly,

$$tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Example

Find without using tables the value of (i) $\sin\frac{5\pi}{12}$ (ii) $\cos\frac{\pi}{12}$

Solution:

(i) $\sin\frac{5\pi}{12} = \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}.\frac{1}{2} = \frac{\sqrt{2}}{2}\left(\sqrt{3}+1\right)$

(ii) $\cos\frac{\pi}{12} = \cos(\frac{\pi}{4}-\frac{\pi}{6}) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}.\frac{1}{2}$

$= \frac{\sqrt{2}}{2}\left(\sqrt{3}+1\right)$.

Note that $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ are complementary angles, hence $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$.

Example

If $sin\alpha = \frac{5}{13}$, and $\cos\beta = \frac{1}{5}$, find the possible values of $\sin(\alpha+\beta)$.

Solution: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

If α and β are acute angles, then $cos\alpha = \frac{12}{13}$, $\sin\beta = \frac{\sqrt{24}}{5}$, hence,

$\sin(\alpha+\beta) = \frac{5}{13}\times\frac{1}{5} + \frac{12}{13}\times\frac{\sqrt{24}}{5} = \frac{5+24\sqrt{6}}{65}$,

otherwise, if α is acute and $\frac{3\pi}{2} \le \beta \le 2\pi$, then $\sin\beta = -\frac{\sqrt{24}}{5}$.

$\therefore$ $\sin(\alpha+\beta) = \frac{5}{13}\times\frac{1}{5} + \frac{12}{13}\times(-\frac{\sqrt{24}}{5}) = \frac{5-24\sqrt{6}}{65}$.

Example

Find without using tables the value of $\sin 15^\circ$.

Solution: $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ$

$= \frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}.\frac{1}{2}$
$= \frac{\sqrt{2}}{4}(\sqrt{3}-1).$

Exercise 6.5 a

1. use the addition formulae to prove the following identities:
(i) $\sin(180^\circ+\theta) = -\sin\theta$ (ii) $\tan(180^\circ-\theta) = -\tan\theta$
(iii) $\sin(\frac{\pi}{2}-\theta) = \cos\theta$ (iv) $\cos(300^\circ-\theta) = \frac{\cos\theta-\sqrt{3}\sin\theta}{2}$
(v) $\tan(225^\circ-\theta) = \frac{1-\tan\theta}{1+\tan\theta}$ (vi) $\cos 3x\cos 2x + \sin 3x \sin 2x = \cos x.$
2. Find without using tables the value of $\tan 15^\circ$.
3. Show that $\sin^2 A + \sin^2(A+120^\circ)+ \sin^2(A-120^\circ) = \frac{3}{2}$.
4. If $\sin\theta = \frac{2}{3}$ and $\cos\phi = -\frac{2}{7}$, find the possible values of $\cos(\theta+\phi)$.

6.5.2 Multiple Angles

By putting $\beta = \alpha$ in the addition formulae, we obtain
$\sin(\alpha+\alpha) = \sin 2\alpha = 2\sin\alpha\cos\alpha,$
$\cos(\alpha+\alpha) = \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2 -1 = 1-\sin^2\alpha.$
$\therefore\ \tan(\alpha+\alpha) = \tan 2\alpha = \frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha}.$
Using the above formulae, we can express $\tan\alpha$ in terms of $\tan\frac{\alpha}{2}$ thus,
$\tan\alpha = \tan 2(\frac{\alpha}{2}) = \frac{\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}.$
If we let $t = \tan\frac{\alpha}{2}$, then,

$$\tan\alpha = \frac{2t}{1-t^2}$$

Similarly, $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2t}{1+t^2}.$

$$\therefore\ \sin\alpha = \frac{2t}{1+t^2}$$

and $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-t^2}{1+t^2}.$
Hence

$$\cos\alpha = \frac{1-t^2}{1+t^2}.$$

Since $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} - 1,$
$\Rightarrow\ 2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$

$$\therefore\ \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}.$$

Similarly, $\cos\alpha = 1-2\sin^2\frac{\alpha}{2},$

$$\therefore\ \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

Also,

$$\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}=\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\frac{\sqrt{1+\cos\alpha}}{2}}=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

From the cosine rule, $\cos A=\frac{b^2+c^2-a^2}{2bc}, \Rightarrow \cos A=2\cos^2\frac{A}{2}-1=\frac{b^2+c^2-a^2}{2bc}$.

$\Rightarrow 2\cos^2\frac{A}{2}=\frac{(b+c)^2-a^2}{2bc}=\frac{(a+b+c)(b+c-a)}{2bc}$.

Let $a+b+c=2s$, where s is the semi-perimeter of triangle ABC, i.e $s=\frac{a+b+c}{2}$.

$\therefore\ 2\cos^2\frac{A}{2}=\frac{4s(s-a)}{2bc}=\frac{2s(s-a)}{bc}$.

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

In the same vein,

$$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}, \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}.$$

Also, $\cos A=1-2\sin^2\frac{A}{2}=\frac{b^2+c^2-a^2}{2bc}$.

$\therefore\ 2\sin\frac{A}{2}=\frac{(a+b-c)(a-b+c)}{2bc}=\frac{4(s-c)(s-b)}{2bc}=\frac{2(s-b)(s-c)}{bc}$.

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Hence,

$$\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}, \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Similarly, $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

$$\therefore\ \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

6.5.3. Sine and Cosine Rules

Sine rule: The sine rule states that in any triangle ABC,

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

$4B$ $6A$ $5C$ dD $1a$ $2c$ $3b$ hh

Proof: Consider a triangle ABC, and construct a perpendicular from A to
meet BC at D. From the figure,
$\sin B = \frac{h}{c}, \Rightarrow h = c\sin B$, and similarly, $\sin C = \frac{h}{b}, \Rightarrow h = b\sin C$.
$\therefore\ h = c\sin B = b\sin C, \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$.
similarly, $\frac{a}{\sin A} = \frac{c}{\sin C}$.

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The formula also holds for obtuse angled triangles.

Cosine Rules: the cosine rules state that in any triangle ABC,
$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$.

Proof: from the triangle above,
$c^2 = (a-x)^2 + h^2 = a^2 - 2ax + x^2 + h^2$, by Pythagoras' rule.
$\therefore\ c^2 = a^2 - 2ax + b^2$, since $b^2 = h^2 + x^2$.
But $x = b\cos C, \Rightarrow\ c^2 = a^2 + b^2 - 2ab\cos C$.
In the same vein, $a^2 = b^2 + c^2 - 2bc\cos A,\ and\ b^2 = a^2 + c^2 - 2ac\cos B$.

$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A,\ b^2 = a^2 + c^2 - 2ac,\ c^2 = a^2 + b^2 - 2ab\cos C.$$

Example
In triangle $ABC, a = 4, b = 5, c = 6$. Find the angles A, B, and C.
Using cosine rule, $4^2 = 5^2 + 6^2 - 2 \times 5 \times 6\cos A$
$\Rightarrow \cos A = 0.75$.
$\therefore\ A = \cos^{-1} 0.75 \simeq 41.4^\circ$
Also, $5^2 = 4^2 + 6^2 - 2 \times 4 \times 6\cos B$.
$\Rightarrow \cos B = 0.5625 \Rightarrow B = \cos^{-1} 0.5625 \simeq 55.8^\circ$.
$\therefore\ C = 82.8^\circ$.

Exercise 6.5b
1. Solve each of the following triangles, given that
$(i)\ a = 12, b = 10, c = 8$ $(ii)\ A = 123^\circ, B = 62^\circ, c = 0.34$
$(iii)\ a = 26.35,\ A = 37^\circ 23', B = 43^\circ 31'$.
2. Show that $a^2 + bc = b^2 + c^2$,if $A = 60^\circ$.
3. Prove that $a^2 + b^2 + c^2 = 2bc\cos A + 2ac\cos B + 2ab\cos C$.

6.1.1

6.6 The Area of a Triangle

Consider the triangle ABC, with height, h, $\sin C = \frac{h}{b} \Rightarrow h = b \sin C$.

$4B$ $6A$ $5C$ dD $1a$ $2c$ $3b$ hh

$\therefore$ Area, $A = \frac{1}{2} ab \sin C = \frac{1}{2} ab.2 \sin \frac{C}{2} \cos \frac{C}{2} = ab \sin \frac{C}{2} \cos \frac{C}{2}$.

$$= ab\sqrt{\frac{(s-a)(s-b)}{ab}} \times \sqrt{\frac{s(s-c)}{ab}}.$$

$$\therefore \ A = \sqrt{s(s-a)(s-b)(s-c)}.$$

The formula above is usually referred to as $Hero's\ formula$.

Example

Express $\tan 3A$ interms of $\tan A$.

Solution: $\tan A = \tan(2A+A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\frac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \frac{2\tan A}{1-\tan^2 A}\tan A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

Example

Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

$cos3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$.

Exercise 6.6

1. Express (i) $\sin 4\theta$ in terms of $\sin\theta$
   (ii) $\tan 4\theta$ in terms of $\tan\theta$
2. Show that $\sin\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{2-\sqrt{3}}{4}$.
3. Given that $\sin\frac{\pi}{10} = \frac{1}{4}(\sqrt{5}-1)$, find the exact value of $\cos\frac{\pi}{5}$.
4. Find without using tables the values of (i) $\cos 22\frac{1}{2}^\circ$ (ii) $\tan 22\frac{1}{2}^\circ$
5. Prove that $\frac{1+\sin\phi-\cos\phi}{1+\sin\phi+\cos\phi} = \tan\frac{\phi}{2}$.
6. Without using tables, determine the value of $\tan 7\frac{1}{2}^\circ$.
7. Using the formulae $\sin 3\beta = 3\sin\beta - 4\sin^3\beta,\ and\ \cos 3\beta = 4\cos^3\beta - 3\cos\beta$,
   express $\cos 5\beta$ in terms of $\cos\beta$. Hence, prove that $\cos 18^\circ = \frac{1}{4}\sqrt{(10+2\sqrt{5})}$,
   and evaluate $\cos 54^\circ$, leaving your result in surd form.
8. Find the area of the triangle with the following dimensions:
   (i) $a = 623cm, b = 913cm, A = 34^\circ$ (ii) $A = 57.9, b = 126.3, c = 137.2$.
9. The sides of atriangle are $13k, 20k,\ and\ 21k$ units, and the area of the triangle
   is 14 square units. Find the numerical value of k.

6.7 Factor Formulae

Recall $\quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$,
and $\quad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.
Adding corresponding members of these equations, we obtain
$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

$$\Rightarrow \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right].$$

Similarly, $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

$$\Rightarrow \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right].$$

By subtracting the expressions, we obtain
$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$

$$\Rightarrow \cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right].$$

and $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$.

$$\Rightarrow \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right].$$

If in these identities, we put $\alpha+\beta = A$ and $\alpha-\beta = B$, then
$\alpha = \frac{1}{2}(A+B)$, and $\beta = \frac{1}{2}(A-B)$, we obtain

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B), \ \cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B).$$

and

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B), \ \cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B).$$

Examples

1. Express these quantities as products: (i) $\cos 50^\circ + \cos 12^\circ$ (ii) $\sin 40^\circ + \sin 20^\circ$
(iii) $\sin 152^\circ - \sin 30^\circ$ (iv) $\cos 2\theta + \cos 4\theta$
2. Prove that $\cos A + 2\cos 3A + \cos 5A = 4\cos^2 A\cos 3A$.
3. If A, B *and* C are the angles of *triangle* ABC, show that
$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.
4. Prove that $\frac{\sin 4\omega - \sin 2\omega}{\cos 4\omega + \cos 2\omega} = \tan\omega$.

Solutions

1.(i) $\cos 50^\circ + \cos 12^\circ = 2\cos\frac{1}{2}(50^\circ + 12^\circ)\cos\frac{1}{2}(50^\circ - 12^\circ)$
$= 2\cos 31^\circ \cos 19^\circ$.

(ii) $\sin 40° + \sin 20° = 2\sin\frac{1}{2}(40° + 20°)\cos\frac{1}{2}(40° - 20°)$
$= 2\sin 30° \cos 10°.$

(iii) $\sin 152° - \sin 30° = 2\cos\frac{1}{2}(152° + 30°)\sin\frac{1}{2}(152° - 30°)$
$= 2\cos 91° \sin 61°.$

(iv) $\cos 2\theta + \cos 4\theta = 2\cos\frac{1}{2}(2\theta + 4\theta)\cos\frac{1}{2}(2\theta - 4\theta)$
$= 2\cos 3\theta \cos\theta.$

2. $LHS = (\cos A + 2\cos 3A) + (\cos 3A + \cos 5A)$
$= 2\cos\frac{1}{2}(A + 3A)\cos\frac{1}{2}(A - 3A) + 2\cos\frac{1}{2}(3A + 5A)\cos\frac{1}{2}(3A - 5A)$
$= 2\cos 2A\cos(-A) + 2\cos 4A\cos(-A)$
$= 2\cos 2A\cos A + 2\cos 4A\cos A = 2\cos A(\cos 2A + \cos 4A)$
$= 2\cos A\left[2\cos\frac{1}{2}(2A + 4A)\cos\frac{1}{2}(2A - 4A)\right] = 2\cos A(2\cos 3A\cos(-A))$
$= 4\cos^2 A\cos 3A.$
$= RHS.$

3. $\cos A + \cos B + \cos C = \cos A + 2\cos\frac{1}{2}(B + C)\cos\frac{1}{2}(B - C)$
$= 1 - 2\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{1}{2}(B - C) = 1 + 2\sin\frac{A}{2}\left[\cos\frac{1}{2}(B - C) - \sin\frac{A}{2}\right]$
$= 1 + 2\sin\frac{A}{2}\left[\cos\frac{1}{2}(B - C) - \cos\frac{1}{2}(B + C)\right] = 1 + 2\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)$
$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

4. $\frac{\sin 4\omega - \sin 2\omega}{\cos 4\omega + \cos 2\omega} = \frac{2\cos\frac{1}{2}(4\omega + 2\omega)\sin\frac{1}{2}(4\omega - 2\omega)}{2\cos\frac{1}{2}(4\omega + 2\omega)\cos\frac{1}{2}(4\omega - 2\omega)} = \frac{2\cos 3\omega\sin\omega}{2\cos 3\omega\cos\omega} = \frac{\sin\omega}{\cos\omega} = \tan\omega.$

Exercise 6.7

1. Express as a product: (i) $\sin 40° + \sin 20°$ (ii) $\cos 50° + \cos 30°$
(iii) $\cos 8\theta - \cos 2\theta$
2. Express as sums or differences: (i) $\sin 30°\cos 50°$ (ii) $\cos 3x\cos 4x$
(iii) $\sin 4x\cos 2x$
3. Prove that (i) $\frac{\sin 6x - \sin 4x}{\cos 5x} = 2\sin x$ (ii) $\frac{\sin\theta - \sin 2\theta + \sin 3\theta}{\cos\theta - \cos 2\theta + \cos 3\theta} = \tan 2\theta.$
4. If A, B and C are angles of a triangle, show that
$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$
5. Prove that $\frac{\cos\alpha + \cos\beta}{\sin\alpha - \sin\beta} = \cot\frac{\alpha - \beta}{2}.$

6.8 Trigonometric Equations

To solve any equation involving trigonometric functions, appropriate trigonometric identities may be employed to simplify such equations before solutions are obtained. We illustrate this by using the following examples:

Examples

Solve the following equations for values of θ between $0°$ and $360°$:
(i) $\sin\theta = \cos 55°$ (ii) $2\sin^2\theta + 2\cos 2\theta = 1$

Solutions: (i) $\sin\theta = \cos(90° - \theta) = \cos 55°$
$\therefore\ 90° - \theta = 55° \Rightarrow \theta = 90° - 55° = 35°$ *or* $\theta = 180° - 35° = 145°.$
$\therefore\ \theta = 35°, 145°.$
(ii) $2\sin^2\theta + 2\cos 2\theta = 1 \Rightarrow 2\sin^2\theta + 2(1 - 2\sin^2\theta) = 1$
$\Rightarrow 2 - 2\sin^2\theta = 1,$

$$\therefore\ \sin^2\theta = \tfrac{1}{2},$$

Hence $\sin\theta = \pm\frac{1}{\sqrt{2}}$.

$\therefore\ \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ, 135^\circ.$

and $\sin\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = 225^\circ, 315^\circ.$

$\therefore$ the solution is $\theta = 45^\circ, 135^\circ, 225^\circ, 315^\circ.$

6.8.1 The General Solution of $\sin\theta = c$

If $\sin\theta = c$, with c a constant such that $-1 \le c \le 1$, we see that there is always a solution of the equation between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Such a solution is usually referred to as the *principal solution*. Suppose the principal solution is α, then the general expression for the solution of the equation is given by $\theta = k\pi + (-1)^k\alpha$, where $k \in Z$, is called the *general solution* of the equation $\sin\theta = c$.

Example

Find the general solution of the equation $\cos 2x + 3\sin x + 1 = 0$.

Solution: $\cos 2x + 3\sin x + 1 = 0 \Rightarrow (1 - 2\sin^2 x) + 3\sin x + 1 = 0.$

$\therefore\ 2 + 3\sin x - 2\sin^2 x = 0$

$\Rightarrow\ 2 + 4\sin x - \sin x - 2\sin x = 0$

$\therefore\ (2 - \sin x)(1 + 2\sin x) = 0$

$\therefore\ 2 - \sin x = 0$ *or* $1 + 2\sin x$, i.e $\sin x = 2$ *or* $\sin x = -\frac{1}{2}$.

But $\sin x = 2$ is impossible (why?).

$\therefore\ \sin x = -\frac{1}{2} \Rightarrow x = -\frac{\pi}{6}$ is the principal solution.

$\therefore$ the general solution is $x = k\pi + (-1)^{k+1}(\frac{\pi}{6})$.

Similarly, the general solution of $\cos\theta = c,\ -1 \le c \le 1$, is given by $\theta = 2n\pi \pm \alpha,\ n \in Z$ and $0 < \alpha < \pi$ is the principal solution.

For the equation $\tan\theta = c$, the principal solution α lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and the general solution is given by $\theta = n\pi + \alpha,\ n \in Z$

Example

Find the general solution of the equation $2\cos^2\theta - 3\cos\theta - 2 = 0$.

Solution $2\cos^2\theta - 3\cos\theta - 2 = 0 \Rightarrow\ 2\cos^2\theta - 4\cos\theta + \cos\theta - 2 = 0$

$\therefore\ (\cos\theta - 2)(2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = -\frac{1}{2}$ *or* $\cos\theta = 2$(which is impossible!)

$\therefore$ the principal solution is $\theta = \frac{2\pi}{3}$.

Thus the general solution is $\theta = 2n\pi \pm \frac{2\pi}{3}$.

Example

Find the general solution of the equation $3\tan^2 x - 1 = 0$.

Solution $3\tan^2 x - 1 = 0 \Rightarrow \tan^2 x = \frac{1}{3}$.

$\therefore\ \tan x = \pm\frac{1}{\sqrt{3}} \Rightarrow \tan x = \frac{1}{\sqrt{3}}$ *or* $-\frac{1}{\sqrt{3}}$.

If $\tan x = \frac{1}{\sqrt{3}}$, the principal solution is $x = \frac{\pi}{6}$ and if $\tan x = \frac{1}{\sqrt{3}}$, the principal
solution is $x = -\frac{\pi}{6}$. Hence the general solution is $x = n\pi \pm \frac{\pi}{6}$.

The Function $y = a\sin x + b\cos x$

The expression $a\sin x+b\cos x$, where a and b are constants can be written as

$R\sin(x+\alpha) \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$,

provided that $R\cos\alpha = a$

and $\quad R\sin\alpha = b$

$\therefore\ (R\cos\alpha)^2 + (R\sin\alpha)^2 = a^2+b^2 \Rightarrow R^2 = a^2+b^2$.

$\therefore\ R = \sqrt{a^2+b^2}$.

$\therefore\ y = a\sin x + b\cos x \equiv R\sin(x+\alpha)$,

has a maximum value R, when $x+\alpha = \frac{\pi}{2}$, and a minimum value $-R$, when $x+\alpha = \frac{3\pi}{2}$.

6.8.2 The Equation $a\sin x + b\cos x = c$

To solve the equation $a\sin x + b\cos x = c$,

where a, b and c are constants, we express $a\sin x+b\cos x \equiv R\sin(x\pm\alpha)$

or $a\sin x + b\cos x \equiv R\cos(x\pm\alpha)$,

where $R > 0$ and $0 \le \alpha \le 2\pi$, R and α are to be determined.

For example, $a\sin x+b\cos x \equiv R\sin(x-\alpha) = R\sin x\cos\alpha - R\cos x\sin\alpha$

where $R\cos\alpha = a$

and $\quad R\sin\alpha = b$

such that $R = \sqrt{a^2+b^2}$, and $\tan\alpha = \frac{b}{a}$.

Hence the original equation becomes $R\sin(x-\alpha) = c$.

The signs of a and b determine the quadrant in which α lies. For instance, if $a < 0$
and $b > 0$, α lies in the second quadrant .

Example

Solve the equation $4\cos x + 3\sin x = 2$, for values of x between $0°$ and $360°$.

Solution $4\cos x + 3\sin x \equiv R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$.

$\therefore\ R\cos\alpha = 4$

$R\sin\alpha = 3$

$\therefore\ R = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$

and $\tan\alpha = \frac{3}{4}$. $\Rightarrow \alpha = \tan^{-1}\frac{3}{4} \simeq 36.9°$.

$\therefore$ the equation becomes $5\cos(x-36.9°) = 2 \Rightarrow \cos(x-36.9°) = 0.4$.

$\therefore\ x - 36.9° = 66.4°, 293.6° \Rightarrow x = 103.3°, 330.5°$.

Example

Express the following functions in the form $A\sin(x+\alpha)$, and hence find the maximum and minimum values of each: (a) $\sin x + \cos x$ (b) $4\sin x - 3\cos x$.

Solution: (a) $\sin x + \cos x \equiv A\sin(x+\alpha) = A\sin x\cos\alpha + \cos x\sin\alpha$
$\therefore\ A\sin\alpha = 1$, and $A\cos\alpha = 1, \Rightarrow A^2 = 2$, hence, $A = \sqrt{2}$,
and $\tan\alpha = 1, \Rightarrow \alpha = \tan^{-1} 1 = \frac{\pi}{4}$.
$\therefore\ \sin x + \cos x \equiv \sqrt{2}\sin(x + \frac{\pi}{4})$. Hence the maximum value$= \sqrt{2}$, and it is attained when $x + \frac{\pi}{4} = \frac{\pi}{2}$, i.e $x = \frac{\pi}{4}$, while a minimum value of $-\sqrt{2}$ is attained when $x + \frac{\pi}{4} = \frac{3\pi}{2}$, *i.e* $x = \frac{5\pi}{4}$.
(b) $4\sin x - 3\cos x \equiv A\sin(x-\alpha) = A\sin x\cos\alpha - A\cos x\sin\alpha$.
$\therefore\ A\sin\alpha = 4,\ \textit{and}\ A\cos\alpha = -3,$
$\therefore\ A = 5,\ \textit{and}\ \tan\alpha = -0.75, \Rightarrow \alpha = \tan^{-1} -0.75 \simeq 143.1°.$
$\Rightarrow 4\sin x - 3\cos x \equiv 5\sin(x - 143.1°).$
$\therefore$ maximum value $= 5$, and minimum value$= -5$.

Example

Solve the equation $4\sin x - 3\cos x = 2$ by means of the substitution $t = \tan\frac{x}{2}$.

Solution

Note that if $t = \tan\frac{x}{2},\ \textit{then},\ \sin x = \frac{2t}{1+t^2},\ \textit{and}\ \cos x = \frac{1-t^2}{1+t^2}$.
Substituting for *sinx* and *cosx* in the equation, we obtain
$4(\frac{2t}{1+t^2}) - 3(\frac{1-t^2}{1+t^2}) = 2$
$\Rightarrow 8t - 3(1-t^2) = 2(1+t^2), \Rightarrow 8t - 3 + 3t^2 = 2 + 2t^2.$
$\therefore\ t^2 + 8t - 5 = 0.$
$\Rightarrow t = \frac{-8\pm\sqrt{64+20}}{2} = \frac{-8\pm\sqrt{84}}{2} = \frac{-8\pm 9.165}{2}.$
$\therefore\ t = 0.5825\ \textit{or}\ t = -8.5825.$
Thus, $\tan\frac{x}{2} = -8.5825, \Rightarrow \frac{x}{2} = \tan^{-1} -8.5825 = 96.65°, 276.65°,$
$\Rightarrow x = 193.3°, 553.3°.$
or $\tan\frac{x}{2} = 0.5825, \Rightarrow \frac{x}{2} = \tan^{-1} 0.5825 = 30.22°, 210.22°,$
$\Rightarrow x = 60.44°, 420.44°.$
$\therefore\ t = 60.44°, 193.3°, 420.44°, 553.3°.$

6.8.3 Equations Involving Factor Formulae

Solve the equation $\sin 3x + \sin 5x = \sin 4x, 0 \leq x \leq 2\pi.$
$\sin 3x + \sin 5x = 2\sin\frac{1}{2}(3x+5x)\cos\frac{1}{2}(3x-5x) = \sin 4x$
$\therefore\ 2\sin 4x\cos x = \sin 4x, \Rightarrow 2\sin 4x\cos x - \sin 4x = 0.$
$\Rightarrow \sin 4x(2\cos x - 1) = 0, \Rightarrow \sin 4x = 0\ \textit{or}\ 2\cos x - 1 = 0,$
$\therefore\ 4x = \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi, \Rightarrow x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$
or $\cos x = 0.5, \Rightarrow x = \cos^{-1} 0.5 = \frac{\pi}{3}, \frac{5\pi}{3}.$
Hence the complete solution is $x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{5\pi}{3}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi.$

Exercise 6.8

1. Solve the following equations for values of θ between $0°$and $360°$:
$(i)\ \cos\theta = -\frac{1}{2}\ \ (ii)\ \sin\theta = \frac{\sqrt{2}}{2}\ \ (iii)\ \sin^2\theta - \sin\theta - 2 = 0$
$(iv)\ 4\sin^2\theta\tan\theta - \tan\theta = 0\ \ (v)\ \cos\theta + \cos 3\theta + \cos 4\theta$
$(vi)\ 3\tan^2\theta - 5\sec\theta + 1 = 0.$

2. Find the general solution of each of the following equations:

(i) $\sin x = \frac{\sqrt{3}}{2}$ (ii) $3\tan^2 x = 1$ (iii) $2\sin^2 x + 3\cos x = 0$

(iv) $\sin 5\theta + \sin 3\theta + \cos\theta = 0$ (v) $\cos\theta - \sin 2\theta = \cos 5\theta$.

3. Express each of the following in the form $R\sin(x+\alpha)$, hence find its maximum and minimum values:

(i) $\sin x + \cos x$ (ii) $2\sin x + 3\sin x$ (iii) $4\sin x - 3\cos x$ (iv) $2\cos x + 3\sin x$.

4. Find the values of x between 0 and 2π for which

(i) $\sin x + \cos x = 1$ (ii) $\cos x = 3\sin x = -2$ (iii) $2\cos x + 3\sin x = 3$

(iv) $\sin x - \sqrt{3}\cos x = 1$.

5. Find all the angles between $0°$ and $360°$ inclusive for which

$\sin^2\theta + 2\cos 2\theta = 2\cos\theta$.

6. Prove that

$\sin x + \sin 2x + \cos x + \cos 2x = 2\sqrt{2}\cos\frac{1}{2}x\sin(\frac{3x}{2} + \frac{\pi}{4})$.

Hence, or otherwise, determine all values of x between 0 and 2π for which

$\sin x + \sin 2x + \cos x + \cos 2x = 0$.

7. Solve the equation $\frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta} = \sqrt{3}$, for $0 \le \theta \le 2\pi$.

8. Solve by means of the substitution $t = \tan\frac{x}{2}$, the equation

$2\sin x + 3\cos x = -1,\ for\ 0 \le x \le 2\pi$.

6.9 Parametric Functions in Trigonometry

A function of the form $X = x(t);\ Y = y(t)$ is said to be a *parametric function* with t as a *parameter*.

Consider the diagram below:

From the diagram,

$$\sin\theta = \frac{y}{r}, \Rightarrow y = r\sin\theta,\ and\ \cos\theta = \frac{x}{r}, \Rightarrow x = r\cos\theta$$

Note that $x^2 + y^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2(\sin^2\theta + \cos^2\theta) = r^2$.

$\therefore\ x^2 + y^2 = r^2$, which is the equation of a circle of radius r with centre at the origin.Observe that a cartesian equation can be obtained from a parametric equation by eliminating the parameter.

Examples

(a) Eliminate t from the equations $x = 2 + 3\cos t;\ y = 1 + 3\sin t$.

(b) Eliminate θ from the equations $x = \cos\theta + \sin\theta;\ y = 1 - \cos\theta$.

(c) If $x = \tan\theta - \sin\theta;\ y = \tan\theta + \sin\theta$, prove that $(x^2 - y^2)^2 = 16xy$.

Solution: (a) $3\cos t = x - 2$ and $3\sin t = y - 1$.

$\therefore\ (3\cos t)^2 + (3\sin t)^2 = (x-2)^2 + (y-1)^2$

$\Rightarrow 9\sin^2 t + 9\cos^2 t = 9(\sin^2 t + \cos^2 t) = (x-2)^2 + (y-1)^2$,

$\therefore\ (x-2)^2 + (y-1)^2 = 9$.

(b) $x + y = \cos\theta + \sin\theta + 1 - \cos\theta = \sin\theta + 1.\times$

$\Rightarrow \sin\theta = x + y - 1,\ and\ \cos\theta = 1 - y,$

$\therefore\ \sin^2\theta + \cos^2\theta = (x + y - 1)^2 + (1 - y)^2 = 1.$

(c) $x^2 - y^2 = (\tan\theta - \sin\theta)^2 + (\tan\theta + \sin\theta)^2$

$= (\tan\theta - \sin\theta + \tan\theta + \sin\theta)(\tan\theta - \sin\theta - \tan\theta - \sin\theta) = (2\tan\theta)(-2\sin\theta).$

$\therefore\ x^2 - y^2 = -4\tan\theta\sin\theta$

$\Rightarrow (x^2 - y^2)^2 = (-4\tan\theta\sin\theta)^2 = 16\tan^2\theta\sin^2\theta = 16\,[\tan^2\theta(1 - \cos^2\theta)]$

$= 16(\tan^2\theta - \sin^2\theta) = 16(\tan\theta + \sin\theta)(\tan\theta - \sin\theta) = 16xy.$

$\therefore\ (x^2 - y^2)^2 = 16xy.$

Exercise 6.9

1. Eliminate θ from each of the following equations:
   (i) $x = \sin\theta,\ y = \cos\theta$
   (ii) $x = \cos\theta + \sin\theta,\ y = 1 - \cos\theta$
   (iii) $x = \sin\theta,\ y = \tan 2\theta$
   (iv) $x = \cos 2\theta,\ y = \cos\theta.$
2. If $x = 2\cos\theta - \cos 2\theta\ and\ y = 2\sin\theta - \sin 2\theta$, show that
   $(x^2 + y^2)^2 - 6(x^2 + y^2) + 8x - 3 = 0.$
3. Express as a cartesian equation the pair of parametric equations
   $x = a\cos^3 t,\ y = b\sin^3 t.$
4. If $u = x\cos\alpha + y\sin\alpha,\ and\ v = x\sin\alpha - y\cos\alpha,$
   show that $u^2 + v^2 = x^2 + y^2.$

Chapter 7

Complex Numbers

Complex nnumbers originated from the inability to find real roots or solutions to quadratic equations of the form $ax^2+bx+c=0$ for which the discriminant $b^2-4ac<0$..

Definition 7.1 (Complex Numbers)

Let x and y be two real numbers,then the ordered pair (x,y) of real numbers is called a complex number.If z denotes a complex number (x,y), then we can write $z=(x,y)$,where the real numbers x and y are called the real and imaginary parts respectively of $z, i.e.$ $\Re z=x$ and $\Im z=y$.

For example, the real and imaginary parts of the complex number $z=(3,4)$ are 3 and 4 respectively.

If the real part of a complex number is zero, i.e. $z=(0,y)$ then the complex number z is said to be pure imaginary. For instance, $(0,5)$ is pure imaginary.

Remark:

Suppose the imaginary part of a complex number z is zero, then we can write $z=(x,0)$ as $x, i.e. z=(x,0)=x$.

Two complex numbers $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$ then $z_1=z_2$ if and only their component parts are equal, i.e iff $x_1=x_2$ and $y_1=y_2$.

7.2 Sum and Multiplication of complex numbers.

Let $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$ be two complex numbers.
thier sum is defined by $z_1+z_2=(x_1+x_2,y_1+y_2)$.... (7.1+) and the product as $z_1z_2=(x_1x_2-y_1y_2,x_1y_2+x_2y_1)$ .... (7.1x) . For instance, if $z_1=(3,4)$ and $z_2=(-5,2)$ then $z_1+z_2=(3+(-5),4+2)=(-2,6)$ and $z_1z_2=(3\times(-5)+4\times 2, 3\times 2+(-5)\times 4)=(-7,-14)$

Remark:

Let i denotes the pure imaginary number $(0,1)$ and $x=(x,0)$ then using the definition of sum and product above, we therefore write a complex number $z=(x,y)$ in Cartesian form as $z=x+iy$ or $z=x+yi$ where i is such that

$i^2 = (-1, 0) = -1$.This representation does not invalidate x and y as the real and imaginary parts of z.With i so defined, we can also write the sum and product of two complex numbers as $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$ and $z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$ respectively.

Remark: The introduction of i with the definition of addition and multiplication as represented by equations (7.1+) and (7.1x), we can say that the complex number system is thus a natural extention of the real number system.

Other properties of addition and multiplication of complex numbers are the same as for real numbers. We list here some of these algebraic properties and verify a few of them while students handle the rest.

The **commutative laws**, $z_1 + z_2 = z_2 + z_1, z_1 z_2 = z_2 z_1$;and the **associative laws**, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), (z_1 z_2) z_3 = z_1 (z_2 z_3)$ follow easily from addition and multiplication of real numbers will obey these laws and the fact that $i^2 = -1$.For instance if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy$ then $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2)$
$= x_2 + x_1 + i(y_2 + y_1) = (x_2 + iy_2) + (x_1 + iy_1) = z_2 + z_1$.

Similarly, $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$
$= x_2 x_1 - y_2 y_1 + i(x_2 y_1 + x_1 y_2) = (x_2 + iy_2)(x_1 + iy_1) = z_2 z_1$. ative

The **distributive law** $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ are left as exercises to students. The numbers $0 = 0 + i0$ and $1 = 1 + i0$ are respectively called the additive and multiplicative identities of the complex numbers.That is for any complex number z, then $z + 0 = z$ and $z1 = z$.Note that 0 and 1 are the only numbers with such properties.

Subtraction: Given any complex number $z = x + iy$, we define $-z = -x - iy$ as its additive inverse which satisfies the equation $z + (-z) = 0$. Moreover, for a given complex number z, there is only one additive inverse namely $-z$.Additive inverses are used to define subtraction: $z_1 - z_2 = z_1 + (-z_2)$. So if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy)$ then $z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$.

Definition 7.3 (Complex Conjugate)

Given a complex number $z = x + iy$ then the complex number $\overline{z} = x - iy$ is called the complex cojugate of z.For example, if $z = 3 + 4i$ then its complex conjugate is $\overline{z} = 3 - 4i$. This number $\overline{z}$ represented by the point $(x, -y)$, is the reflection of the point (x, y) representing z along the real axis.Students can easily verify that for any complex number z, $\overline{\overline{z}} = z$.

For any nonzero complex number $z = x + iy$,there is a number z^{-1}such that $zz^{-1} = 1$.This number z^{-1} is called the multiplicative inverse of z.To find the inverse $w = u + iv$ of $z = x + iy$,we need to solve the equation $(x + iy)(u + iv) = 1$ for u and v in terms of x and y.This gives $u = \frac{x}{x^2+y^2}$ and

$v=\frac{-y}{x^2+y^2}$. Therefore the multiplicative inverse of z is $w=u+iv=\frac{x-iy}{x^2+y^2}$

Division: Given two complex numbers z_1 and $z_2\neq 0$ then the quotient $\frac{z_1}{z_2}$is a complex number defined by $\frac{z_1}{z_2}=z_1z_2^{-1}=\frac{x_1x_2+y_1y_2+i(y_1x_2-y_2x_1)}{x_2^2+y_2^2}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}$ ...(7.4) where $z_1=x_1+iy_1$ and $z_2=x_2+iy)$, $z_2\neq 0$ This quotient is not defined when $z_2=0$. Note that $z_2=0$ means that $x_2^2+y_2^2=0$ and this is not permitted in expression (7.4) .

Example

(a) Evaluate the following sums:(i)$(3+i)+(1+2i)$ (ii) $(-5+3i)+(1-i)$

(b)Perform the following difference(i)$(2-3i)-(1+2i)$

(ii)$(-5+3i)-(3+i)$

(c)Find the following products (i)$(2-3i)\,(4+5i)$ (ii)$(3-4i)\,(3+4i)$
hence evaluate (iii)$(p+iq)(p-iq)$

(d)Simplify the following quotients (i)$\frac{3i-2}{1+2i}$ (ii)$(\frac{1}{2-3i})(\frac{1}{1+i})$ (iii)$\frac{\cos\theta+i\sin\theta}{\cos\phi-i\sin\phi}$

Solution

(a)i $\quad (3+i)+(1+2i)=3+1+i(1+2)=4+3i$

(ii) $\quad (-5+3i)+(1-i)=-5+1+3i-i=-4+2i$

(b)i $\quad (2-3i)-(1+2i)=2-1-3i-2i=1-5i$

(ii) $\quad (-5+3i)-(3+i)=-5-3+3i-i=-8+2i$

(c)i $\quad (2-3i)\,(4+5i)=2.4+2.5i-3i.4-3i.5i$

$$=8+10i-12i-15i^2=8+15-2i=23-2i$$

(ii) $\quad (3-4i)\,(3+4i)=3.3+3.4i-4i.3-4i.4i$

$$=9+12i-12i-16i^2=9+16=25$$

(iii)From the second example we can conclude that

$$(p+iq)(p-iq)=p^2+q^2$$

(c)i$\frac{3i-2}{1+2i}=\frac{3i-2}{1+2i}.\frac{1-2i}{1-2i}=\frac{3i.1+3i(-2i)-2.1-2(-2i)}{1.1+1(-2i)+2i.1+2i(-2i)}=\frac{3i-6i^2-2+4i}{1-2i+2i-4i^2}=\frac{3i+6-2+4i}{1+4}=\frac{4+7i}{5}=\frac{4}{5}+\frac{7i}{5}$

(ii) $(\frac{1}{2-3i})(\frac{1}{1+i})=\frac{1.1}{2.1+2.i-3i.1-3i.i}=\frac{1}{2+2i-3i-3i^2}=\frac{1}{2-i+3}=\frac{1}{5-i}$

$$=\frac{1.5+i}{5-i(5+i)}=\frac{5+i}{5.5+5.i-i.5-i.i}=\frac{5+i}{25+5i-5i+1}=\frac{5+i}{26}=\frac{5}{26}+\frac{i}{26}$$

(iii)$\frac{\cos\theta+i\sin\theta}{\cos\phi-i\sin\phi}=\frac{\cos\theta+i\sin\theta}{\cos\phi-i\sin\phi}.\frac{\cos\phi+i\sin\phi}{\cos\phi+i\sin\phi}=\frac{\cos\theta\cos\phi+i\cos\theta\sin\phi+i\sin\theta\cos\phi+i^2\sin\theta\sin\phi}{\cos\phi\cos\phi+i\cos\phi\sin\phi-i\sin\phi\cos\phi-i^2\sin\phi\sin\phi}$

$=\frac{\cos\theta\cos\phi+i(\cos\theta\sin\phi+\sin\theta\cos\phi)-\sin\theta\sin\phi}{\cos\phi\cos\phi+i\cos\phi\sin\phi-i\sin\phi\cos\phi+\sin\phi\sin\phi}=\frac{\cos\theta\cos\phi-\sin\theta\sin\phi+i(\cos\theta\sin\phi+\sin\theta\cos\phi)}{\cos^2\phi+\sin^2\phi}=\frac{\cos(\theta-\phi)+i\sin(\theta-\phi)}{1}$

$= \cos(\theta - \phi) + i\sin(\theta - \phi)$ .This follows from the fact that $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ and $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

It is natural to associate the complex number $z = x + iy$ with a point in the plane whose cartesian coordinates are x and y.Each complex number corresponds to just one point, and conversely.The number $-1+i$,for instance, is represented by the point $(-1, 1)$ in the figure below.

The number z can also be thought of as a directed line segment, or vector, from the origin to the point (x, y). In fact, we often refer to a complex number z as the point z or the vector z.When used for the purpose of displaying the numbers $z = x + iy$ geometrically,the xy-plane is called the complex plane or the z-plane.The x-axis is called the real axis, and the y-axis is known as the imaginary axis.

Given any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ the sum z_1+z_2 corresponds to the point (x_1+x_2, y_1+y_2). It also corresponds to a vector with those coordinates as its components.Hence $z_1 + z_2$ may be obtained vectorially as shown in the figure below. The difference $z_1 - z_2 = z_1 + (-z_2)$ corresponds to the sum of the vectors for $z_1$and $-z_2$.(See the figure) Note that the number $z_1 - z_2$ can be interpreted as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1).

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently,then, this product is neither the scalar nor the vector product used in ordinary vector analysis.We shall discuss the geometrical interpretation of the product of z_1 and z_2 later.

Definition 7.5 (Absolute Value)

Given a complex number $z = x+iy$, its modulus or magnitude or absolute value $|z|$ is defined as the nonnegative real number

$|z| = \sqrt{x^2 + y^2} = (x^2+y^2)^{\frac{1}{2}}$ Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin $(0, 0)$ or the length of the vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. It should be noted that there is no ordering in the complex number system i.e. it is meaningless to say $z_2 < z_1$ unless both $z_1$and z_2 are real.But the statement $\bar{z}_2| < |z_1|$ means that the point z_2 is closer to the origin than the point z_1 is.

Example

(a) Find the modulus of (i) $3 - 4i$ (ii) $-1 + i$ (iii) $1 - i\sqrt{3}$

(b) Which of these points $-3 + i2$ and $1 + i4$ is closer to the origin?

Solution

a(i) $|z| = |x + iy| = (x^2 + y^2)^{\frac{1}{2}} = (3^2 + (-4)^2)^{\frac{1}{2}} = (9 + 16)^{\frac{1}{2}} = (25)^{\frac{1}{2}} = 5$

(ii) $|z| = |x+iy| = (x^2+y^2)^{\frac{1}{2}} = \left((-1)^2+1^2\right)^{\frac{1}{2}} = \sqrt{1+1} = \sqrt{2}$

(iii)$|z| = |x+iy| = (x^2+y^2)^{\frac{1}{2}} = \left[1^2+\left(\sqrt{-3}\right)^2\right]^{\frac{1}{2}} = \sqrt{1+3} = \sqrt{4} = 2$

(b) $|z_1| = |x+iy| = (x_1^2+y_1^2)^{\frac{1}{2}} = \sqrt{(-3)^2+2^2} = \sqrt{9+4} = \sqrt{13}$ and

$|z_2| = |x_2+iy_2| = (x_2^2+y_2^2)^{\frac{1}{2}} = \sqrt{1^1+4^2} = \sqrt{1+16} = \sqrt{17}$

Therefore z_1 is closer to the origin than z_2 is.

The distance between two points represented by the complex numbers $z_1 = x_1+iy_1$and $z_2 = x_2+iy_2$ is given by $z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = |z_2-z_1|$ The complex numbers z corresponding to the points lying on the circle with centre z_0 and radius r thus satisfy the equation $|z-z_0| = r$,or

$z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2} = r$ and conversely.This set of points is simply referred to as a circle.

Example

Show that the equation $|z-1+3i| = 2$ represents a circle.

Solution. Let $z = x+iy$,then $|z-1+3i| = 2 \Longrightarrow |x+iy-1+3i|$

$= (x-1)+(y+3)\,i = \sqrt{(x-1)^2+(y+3)^2} = 2$

$\Longrightarrow (x-1)^2+(y+3)^2 = 2^2 \Longrightarrow (x-1)^2+(y+3)^2 = 4$. This last expression is the equation of a circle centred at $(1,-3)$ with radius of 2 units.

Recall that if $z = x+iy$ then $x = \Re z$ and $y = \Im z$ and we can from the definition of absolute value write $|z|^2 = x^2+y^2 = (\Re z)^2+(\Im z)^2$ and also state that $\Re z \leq |\Re z| \leq |z|$ and $\Im z \leq |\Im z| \leq |z|$.

,and $|\overline{z}| = |z|$

Theorem 7.1

Given two complex numbers $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ we can show that (i) $\overline{z_1+z_2} = \overline{z}_1+\overline{z}_2$;conjugate of sums is equal to sum of conjugates.

(ii) $\overline{z_1-z_2} = \overline{z}_1-\overline{z}_2$;conjugate of difference is equal to difference of conjugates.(iii) $\overline{z_1z_2} = \overline{z}_1\overline{z}_2$;conjugate of products is equal to product of conjugates.(iv)$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}, z_2 \neq 0$;conjugate of quotients is equal to quotient of conjugates.

Proof

(i) $\overline{z_1+z_2} = \overline{(x_1+iy_1)+(x_2+iy_2)} = \overline{x_1+x_2+i(y_1+y_2)}$

$= x_1+x_2-i(y_1+y_2) = x_1+x_2-iy_1-iy_2$

$= x_1-iy_1+x_2-iy_2 = \overline{z}_1+\overline{z}_2$

(ii)- (iv) are left as exercises for students to verify.

The sum $z+\overline{z}$of a complex number z and its conjugate $\overline{z}$ is the real number

$2x = 2\Re z$, and the difference $z-\overline{z}$ is the pure imaginary number

$2yi = 2i\Im z$.Hence we have the two identities: $\Re z = \frac{z+\overline{z}}{2}$ and $\Im z = \frac{z-\overline{z}}{2i}$.

An important identity relating the conjugate of a complex number $z = x+iy$ to its modulus is $z\overline{z} = |z|^2 = x^2+y^2$. This provides another way of simplifying the quotient $\frac{z_1}{z_2}$. The procedure is to rationalise the denominator by multiplying both numerator and denominator by the conjugate $\overline{z}_2$ of the denominator z_2 so that the denominator becomes the real number $|z_2|^2 = x_2^2+y_2^2$.

Example

Simplify the quotient $\frac{-1+3i}{2-i}$.

Solution:$\frac{-1+3i}{2-i} = \frac{(-1+3i)}{(2-i)}\frac{(2+i)}{(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1+i$

We shall now state and prove some of the following results:(i) $|z_1z_2| = |z_1||z_2|$ i.e.absolute value of products is equal to product of absolute values. (ii) $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$ i.e.absolute value of quotients is equal to quotient of absolute values. (iii) $z_1+z_2| \leq |z_1|+|z_2|$. This is called the triangular inequality.

Proof: (i) $|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2})\ \ (\because |z|^2 = z\overline{z})$

$$= (z_1z_2)(\overline{z}_1\overline{z}_2) = z_1z_2\overline{z}_1\overline{z}_2\ (\because \overline{z_1z_2} = \overline{z}\overline{z})$$
$$= z_1\overline{z}_1z_2\overline{z}_2 = |z_1|^2|z_2|^2\ (\because |z|^2 = z\overline{z})$$

$$\text{i.e. } |z_1z_2|^2 = |z_1|^2|z_2|^2$$

and taking square roots of both sides yields the results,

$$|z_1z_2| = |z_1||z_2|$$

(ii) Students should follow the steps used in (i) above to verify the second result.

(iii).$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})\quad (\because |z|^2 = z\overline{z})$

$$= (z_1+z_2)(\overline{z}_1+\overline{z}_2)\ \ (\because \overline{z_1+z_2} = \overline{z}+\overline{z})$$
$$= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2$$
$$= |z_1|^2 + 2\Re(z_1\overline{z}_2) + |z_2|^2 \quad (\because z_1\overline{z}_2 + z_2\overline{z}_1 = 2\Re(z_1\overline{z}_2)$$
$$\leq |z_1|^2 + 2|\Re(z_1\overline{z}_2)| + |z_2|^2\ \ (\because \Re z_1 \leq |\Re z_1|)$$
$$\leq |z_1|^2 + 2|z_1\overline{z}_2| + |z_2|^2 \qquad (\because |\Re z_1| \leq |z_1|)$$
$$\leq |z_1|^2 + 2|z_1||\overline{z}_2| + |z_2|^2 \qquad (\because |z_1z_2| = |z_1||z_2|)$$
$$\leq |z_1|^2 + 2|z_1||z_2| + |z_2| \qquad (\because |z| = |\overline{z}|)$$

i.e. $|z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$and taking the square roots of both sides and knowing that since moduli are nonnegative we therefore have the result,

$|z_1 + z_2| \leq |z_1|| + |z_2|$

We can generalise the triangular inequality to any finite number of terms. That is one can write $|z_1+z_2+...+z_n| = |z_1|+|z_2|+...+|z_n|$ or $\left|\sum_{i=1}^{n} z_i\right| = \sum_{i=1}^{n} |z_i|$, which can be verified by means of mathematical induction, for $n = 2, 3, ...$

7.6 Polar or trigonometry form of complex numbers

Let (r, θ) be the polar coordinates of a point (x, y) corresponding to the complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$ then z can be written as $z = r(\cos\theta + i\sin\theta)$; for instance,$1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2}(\cos\frac{-7\pi}{4} + i\sin\frac{-7\pi}{4})$. The number or angle θ is called the argument or amplitude of the complex number $z = x + iy$ and given by the formula $\theta = \arctan\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right)$.

Geomerically, argz is the angle(measured in radians) that z makes with the positive real axis when z is interpreted as a directed line segment from the origin.The argument is not unique since if θ is arg$z$ so is $\theta + 2\pi n, n = 0, \pm1, \pm2, ...$Thus for any nonzero complex number z,the value of arg z that lies in the range $-\pi < \theta \leq \pi$ is called the the principal value denoted by Arg z.This principal value is usually unique.

If $z = 0$, θ is not defined and so, we shall always assume that the complex number z is nonzero when using polar form.

If $z \neq z_0$,the representation $z - z_0 = \rho(\cos\phi + i\sin\phi)$ of $z - z_o$ in polar form can be interpreted geometrically as below.

$\rho = |z - z_0|$ is the distance between the points z and z$_0$ while $\arg(z - z_0) = \phi$ is the angle of inclination of the vector representing $z - z_0$.

7.2.1 Identities involving Arguments

Theorem 7.2

Given two complex numbers z_1 and z_2,then (i) $\arg(z_1z_2) = argz_1 + \arg z_2$ and (ii) $arg(\frac{z_1}{z_2}) = argz_1 - argz_2$ provided $z_2 \neq 0$

Proof

(i) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $\arg z_1 = \theta_1$,and $\arg z_2 = \theta_2$ so $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ which implies that $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.Hence the result. Note that these statements about the argument is not true for the principal value. For example, if $z_1 = -1$ and $z_2 = i$ then $Argz_1 = \pi$, and $Argz_2 = \frac{\pi}{2}$ but $Argz_1z_2 = -\frac{\pi}{2}$ and not $\frac{3\pi}{2}$

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ which shows that $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

$= \arg z_1 - \arg z_2$ concluding the proof. (Details to be filled in by the student)

7.7 Exponential Form of Complex Numbers

It is sometimes more convenient to express $e^{i\theta}$ as $\cos\theta + i\sin\theta$. This is known as Euler's Formula, i.e.$e^{i\theta} = \cos\theta + i\sin\theta$.

If $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ then using Euler's formula we can show that $z_1 z_2 = e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Similarly we also show that $\frac{z_1}{z_2} = \frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$ This is equivalent to the case of e^x for real x.Thus,if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then $z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

7.8 Powers and roots of unity of Complex Numbers

The intergral powers of a nonzero complex number $z = re^{i\theta}$is given by $z^n = r^n e^{in\theta}$ (7.9) which is valid for all $n = 0, \pm1, \pm2, ...$ For $r = 1$, $\left(e^{i\theta}\right)^n = e^{in\theta}, n = 0, \pm1, \pm2, ...$ or $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.This is known as De Moivre's theorem. Equation(7.9) is useful in the computation of the roots of nonzero complex numbers.

Example

Solve the equation $z^n = 1$.

Solution

For nonzero $z = re^{i\theta}$,we look for r and θ such that $\left(re^{i\theta}\right)^n = 1$ or $r^n e^{in\theta} = 1e^{i(0+2\pi k)}$. Hence $r^n = 1$ and $n\theta = 0 + 2\pi k, k = 0, \pm1, \pm2, ...$Consequently, $r = 1$, and $\theta = \frac{2\pi k}{n}$ and so we have n distinct solutions given by $c_k = e^{i\left(\frac{2\pi k}{n}\right)} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, k = 0, 1, ..., n-1$.This fact demostrate the Fundamental Theorem of Algebra which states that an equation of n degree has exactly n (roots)solutions.The complex numbers, $c_k = e^{i\left(\frac{2\pi k}{n}\right)} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, k = 0, 1, ..., n-1$ are the nth distinct roots of unity.The other roots are just a repitition of these ones.

Geometrically, the n roots of unity represent the vertices of a regular polygon with n sides inscribed in a unit circle centred at the origin and one of the roots is $c_0 = 1$.

Note that if $c \neq 1$ is one of the root of unity, the it is easy to show that $1 + c + c^2 + c^3 + ... + c^{n-1} = 0$

Examlpe

Solve the equation $z^3 = 1$ and sketch the roots of the equation.

Solution.

$c_k = e^{i\left(\frac{2\pi k}{3}\right)} = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ are the solutions of the given equation i.e.$c_0 = 1, c_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, c_3 = \cos\frac{2\pi 2}{3} + i\sin\frac{2\pi 2}{3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.The sketch of the graph shows that the roots represent the vertices of an equilateral triangle inscribed in a unit circle centred at the origin.Using either c_2 or c_3 from the above given example, we can easily verify that $1 + c_2 + c_2^2 = 1 + c_3 + c_3^2 = 0$.

Exercise 7.1

Students to demostrate the above example with $z^4 = 1$.

7.10 Roots of Nonzero complex Numbers

The concept of roots of unity can be used to find the roots of the nonzero complex number $z_0 = r_0 \exp i\theta_0$. The roots are obtained by solving the equation $z^n = z_0$ for z, and these roots are given the numbers $c_k = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \quad (7.4), \ k = 0, 1, 2, ..., n-1,$ where the number $\sqrt[n]{r_0}$ is the length of each radius vector representing the n roots.An argument of the first root, c_0 is $\frac{\theta_0}{n}$ and the other roots are obtained by adding integral multiples of $\frac{2\pi}{n}$.Consequently, as the case with the nth root of unity,the roots when $n = 2$,always lie at the opposite ends of a diameter of a circle, one root being the negative of the other; and when $n \geq 3$,they lie at the vertices of a regular polygon of n sides.If c is any particular nth root of z_0,the set of all other nth roots can be written as $c, cw_n, cw_n^2, cw_n^3, ..., cw_n^{n-1}$ where $w_n = \exp i(\frac{2\pi}{n})$.This is true because multiplication of any nonzero complex number by w_n corresponds to increasing the argument of that number by $\frac{2\pi}{n}$. We shall denote the set of nth roots of a nonzero complex number z_0 by $z_0^{\frac{1}{n}}$. If in particular z_0 is a positive real number r_0, then the symbol $r_0^{\frac{1}{n}}$ denotes a set of roots while $\sqrt[n]{r_0}$ is reserved for the one positive root.When the value of θ_0 that is used in the expression (7.4) is the principal value of the $\arg z_0 (-\pi < \theta \leq \pi)$, then the number c_0 is often referred to as the principal nth root of z_0. Thus, when z_0 is a positve real number r_0, its principal root is $\sqrt[n]{r_0}$. If $z_0 = 0$, then $z^n = z_0$ has only one solution $z = 0$.Hence the only nth root of zero is zero.

It is sometimes more convenient to write z_0 in its general exponential form $z_0 = r_0 \exp\left[i\left(\theta_0 + 2k\pi\right)\right], k = 0, \pm 1, \pm 2, ..$and then we write $z_0^{\frac{1}{n}} = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0 + 2k\pi}{n}\right)\right]$ where $k = 0, 1, 2, ...n-1$.This formula is valid where $n = 1, 2, 3, ...$

Example Find all the values (roots) of $(-8i)^{\frac{1}{3}}$ or the cube roots of $-8i$

Solution. $r_0 = |-8i| = \sqrt{(-8)^2} = \sqrt{64} = 8, \theta_0 = \arctan \frac{y}{x} = \arctan\left(\frac{-8}{0}\right) = -\frac{\pi}{2}$. Therefore $-8i = 8 \exp\left[i\left(\frac{-\pi}{2} + 2k\pi\right)\right]$ where $k = 0, \pm 1, \pm 2, ...$ so that the desired distinct roots are $c_k = 2\exp\left[i\left(\frac{-\pi}{6} + \frac{2\pi k}{3}\right)\right]$ where $k = 0, 1, 2$. In cartesian coordinates, the roots are

$c_0 = 2\exp\left[i\left(\frac{-\pi}{6}\right)\right] = 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right) = \sqrt{3} - i,$

$c_1 = 2\exp\left[i\left(\frac{-\pi}{6} + \frac{2\pi}{3}\right)\right] = 2\exp\left[i\left(\frac{3\pi}{6}\right)\right] = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i,$

and

$c_2 = 2\exp\left[i\left(\frac{-\pi}{6} + \frac{2\pi 2}{3}\right)\right]$

$= 2\exp\left[i\left(\frac{11\pi}{6}\right)\right] = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = -\sqrt{3} - i$. These roots lie at the vertices of an equilateral triangle inscribed in a circle of radius 2 units centred at the origin. The principal root is $c_0 = \sqrt{3} - i$.

Chapter 8

Matrices

8.1 Matrices and their Applications to Solution of Simultaneous Equations

Rectangular arrays such as

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \qquad B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_3 & c_3 \end{pmatrix}$$

are called matrices.

A matrix is therefore any rectangular (or square) array of elements arranged in a definite number of rows and columns.A vector is a special case of a matrix which has only one row or one column.

The dimension of a matrix is determined by its number of rows and columns. Thus a matrix with m rows and n columns is called an $(m \times n)$ matrix. The matrices A and B above are of dimensions 2×2 and 3×3 respectively.
Equality: Two matrices are equal if and only if they have the same dimensions and corresponding elements are equal. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

if and only if $a = x, b = y, c = z$, and $d = w$.
Matrix Addition(Subtraction): Two matrices of the same dimensions can be added (or subtracted), and it is done by adding (subtracting) the corresponding elements. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a \pm x & b \pm y \\ c \pm z & d \pm w \end{pmatrix}$$

Zero Matrices: A matrix (of any pair of dimensions) each of whose elements is zero is called the zero matrix (for that pair of dimensions). Examples of zero matrices are:

$\begin{pmatrix} 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Product: The product of a scalar k times a matrix is a matrix whose elements are k times the corresponding elements of the given matrix. For example,

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

3. **PRODUCT OF MATRICES**

Two matrices can be multiplied if and only if the number of columns of the first matrix is equal to the number of rows of the second matrix. That is $A_{m\times n}$ and $B_{r\times s}$ are compartible for multiplication if $m = s$ and the product C is a matrix of $n \times r$ dimension i.e.$C_{n\times r}$.The product of the two matrices below

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

is given as $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{pmatrix}$

From this we observe that:

1.The element $ap + cq$ in the first row and first column of the product is equal to the product of the first row $(p,\ q)$ of the first matrix with the first column $\begin{pmatrix} a \\ c \end{pmatrix}$ of the second matrix.

2. The element $bp+dq$ in the first row and second column of the product is equal to the product of the first row $(p\ q)$ of the first matrix with the second column $\begin{pmatrix} b \\ d \end{pmatrix}$ of the second matrix.

3. Similarly, $ar + cs = (r\ s)\begin{pmatrix} a \\ c \end{pmatrix}$ and $br + ds = (r\ s)\begin{pmatrix} b \\ d \end{pmatrix}$.

Let us consider again how we can combine (by multiplication) the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{and} \begin{pmatrix} x \\ y \end{pmatrix}$$

to obtain the column vector (or 2×1 matrix)

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

We shall write the product of these two as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Notice that $ax+by=(a\ b)\begin{pmatrix} x \\ y \end{pmatrix}$ and $cx+dy=(c\ d)\begin{pmatrix} x \\ y \end{pmatrix}$

Let us take some more examples.
Example 1.

a) $$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}=\begin{pmatrix} 1\times 2+-2\times 1+3\times 5 \\ 1\times 2+1\times 1+4\times 5 \end{pmatrix}=\begin{pmatrix} 15 \\ 23 \end{pmatrix}$$

b) $$\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}=\begin{pmatrix} 2\times 3+-3\times -2 \\ 1\times 3+4\times -2 \end{pmatrix}=\begin{pmatrix} 12 \\ -5 \end{pmatrix}$$

c) $$\begin{pmatrix} 4 & 1 & -2 \\ 3 & 2 & 5 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 14 \\ 6 \end{pmatrix}$$

Product of Matrices. Let A be a $p\times q-$dimensional matrix and **B** a $q\times r$-dimensional matrix. Their product $AB=C$ is a $p\times r$-dimensional matrix whose elements are as follows: The element in the ith row and the jth column of C is the product of the ith row of A with the jth column of B.
Example 3.

a) $$\begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 7 \end{pmatrix}=\begin{pmatrix} 17 & 5 \\ 11 & 16 \end{pmatrix}$$

b) $$\begin{pmatrix} 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 & -2 \\ 6 & 3 & 4 \end{pmatrix}=\begin{pmatrix} -6 & -9 & -14 \end{pmatrix}$$

c) $$\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix}=\begin{pmatrix} 9 & 21 \\ -7 & -5 \end{pmatrix}$$

Remarks.

1. The product AB is in this order. A first, B second, we multiply rows of A by columns of B.
2. The product AB is defined only when the number of columns in A is equal to the number of rows in B.
3. When A and B are square of the same dimension, both AB and BA are defined. However, in general, AB does not equal BA, that is, multiplication of square matrices is not commutative.

Example 4.
$$\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}=\begin{pmatrix} 7 & 5 \\ 17 & -13 \end{pmatrix}$$
But
$$\begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}=\begin{pmatrix} -7 & 13 \\ 13 & 1 \end{pmatrix}$$

4. DETERMINANTS

Let A be the 2 x 2 matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

Then we define the expression $a_1b_2 - a_2b_1$ to be the determinant of A and write

$$\det A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Remarks:

1. We use parentheses for matrices, and parallel lines for the corresponding determinant.

2. A determinant is a single number associated with a square matrix. The determinant is not the array; the array is the matrix. For 3 x 3 matrices we define the determinant in the following way:

Let $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ be a given 3 x 3 matrix. If we strike out the row and column containing any element, we are left with a 2 x 2 matrix whose determinant has already been defined. This determinant is called the minor of the corresponding element. We list a few examples of these:

Element	*Minor*
a_1	$b_2c_3 - b_3c_2$
b_1	$a_2c_3 - a_3c_2$
c_1	$a_2b_3 - a_3b_2$

We now attach an algebraic sign to each minor in the following way: Consider the corresponding element, and move it by a series of horizontal and/or vertical steps to the upper left-hand corner. The sign is $+$ if the number of steps required is even, and the sign is $-$ if the cofactor of the corresponding element. The cofactor of any element will be denoted by the corresponding capital letter, for instance, the cofactor of a_1 is A_1

We list few examples:

Element	*Minor*
a_1	$A_1 = b_2c_3 - b_3c_2$
b_1	$B_1 = -(a_2c_3 - a_3c_2)$
c_2	$C_2 = -(a_1b_3 - a_3b_1)$
c_3	$C_3 = a_1b_2 - a_2b_1$

To define the determinant, we now consider the first row and define

$$\det A = a_1A_1 + b_1B_1 + c_1C_1$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_2 - a_3b_2)$$
$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

We might equally well have done this for any row or column, and at first sight you would expect the results to be six different numbers. They are in fact, all equal.

Example 5. Find

$$\det\begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 4 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

Choosing the first row we find that the cofactors are

$$\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3, \quad \begin{vmatrix} 4 & -1 \\ 1 & 1 \end{vmatrix} = -5, \quad \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 7;$$

So the answer is $(1)(3) + (-2)(-5) + (3)(7) = 34$.

As an alternative solution, choose the first column. The cofactors are respectively

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 3, \quad -\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = 8, \quad \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} = -1;$$

So the answer is $(1)(3) + (4)(8) + (1)(-1) = 34$.

Definitions

1. The transpose of a matrix $\mathbf{A}$ (within an A^T) is the matrix obtained by writing the rows of the matrix A in order as columns of A^T.

$$\text{If } A = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 4 & 2 \\ 5 & -1 & 2 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 4 & -1 \\ 6 & 2 & 2 \end{pmatrix}$$

We observe that if A is an m x n matrix, then A^T is an n x m matrix.

Note that: $(A+B)^T = A^T + B^T$

$(A^T)^T = A$

$(A+B)^T = B^T + A^T$

$(AB)^T = B^T A^T$

2.The adjoint of a matrix A written as adj A is the transpose of the cofactors of the matrix.

Exercises. (1) Find the cofactors and adjoint of the square matrix

$$\text{(i) } A = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$$

Solution: Cofactors are:

$$A_1 = \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}, B_1 = -\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}, C_1 = \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix}$$

$$A_2 = -\begin{vmatrix} -3 & 0 \\ 1 & 3 \end{vmatrix}, B_2 = \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix}, C_2 = -\begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix}$$

$$A_3 = \begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix}, B_3 = -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}, C_3 = \begin{vmatrix} 1 & -3 \\ 2 & 0 \end{vmatrix}$$

$A_1 = 0 \times 3 - 1 = -1, B_1 = -(2 \times 3 - 4) = -2, C_1 = 2 \times 1 - 0 = 2$

$A_2 = -(-9 - 0) = 9, B_2 = 1 \times 3 - 4 \times 0 = 3, C_2 = -(1 + 12) = -13$

$A_3 = (-3 - 0) = -3, B_3 = -(1 - 0) = -1, C_3 = (1 \times 0 + 2 \times 3) = 6$

3. The cofactor of **A** written as cof A, is the matrix obtained by replacing elements of A by their corresponding cofactors.

$$\text{Cof } A = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 2 \\ 9 & 3 & 13 \\ -3 & -1 & 6 \end{pmatrix}$$

$$\text{Adj } A = (\text{Cof } A)^T$$

$$= \begin{pmatrix} -1 & 9 & -3 \\ -2 & 3 & -1 \\ 2 & 13 & 6 \end{pmatrix}$$

5. **INVERSE OF A SQUARE MATRIX**

The multiplication of square matrices has many, not all, of the properties of ordinary multiplication of real numbers. For instance, we have seen that it is associative, but not commutative, matrices have identity elements for multiplication. Some have inverses but not all of them have inverses. Consider the identity for multiplication of 3 x 3 square matrices, namely the matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Its multiplication by a matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

and moreover,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Remarks

1. For 2 x 2 matrices, the identity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. We denote identity matrices by the common symbol I. Finally, we ask whether there is a multiplicative inverse for square matrices.

Inverse of a Matrix.

If A is a square matrix, its inverse if it exists is a square matrix A^{-1} (read "A inverse") which satisfies the conditions

$$AA^{-1} = I, \text{ and } A^{-1}A = I.$$

Remarks.

1. In the notation A^{-1} the -1 is not an exponent, it is merely a symbol indicating the inverse. Do not write $A^{-1} = \frac{1}{A}$. We shall not define the quotient of two matrices.

2. Some square matrices do not have inverses. In the 2 x 2 case we are given a matrix A and are looking for an A^{-1} which satisfies the definition. First let us require that $AA^{-1} = I$

If **A** is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ we are asked to solve the matrix equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We take the product of the left to obtain

$$\begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

>From the definition of the equality of two matrices, this gives us the two simultaneous systems

1. $aw + by = 1$ 2. $ax + bz = 0$

$cw + dy = 0$ $cx + dz = 1$

Writing $\Delta = ad - bc$, and supposing this not to be zero, we find that the solution is

$$w = \frac{d}{\Delta},\ y = \frac{-c}{\Delta}, \quad x = \frac{-b}{\Delta}, \quad z = \frac{a}{\Delta}$$

Therefore,

$$A^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We find that

$$A^{-1}A = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

In the following remarks,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Remarks:

1. If $ad - bc \neq 0$, the matrix A has an inverse A^{-1}such that $AA^{-1} = A^{-1}A = I$.

2. If $ad - bc = 0$, the inverse of A does not exist. To find the inverse of a particular matrix, you may either use the formula just given or you may solve equations (1) and (2).

Example 4.

(a) If $A = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$, $A^{-1} = \frac{1}{13} \begin{pmatrix} 4 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{13} & \frac{-5}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix}$

.

(b) To find the inverse of $\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$ from first principles we write

$$\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus we derive

$2w + 5y = 1$ $\quad\quad$ $2x + 5z = 0$

$-w + 4y = 0$ $\quad\quad$ $-x + 4z = 1$

These are equivalent to

$2w + 5y = 1$ $\quad\quad$ $2x + 5z = 0$

$-2w + 8y = 0$ $\quad\quad$ $-2x + 8z = 2$

Hence $13y = 1 \Rightarrow y = \frac{1}{13}$, $w = \frac{4}{13}$, $13z = 2 \Rightarrow \frac{2}{13}$, $x = \frac{-5}{13}$

Therefore

$$A^{-1} = \begin{pmatrix} \frac{4}{13} & \frac{-5}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix}$$

(b) If $A = \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix}$, A^{-1}does not exist.

A 2 x 2 matrix for which $ad - bc \neq 0$ is called nonsingular; if $ad - bc = 0$, it is singular. Hence a matrix has an inverse if and only if it is nonsingular.

Inverse Formula

If A is a square matrix and its inverse A^{-1} exists, then

$$A^{-1} = \frac{adj\ A}{\det A}$$

Remark. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$$

Solution.

$$\text{Cof } A = \begin{pmatrix} -1 & -2 & 2 \\ 9 & 3 & -13 \\ -3 & -1 & 6 \end{pmatrix} \text{ and Adj } A = \begin{pmatrix} -1 & 9 & -3 \\ -2 & 3 & -1 \\ 2 & -13 & 6 \end{pmatrix}$$

$\det A = (0 - 1) + 3(6 - 4) + 0(2 - 0) = 5$

$$A^{-1} = \frac{adj A}{\det A} = \frac{1}{5}\begin{pmatrix} -1 & 9 & -3 \\ -2 & 3 & -1 \\ 2 & -13 & 6 \end{pmatrix} = \begin{pmatrix} \frac{-1}{5} & \frac{9}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{3}{5} & \frac{-1}{5} \\ \frac{2}{5} & \frac{-13}{5} & \frac{6}{5} \end{pmatrix}$$

Exercise.

Given $A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & 1 & -2 \\ -5 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & 2 \\ 0 & -1 & 4 \\ 6 & 2 & 5 \end{pmatrix}$

Find A^{-1}, $B^{-1}$and show that $(AB)^{-1} = B^{-1}A^{-1}$.

2. **Gauss-Jordan Method**

This method of finding the inverse of matrix uses row operations.

Consider a 2x2 square matrix

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

Suppose we place the identity matrix of the same dimensions 2 x 2 by the right of matrix A above to get the augmented matrix

$$A : I = \begin{pmatrix} a_1 & b_1 & . & 1 & 0 \\ a_2 & b_2 & . & 0 & 1 \end{pmatrix}. \qquad \ldots(2)$$

To get the inverse of A above, we shall perform elementary row operations to the augmented matrix (2) until the left side of the matrix becomes an identity matrix of dimension 2 x 2 as in (3).

$$\begin{pmatrix} 1 & 0 & . & c_1 & d_1 \\ 0 & 1 & . & c_2 & d_2 \end{pmatrix} \qquad \ldots(3)$$

The right side of (3) become the inverse and this implies that

$$A^{-1} = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}$$

This method of finding the inverse is called the Gauss-Jordan method.

Exercise.

If $\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$, find the inverse of A using the Gauss-Jordan method

Solution: Let R_1 and R_2 denote rows of the augmented matrix.

The elementary row operations are shown on the right

The augmented

$$A : I = \begin{pmatrix} 2 & 5 & . & 1 & 0 \\ -1 & 4 & . & 0 & 1 \end{pmatrix} \quad \begin{matrix} \to R_1 \quad R_2+R_1 \to R_2 \quad 1\ 9\ 1\ 1 \\ \to R_2 \ -\ 1\ 4\ 0\ 1 \end{matrix}$$

$$= \begin{pmatrix} 1 & 9 & . & 1 & 1 \\ -1 & 4 & . & 0 & 1 \end{pmatrix} \quad \begin{matrix} \to R_1 \quad R_1+R_2 \to R_1 \quad 1\ 9\ 1\ 1 \\ \to R_2 \ 0\ 13\ 1\ 2 \end{matrix}$$

$$= \begin{pmatrix} 1 & 9 & . & 1 & 1 \\ 0 & 13 & . & 1 & 2 \end{pmatrix} \quad \begin{matrix} \to R_1 \quad \frac{R_2}{13} \to R_2 \quad 1\ 9\ 1\ 1 \\ \to R_2 \ 0\ 1\ \frac{1}{13}\ \frac{2}{13} \end{matrix}$$

$$= \begin{pmatrix} 1 & 9 & . & 1 & 1 \\ 0 & 1 & . & \frac{1}{13} & \frac{2}{13} \end{pmatrix} \quad \begin{matrix} \to R_1 \quad R_1-9R_2 \to R_1 \quad 1\ 0\ \frac{4}{13}\ \frac{-5}{13} \\ \to R_2 \ 0\ 1\ \frac{1}{12}\ \frac{2}{13} \end{matrix}$$

$$= \begin{pmatrix} 1 & 9 & . & \frac{4}{13} & \frac{-5}{13} \\ 0 & 1 & . & \frac{1}{13} & \frac{2}{13} \end{pmatrix}$$

Thus

$$A^{-1} = \begin{pmatrix} \frac{4}{13} & \frac{-5}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix}$$

Exercise: Find the inverse of the matrix A using the Gauss-Jordan method.

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$$

Remark:

$$A^{-1} = \begin{pmatrix} -\frac{1}{5} & \frac{9}{5} & \frac{-3}{5} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{-13}{5} & \frac{6}{5} \end{pmatrix}$$

3. Gaussian Elimination:

Consider the system of linear equations

$2x + 5y - 3z = 5$
$x - 4y + z = -5$
$4x + 3y - z = 10$

This system can be put in a matrix form as:

$$\begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 1 \\ 4 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -15 \\ 10 \end{pmatrix}$$

Next we perform elementary row operations on the system to eliminate one or two of the unknowns x, y and z. The operations are shown on the right:

$$\begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 1 \\ 4 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -15 \\ 10 \end{pmatrix} \to \begin{matrix} \to\boxed{\mathrm{R}_1} \\ \boxed{\mathrm{R}_2} \\ \to\boxed{\mathrm{R}_3} \end{matrix} \overset{R_1 \to R_2}{\begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 1 \\ 4 & 3 & -1 \end{pmatrix}} \overset{R_2 \to R_1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 5 \\ -15 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 & 1 \\ 0 & 13 & -5 \\ 0 & 19 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -15 \\ 30 \end{pmatrix} \to \begin{matrix} \to\boxed{\mathrm{R}_1} \\ \boxed{\mathrm{R}_2} \\ \to\boxed{\mathrm{R}_3} \end{matrix} \overset{R_{2-}2R_2 \to R_2}{\begin{pmatrix} 1 & -4 & 1 \\ 2 & 5 & -3 \\ 4 & 3 & -1 \end{pmatrix}} \overset{R_1 - 4R_1 \to R_2}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 5 \\ -15 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 & 1 \\ 0 & 13 & -5 \\ 0 & 19 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -15 \\ 30 \end{pmatrix} \begin{matrix} \to \boxed{\mathrm{R}_1} \\ \to \boxed{\mathrm{R}_2} \\ \to \boxed{\mathrm{R}_3} \end{matrix} \overset{R_1 - R_2 \to R_3}{\begin{pmatrix} 1 & -4 & 1 \\ 0 & 13 & -5 \\ 0 & 6 & 0 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 10 \end{pmatrix}$$

This last system is equivalent to the system of equations.

$$x - 4y + z = -5 \qquad \ldots \text{(i)}$$
$$13y - 5z = 15 \qquad \ldots \text{(ii)}$$
$$6y = 15 \qquad \ldots \text{(iii)}$$

>From (iii) $6y = 15 \Rightarrow y = \frac{5}{2}$

So from, (ii) $13.\frac{5}{2} - 5z = 15 \Rightarrow -z = 3 - \frac{13}{2} \Rightarrow z = \frac{7}{2}$

Finally, (i) $x - 4(\frac{5}{2}) + \frac{7}{2} = -5 \Rightarrow x = \frac{13}{2} - 5 = \frac{3}{2}$.

Thus the solutions to the linear system of equations are

Exercises. Use Gaussian elimination method to solve the system of linear equations:

1. $2x + y + 3z = 11$ 2. $x + 2y + 4z = 2$

6. APPLICATIONS OF MATRICES TO SIMULTANEOUS EQUATIONS

We have seen above that the simultaneous system

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

can be written in the compact form $Ax + D = \mathbf{0}$,

where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Similarly $a_1x + b_1y + c_1z + d_1 = 0$

$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

can be written in the form $AX + D = \mathbf{0}$, where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

1. Cramer's Rule:

This suggests that the problem of solving the simultaneous equations is really that of solving the matrix equation $AX + D = \mathbf{0}$.

But this is now an easy problem for us. For $AX + D = \mathbf{0}$ is equivalent to

$AX = -D$.

Multiplying both sides, on the left, by A^{-1}, we have

$A^{-1}AX = -A^{-1}D$

or $\mathbf{IX} = -\mathbf{A}^{-1}\mathbf{D}$, since $A^{-1}A = I$

$\mathbf{X} = -\mathbf{A^{-1}D}$is the solution.

A possible method of solution, therefore, is to compute A^{-1} by the methods of sections 4 and 5 and then to find $\mathbf{A}^{-1}\mathbf{D}$. The above formula is known as "Cramer's Rule".

Example 6. Solve $2x + 5y - 6 = 0$

$x - 2y + 5 = 0$

Solution:

$A = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}$, $\det A = -9$

$\mathbf{A}^{-1} = \frac{-1}{9}\begin{pmatrix} -2 & -5 \\ -1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$

$X = -A^{-1}D = \frac{-(-1)}{9}\begin{pmatrix} -2 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -6 \\ 5 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -13 \\ 16 \end{pmatrix} =$
$\begin{pmatrix} -\frac{13}{9} \\ \frac{16}{9} \end{pmatrix}$

So

$x = -\frac{13}{9}$, $y = \frac{16}{9}$.

Cramer's rule is sometimes written in a different form. To illustrate this, let us consider the system

$a_1x + b_1y + d_1 = 0$

$a_2x + b_2y + d_2 = 0$

Then

$$A^{-1}D = \frac{1}{\Delta}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$= \frac{1}{\Delta}\begin{pmatrix} b_2d_1 - b_1d_2 \\ -a_2d_1 + a_1d_2 \end{pmatrix}$$

Therefore

$$x = \frac{-(b_2d_1 - b_1d_2)}{\Delta} = -\frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{-(a_1d_2 - a_2d_1)}{\Delta} = -\frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Remarks

1. The numerator for x is obtained from the denominator by replacing the column of a's (coefficient of x) by the column of d's.

2. To obtain the numerator of y we replace the column of b's in the denominator by the column of d's.

Example 7. Applying this method to the system in Example 6, we obtain

$$x = -\frac{\begin{vmatrix} -6 & 5 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix}} = \frac{-(-13)}{-9} = \frac{-13}{9} \quad \text{and} \quad y = -\frac{\begin{vmatrix} 2 & -6 \\ 1 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix}} =$$

$\frac{-16}{-9} = \frac{16}{9}$

Finally let us consider the following "homogeneous" system of two equations in three unknowns:

$$a_1x + b_2y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

Geometrically these equations represent two planes through the origin, and so we expect to find a line of solutions. By the use of determinants we can express the solution in a mere elegant form. For instance, solutions of

$$a_1x + b_2y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

are

$$x = k\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad y = -k\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad z = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

where k is an arbitrary scalar (provided that at least one of these is different from zero).

Example 8. Solve

$$3x - 2y + z = 0$$
$$x + 4y + 2z = 0$$

Solution:

$$x = k\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix}, \quad y = -k\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}, \quad z = k\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix}$$

or $x = -8k, y = -5k$ and $z = 14k$.

EXERCISES

1. Show that for 2 x 2 matrices, det $A = 0$ if the two columns (or rows) are proportional or equal.
2. By direct calculation show that
   Det $A = b_1B_1 + b_2B_2 + b_3B_3$.
3. Show that if two rows (columns) of a matrix are proportional (or equal), its determinant is zero.
4. Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 4 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

What is the relationship between $\det(AB)$ and $(\det A)(\det B)$?

5. Given the system

$a_1x + b_1y + c_1z + d_1 = 0$
$a_2x + b_2y + c_2z + d_2 = 0$
$a_3x + b_3y + c_3z + d_3 = 0$

Use Cramer's rule to show that:

$$x = \frac{-d_1A_1 + d_2A_2 + d_3A_3}{\Delta}$$
$$y = \frac{-d_1B_1 + d_2B_2 + d_3B_3}{\Delta}$$
$$z = \frac{-d_1C_1 + d_2C_2 + d_3C_3}{\Delta}$$

Hence write formulae for x, y, and z as quotients of determinants.