1) Seja $Y \sim \text{Pascal}(r, \pi)$. A função densidade de Y é

$$f(y; r, \pi) = \binom{y-1}{r-1} \pi^r (1-\pi)^{y-r}, \quad y = r, r+1, r+2, \dots, \quad 0 < \pi < 1.$$

a) Seja $Y^* = Y/r \iff Y = rY^*$. Assim, temos que

$$f(y^*; r, \pi) = \binom{ry^*-1}{r-1} \pi^r (1-\pi)^{ry^*-r}$$
$$= \exp\left\{ \log\binom{ry^*-1}{r-1} + r\log(\pi) + (ry^*-r)\log(1-\pi) \right\}$$
$$= \exp\left\{ r\left[y^*\log(1-\pi) + \log\left(\frac{\pi}{1-\pi}\right)\right] + \log\binom{ry^*-1}{r-1} \right\}.$$

Tome $\phi = r$, $\theta = \log(1-\pi) \iff \pi = 1-e^\theta$, $\theta < 0$,

$b(\theta) = -\log\left(\frac{\pi}{1-\pi}\right) = -\log\left(\frac{1-e^\theta}{e^\theta}\right) = \theta - \log(1-e^\theta)$,

e $c(y^*; \phi) = \log\binom{ry^*-1}{r-1} = \log\binom{\phi y^*-1}{\phi-1}$. Logo,

$Y^* = \frac{Y}{r} \in$ FE uniparamétrica canônica.

b) Segue que

$$E(Y^*) = \mu = b'(\theta) = 1 - \frac{1}{1-e^\theta} \cdot -e^\theta = \frac{1}{1-e^\theta}.$$

Daí, como $\pi = 1 - e^{\theta}$,

$$\mu = E(Y^*) = \frac{1}{\pi} \; . \; \Rightarrow \pi = 1/\mu .$$

A função de variância é obtida de

$$V(\mu) = b''(\theta) = \frac{d}{d\theta}\left[(1 - e^{\theta})^{-1} \right] = -(1-e^{\theta})^{-2} \cdot -e^{\theta}$$

$$= \frac{e^{\theta}}{(1-e^{\theta})^2} = \frac{1-\pi}{\pi^2} = \frac{1}{\pi^2} - \frac{1}{\pi} = \left(\frac{1}{\pi}\right)^2 - \frac{1}{\pi}$$

$$= \mu^2 - \mu = \mu(\mu - 1).$$

Finalmente,

$$Var(Y^*) = \phi^{-1} V(\mu) = \frac{\mu(\mu-1)}{r}.$$

c) Fazendo $r = 1$, temos $Y^* = Y \sim \text{Pascal}(1, \pi) = \text{geo}(\pi)$

A densidade fica dada por

$$f(y^*; \pi) = \exp\left\{ y \log(1-\pi) + \log\left(\frac{\pi}{1-\pi}\right) + \overbrace{\log\binom{y-1}{0}}^{0} \right\}$$

$$= \exp\left\{ y \log(1-\pi) + \log\left(\frac{\pi}{1-\pi}\right) \right\}.$$

Tome $\phi = 1$, $\theta = \log(1-\pi) \Leftrightarrow \pi = 1 - e^{\theta}$, $b(\theta) = -\log\left(\frac{\pi}{1-\pi}\right)$ $= \theta - \log(1 - e^{\theta})$, e $c(y;\phi) = 0$. Portanto, $Y \sim FE$ uniparamétrica canônica. Em outras palavras, a distribuição geométrica de parâmetro π $(0 < \pi < 1) \in FE$.

Analogamente, $\mu = E(Y) = b'(\theta) = \frac{1}{\pi}$, $V(\mu) = \mu(\mu - 1)$,

e $Var(Y) = \mu(\mu-1)$.

2) Temos que $Y \sim ES(\mu, \phi)$. A função densidade de Y é

$$f(y;\theta,\phi) = a(y,\phi)\exp\{\phi[\theta(y+1) - \theta\log(\theta)]\}, \quad \theta>0, \; y \in \mathbb{R}, \; \phi>0.$$

a) Veja que

$$f(y;\theta,\phi) = \exp\{\phi[\theta y + \theta(1-\log(\theta))] + \log[a(y,\phi)]\}.$$

Tome $\phi=\phi$, $\theta=\theta$, $b(\theta) = -\theta[1-\log(\theta)]$, e $c(y,\phi) = \log[a(y,\phi)]$. Assim, $Y \in$ FE uniparamé-trica canônica.

b) Segue que

$$\mu = E(Y) = b'(\theta) = -(1-\log(\theta)) - \theta \cdot -\frac{1}{\theta} = \log(\theta), \quad \theta>0.$$

$$\Rightarrow \underline{\theta = e^{\mu}}.$$

A função de variância é obtida de

$$V(\mu) = b''(\theta) = \frac{1}{\theta}, \quad \theta>0. \Rightarrow V(\mu) = e^{-\mu}.$$

Por fim, $Var(Y) = \phi^{-1}V(\mu) = \dfrac{e^{-\mu}}{\phi}$.

3) Suponha que $Y_i \overset{ind}{\sim} NI(\mu_i, \phi)$, $\mu_i = g^{-1}(\eta_i)$, $i = 1, 2, \cdots, n$.

Sabemos que $\hat{\phi}$ é obtido tal que

$$\sum_{i=1}^{n} c'(y_i, \hat{\phi}) = \frac{1}{2} D(\underset{\sim}{y}; \underset{\sim}{\hat{\mu}}) - \sum_{i=1}^{n} \{ y_i \tilde{\theta}_i - b(\tilde{\theta}_i) \}.$$

Como $Y_i \overset{ind}{\sim} NI(\mu_i, \phi)$, então

$$c(y_i;\phi) = \frac{1}{2}\log\left(\frac{\phi}{2\pi y^3}\right) - \frac{\phi}{2y}.$$

$$\Rightarrow c'(y_i;\phi) = \frac{1}{2\phi} - \frac{1}{2y_i}.$$

Também, $\theta_i = -\frac{1}{2\mu_i^2} \Rightarrow \tilde{\theta}_i = -\frac{1}{2y_i^2}$, $b(\theta_i) = -\frac{1}{\mu_i}$

$\Rightarrow b(\tilde{\theta}_i) = -\frac{1}{y_i}$. Daí, $\hat{\phi}$ é tal que

$$\sum_{i=1}^{n}\left\{\frac{1}{2\hat{\phi}} - \frac{1}{2y_i}\right\} = \frac{1}{2}D(\underset{\sim}{y};\underset{\sim}{\hat{\mu}}) - \sum_{i=1}^{n}\left\{y_i \cdot -\frac{1}{2y_i^2} + \frac{1}{y_i}\right\} \iff$$

$$\frac{n}{2\hat{\phi}} - \sum_{i=1}^{n}\frac{1}{2y_i} = \frac{1}{2}D(\underset{\sim}{y};\underset{\sim}{\hat{\mu}}) - \sum_{i=1}^{n}\frac{1}{2y_i} \iff$$

$$\hat{\phi} = \frac{n}{D(\underset{\sim}{y};\underset{\sim}{\hat{\mu}})}, \text{ com } D(\underset{\sim}{y};\underset{\sim}{\hat{\mu}}) = \sum_{i=1}^{n}\frac{(y_i-\hat{\mu}_i)^2}{y_i\hat{\mu}_i^2}.$$

Agora, note que o logaritmo da função de verossimilhança avaliado em $\underset{\sim}{\hat{\theta}} = (\underset{\sim}{\hat{\beta}}^T, \hat{\phi})^T$ é

$$\ell(\underset{\sim}{\hat{\theta}}) = \hat{\phi}\sum_{i=1}^{n}[y_i\hat{\theta}_i - b(\hat{\theta}_i)] + \sum_{i=1}^{n}c(y_i;\hat{\phi})$$

$$= \hat{\phi}\sum_{i=1}^{n}\left[y_i \cdot -\frac{1}{2\hat{\mu}_i^2} + \frac{1}{\hat{\mu}_i}\right] + \sum_{i=1}^{n}\left[\frac{1}{2}\log\left(\frac{\hat{\phi}}{2\pi y_i^3}\right) - \frac{\hat{\phi}}{2y_i}\right]$$

$$= -\hat{\phi}\sum_{i=1}^{n}\frac{(y_i-\hat{\mu}_i)^2}{2\hat{\mu}_i^2 y_i} + \frac{n}{2}\log(\hat{\phi}) - \frac{1}{2}\sum_{i=1}^{n}\log(2\pi y_i^3)$$

$$= -\frac{n}{D(\underset{\sim}{y};\underset{\sim}{\hat{\mu}})}\frac{1}{2}D(\underset{\sim}{y};\underset{\sim}{\hat{\mu}}) + \frac{n}{2}\log\left(\frac{n}{D(\underset{\sim}{y};\underset{\sim}{\hat{\mu}})}\right) - \frac{1}{2}\sum_{i=1}^{n}\log(2\pi y_i^3)$$

$$= -\frac{n}{2} - \frac{n}{2}\log\left(\frac{D(\underset{\sim}{y};\hat{\underset{\sim}{\mu}})}{n}\right) + C,$$

em que $C = -\frac{1}{2}\sum_{i=1}^{n}\log(2\pi y_i^3)$ não depende de $\hat{\underset{\sim}{\theta}}$.

Daí, obtemos que

$$AIC = -2\ell(\hat{\underset{\sim}{\theta}}) + 2p$$
$$= n + n\log\left(\frac{D(\underset{\sim}{y},\hat{\underset{\sim}{\mu}})}{n}\right) - 2C + 2p.$$

Logo, o método de Akaike é equivalente a relacionar o modelo (NI) com menor valor para

$$n\log\left(\frac{D(\underset{\sim}{y};\hat{\underset{\sim}{\mu}})}{n}\right) + 2p.$$

4) Suponha que $Y_i \overset{ind}{\sim} Pascal(r,\pi_i)$, $i = 1, 2, \dots, n$, com componente sistemático dado por

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \alpha.$$

a) Primeiro, note que

$$1-\pi_i = \frac{1}{1+e^{\alpha}}$$

$$e^{\alpha} = \frac{\pi_i}{1-\pi_i} \iff \pi_i = \frac{e^{\alpha}}{1+e^{\alpha}}, \quad i = 1, 2, \dots, n.$$

o logaritmo da função de verossimilhança é

$$
\begin{aligned}
\ell(\alpha) &= \sum_{i=1}^{n} \log\left[f(y_i; r, \pi_i)\right] \\
&= \sum_{i=1}^{n} \log\left[\binom{y_i-1}{r-1} \pi_i^{r} (1-\pi_i)^{y_i - r}\right] \\
&= \sum_{i=1}^{n} \log\binom{y_i-1}{r-1} + r\sum_{i=1}^{n} \log\left(\frac{\pi_i}{1-\pi_i}\right) + \sum_{i=1}^{n} \log(1-\pi_i)\, y_i \\
&= \sum_{i=1}^{n} \log\binom{y_i-1}{r-1} + r\sum_{i=1}^{n} \alpha + \sum_{i=1}^{n} y_i \log\left(\frac{1}{1+e^{\alpha}}\right) \\
&= \sum_{i=1}^{n} \log\binom{y_i-1}{r-1} + nr\alpha - \sum_{i=1}^{n} y_i \log(1+e^{\alpha}) \\
&= \sum_{i=1}^{n} \log\binom{y_i-1}{r-1} + nr\alpha - n\bar{y} \log(1+e^{\alpha}).
\end{aligned}
$$

A função escore p/ α é

$$U_\alpha = \frac{d\ell(\alpha)}{d\alpha} = nr - \frac{n\bar{y}\, e^{\alpha}}{1+e^{\alpha}}.$$

Então, o EMV de α, $\hat{\alpha}$, é obtido de $U_{\hat{\alpha}} = 0$.

$$U_{\hat{\alpha}} = 0 \iff r = \frac{\bar{y}\, e^{\hat{\alpha}}}{1+e^{\hat{\alpha}}} \iff r + r e^{\hat{\alpha}} = \bar{y} e^{\hat{\alpha}}$$

$$\iff e^{\hat{\alpha}}(r - \bar{y}) = -r \implies \hat{\alpha} = \log\left(\frac{r}{\bar{y}-r}\right),$$

desde que $\bar{y} > r$.

Para obter a variância assintótica $var(\hat{\alpha})$, calcu-laremos a 2ª derivada de $\ell(\alpha)$:

$$U_\alpha' = \frac{d}{d\alpha} U_\alpha = -n\bar{y}\, \frac{e^\alpha(1+e^\alpha) - e^\alpha e^\alpha}{(1+e^\alpha)^2}$$ (1)

$$= -\frac{n\bar{y}\, e^\alpha}{(1+e^\alpha)^2}.$$

Como as condições de regularidade são satisfeitas p/ a FE, temos que a informação de Fisher p/ α é

$$K_{\alpha\alpha} = E\{-U_\alpha'\} = E\left\{\frac{n\bar{Y}\, e^\alpha}{(1+e^\alpha)^2}\right\} = \frac{n\, e^\alpha}{(1+e^\alpha)^2} E(\bar{Y}).$$

Já vimos que $E(Y_i^*) = E\left(\frac{Y_i}{r}\right) = \frac{1}{\pi_i}$. Então, $E(Y_i) = r E(Y_i^*) = r/\pi_i$. Assim,

$$E(Y_i) = \frac{r}{\frac{e^\alpha}{1+e^\alpha}} = \frac{r(1+e^\alpha)}{e^\alpha} \Rightarrow$$

$$E(\bar{Y}) = E\left(\frac{1}{n}\sum_{i=1}^n Y_i\right) = \frac{1}{n}\sum_{i=1}^n E(Y_i) = \frac{1}{n}\sum_{i=1}^n \frac{r(1+e^\alpha)}{e^\alpha}$$

$$= \frac{r(1+e^\alpha)}{e^\alpha}.$$

Portanto,

$$K_{\alpha\alpha} = \frac{n\, e^\alpha}{(1+e^\alpha)^2} \frac{r(1+e^\alpha)}{e^\alpha} = \frac{nr}{1+e^\alpha}.$$

Implicando que $Var(\hat{\alpha}) = \frac{1+e^\alpha}{nr}$.

2) Suponha que $Y_i \overset{ind}{\sim} \text{Poisson}(\mu_i)$, em que $\sqrt{\mu_i} = \eta_i$ com $\eta_i = \alpha + \beta(x_i - \bar{x})$, $i = 1, 2, \dots, n$.
Queremos testar $H_0: \beta = 0$ contra $H_1: \beta \neq 0$.

a) Queremos escrever a matriz X do modelo tal que $X\underset{\sim}{\beta} = \underset{\sim}{\eta}$, com $\underset{\sim\, n\times 1}{\eta} = (\eta_1 \; \eta_2 \cdots \eta_n)^T$, $\underset{\sim\, 2\times 1}{\beta} = (\alpha \;\; \beta)^T$ e

$$\underset{n\times 2}{X} = \begin{bmatrix} 1 & x_1 - \bar{x} \\ 1 & x_2 - \bar{x} \\ \vdots & \vdots \\ 1 & x_n - \bar{x} \end{bmatrix} \longrightarrow \text{matriz do modelo}.$$

b) Queremos obter a matriz de covariâncias assintó- (4)
-ticas $Var(\hat{\beta})$.

Para os MLGs vimos que a matriz de Informação p/ β é

$$K_{\beta\beta} = \phi X^{+} W X \Rightarrow Var(\hat{\beta}) = \phi^{-1}(X^{+}WX)^{-1}.$$

Como $Y_i \overset{ind}{\sim} Poisson(\mu_i)$, temos que $\phi = 1$. Ainda, como $\sqrt{\mu_i} = \eta_i$ (função de ligação raiz quadrada), sabemos que $w_i = 4, \forall i = 1, 2, \dots, n$. Assim, obtemos que

$$W = diag\{4, 4, \dots, 4\} = 4 I_n \Rightarrow Var(\hat{\beta}) = \frac{1}{4}(X^{+}X)^{-1}.$$

Ainda

$$X^{+}X = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1-\bar{x} & x_2-\bar{x} & \dots & x_n-\bar{x} \end{bmatrix} \begin{bmatrix} 1 & x_1-\bar{x} \\ 1 & x_2-\bar{x} \\ \vdots & \vdots \\ 1 & x_n-\bar{x} \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum_{i=1}^{n}(x_i-\bar{x}) \\ \sum_{i=1}^{n}(x_i-\bar{x}) & \sum_{i=1}^{n}(x_i-\bar{x})^2 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \sum_{i=1}^{n}(x_i-\bar{x})^2 \end{bmatrix},$$

pois $\sum_{i=1}^{n}(x_i-\bar{x}) = 0$. A inversa de $X^{+}X$ é

$$(X^{+}X)^{-1} = \begin{bmatrix} 1/n & 0 \\ 0 & 1/\sum_{i=1}^{n}(x_i-\bar{x})^2 \end{bmatrix}.$$

Portanto, a matriz de covariâncias assintótica ⑤ fica dada por

$$Var(\hat{\beta}) = \begin{bmatrix} 1/4n & 0 \\ 0 & 1/4\sum_{i=1}^{n}(x_i-\bar{x})^2 \end{bmatrix} . \Rightarrow$$

$$Var(\hat{\alpha}) = \frac{1}{4n}$$

$$Var(\hat{\beta}) = \frac{1}{4\sum_{i=1}^{n}(x_i-\bar{x})^2}.$$