Buffon's Needle: One of the First Examples of Monte Carlo

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Abstract

In this brief report, we present results from a Monte Carlo experiment to estimate π. The problem of Buffon's needle is stated. Then, an unbiased estimator of $1/\pi$ and its variance are derived based on this problem statement. The Monte Carlo results show that this estimator converges very slowly to $1/\pi$ and is a poor choice for obtaining estimates of π.

Figure 1: Georges Louis Leclerc Comte de Buffon, born September 7, 1707 in Montbard, France.

1 Introduction

Looking at the figure of Georges Buffon (figure 1) one wonders if Bernoulli could have found him attractive given the large distance from the tip of his nose to the top of his forehead. From 1749 until his death, Buffon wrote 36 volumes of of Natural History. However, he is most well know for his experimental suggestion of how to estimate the value of π.

This experiment is know until today as "Buffon's Needle". To perform the experiment we throw a needle or pencil of length l onto a piece of paper or floor with equally spaced lines with spacing $d \geq l$. Buffon related the fraction of throws which crossed a line to the number π and suggested the use of such an experiment to estimate π.

2 The Estimator and its Variance

For n throws of the needle let S_n denote the number of times the needle crossed a line. In this case[1], the estimator $\hat{\theta} = dS_n/2ln \to \theta \equiv 1/\pi$. This can be seen by considering the geometry and generation of the experiment. One may consider that throwing the needle is equivalent to randomly choosing a) how close the center of the needle is to the nearest line and b) the angle that the needle makes with an axis parallel to the lines.

In this case, the proximity of the center of the needle to a line, y, can be considered as drawn from a $\mathrm{U}(0, d/2)$ distribution while the angle, x, can be considered drawn independantly from a $\mathrm{U}(0, \pi)$ distribution. The needle crosses a line if $y < l/2 \sin(x)$. The rectangle defined by $0 < x < \pi$ and $0 < y < d/2$ has area $\pi d/2$. The integral $\int_0^\pi l/2 \sin(x) \mathrm{d}x = l$ gives the area of the region within the rectangle for which $y < l/2 \sin(x)$. Hence, as $n \to \infty$ the number of throws which cross a line divided by the total number of throws, $S_n/n \to 2l/\pi d$ and $\hat{\theta} \to \theta$.

Note that $S_n \sim \mathrm{Bi}(2l/\pi d, n)$. This facilitates the calculation of the variance of the estimator,

$$\begin{aligned}
\mathrm{Var}(\hat{\theta}) &= \left(\frac{d}{2nl}\right)^2 \mathrm{Var}(S_n) \\
&= \left(\frac{d}{2nl}\right)^2 n \left(\frac{2l}{\pi d}\right) \left(1 - \frac{2l}{\pi d}\right) .
\end{aligned}$$

This variance increases without bound as a function of d/l, hence we choose the smallest value which respects $d \geq l$, $d = l$. Hence the variance of the estimator reduces to $\mathrm{Var}(\hat{\theta}) = \frac{1}{n}\left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right)$.

3 Monte Carlo Experiment

We choose $d = l = 2$ for our experiment since the first minimizes the variance while the second simplifies the random number generation. In this case, the condition for the needle crossing a line, $y < l/2 \sin(x)$, can be rewritten as $u_1 < \sin(u_2\pi)$ where u_1 and u_2 are independant random variables distributed uniformly over the interval $[0, 1]$ (see [1]).

If we wish to estimate π by the estimator $\hat{\pi} = 1/\hat{\theta}$ to within an accuracy δ with a certain confidence, we must estimate $1/\pi$ to an accuracy of $\varepsilon = \delta/(1 + \frac{\delta}{\pi})$ with the same confidence. The number of needle throws necessary for a given accuracy and confidence can by estimated by the use of Chebyshev's inequality,

$$\mathrm{Prob}\left(\left|\frac{S_n}{2n} - \frac{1}{\pi}\right| > \varepsilon\right) \leq \frac{\mathrm{Var}(S_n)}{4n^2\varepsilon^2} = \frac{1}{n\varepsilon^2}\left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right) .$$

To guarantee a confidence of at least 95%, the probability in the above equation must be less than or equal to 0.05. For our estimate of π to be within $\delta = 10^{-4}$ of the true value with a probability of 95% we see that $n > 1.16 \times 10^8$.

Confidence	95%	95%
δ	10^{-4}	10^{-3}
n	1.17×10^8	1.17×10^6
π	3.14159265	3.14159265
$\bar{\hat{\pi}}$	3.14178384	3.14192562
Std. Dev.	0.00014979	0.00197464

Table 1: The average and standard deviation of the 100 estimates of π obtained from the Monte Carlo experiments of Buffon's needle.

See the appendix for a listing of the source code and details of the compilation. The experiment was conducted as follows. A given accuracy and confidence level were chosen and the required number of throws necessary to achieve such accuracy was calculated. One hundred different estimates of π were calculated based on this number of throws. The arithmetic mean of these estimates, $\bar{\hat{\pi}}$, is reported in table 1 along with the standard deviation of these estimates. The full output from the program can be found in the appendix.

The results in table 1 confirm the slow convergence of our estimate of π. One sees that the standard deviation is slightly larger than the the desired accuracy. Hence, Chebyshev's inequality was insufficient to guarantee our required accuracy. For example, one can see in the full output that many more than 5% of the estimates are more than δ from the true value of π.

4 Conclusion

George Buffon was quoted as saying "Genius is only a greater aptitude for patience". This may be so, but it is certain that patience is necessary to estimate π accurately by his needle experiment. Who among us would have the patience to throw a needle 1.7×10^8 times to estimate π to three decimal places.

A Source Code

The following code was compiled on an AMD Athlon XP1500+ running Windows 2000 Professional using the command 'gcc -O2 -o buffon.exe buffon.c -lm'.

A.1 buffon.c

```
/*********************************************************
 * PROGRAM: buffon.c
 *
 * Bonus Track for Computao Cientfica.
 *
 * DATE: 19-05-2004
 *
 * USE: Estimate PI by throwing needles.
 *
 ********************************************************/

#include <time.h>   /* for timing the code */
#include <stdio.h>  /* for writing to files */
#include <math.h>   /* for sines */
```

```
/* George Marsaglias uniform random number generator */
/* (has a very large period, > 2^60, and passes Diehard tests; */
/* uses a multiply-with-carry method) */
#define s1new (s1=(18000*(s1&0xFFFF)+(s1>>16)))
#define s2new (s2=(30903*(s2&0xFFFF)+(s2>>16)))
#define UNI (((s1new<<16)+(s2new&0xFFFF))*2.32830643708e-10)
unsigned long s1=362436069, s2=521288629;
#define setseed(seed1,seed2) {s1=seed1;s2=seed2;}

#define N_REPS 100
#define N_THROWS 1170000  /* to achieve delta = 10^-3 */
#define OUTPUT_FILENAME "buffon3.out" /* number refers to delta */
#define PI 3.14159265358979323846264338327950

int main(void)
{
  /* timing variables */
  clock_t start, end;

  /* calculation variables */
  int i,r,S[N_REPS];
  double pi_est[N_REPS],pi_avg=0.0,pi_var=0.0;

  /* output file pointer */
  FILE *ofp;

  /* start the clock */
  start = clock();

  for (r=0;r<N_REPS;r++) {
    /* initialize the number of hits to zero */
    S[r]=0;
    /* for each throw, if the needle crosses the line,
       increment the number of hits */
    for (i=0;i<N_THROWS;i++)
      if (UNI < sin(UNI*PI)) S[r]++;
    /* calculate the estimated value of pi and sum it
       to the variable holding the average of the estimates */
    pi_avg += (pi_est[r] = ((double)(2*N_THROWS)) / ((double)S[r]));
  }
  /* divide the sum of the estimates by the number of
     replications to obtain the average */
  pi_avg /= N_REPS;
  /* calculate the variance of the estimates */
  for (r=0;r<N_REPS;r++)
    pi_var += (pi_est[r]-pi_avg)*(pi_est[r]-pi_avg);
  pi_var /= N_REPS-1;

  /* stop the clock */
  end = clock();

  ofp=fopen(OUTPUT_FILENAME,"w");
  fprintf(ofp,"ELAPSED TIME = %6.2f seconds\n\n",
  ((double)(end-start))/CLOCKS_PER_SEC);
  fprintf(ofp,"TRUE PI VALUE = %10.8f\n\n",PI);
  for (r=0;r<N_REPS;r++)
    fprintf(ofp,"PI ESTIMATES  = %10.8f\n",pi_est[r]);
  fprintf(ofp,"\nTRUE PI VALUE = %10.8f\n",PI);
  fprintf(ofp,  "PI AVERAGE    = %10.8f\n",pi_avg);
  fprintf(ofp,  "PI STD DEV    = %10.8f\n",sqrt(pi_var));
  fprintf(ofp,  "PI VARIANCE   = %10.8f\n",pi_var);
  fclose(ofp);
  return(0);
}
```

B Program Output

B.1 $n = 1.17 \times 10^8$ Experiment

```
ELAPSED TIME = 2221.49 seconds
```

```
TRUE PI VALUE = 3.14159265

PI ESTIMATES  = 3.14192439
PI ESTIMATES  = 3.14181956
PI ESTIMATES  = 3.14190709
PI ESTIMATES  = 3.14166235
PI ESTIMATES  = 3.14157525
PI ESTIMATES  = 3.14189566
PI ESTIMATES  = 3.14185786
PI ESTIMATES  = 3.14191330
PI ESTIMATES  = 3.14162515
PI ESTIMATES  = 3.14161119
PI ESTIMATES  = 3.14198949
PI ESTIMATES  = 3.14179324
PI ESTIMATES  = 3.14192431
PI ESTIMATES  = 3.14164004
PI ESTIMATES  = 3.14151456
PI ESTIMATES  = 3.14197801
PI ESTIMATES  = 3.14183268
PI ESTIMATES  = 3.14190473
PI ESTIMATES  = 3.14169356
PI ESTIMATES  = 3.14146201
PI ESTIMATES  = 3.14199223
PI ESTIMATES  = 3.14181821
PI ESTIMATES  = 3.14202349
PI ESTIMATES  = 3.14164067
PI ESTIMATES  = 3.14145475
PI ESTIMATES  = 3.14201151
PI ESTIMATES  = 3.14181243
PI ESTIMATES  = 3.14190262
PI ESTIMATES  = 3.14161844
PI ESTIMATES  = 3.14159803
PI ESTIMATES  = 3.14197434
PI ESTIMATES  = 3.14186208
PI ESTIMATES  = 3.14181484
PI ESTIMATES  = 3.14168036
PI ESTIMATES  = 3.14153940
PI ESTIMATES  = 3.14199632
PI ESTIMATES  = 3.14191465
PI ESTIMATES  = 3.14166796
PI ESTIMATES  = 3.14181479
PI ESTIMATES  = 3.14153632
PI ESTIMATES  = 3.14194780
PI ESTIMATES  = 3.14193494
PI ESTIMATES  = 3.14172410
PI ESTIMATES  = 3.14178417
PI ESTIMATES  = 3.14150735
PI ESTIMATES  = 3.14199219
PI ESTIMATES  = 3.14193806
PI ESTIMATES  = 3.14170596
PI ESTIMATES  = 3.14173330
PI ESTIMATES  = 3.14157803
PI ESTIMATES  = 3.14193072
PI ESTIMATES  = 3.14191730
PI ESTIMATES  = 3.14178189
PI ESTIMATES  = 3.14171756
PI ESTIMATES  = 3.14154902
PI ESTIMATES  = 3.14195114
PI ESTIMATES  = 3.14196662
PI ESTIMATES  = 3.14166366
PI ESTIMATES  = 3.14178716
PI ESTIMATES  = 3.14146888
PI ESTIMATES  = 3.14204703
PI ESTIMATES  = 3.14194620
PI ESTIMATES  = 3.14167070
PI ESTIMATES  = 3.14174692
PI ESTIMATES  = 3.14160279
PI ESTIMATES  = 3.14196211
PI ESTIMATES  = 3.14187335
PI ESTIMATES  = 3.14175354
PI ESTIMATES  = 3.14169133
PI ESTIMATES  = 3.14164797
PI ESTIMATES  = 3.14188279
PI ESTIMATES  = 3.14193650
PI ESTIMATES  = 3.14184027
PI ESTIMATES  = 3.14160532
```

```
PI ESTIMATES  = 3.14175713
PI ESTIMATES  = 3.14170398
PI ESTIMATES  = 3.14199919
PI ESTIMATES  = 3.14178391
PI ESTIMATES  = 3.14168723
PI ESTIMATES  = 3.14170276
PI ESTIMATES  = 3.14174882
PI ESTIMATES  = 3.14195207
PI ESTIMATES  = 3.14181964
PI ESTIMATES  = 3.14166509
PI ESTIMATES  = 3.14173233
PI ESTIMATES  = 3.14173152
PI ESTIMATES  = 3.14189208
PI ESTIMATES  = 3.14191245
PI ESTIMATES  = 3.14165117
PI ESTIMATES  = 3.14176565
PI ESTIMATES  = 3.14162599
PI ESTIMATES  = 3.14197940
PI ESTIMATES  = 3.14186951
PI ESTIMATES  = 3.14168361
PI ESTIMATES  = 3.14178640
PI ESTIMATES  = 3.14161009
PI ESTIMATES  = 3.14191393
PI ESTIMATES  = 3.14188161
PI ESTIMATES  = 3.14178497
PI ESTIMATES  = 3.14175932

TRUE PI VALUE = 3.14159265
PI AVERAGE    = 3.14178384
PI STD DEV    = 0.00014979
PI VARIANCE   = 0.00000002
```

B.2 $n = 1.17 \times 10^6$ Experiment

```
ELAPSED TIME = 2221.49 seconds

TRUE PI VALUE = 3.14159265

PI ESTIMATES  = 3.14192439
PI ESTIMATES  = 3.14181956
PI ESTIMATES  = 3.14190709
PI ESTIMATES  = 3.14166235
PI ESTIMATES  = 3.14157525
PI ESTIMATES  = 3.14189566
PI ESTIMATES  = 3.14185786
PI ESTIMATES  = 3.14191330
PI ESTIMATES  = 3.14162515
PI ESTIMATES  = 3.14161119
PI ESTIMATES  = 3.14198949
PI ESTIMATES  = 3.14179324
PI ESTIMATES  = 3.14192431
PI ESTIMATES  = 3.14164004
PI ESTIMATES  = 3.14151456
PI ESTIMATES  = 3.14197801
PI ESTIMATES  = 3.14183268
PI ESTIMATES  = 3.14190473
PI ESTIMATES  = 3.14169356
PI ESTIMATES  = 3.14146201
PI ESTIMATES  = 3.14199223
PI ESTIMATES  = 3.14181821
PI ESTIMATES  = 3.14202349
PI ESTIMATES  = 3.14164067
PI ESTIMATES  = 3.14145475
PI ESTIMATES  = 3.14201151
PI ESTIMATES  = 3.14181243
PI ESTIMATES  = 3.14190262
PI ESTIMATES  = 3.14161844
PI ESTIMATES  = 3.14159803
PI ESTIMATES  = 3.14197434
PI ESTIMATES  = 3.14186208
PI ESTIMATES  = 3.14181484
PI ESTIMATES  = 3.14168036
PI ESTIMATES  = 3.14153940
PI ESTIMATES  = 3.14199632
```

```
PI ESTIMATES  = 3.14191465
PI ESTIMATES  = 3.14166796
PI ESTIMATES  = 3.14181479
PI ESTIMATES  = 3.14153632
PI ESTIMATES  = 3.14194780
PI ESTIMATES  = 3.14193494
PI ESTIMATES  = 3.14172410
PI ESTIMATES  = 3.14178417
PI ESTIMATES  = 3.14150735
PI ESTIMATES  = 3.14199219
PI ESTIMATES  = 3.14193806
PI ESTIMATES  = 3.14170596
PI ESTIMATES  = 3.14173330
PI ESTIMATES  = 3.14157803
PI ESTIMATES  = 3.14193072
PI ESTIMATES  = 3.14191730
PI ESTIMATES  = 3.14178189
PI ESTIMATES  = 3.14171756
PI ESTIMATES  = 3.14154902
PI ESTIMATES  = 3.14195114
PI ESTIMATES  = 3.14196662
PI ESTIMATES  = 3.14166366
PI ESTIMATES  = 3.14178716
PI ESTIMATES  = 3.14146888
PI ESTIMATES  = 3.14204703
PI ESTIMATES  = 3.14194620
PI ESTIMATES  = 3.14167070
PI ESTIMATES  = 3.14174692
PI ESTIMATES  = 3.14160279
PI ESTIMATES  = 3.14196211
PI ESTIMATES  = 3.14187335
PI ESTIMATES  = 3.14175354
PI ESTIMATES  = 3.14169133
PI ESTIMATES  = 3.14164797
PI ESTIMATES  = 3.14188279
PI ESTIMATES  = 3.14193650
PI ESTIMATES  = 3.14184027
PI ESTIMATES  = 3.14160532
PI ESTIMATES  = 3.14175713
PI ESTIMATES  = 3.14170398
PI ESTIMATES  = 3.14199919
PI ESTIMATES  = 3.14178391
PI ESTIMATES  = 3.14168723
PI ESTIMATES  = 3.14170276
PI ESTIMATES  = 3.14174882
PI ESTIMATES  = 3.14195207
PI ESTIMATES  = 3.14181964
PI ESTIMATES  = 3.14166509
PI ESTIMATES  = 3.14173233
PI ESTIMATES  = 3.14173152
PI ESTIMATES  = 3.14189208
PI ESTIMATES  = 3.14191245
PI ESTIMATES  = 3.14165117
PI ESTIMATES  = 3.14176565
PI ESTIMATES  = 3.14162599
PI ESTIMATES  = 3.14197940
PI ESTIMATES  = 3.14186951
PI ESTIMATES  = 3.14168361
PI ESTIMATES  = 3.14178640
PI ESTIMATES  = 3.14161009
PI ESTIMATES  = 3.14191393
PI ESTIMATES  = 3.14188161
PI ESTIMATES  = 3.14178497
PI ESTIMATES  = 3.14175932

TRUE PI VALUE = 3.14159265
PI AVERAGE    = 3.14178384
PI STD DEV    = 0.00014979
PI VARIANCE   = 0.00000002
```

References

[1] Koenker R. (2004) Economics 476 Problem Set 2. http://www.econ.uiuc.edu/ roger/courses/476/problems/ps2.pdf