Expected Value of the Conditional Variance: Since Var(Y|X) is a random variable, we can talk about its expected value. Using the formula $Var(Y|X) = E(Y^2|X) - [E(Y|X)]^2$, we have

$$E(Var(Y|X)) = E(E(Y^2|X)) - E([E(Y|X)]^2)$$

We have already seen that the expected value of the conditional expectation of a random variable is the expected value of the original random variable, so applying this to Y^2 gives

$$(*) \qquad E(Var(Y|X)) = E(Y^2) - E([E(Y|X)]^2)$$

Variance of the Conditional Expected Value: For what comes next, we will need to consider the variance of the conditional expected value. Using the second formula for variance, we have

$$Var(E(Y|X)) = E([E(Y|X)]^2) - [E(E(Y|X))]^2$$

Since $E(E(Y|X)) = E(Y)$, this gives

$$(**) Var(E(Y|X)) = E([E(Y|X)]^2) - [E(Y)]^2.$$

Putting It Together:

Note that (*) and (**) both contain the term $E([E(Y|X)]^2)$, but with opposite signs. So adding them gives:

$$E(Var(Y|X)) + Var(E(Y|X)) = E(Y^2) - [E(Y)]^2,$$

which is just Var(Y). In other words,

$$(***) \qquad Var(Y) = E(Var(Y|X)) + Var(E(Y|X)).$$

In words: The marginal variance is the sum of the expected value of the conditional variance and the variance of the conditional means.