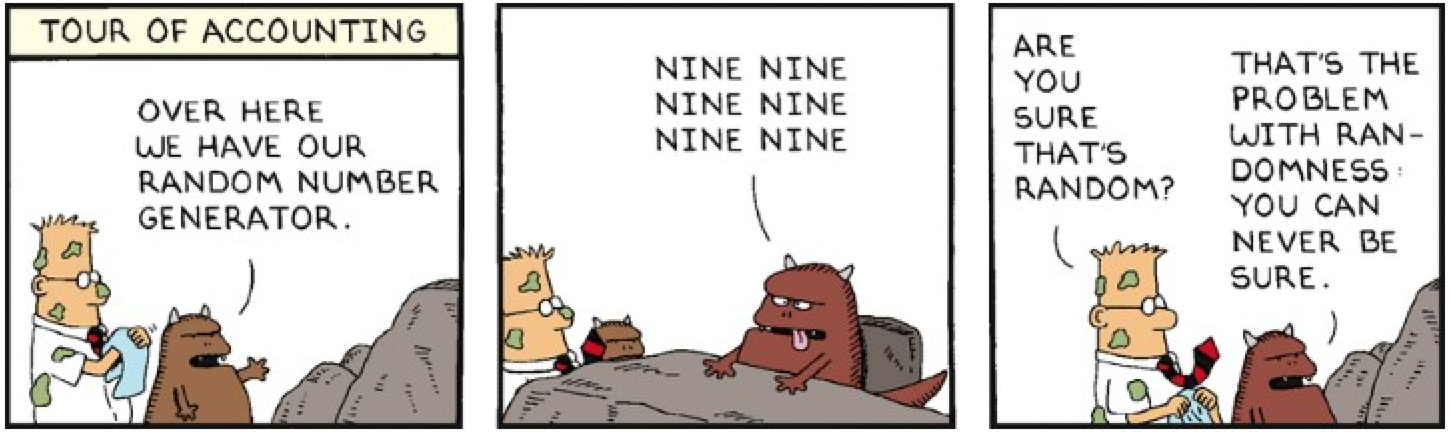
Inverse Transform Method



Things to remember

✓If R ~ U(0,1), A Random variable X will assume value x whenever,

$$F(x-1) < R \le F(x)$$
$$\Leftrightarrow x-1 < F^{-1}(R) \le x$$
$$\Leftrightarrow x = \lceil F^{-1}(R) \rceil$$

✓Also we should know the identities :

$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \le x < n+1$$
$$\lceil x \rceil = n \quad \Leftrightarrow \quad n-1 < x \le n$$
$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad x-1 < n \le x$$
$$\lceil x \rceil = n \quad \Leftrightarrow \quad x \le n < x+1$$

We could use this identity also. It results in $F^{-1}(R)$ = floor(x) which is not useful because x is already an integer for discrete distribution.

Discrete Distributions

Empirical Discrete Distribution

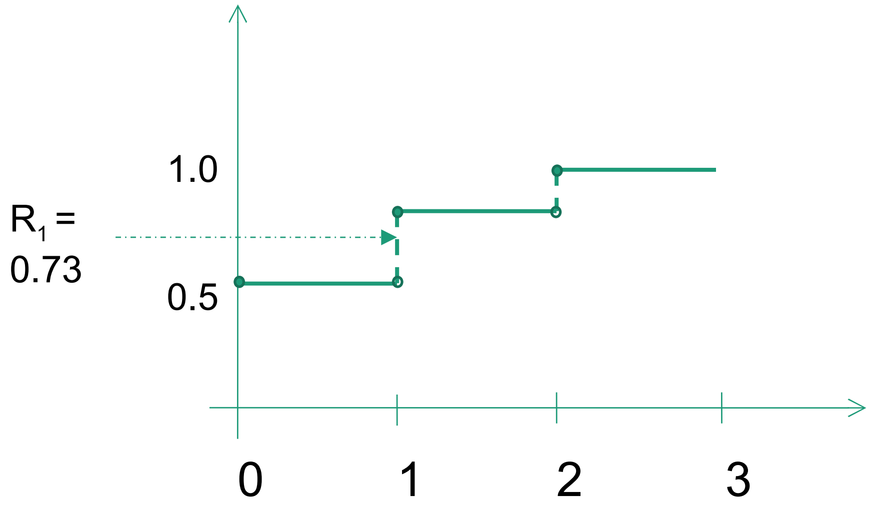
Say we want to find random variate for the following discrete distribution.

x	p(x)
0	0.50
1	0.30
2	0.20

At first find the F

x	p(x)	F(x)
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \le x < 1 \\ 0.8, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

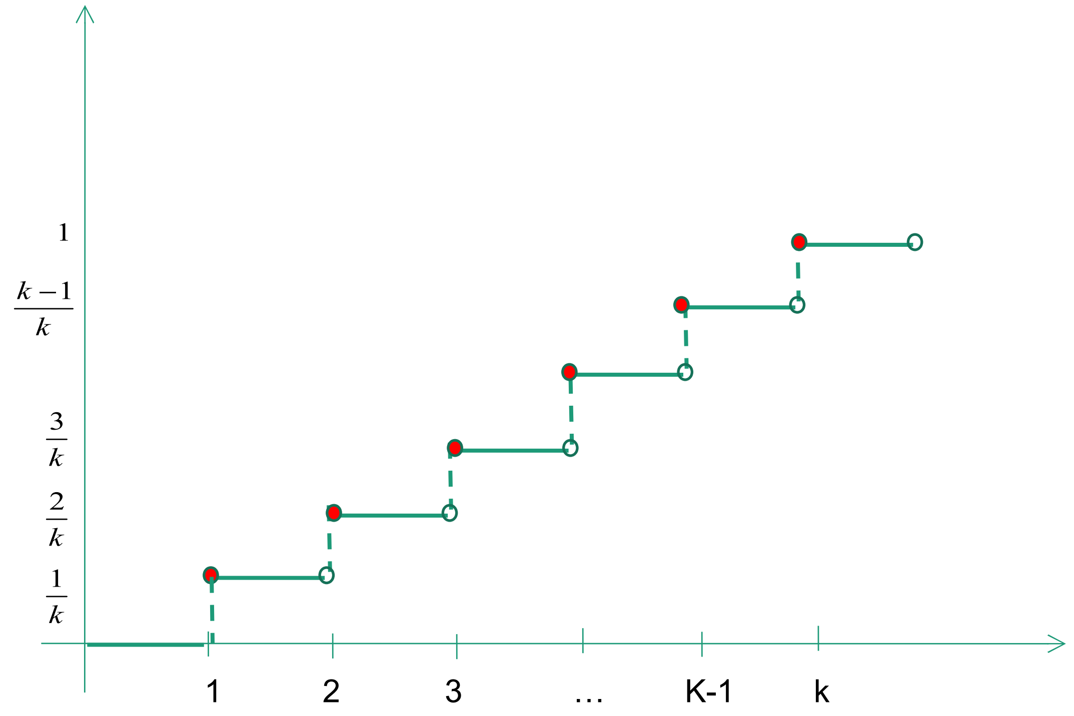


$$X = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$

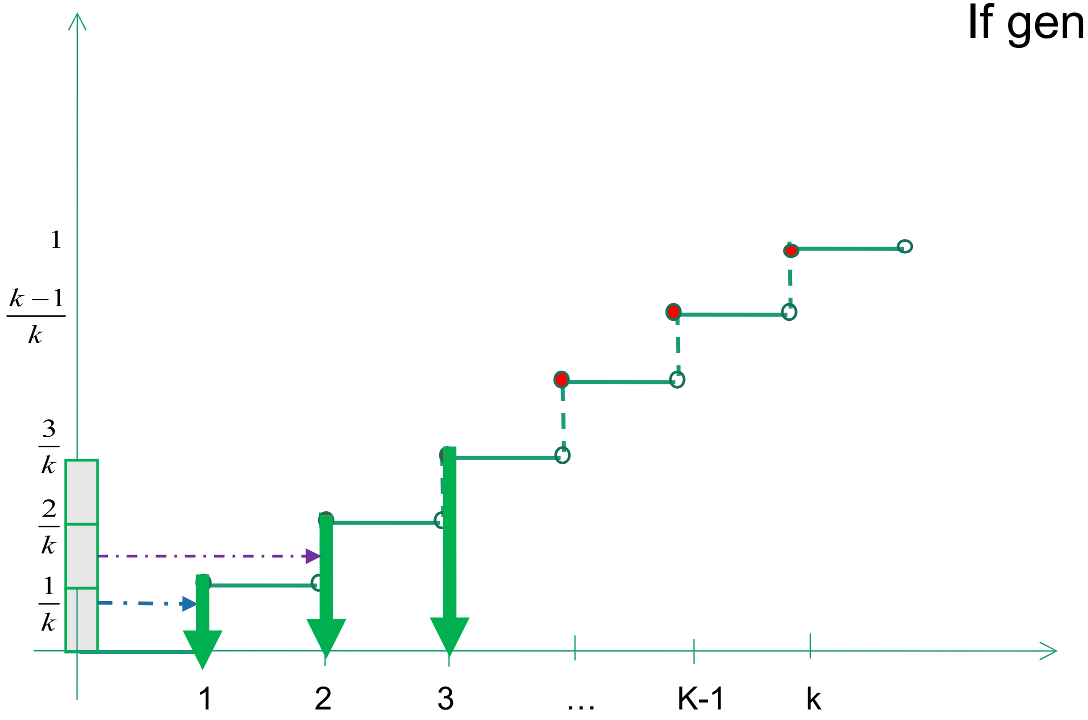
Discrete Uniform Distribution

Consider $p(x) = \frac{1}{k},\ x = 1, 2, 3 \ldots k$

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{k}, & 1 \le x < 2 \\ \frac{2}{k}, & 2 \le x < 3 \\ \vdots & \vdots \\ \frac{k-1}{k}, & k-1 \le x < k \\ 1, & k \le x \end{cases}$$



Generate R~U(0,1)



If generated $0 \le R \le \frac{1}{k}$, output X= 1

$\frac{1}{k} < R \le \frac{2}{k}$ output X= 2

$\frac{2}{k} < R \le \frac{3}{k}$ output X= 3

$\frac{3}{k} < R \le \frac{4}{k}$ output X= 4

$\frac{i-1}{k} < R \le \frac{i}{k}$, output X= i

Discrete Uniform Distribution

If generated $0 \le R \le \frac{1}{k}$, output X=1

$\frac{1}{k} < R \le \frac{2}{k}$, output X=2

$\frac{2}{k} < R \le \frac{3}{k}$, output X=3

$\frac{3}{k} < R \le \frac{4}{k}$, output X=4

$\frac{i-1}{k} < R \le \frac{i}{k}$, output X = i

$$\frac{i-1}{k} < R \le \frac{i}{k}$$
$$\Rightarrow i-1 < Rk \le i$$
$$\Rightarrow i-1 < Rk \text{ and } Rk \le i$$
$$\Rightarrow i < Rk+1 \text{ and } Rk \le i$$
$$\Rightarrow Rk \le i \text{ and } i < Rk+1$$
$$\Rightarrow Rk \le i < Rk+1$$
$$\Rightarrow i = \lceil Rk \rceil = \text{output } X$$
$$\therefore X = \lceil Rk \rceil$$

Important

Expand this method for a general discrete uniform distribution *DU(a,b)*

Algorithm to generate random variate for p(x)=1/k where x = 1, 2, 3, … k

Generate R~U(0,1) uniform random number
Return ceiling(R*k)

Geometric Distribution

$$p(x) = p(1-p)^x, \text{ where x} = 0, 1, 2, \ldots\ldots$$

$$F(x) = \sum_{j=0}^{x} p(1-p)^x = 1-(1-p)^{x+1}$$

$$R = 1-(1-p)^{x+1}$$

$$\Leftrightarrow (1-p)^{x+1} = 1-R$$

$$\Leftrightarrow (x+1)\ln(1-P) = \ln(1-R)$$

$$\Leftrightarrow x = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

$$F^{-1}(R) = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

Recall :

$$X = R \Leftrightarrow F(x-1) < R \leq F(x)$$

$$X = \lceil F^{-1}(R) \rceil = \left\lceil \frac{\ln(1-R)}{\ln(1-p)} - 1 \right\rceil$$

Algorithm to generate random variate for

Generate R~U(0,1) uniform random number
Return ceiling(ln(1-R)/ln(1-p)-1)

Proof of inverse transform method for Discrete case

- ✓ Kelton 8.2.1 Page 444
- ✓ We need to show that $P(X = x_i) = p(x_i) \text{ for all } i$

$$\text{for } i = 1,\ X = x_1 \text{ iff } U \le F(x_1) = p(x_1)$$

$$\text{(since the } x_i \text{ are in increasing order)}$$

$$\text{since, } U \sim U(0,1)\ \ P(X = x_1) = p(x_1)$$

$$\text{When, } \ i \ge 2,\ X = x_i \text{ iff } \ F(x_{i-1}) < U \le F(x_i),$$

$$\text{since, the algorithm chooses i such that } U \le F(x_i)$$

$$0 \le F(x_{i-1}) < F(x_i) \le 1$$

$$P(X = x_i) = P[F(x_{i-1}) < U \le F(x_i)] = F(x_i) - F(x_{i-1}) = p(x_i)$$

Inverse Transform Method

Disadvantage

- ✓ Need to evaluate F^{-1}. We may not have a closed form. In that case numerical methods necessary. Might be hard to find stopping conditions
- ✓ For a given distribution, Inverse transform method may not be the fastest way.

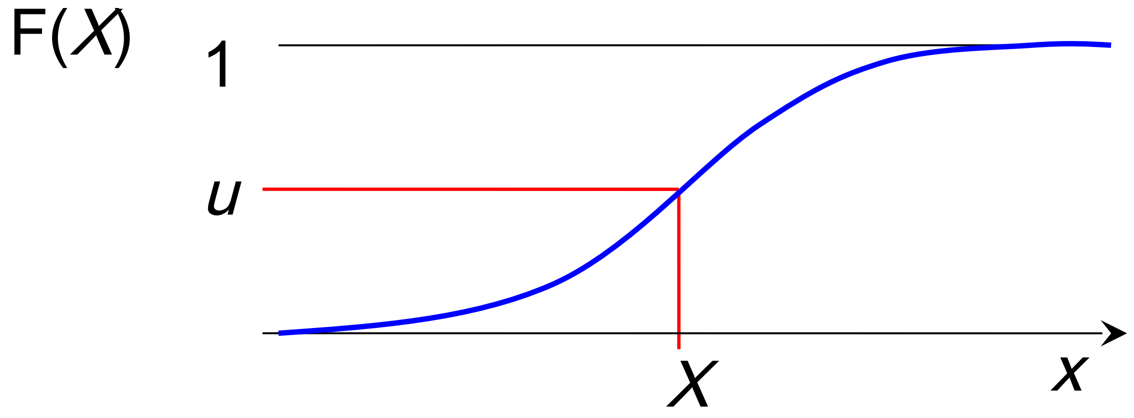
Advantage

- ✓ Straightforward method, easy to use
- ✓ Variance reduction techniques has an advantage if inverse transform method is used
- ✓ Facilitates generation of order statistics.

Inverse Transform Method

- ✓Suppose we would like to generate a sequence of continuous random variates having density function $F(X)$
- ✓Let U be a random variable uniformly distributed in the interval (0,1). For any continuous distribution function, the random variate X is given by

$$X = F^{-1}(U).$$



Example: Uniform Distibution Random Variable

Step 1 – compute *cdf* of the desired random variable *X*

$$F(x)=\begin{cases} 0, & x<a \\ \dfrac{x-a}{b-a}, & a\le x<b \\ 1, & x\ge b \end{cases}$$

Step 2 – Set F(*X*) = *R* where *R* is a random number ~U[0,1)

$$F(x)=R=\frac{x-a}{b-a}$$

Step 3 – Solve F(*X*) = *U* for *X* in terms of *R*. $X = F^{-1}(U)$.

$$U(b-a)=X-a, \quad X=U(b-a)+a$$

Step 4 – Generate random numbers R_i and compute desired random variates:

$$X_i = U_i(b\text{-}a) + a$$

Example: Uniform Distibution Random Variable

$$X_i = F^{-1}(U) = U_i(b-a) + a$$

If $X_i \sim U[5,10)$

$a = 5$

$b = 10$

$\underline{U_i}$	$\underline{X_i}$
.5	.5(10 - 5)+5 = 7.5
.7	.7(10 – 5) + 5 = 8.5
.1	.1(10 – 5) + 5 = 5.5

Example: Exponentially Distributed Random Variable

Suppose we would like to generate a sequence of random variates having density function

$$f(x) = \lambda e^{-\lambda x}$$

Solution

Find the cumulative distribution

$$F(x) = \int_0^x \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x}$$

Let a uniformly distributed random variable *u*

$$u = F(x) = 1 - e^{-\lambda x} \Rightarrow \ln(1-u) = -\lambda x \Rightarrow x = -\frac{1}{\lambda}\ln(1-u)$$

Equivalently, since 1-*u* is also uniformly distributed in (0,1)

Exemplo no R

Exponencial(lambda)

```
n=1000
lambda=0.5

u=runif(n)
x=-log(u)/lambda

hist(x,breaks=50, freq=FALSE,col="blue",main="Distribuição Exponencial-lambda")
curve(lambda*exp(-lambda*x), 0, to=NULL, col="red", lwd=2, add=T)
```

OBRIGADO!