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Explicit meshfree $u - p_w$ solution of the dynamic Biot formulation at large strain

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Abstract In this paper an efficient and robust methodology to simulate saturated soils subjected to low frequency dynamic loadings under large deformation regime is presented. The coupling between solid and fluid phases is solved through the dynamic reduced formulation of the Biot's equations. The additional novelty lies in the employment of an explicit time integration scheme of the $u - p_w$ (solid displacement – pore water pressure) formulation which enables us to take advantage of such explicit schemes. Shape functions based on the elegant Local Maximum Entropy approach, through the framework of Optimal Transportation Meshfree schem, are utilized to solve fluid saturated dynamic problems.

Keywords Biot's equations · Meshfree · Newmark Predictor-Corrector · Explicit approach · Large strains

1 Introduction

Modeling saturated soils under dynamic loads is an interesting issue, particularly when dynamic consolidation or quick settlements of soils under large deformations are concerned. However, the research focused on this aspect is scant, the literature being even more limited when finite deformations are involved. This is mainly due to the fact that, on the one hand, the $u - p_w$ (solid displacement – fluid pressure) formulation is widely used in dynamics to solve the coupled problem due to its simplicity (e.g. [7,26,27]), and on the other hand, since fluid accelerations are neglected in this formulation, this makes it impossible to capture high frequency movements when the coupling between soil and water needs to be dealt with [28].

Along the years, depending on the employed formulation for coupled problems (either simplified or complete), on the assumptions (if the accelerations are considered or not) and on the way that the equations are solved (explicit or implicit), different techniques to solve the coupled problem have been developed. The governing equations of the coupled problem were first introduced by Biot [**?**], then reviewed by Zienkiewicz and co-workers [27–29,26]. There were two alternatives to achieve the same set of equations: one by Zienkiewicz, Chang and Bettes [28], or Zienkiewicz and Shiomi [29, 26] applied at macroscopic scale, the other by Lewis and Schrefler [13] within the Hybrid Mixture Theory starting from the microscopic scale. Both showed that an accurate enough solution can be achieved for low frequency dynamic problems by neglecting the convective and acceleration terms in the complete formulation, deriving the $u - p_w$ formulation.

Regarding the application of $u - p_w$ formulation under large deformation regime, the first works were carried out by Diebels and Ehlers [10], Borja *et al.* [4,5] and Armero [1] who tested their models by simulating the constitutive behavior of the the solid phases with linear elastic, Cam-Clay and Drucker-Prager theories respectively. Around the same period of time, Ehlers and Eipper [11] applied a new Neo-Hookean constitutive model to represent the compaction of the soil up to the solid compaction point. All of these researches were solved using implicit schemes where the linearization of

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the derivatives of the $u - p_w$ equations was necessary. This linearization was also made by Sanavia *et al.*[21] who considered several neglected terms of the previous works and extended the methodology to unsaturated soils [23].

By contrast, the complete formulation valid for all frequencies movements is known to be essential for solving dynamic problems [12,19]. Nevertheless, the formulation employing the total displacement of the water, U, as a nodal unknown is unstable when large deformations of the fluid phase occur. As an alternative, the employment of the relative water displacements, w, has been proved to be successful [15,17]. Traditional manner to solve the complete formulation is the utilization of implicit schemes [4,5,1,11] except the recent work of Ye *et al.* [25]. Thus, the current work represents the first one that solves the complete formulation with relative water displacements, $u - w$, using an explicit scheme. Since there is no necessity in formulating the tangent stiffness matrix in an explicit procedure, the complex process of linearization of the governing equations is avoided. In addition, as no matrix inversion is involved, the computational effort is minimized and code parallelization is facilitated.

Moreover, it bears emphasis that the proposed methodology, as it is thought for the finite strain regime, is carried out within a meshfree scheme due to its numerous advantages when large deformations are involved. In particular, the shape functions developed by Arroyo and Ortiz [2] based on the principle of maximum entropy [18] are employed. The spacial domain has been discretized into nodes and material points following the Optimal Transportation Meshfree (OTM) scheme of Li *et al.* [14]. The Drucker-Prager yield criterion, the good performance of which has been demonstrated for large deformation problems [**?**], is herein adopted.

In contrast to the work of Bandara and Soga [**?**] or Ceccato and Simonini [8], who made use of two material sets for solid and water phases in their Material Point Method (MPM) schemes, a single set of materials for the coupling between water and solid phases is employed in this work since the relative water displacement is considered. This leads to significant savings on the computational effort. In addition, this formulation is stronger than some others such as the Smooth Particle Hydrodynamics (SPH) since the pore pressure is also computed in the material points. The SPH formulation presents a tensile instability since only one nodal set is employed to contain displacement and stress fields.

The rest of the paper is organized as follows. The Biot's equations are presented next. The constitutive models employed to model the solid behavior are summarized in Section 3. The explicit methodology implemented is elucidated in Section 4. Applications to various problems are illustrated in Section **??**. Relevant conclusions are drawn in Section **??**. The definitions of all symbols used in the equations are provided in the nomenclature appendix.

2 Biot's equations: u-p_w formulation

The Biot's equations [3] are based on formulating the mechanical behavior of a solid-fluid mixture, the coupling between different phases, and the continuity of flux through a differential domain of saturated porous media. Hereinafter, the balance equations will be derived from Lewis and Schrefler [13] in the spatial setting (see [13] or [22,23] for the kinematic equations), departing from the more general equation, and, in order to reach the compact $u - p_w$ form, making the necessary hypotheses.

Concerning the notation, bold symbols are employed herein for vectors and matrices as well as regular letters for scalar variables. Let $\boldsymbol{u}$ and $\boldsymbol{U}$ represent the displacement vector of the solid skeleton and the absolute displacement of the fluid phase respectively. Since in porous media theory is common to describe the fluid motion with respect to the solid, the relative displacement of the fluid phase with respect to the solid one, $\boldsymbol{w}$, is introduced and expressed as [16]

$$\boldsymbol{w} = nS_w\,(\boldsymbol{U} - \boldsymbol{u}), \tag{1}$$

where S_w is the degree of water saturation and n the soil porosity. Note that $(\boldsymbol{U} - \boldsymbol{u})$ is usually termed as $\boldsymbol{u}^{ws}$ in the literature [13].

In the calculation of the internal forces of the soil, the Terzaghi's effective stress theory [24] will be followed, which is defined as follows:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma'} - p_w\mathbf{I}, \tag{2}$$

where $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma}$ are the respective effective and total Cauchy stress tensors (positive in tension), whereas $\mathbf{I}$ is the second order unit tensor. Contrary, pore pressure p_w, is assumed positive for compression.

Let ρ, ρ_w and ρ_s respectively represent the mixture, fluid phase and solid particle densities, the mixture density can be defined as function of the porosity:

$$\rho = nS_w\rho_w + (1 - n)\rho_s. \tag{3}$$

In the above equations, the porosity, n, is the ratio between the voids volume, V_v, and the total volume, V_T:

$$n = \frac{V_v}{V_T} = \frac{V_v}{V_v + V_s}, \tag{4}$$

where V_s is the volume of the solid grains.

In the current work, the soil is assumed to be totally saturated, i.e. V_v coincides with the water volume, which results S_w equals to one. Meanwhile, the volumetric compressibility of the mixture, Q [27] is calculated as

$$Q = \left[\frac{1-n}{K_s} + \frac{n}{K_w}\right]^{-1}, \tag{5}$$

where K_s is the bulk modulus of the solid grains, whereas K_w is the compressive modulus of the fluid phase (usually water).

Next, we first explain in detail the derivation of mass balance and linear momentum equations for a fluid saturated multiphase media. Then the final $u-p_w$ formulation is presented. The following equations are first given by Lewis and Schrefler [13]. In this research, D^s/Dt denotes the material time derivative with respect to the solid, considering:

$$\boldsymbol{a}^s = \ddot{\boldsymbol{u}} = \frac{D^s\dot{\boldsymbol{u}}}{Dt} = \frac{D^{2s}\boldsymbol{u}}{Dt^2}$$

$$nS_w\boldsymbol{a}^{ws} = \ddot{\boldsymbol{w}} = \frac{D^s\dot{\boldsymbol{w}}}{Dt} = \frac{D^{2s}\boldsymbol{w}}{Dt^2}$$

where $\boldsymbol{a}^s$ and $\boldsymbol{a}^{ws}$ are the solid acceleration and the relative water acceleration with respect to the solid respectively, being the proposed expressions based on the relationships $\dot{\boldsymbol{u}} \equiv \boldsymbol{v}^s$ and $\dot{\boldsymbol{w}} \approx nS_w\boldsymbol{v}^{ws}$.

2.1 Derivation of the mass balance equation

The general mass balance equation in a multiphase media for compressible grains is presented next. Let p_w, p_g represent the water and gas pressures respectively, T, the temperature, then this general mass balance equation is written as follows,

$$\left(\frac{\alpha-n}{K_s}S_w^2 + \frac{nS_w}{K_w}\right)\frac{D^sp_w}{Dt} + \frac{\alpha-n}{K_s}S_wS_g\frac{D^sp_g}{Dt} - \beta_{sw}\frac{D^sT}{Dt} + \left(\frac{\alpha-n}{K_s}S_wp_w - \frac{\alpha-n}{K_s}S_wp_g + n\right)\frac{D^sS_w}{Dt} + \alpha S_w\text{div }\dot{\boldsymbol{u}} + \frac{1}{\rho_w}\text{div }(\rho_w\dot{\boldsymbol{w}}) = -ne^w \tag{6}$$

where the right hand side term represents the quantity of water lost through evaporation for unit time and volume. The thermal expansion coefficient of the solid-fluid mixture, β_{sw}, is a combination of that of the solid, β_s, and the fluid, β_w:

$$\beta_{sw} = S_w[(\alpha-n)\beta_s + n\beta_w]. \tag{7}$$

In addition, α is the Biot's coefficient:

$$\alpha = 1 - \frac{K_T}{K_s}. \tag{8}$$

where K_T denotes the bulk modulus of the solid skeleton. Biot's coefficient may be usually assumed equal to one in soils as the grains are much more rigid than the mixture.

As we consider a totally saturated, iso-thermal multiphase media, $D^sT/Dt = 0, S_g = 0, S_w = 1, k^{rw} = 1$, $e^w = 0$, consequently, $D^sS_w/Dt = 0$. If additionally the fluid density variation is neglected and we take into consideration Eq. (5), Eq. (6) is simplified as,

$$\frac{\dot{p_w}}{Q} + \text{div }\dot{\boldsymbol{u}} + \text{div }\dot{\boldsymbol{w}} = 0, \tag{9}$$

2.2 Linear momentum balance equations

On the one hand, the relative velocity of the fluid, $\dot{\boldsymbol{w}}$, in Eq. (6) is defined through the generalized Darcy law as [13]

$$\dot{\boldsymbol{w}} = \frac{k^{rw}\boldsymbol{k}}{\mu_w}\left[-\text{grad}\,p_w + \rho_w(\boldsymbol{g} - \ddot{\boldsymbol{u}} - \frac{\ddot{\boldsymbol{u}}}{n})\right], \tag{10}$$

where $\boldsymbol{g}$ represents the gravity acceleration vector, $\boldsymbol{k}$, the intrinsic permeability tensor of the porous matrix in water saturated condition, k^{rw} is the water relative permeability parameter (a dimensionless parameter varying from zero to one) and μ_w is the dynamic viscosity of the water [Pa · s]. For the case of isotropic permeability, the *intrinsic* permeability, expressed in [m^2], is related with the notion of hydraulic conductivity, κ [m/s], by the following equation

$$\frac{k}{\mu_w} = \frac{\kappa}{\rho_w g}. \tag{11}$$

On the other hand, according to Lewis and Schrefler [13], the linear momentum balance equation for the multiphase system can also be expressed as the summation of the dynamic equations for the individual constituents relative to the solid as, i.e.,

$$-\rho\ddot{\boldsymbol{u}} - \rho_w\ddot{\boldsymbol{w}} - nS_g\rho_g\boldsymbol{a}^{gs} + \text{div }\boldsymbol{\sigma} + \rho\boldsymbol{g} = \boldsymbol{0}, \tag{12}$$

where the convective terms, related to the acceleration terms, have been neglected, which is normal in soils. Since in the present research there is no gassy phase, as the soil will be considered as totally saturated ($S_g = 0$), and plugging Eq. (2) into Eq. (12), the linear momentum equation can be written as follows

$$\text{div }[\boldsymbol{\sigma'} - p_w\,\mathbf{I}] - \rho\ddot{\boldsymbol{u}} - \rho_w\ddot{\boldsymbol{w}} + \rho\boldsymbol{g} = \boldsymbol{0}. \tag{13}$$

2.3 The $u - p_w$ formulation

Considering the three Biot's equations, the $\boldsymbol{u} - p_w$ assumes that accelerations of the fluid phase are negligible. Thus, Eq. (13) yields:

$$\text{div}\ [\boldsymbol{\sigma'} - p_w\,\mathbf{I}] - \rho\ddot{\boldsymbol{u}} + \rho\boldsymbol{g} = \mathbf{0}. \tag{14}$$

Moreover, in order to avoid the employment of $\boldsymbol{w}$ as a degree of freedom of our problem, Eqs. (9) and (10) can be combined and the mass equation can be expressed as

$$\dot{p_w} = -Q\left[\text{div}\ \dot{\boldsymbol{u}} + \frac{\boldsymbol{k}}{\mu_w}\text{div}\left(\rho_w\boldsymbol{g} - \rho_w\ddot{\boldsymbol{u}} - \text{grad}\ p_w\right)\right]. \tag{15}$$

3 Constitutive models for the solid phase

In this Section, we describe the two types of material models implemented to assess the performance of the formulation presented in Section 2. One is for elastic behavior, the other one involves plastic deformation which follows the Drucker-Prager failure criterion.

3.1 Neo-Hookean material model extended to compressible range

One of the widely used material model for predicting non-linear elastic behavior for solids undergoing large deformations is the Neo-Hookean model extended to compressible range. Under spacial configurations, it is expressed as follows [**?**]:

$$\boldsymbol{\tau}' = J\boldsymbol{\sigma}' = G(\boldsymbol{b} - \mathbf{I}) + (\lambda\,\ln J)\mathbf{I}, \tag{16}$$

where $\boldsymbol{\tau}'$ and $\boldsymbol{b}$ are the effective Kirchhoff stress tensor and the left Cauchy-Green tensor respectively, whereas J is the Jacobian determinant, G and λ are the Lamé constants.

In order to take into consideration the compaction point of the soil, Ehlers and Eipper [11] presented a modification of the Neo-Hookean law taking into account the influence of the initial porosity n_0 and the Jacobian, i.e.

$$\boldsymbol{\tau}' = G(\boldsymbol{b} - \mathbf{I}) + \lambda\,n_0^2\left(\frac{J}{n_0} - \frac{J}{J - 1 + n_0}\right)\mathbf{I}, \tag{17}$$

which is going to be used for the validation examples in Section **??**.

Table 1 Equivalent plastic strain

Classical	$\bar{\varepsilon}^p_{k+1} = \bar{\varepsilon}^p_k + \Delta\gamma\sqrt{3\alpha_Q^2 + 1}$
Apex	$\bar{\varepsilon}^p_{k+1} = \bar{\varepsilon}^p_k + \sqrt{\Delta\gamma_1^2 + 3\alpha_Q^2\left(\Delta\gamma_1 + \Delta\gamma_2\right)^2}$

3.2 Drucker-Prager yield criterion

For the calculation of plastic deformations, we follow the work of Cuitiño and Ortiz [9] to relate the right Cauchy-Green strain tensor $\mathbf{C}$ and the small strain tensor $\boldsymbol{\varepsilon}$, during the trial step. In other words, for the current loading step, $k+1$, the trial elastic deformations, pressure (p^{trial}_{k+1}) and the deviatoric stress tensor ($\mathbf{s}^{trial}_{k+1}$) are computed as the elastic deformations, pressure and the deviatoric stress tensor are computed as:

$$\mathbf{C}^{e\ trial}_{k+1} = (\mathbf{F}^p_k)^{-T}\mathbf{C}_{k+1}(\mathbf{F}^p_k)^{-1}, \tag{18}$$

$$\boldsymbol{\varepsilon}^{e\ trial}_{k+1} = \frac{1}{2}\log\mathbf{C}^{e\ trial}_{k+1}, \tag{19}$$

$$p^{trial}_{k+1} = K\left(\varepsilon^e_{vol}\right)^{trial}_{k+1}, \tag{20}$$

$$\mathbf{s}^{trial}_{k+1} = 2G\left(\boldsymbol{\varepsilon}^e_{dev}\right)^{trial}_{k+1}. \tag{21}$$

where K and G represent the bulk and shear moduli of the solid respectively. Once the incremental plastic strain tensor is known, the plastic deformation gradient can be derived as:

$$\Delta\mathbf{F}^p_{k+1} = \exp(\Delta\boldsymbol{\varepsilon}^p_{k+1}), \tag{22}$$

$$\mathbf{F}^p_{k+1} = \Delta\mathbf{F}^p_{k+1}\mathbf{F}^p_k. \tag{23}$$

Regarding the Drucker-Prager yield criterion, the methodology of Sanavia *et al.* [23,20] is employed for its reduced computational effort and its capacity to distinguish if the location of the stress state is on the cone or apex before calculating the plastic strain. The current cohesion, c_{k+1}, and its derivative, the hardening modulus, H, are calculated following Camacho and Ortiz research [6] from the reference value, c_0, the reference plastic strain, ε_0, and the hardening exponent, N^ε, as follows:

$$c_{k+1} = c_0\left(1 + \frac{\bar{\varepsilon}^p_{k+1}}{\varepsilon_0}\right)^{\frac{1}{N^\varepsilon}},$$

$$\frac{\partial c}{\partial\bar{\varepsilon}^p} = H = \frac{c_0}{N^\varepsilon\varepsilon_0}\left(1 + \frac{\bar{\varepsilon}^p_{k+1}}{\varepsilon_0}\right)^{\left(\frac{1}{N^\varepsilon} - 1\right)}, \tag{24}$$

where $\bar{\varepsilon}^p_{k+1}$ is the current equivalent plastic strain, calculated in different ways depending on the fact that if the stress state is in the classical or apex region, see Table 1.

Table 2 Parameters for Drucker-Prager and von-Mises yield criteria

	DP: Plane strain	DP: Outer cone	von-Mises
α_F	$\frac{\tan\phi}{\sqrt{3+4\tan^2\phi}}\sqrt{\frac{2}{3}}$	$\frac{2\sin\phi}{3-\sin\phi}\sqrt{\frac{2}{3}}$	0
α_Q	$\frac{\tan\psi}{\sqrt{3+4\tan^2\psi}}\sqrt{\frac{2}{3}}$	$\frac{2\sin\psi}{3-\sin\psi}\sqrt{\frac{2}{3}}$	0
β	$\frac{3}{\sqrt{3+4\tan^2\phi}}\sqrt{\frac{2}{3}}$	$\frac{6\cos\phi}{3-\sin\phi}\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$

In order to know which algorithm to employ, a limit value for the pressure, p_{lim} is calculated:

$$p_{lim} = \frac{3\alpha_Q K}{2G}\|\mathbf{s}_{k+1}^{trial}\| + \frac{\beta}{3\alpha_F}\left(\frac{\|\mathbf{s}_{k+1}^{trial}\|}{2G}H\sqrt{1+3\alpha_Q^2}+c_k\right). \quad (25)$$

If the trial pressure is lower than this limit, classical return-mapping algorithm is employed, otherwise the apex algorithm is adopted.

The yield conditions for the classical and apex regions respectively are:

$$\Phi^{cl} = \|\mathbf{s}_{k+1}^{trial}\| - 2G\Delta\gamma + 3\alpha_F[p_{k+1}^{trial} - 3K\alpha_Q\Delta\gamma] - \beta c_{k+1}, \quad (26)$$

$$\Phi^{ap} = \frac{\beta}{3\alpha_F}\left[c_k + H\sqrt{\Delta\gamma_1^2 + 3\alpha_Q^2(\Delta\gamma_1+\Delta\gamma_2)^2}\right] - p_{k+1}^{trial} + 3K\alpha_Q(\Delta\gamma_1+\Delta\gamma_2), \quad (27)$$

where $\Delta\gamma_1 = \frac{\|\mathbf{s}_{k+1}^{trial}\|}{2G}$, $\Delta\gamma$ and $\Delta\gamma_2$ are the objective functions to be calculated in the Newton-Raphson scheme for the classical or apex regions accordingly.

For the calculation of the Drucker-Prager parameters from the friction angle, ϕ, and the dilatancy angle, ψ, the plane strain case is presented in Table 2. Additionally the parameters for the out cone are shown in Table 2. This cone circumscribes the Mohr-Coulomb plastic region, and the corresponding values for a von Mises criterion.

4 Discretization of the solution: Explicit scheme

To solve the aforementioned coupled problem in the time domain, the standard central difference explicit Newmark time integration scheme is employed. If the current time step is numbered as $k+1$, and assuming the solution in the previous step k has been already obtained (hence it is known), a relationship between $\mathbf{u}_{k+1}$, $\dot{\mathbf{u}}_{k+1}$ and $\ddot{\mathbf{u}}_{k+1}$ is established according to a finite difference scheme, as follows:

$$\ddot{\boldsymbol{u}}_{k+1} = \ddot{\boldsymbol{u}}_k + \Delta\ddot{\boldsymbol{u}}_{k+1},$$
$$\dot{\boldsymbol{u}}_{k+1} = \dot{\boldsymbol{u}}_k + \ddot{\boldsymbol{u}}_k\Delta t + \gamma\Delta t\Delta\ddot{\boldsymbol{u}}_{k+1},$$
$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + \dot{\boldsymbol{u}}_k\Delta t + \frac{1}{2}\Delta t^2\ddot{\boldsymbol{u}}_k + \beta\Delta t^2\Delta\ddot{\boldsymbol{u}}_{k+1}. \quad (28)$$

Similarly, the pore pressure, evaluated at material point level, can be expressed in terms of its derivative.

$$p_{w_{k+1}} = p_{w_k} + \dot{p}_{w_k}\Delta t + \theta\Delta t\Delta\dot{p}_{wk+1}. \quad (29)$$

When the Newmark scheme parameters, γ and β are set to 0.5 and 0 respectively, the central difference scheme is obtained. In the present research, $\theta = \gamma = 0.5$. Rearranging terms, *Predictor* and *Corrector* terms can be obtained:

$$\dot{u}_{k+1} = \underline{\dot{u}_k + (1-\gamma)\Delta t\,\ddot{u}_k} + \gamma\Delta t\,\ddot{u}_{k+1}, \quad (30)$$
$$p_{w_{k+1}} = \underline{p_{w_k} + (1-\gamma)\Delta t\,\dot{p}_{wk}} + \gamma\Delta t\,\dot{p}_{wk+1}; \quad (31)$$

being the underlined terms the ones of the predictor step, which will be called $\dot{u}_{k+*}$ and $p_{w_{k+*}}$.

About the numerical stability of the proposed methodology, it is guaranteed when the Courant-Friedrichs-Lewy (CFL) condition is satisfied. In particular, the time step, Δt, should be small enough to ensure that the compressive wave can travel between nodes, i.e.

$$\Delta t < \frac{h}{V_c}, \quad (32)$$

where h represents the discretization size and V_c is the p-wave velocity (see [28]), which is defined by

$$V_c = \sqrt{\left(D + \frac{K_f}{n}\right)\frac{1}{\rho}}, \quad \text{where} \quad D = \frac{2G(1-\nu)}{1-2\nu}. \quad (33)$$

4.1 Spatial discretization

The Optimal Transportation Meshfree (CITAS) has been demonstrated to perform reasonably well in geotechnical problems and, specifically, in multiphase problems. CITASSSS It is based in the conjunction of material points and nodes. As mentioned before, the shape functions are based on the work of Arroyo and Ortiz [2], who defined the Local Max-Ent shape function (LME) of the material point (x) with respect to the neighborhood (x_a) as follows:

$$N_a(\mathbf{x}) = \frac{\exp\left[-\beta\,|\mathbf{x}-\mathbf{x_a}|^2 + \boldsymbol{\lambda}^*\cdot(\mathbf{x}-\mathbf{x_a})\right]}{Z(\mathbf{x},\boldsymbol{\lambda}^*(\mathbf{x}))}, \quad (34)$$

where the computation is done along a neighborhood N_b and

$$Z(\mathbf{x},\boldsymbol{\lambda}) = \sum_{a=1}^{Nb}\exp\left[-\beta\,|\mathbf{x}-\mathbf{x_a}|^2 + \boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{x_a})\right]. \quad (35)$$

The first derivatives of the shape function can be obtained from the own shape function and its Hessian matrix $\mathbf{J}$ by employing the following expression:

$$\nabla N_a^* = -N_a^* \left(\mathbf{J}^*\right)^{-1} (\mathbf{x} - \mathbf{x_a}), \tag{36}$$

The parameter β defines the shape of the neighborhood and it is related with the discretization size (or nodal spacing), h, and the constant, γ, which controls the locality of the shape functions, as follows,

$$\beta = \frac{\gamma}{h^2}. \tag{37}$$

It bears emphasis that $\boldsymbol{\lambda}^*(\mathbf{x})$ comes from the minimization of the function $g(\boldsymbol{\lambda}) = \log Z(\mathbf{x}, \boldsymbol{\lambda})$ to guarantee the maximum entropy. Moreover, in the remapping of the shape function, before recomputing the aforementioned minimization process, it is necessary to update the neighborhood and the parameter $\beta_{k+1}^p < \beta_k^p$ in order to improve the stability.

By employing the outlined shape functions and applying Galerkin procedure to the weak form of Eqs. (12) and (13) (See [23,20] for details), the following matrix equations appear:

$$\boldsymbol{R}^s - \boldsymbol{R}^w - \boldsymbol{M}^s \ddot{\boldsymbol{u}} + \boldsymbol{F}^{ext,\,s} = \mathbf{0} \tag{38}$$

$$-\boldsymbol{R}^w - \boldsymbol{C}^w \dot{\boldsymbol{w}} - \boldsymbol{M}^w \ddot{\boldsymbol{u}} + \boldsymbol{F}^{ext,\,w} = \mathbf{0} \tag{39}$$

$$\dot{p_w} - Q\left[\mathrm{tr}(\dot{\boldsymbol{\varepsilon}}) + \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^w)\right] = 0 \tag{40}$$

where the internal and external forces are defined as:

$$\boldsymbol{R}^s = \sum_{P=1}^{N_P} V_P \boldsymbol{\sigma'} \nabla \mathbf{N}$$

$$\boldsymbol{R}^w = \sum_{P=1}^{N_P} V_P p_w \nabla \mathbf{N}$$

$$\boldsymbol{F}^{ext,\,s} = \boldsymbol{M}^s \boldsymbol{g} - + \int_{\partial\Omega_\tau} \boldsymbol{\sigma'} \boldsymbol{n} \mathbf{N} d\Gamma$$

$$\boldsymbol{F}^{ext,\,w} = \boldsymbol{M}^w \boldsymbol{g} - \int_{\partial\Omega_{p_w}} p_w \boldsymbol{n} \mathbf{N} d\Gamma,$$

and the mass and damping matrices, constructed as lumped matrices in order to alleviate the computational effort of the explicit scheme, are written as follows:

$$\boldsymbol{M}^s = \sum_{P=1}^{N_P} V_P \rho \mathbf{N}$$

$$\boldsymbol{M}^w = \sum_{P=1}^{N_P} V_P \rho_w \mathbf{N}$$

$$\boldsymbol{C}^w = \sum_{P=1}^{N_P} V_P \frac{\mu_w}{k} \mathbf{N}$$

being V_p and N_p the volume and the neighborhood of a material point P respectively.

4.2 Explicit integration

The proposed scheme seeks the value of the solid acceleration, $\ddot{\boldsymbol{u}}$, calculated from equation (38). In this calculation, it is necessary to predict the internal forces from the values of the predicted solid displacement, $\boldsymbol{u}_{k+*}$, and the predicted pore pressure, $p_{w_{k+*}}$. The stress has to be calculated in this predicted step as well:

$$\boldsymbol{\sigma'}_{k+*} = \boldsymbol{\sigma'}(\boldsymbol{F}_{k+*}) = \boldsymbol{\sigma'}(\boldsymbol{F}(\boldsymbol{u}_{k+*}))$$

From Eq. (39), once the solid acceleration is reached, the fluid velocity can be calculated and, finally, the pore pressure rate can be updated from Eq. (40), taking into account the definition of the divergence of both fluid and solid velocities:

$$\mathrm{div}\ (\dot{\boldsymbol{u}}) = \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}_{k+1}) = \mathrm{tr}\left(\frac{1}{2}\ \log \dot{\mathbf{b}}_{k+1}\right), \tag{41}$$

$$\mathrm{div}\ (\dot{\boldsymbol{w}}) = \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}_{k+1}^w) = \mathrm{tr}\left(\frac{1}{2}\ \log \dot{\mathbf{b}}_{k+1}^w\right). \tag{42}$$

$\mathbf{b}$ is the Left Cauchy-Green strain tensor, calculated as:

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T$$

The approximation of the logarithmic strain as the measure to be employed in the reference configuration has been demonstrated to provide good performance when large deformations are modeled. (CITAS CUITIÑO; SANAVIA, TAMAGNINI). In the present research, the rate of tensor $\mathbf{b}$, for each phase, can be calculated as:

$$\dot{\mathbf{b}} = \mathbf{b}(\dot{\mathbf{F}}) = \mathbf{b}(\mathbf{F}(\dot{\boldsymbol{u}}_{k+1}))$$

$$\dot{\mathbf{b}}^w = \mathbf{b}(\dot{\mathbf{F}}^w) = \mathbf{b}(\mathbf{F}(\dot{\boldsymbol{w}}_{k+1}))$$

It is worth mentioning that the subscript $k+1$ is employed in this calculation. All these ingredients are those which integrate the Newmark Predictor-Corrector explicit algorithm for the $\boldsymbol{u} - p_w$ formulation at large strain. Its numerical implementation is explained in the following section.

4.2.1 Explicit algorithm within the OTM framework

The pseudo-algorithm of the whole model can be written as follows. The employment of the superscript p for material point calculations has to be pointed out.

1. Explicit Newmark Predictor ($\gamma = 0.5$, $\beta = 0$)

$$u_{k+1} = u_k + \Delta t \dot{u}_k + 0.5 \Delta t^2\, \ddot{u}_k,$$
$$\dot{u}_{k+*} = \dot{u}_k + (1-\gamma)\Delta t\, \ddot{u}_k,$$
$$p_{w_{k+*}}^p = p_{w_k}^p + (1-\gamma)\Delta t\, \dot{p}_{w_k}^p.$$

2. Nodes and Material points position update

$$x_{k+1} = x_k + \Delta u_{k+1},$$

$$x^p_{k+1} = x^p_k + \sum_{a=1}^{Nb} \Delta u^a_{k+1} N^a(x^p_k).$$

3. Deformation gradient calculation and related parameters

$$\begin{aligned} \Delta \mathbf{F}_{k+1} &= I + \sum_{a=1}^{Nb} \Delta u^a_{k+1} \nabla N^a(x^p_k), \\ \mathbf{F}_{k+1} &= \Delta \mathbf{F}_{k+1} \mathbf{F}_k, \\ V &= J V_0 = \det \mathbf{F}\, V_0, \\ n &= 1 - \frac{1 - n_0}{J}. \end{aligned}$$

4. Update density and recompute lumped mass

$$\rho_{k+1} = n_{k+1}\rho_w + (1 - n_{k+1})\rho_s.$$

5. Remapping loop, reconnect the nodes with their new material neighbors.
6. Constitutive relations from the Elasto-Plastic model, $\boldsymbol{\sigma'}_{k+*}$ and internal forces $\boldsymbol{R}^s_{k+*}$ and $\boldsymbol{R}^w_{k+*}$.
7. Calculate $\ddot{\boldsymbol{u}}_{k+1}$ from Eq. (38):

$$\ddot{\boldsymbol{u}}_{k+1} = [\boldsymbol{M}^s]^{-1} \left[\boldsymbol{R}^s_{k+*} - \boldsymbol{R}^w_{k+*} + \boldsymbol{F}^{ext,\,s}_{k+1} \right]$$

8. Calculate $\dot{\boldsymbol{w}}_{k+1}$ from Eq. (39):

$$\dot{\boldsymbol{w}}_{k+1} = [\boldsymbol{C}^w]^{-1} \left[-\boldsymbol{R}^w_{k+*} - \boldsymbol{M}^w \ddot{\boldsymbol{u}}_{k+1} + \boldsymbol{F}^{ext,\,w}_{k+1} \right]$$

9. Explicit Newmark Corrector

$$\dot{u}_{k+1} = \dot{u}_{k+*} + \gamma \Delta t\, \ddot{u}_{k+1},$$

10. Pore pressure update:

$$\dot{p}^p_{w_{k+1}} = -Q \left(\text{div } \dot{\boldsymbol{u}}_{k+1} + \text{div } \dot{\boldsymbol{w}}_{k+1} \right),$$

$$p^p_{w_{k+1}} = p^p_{w_{k+*}} + \gamma \Delta t\, \dot{p}^p_{w_{k+1}}.$$

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Conflict of interest

The authors declare that they have no conflict of interest.

Author contributions

Conceptualization and mathematical methodology, P. Navas and M. M. Stickle. Implementation, M. Molinos and P.Navas. Validation, A. Yagüe and D.Manzanal. Meanwhile supervision, project administration and funding acquisition belongs to M. Pastor. All authors have contributed to the writing and original draft preparation and they read and agreed to the published version of the manuscript.

A Nomenclature

- $\boldsymbol{a}^s \equiv \ddot{\boldsymbol{u}}$: acceleration vector of the solid = material time derivative of $\boldsymbol{v}^s$
- $\boldsymbol{a}^{ws}$: relative water acceleration vector with respect to the solid = material time derivative of $\boldsymbol{v}^{ws}$ with respect to the solid
- $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$: left Cauchy-Green tensor
- $\bar{\boldsymbol{b}}$: body forces vector
- c: cohesion (equivalent to the yield stress, σ_Y)
- $\boldsymbol{C}$ (time integration scheme): damping matrix
- $\frac{D^s \square}{Dt} \equiv \dot{\square}$: material time derivative of $\square$ with respect to the solid
- $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$: deformation gradient
- $\boldsymbol{g}$: gravity acceleration vector
- G: shear modulus
- h: nodal spacing
- H: hardening modulus, derivative of the cohesion against time.
- $\boldsymbol{I}$: second order unit tensor
- $J = \det \boldsymbol{F}$: Jacobian determinant
- k: intrinsic permeability
- $\boldsymbol{k}$: permeability tensor
- K: bulk modulus
- K_s: bulk modulus of the solid grains
- K_w: bulk modulus of the fluid
- $\boldsymbol{M}$: mass matrix
- n: porosity
- $N(\boldsymbol{x})$, $\nabla N(\boldsymbol{x})$: shape function and its derivatives
- p: solid pressure
- p_w: pore pressure
- $\boldsymbol{P}$ (time integration scheme): external forces vector
- Q: volumetric compressibility of the mixture
- $\boldsymbol{R}$: internal forces vector
- $\boldsymbol{s} = \boldsymbol{\sigma}^{dev}$: deviatoric stress tensor
- t: time
- $\boldsymbol{u}$: displacement vector of the solid
- $\boldsymbol{U}$: displacement vector of the water
- $\boldsymbol{v}^s = \dot{\boldsymbol{u}}$: velocity vector of the solid
- $\boldsymbol{v}^{ws}$: relative velocity vector of the water with respect to the solid
- $\boldsymbol{w}$: relative displacement vector of the water with respect to the solid
- $Z(\boldsymbol{x}, \boldsymbol{\lambda})$: denominator of the exponential shape function
- α_F, α_Q and β: Drucker-Prager parameters
- β, γ: time integration schemes parameters
- β, γ: LME parameters related with the shape of the neighborhood
- $\Delta\gamma$: increment of equivalent plastic strain
- $\bar{\varepsilon}^p$: equivalent plastic strain
- $\boldsymbol{\varepsilon}$: small strain tensor
- ε_0: reference plastic strain

- κ: hydraulic conductivity
- λ: Lamé constant
- $\boldsymbol{\lambda}$: minimizer of $\log Z(\boldsymbol{x}, \boldsymbol{\lambda})$
- μ_w: viscosity of the water
- ν: Poisson's ratio
- ρ: current mixture density
- ρ_w: water density
- ρ_s: density of the solid particles
- $\boldsymbol{\sigma}$: Cauchy stress tensor
- $\boldsymbol{\sigma'}$: effective Cauchy stress tensor
- $\boldsymbol{\tau}$: Kirchhoff stress tensor
- $\boldsymbol{\tau'}$: effective Kirchhoff stress tensor
- Φ: plastic yield surface
- ϕ: friction angle
- ψ: dilatancy angle

Superscripts and subscripts

- ^{dev}: superscript for deviatoric part
- ^{e}: superscript for elastic part
- $_{k}$: subscript for the previous step
- $_{k+1}$: subscript for the current step
- ^{p}: superscript for plastic part
- ^{s}: superscript for the solid part
- ^{trial}: superscript for trial state in the plastic calculation
- ^{vol}: superscript for volumetric part
- ^{w}: superscript for the fluid part relative to the solid one

References

1. Armero, F.: Formulation and finite element implementation of a multiplicative model of coupled poro-plasticity at finite strains under fully saturated conditions. Computer Methods in Applied Mechanics and Engineering **171**(3-4), 205–241 (1999). DOI 10.1016/S0045-7825(98)00211-4
2. Arroyo, M., Ortiz, M.: Local maximum-entropy approximation schemes: a seamless bridge between finite elements and meshfree methods. International Journal for Numerical Methods in Engineering **65**(13), 2167–2202 (2006). DOI 10.1002/nme.1534
3. Biot, M.A.: Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-Frequency range. Journal of the Acoustical Society of America **28**(2), 168–178 (1956). DOI 10.1121/1.1908239
4. Borja, R.I., Alarcón, E.: A mathematical framework for finite strain elastoplastic consolidation. Part1: balance laws, variational formulation, and linearization. Computer Methods in Applied Mechanics and Engineering **122**(94), 145–171 (1995)
5. Borja, R.I., Tamagnini, C., Alarcón, E.: Elastoplastic consolidation at finite strain. Part 2: finite element implementation and numerical examples. Computer Methods in Applied Mechanics and Engineering **159**, 103–122 (1998). URL http://www.sciencedirect.com/science/article/pii/S0045782598801059
6. Camacho, G.T., Ortiz, M.: Adaptive Lagrangian modelling of ballistic penetration of metallic targets. Computer Methods in Applied Mechanics and Engineering **142**(3-4), 269–301 (1997). DOI 10.1016/S0045-7825(96)01134-6
7. Cao, T.D., Sanavia, L., Schrefler, B.A.: A thermo-hydro-mechanical model for multiphase geomaterials in dynamics with application to strain localization simulation. International Journal for Numerical methods in Engineering **107**(January), 312–337 (2016). DOI 10.1002/nme
8. Ceccato, F., Simonini, P.: Numerical study of partially drained penetration and pore pressure dissipation in piezocone test. Acta Geotechnica **12**, 195–209 (2016)
9. Cuitiño, A., Ortiz, M.: A material-independent method for extending stress update algotithms from small-strain plasticity to finite plasticity with multiplicative kinematics. Engineering computations **9**, 437–451 (1992)
10. Diebels, S., Ehlers, W.: Dynamic analysis of fully saturated porous medium accounting for geometrical and material non-linearities. International Journal for Numerical Methods in Engineering **39**(1), 81–97 (1996). DOI 10.1002/(SICI)1097-0207(19960115)39:1¡81::AID-NME840¿3.0.CO;2-B
11. Ehlers, W., Eipper, G.: Finite Elastic Deformations in Liquid-Saturated and Empty Porous Solids. Transport in Porous Media **34**(1986), 179–191 (1999). DOI 10.1007/978-94-011-4579-4_11
12. Jeremić, B., Cheng, Z., Taiebat, M., Dafalias, Y.F.: Numerical simulation of fully saturated porous materials. International Journal for Numerical and Analytical Methods in Geomechanics **32**, 1635–1660 (2008). DOI 10.1002/nag.2347
13. Lewis, R.W., Schrefler, B.A.: The finite element method in the static and dynamic deformation and consolidation of porous media. John Wiley & Sons Ltd. (1998)
14. Li, B., Habbal, F., Ortiz, M.: Optimal transportation meshfree approximation schemes for fluid and plastic flows. International Journal for Numerical Methods in Engineering **83**(June), 1541–1579 (2010). DOI 10.1002/nme
15. López-Querol, S., Blázquez, R.: Liquefaction and cyclic mobility model in saturated granular media. International Journal for Numerical and Analytical Methods in Geomechanics **30**(5), 413–439 (2006). DOI 10.1002/nag.488
16. López-Querol, S., Fernández Merodo, J.A., Mira, P., Pastor, M.: Numerical modelling of dynamic consolidation on granular soils. International Journal for Numerical and Analytical Methods in Geomechanics **32**, 1431–1457 (2008). DOI 10.1002/nag
17. Navas, P., López-Querol, S., Yu, R.C., Li, B.: B-bar based algorithm applied to meshfree numerical schemes to solve unconfined seepage problems through porous media. International Journal for Numerical and Analytical Methods in Geomechanics **40**(6), 962–984 (2016). DOI 10.1002/nag.2472
18. Ortiz-Bernardin, A., Puso, M.A., Sukumar, N.: Construction of Polygonal Interpolants: A Maximum Entropy Approach. International Journal for Numerical Methods in Engineering **61**(12), 2159–2181 (2004). DOI 10.1002/nme.1193
19. Ravichandran, N., Muraleetharan, K.K.: Dynamics of unsaturated soils using various finite element formulations. International Journal for Numerical and Analytical Methods in Geomechanics **33**, 611–631 (2009). DOI 10.1002/nag
20. Sanavia, L., Pesavento, F., Schrefler, B.A.: Finite element analysis of non-isothermal multiphase geomaterials with application to strain localization simulation. Computational Mechanics **37(4)**(4), 331–348 (2006). DOI 10.1007/s00466-005-0673-6
21. Sanavia, L., Schrefler, B.A., Stein, E., Steinmann, P.: Modelling of localisation at finite inelastic strain in fluid saturated porous media. Proc. In: Ehlers W (ed.), IUTAM Symposium on Theoretical and Numerical Methods in Continuum Mechanics of Porous Materials, Kluwer Academic Publishers pp. 239–244 (2001)

22. Sanavia, L., Schrefler, B.A., Steinmann, P.: A Mathematical and numerical model for finite elastoplastic deformations in fluid saturated porous media. In: G. Capriz, V. Ghionna, P. Giovine (ed.) Modeling and Mechanics of Granular and Porous Materials, Series of Modeling and Simulation in Science, Engineering and Technology pp. 297–346 (2001)
23. Sanavia, L., Schrefler, B.A., Steinmann, P.: A formulation for an unsaturated porous medium undergoing large inelastic strains. Computational Mechanics **28**(2), 137–151 (2002). DOI 10.1007/s00466-001-0277-8
24. Terzaghi, K.V.: Principles of Soil Mechanics. Engineering News-Record **95**, 19–27 (1925)
25. Ye, F., Goh, S.H., Lee, F.H.: A method to solve Biot's u-U formulation for soil dynamic applications using the ABAQUS/explicit platform. Numerical Methods in Geotechnical Engineering pp. 417–422 (2010). URL http://ahajournals.org
26. Zienkiewicz, O.C., Chan, A.H.C., Pastor, M., Paul, D.K., Shiomi, T.: Static and Dynamic Behaviour of Geomaterials: A rational approach to quantitative solutions. Part I: Fully saturated problems. Proc. Roy. Soc. Lond. **A429**, 285–309 (1990)
27. Zienkiewicz, O.C., Chan, A.H.C., Pastor, M., Schrefler, B.A., Shiomi, T.: Computational Geomechanics With Special Reference To Earthquake Engineering. Wiley (1999)
28. Zienkiewicz, O.C., Chang, C.T., Bettes, P.: Drained, undrained, consolidating and dynamic behaviour assumptions in soils. Géotechnique **30**(4), 385–395 (1980). DOI 10.1016/j.ocecoaman.2012.02.008
29. Zienkiewicz, O.C., Shiomi, T.: Dynamic Behaviour of saturated porous media: The generalized Biot formulation and its numerical solution. International Journal for Numerical and Analytical Methods in Geomechanics **8**(1), 71–96 (1984). DOI 10.1002/nag.1610080106