Accept / Reject

Method

Accept/Reject Method – Discrete Case

Suppose that we would like to generate random variates from a pmf $\{p_j, j\geq 0\}$ and we have an efficient way to generate variates from a pmf $\{q_j, j\geq 0\}$.

Let a constant c such that $$\frac{p_j}{q_j} \leq c \quad \text{for all } j \text{ such that } p_j > 0$$

In this case, use the following algorithm

1. Generate a random variate Y from pmf $\{q_j, j\geq 0\}$.
2. Generate u=U(0,1)
3. If u< pY/(cqY), set X=Y and return;
4. Else repeat from Step 1.

Show that the algorithm generates a random variate with the required pmf $\{p_j, j \geq 0\}$.

$$p_i = \Pr\{X = i\} = \sum_{k=1}^{\infty} \Pr\{\text{not stop for } 1,\ldots,k-1\} \Pr\{X = i \text{ and stop at iteration } k\}$$

$$\Pr\{X = i \text{ and stop after } k\} =$$

$$\underbrace{\Pr\{X = i \mid Y = i \text{ and is accepted}\}}_{1} \underbrace{\Pr\{\text{Y accepted} | Y = i\}}_{p_i/cq_i} \underbrace{\Pr\{Y = i\}}_{q_i} = q_i \frac{p_i}{cq_i} = \frac{p_i}{c}$$

$$\Pr\{\text{Not stop up to } k-1\} = \left(\sum_j \Pr\{Y \text{ not accepted} | Y = j\} \Pr\{Y = j\} \right)^{k-1}$$

$$= \left(\sum_j \left(1 - \frac{p_j}{cq_j}\right) q_j \right)^{k-1} = \left(1 - \frac{1}{c}\right)^{k-1}$$

Therefore

$$p_i = \sum_{k=1}^{\infty} \left(1 - \frac{1}{c}\right)^{k-1} \frac{p_i}{c} = p_i$$

Exemple:Accept/Reject Method – Discrete Case

Determine an algorithm for generating random variates for a random variable that take values 1,2,..,10 with probabilities 0.11, 0.12, 0.09, 0.08, 0.12, 0.10, 0.09, 0.09, 0.10, 0.10 respectively.

$$c = \max_i \left\{ \frac{p_i}{q_i} \right\} = 1.2$$

```
u1=U(0,1), u2=U(0,1)
Y=floor(10*u1 + 1);
while(u2 > p(Y)/c)
   u1= U(0,1); u2=U(0,1);
   Y=floor(10*rand + 1);
y(i)=Y;
```

Accept/Reject Method – Continue Case

Suppose that we would like to generate random variates from a pdf $f_X(x)$ and we have an efficient way to generate variates from a pdf $g_X(x)$.

Let a constant *c* such that

$$\frac{f_X(x)}{g_X(x)} \leq c \quad \text{for all } x$$

In this case, use the following algorithm

```
1. Generate a random variate Y from density gX(x).
2. Generate u=U(0,1)
3. If u< fX(Y)/(cgX(Y)), set X=Y and return;
4. Else repeat from Step 1.
```

Accept/Reject Method – Continue Case

The continue case is similar to the dicrete algorithm except the comparison step where rather than comparing the two probabilities we compare the values of the density functions.

Theorem

- ✓ The random variates generated by the Accept/Reject method have density $f_X(x)$.
- ✓ The number of iterations of the algorithm that are needed is a geometric random variable with mean c

Note: The constant c is important since is implies the number of iterations needed before a number is accepted, therefore it is required that it is selected so that it has its minimum value.

Example: Accept/Reject Method – Continue Case

Use the method to generate random variates *X* that are normally distributed with mean 0 and variance 1, *N*(0,1).

- First consider the pdf of the absolute value of |*X*|.

$$f_{|X|}(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

- We know how to generate exponentially distributed random variates *Y* with rate λ=1.

$$g_Y(x) = e^{-x}, \quad x \geq 0$$

- Determine *c* such that it is equal to the maximum of the ratio

$$\frac{f_{|X|}(x)}{g_Y(x)} = \frac{2}{\sqrt{2\pi}} e^{x-\frac{1}{2}x^2} \quad \Rightarrow c = \sqrt{\frac{2e}{\pi}}$$

Example: Accept/Reject Method – Continue Case


```
u1=U(0,1), u2=U(0,1);
Y= -log(u1);
while(u2 > exp(-0.5(Y-1)*(Y-1)))
    ▪ u1= U(0,1); u2=U(0,1);
    ▪ Y= -log(u1);
u3= U(0,1);
If u3 < 0.5 X=Y;
Else X= -Y;
```

Suppose we would like $Z \sim N(\mu, \sigma^2)$, then $Z := \sigma X + \mu$

Example:

On average, how many random numbers must be simulated to generate 1000 variates from the Beta($\alpha = 2$, $\beta = 2$) distribution.

```
n=1000
k=0       #conter for accepted
j=0       #iterations
y=numeric(n)

while(k<n){
  u=runif(1)
  j=j+1
  x=runif(1)
  if (x*(1-x)>u){
      #we accept x
    k=k+1
    y[k]=x
  }
}
j
```

OBRIGADO!