

Eigendegradation algorithm applied to visco-plastic weak layers

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Abstract: The time dependence of soil behavior is of particular interest for embankments in soft clays, shear band progression in slopes or where the speed of the load application affects the bearing capacity of the material. This study extends non-local failure, algorithms such as Eigenerosion and Eigensoftening, in order to evaluate the failure of weak layers. In particular, the time dependence of shear band progression is analyzed through the integration of a Perzyna-type visco-plastic model with a degradation algorithm within the Optimal Transportation Meshfree (OTM) framework. The validation of the proposed algorithm is carried out through three practical cases.

Keywords: Meshfree numerical modeling; Finite deformation; Degradation; Sensitive Clays; Visco-Plastic behavior

1. Introduction

The failure of geomaterials has been a topic of study since the early work of Coulomb in 1877. It has been developed theoretically in soil mechanics since the work of Drucker and Prager [1]. Failures are idealized into two types depending on the state of the material: diffuse and localized.

Diffuse failures are associated with the instability of loose and saturated materials (e.g. [2], [3]) under monotonic and dynamic loads. The increase of the pore pressure leads the material to liquefy, reduces the material's effective stresses to zero, and provokes landslides involving large volumes of material. Numerical modeling of diffuse failure initiation has been developed in finite elements to reproduce failures in mine tailing dams ([4], [5]), to reproduce landslide failures ([6], [7]) or the study of loose saturated material under seismic action ([8], [9]). Recently, these phenomena have been studied with meshfree techniques, allowing to evaluate the transition from solid to fluidized materials with SPH techniques with one (e.g. [3], [10], [11], [12], [13]) and two phases ([14], [15]), Material Point Method, MPM, (e.g. [16–18]) and Discrete Element Methods, DEM, for the liquefaction on granular materials (e.g. [19], [20]).

In contrast, localized failures are associated with strain concentration in a narrow zone and a limited region, which produces a discontinuity in the deformation as well as in the strain rate. This failure is related to a weak discontinuity, also known as a weak layer. The progress of the failure of overconsolidated clays that present a softening behavior has been studied extensively. It consists of a reduction of the shear strength from the peak to the residual value. Early work by Rice [21] provided the theoretical basis for understanding the physics and mathematics of the problem. The formation and propagation of shear bands have been studied through theoretical and lab-scale models (e.g. [22], [23]). Numerically, the localization of shear deformations associated with softening behaviors, which is related to negative slopes in the loading-displacement curve, traditionally depends on the mesh discretization (e.g. [24]). The employment of visco-plastic models has allowed the regularization of the problem (e.g. [25]).

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Visco-plastic formulations have been developed to evaluate the rate-dependent failure of geomaterials, shear bands, creep and stress relaxation. In general, such constitutive formulations include empirical models focused on the evaluation of creep and stress relaxation for soft clays ([26], [27]), rheological models for a wide range of deformations (e.g. [28], [11], [29]) and general stress-strain-time constitutive models (e.g. [30], [31]). Elasto-plastic constitutive models incorporating rate dependence are based mainly on the over-stress theory ([32]) and on the concept of a non-stationary flow surface (e.g. [33], [34]). The over-stress theory assumes no viscous strains occur inside the static yield surface (elastic region). However, the inelastic region is rate dependent. This dependency is defined by an over-stress factor. This theory has been implemented in finite element and meshfree methods to study different geotechnical problems (e.g. [29,35]).

In this paper, we propose a methodology for evaluating the progression of a shear band or a weak layer through a phenomenological viscoplastic model, taking the benefits of a non-local failure approach. The proposed algorithm is based on the same principles of both Eigenerosion and Eigensoftening approaches, which have been widely employed through material point-based frameworks in order to assess the fracture evolution in quasi brittle materials ([36], [37]) once specific energy (Eigenerosion) or stress (Eigensoftening) thresholds are reached when the material point fails. Within the aforementioned algorithms, the failure can be modeled instantaneously (Eigenerosion) or following a material-dependent softening law (Eigensoftening). In order to achieve a diffuse failure through the degradation of the material, a combination of these algorithms within a Perzyna-type viscoplastic model is proposed. This methodology is validated together with a particle-based discretization (OTM) with three practical cases: a) the progression of a shear band, b) the effect of the velocity on the ultimate load of a footing and c) the failure of a vertical slope with a weak layer.

Therefore, with the aim of verifying the good performance of the aforementioned algorithm, the rest of the paper is organized as follows. In section 2, the constitutive model with all its ingredients (rate dependent plasticity and non-local failure procedure) is presented; in section 3, time and spatial discretizations are outlined; in section 4, several applications are depicted; and, finally, conclusions are provided in section 5.

2. Constitutive model

2.1. Rate dependent plasticity

Along with the mathematical and numerical models described so far, it is essential to choose an appropriate constitutive model for the materials. It has been studied how soft clays suffer a viscous phenomenon and delayed deformation due to creep behavior ([26]). The concept of the viscoplastic model described in this section and used in the further simulations is based on Perzyna's theory [32]. Perzyna's theory is a modification of classical plasticity, wherein viscous-like behavior is introduced by a time-rate flow rule employing a dynamic plasticity yield function. Similar to the rate-independent theory, the strain rate is decomposed into an elastic and a viscoplastic strain rate:

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^{vp} \quad (1)$$

The stress rate tensor $\dot{\sigma}$ is related to the elastic strain rate via a constitutive tensor D_e , which is constant in the case of linear elasticity and variable (stress dependent) in the case of hypo or hyperelasticity:

$$\dot{\sigma} = D_e(\dot{\epsilon} - \dot{\epsilon}^{vp}) \quad (2)$$

In the theory proposed by Perzyna [32], and later modified by Souza-Neto *et al.* [38], the viscoplastic strain rate is defined in a similar fashion as in the rate independent plasticity theory:

$$\dot{\epsilon}^{vp} = \langle \dot{\lambda} \rangle \frac{\partial g}{\partial \sigma} \quad (3)$$

where $\langle \dot{\lambda} \rangle$ is the viscous flow function, which represents the current magnitude of viscoplastic strain rate. g denotes the viscoplastic potential function and f any valid plasticity function playing the role of loading surface. A von Mises yield surface, with a degradation curve for the undrained shear strength, has been adopted for the failure criteria. Associative flow is invoked by $f = g \cdot \frac{\partial g}{\partial \sigma}$ that represents the current direction of the viscoplastic strain rate. The viscous flow function is defined by:

$$\langle \dot{\lambda} \rangle = \begin{cases} \gamma \left[\left(\frac{q}{\sigma_y} \right)^\alpha - 1 \right] & , \phi > 0 \\ 0 & , \phi \leq 0 \end{cases} \quad (4)$$

where $\langle \cdot \rangle$ denotes Macauley brackets, γ is the fluidity parameter (which is the reciprocal of viscosity) and α is a material constant. Function ϕ is defined as:

$$\phi = q - \sigma_y \quad (5)$$

where q is the deviatoric stress invariant and σ_y is defined by the degradation law which will be detailed later. Concerning algorithm aspects, in displacement-based finite element formulations, stress updates take place at the Gauss points for a known nodal displacement. We start from time t_n with the known converged state:

$$\left[\varepsilon_n, \varepsilon_n^{vp}, \sigma_n, \kappa_n \right] \quad (6)$$

(namely total strain, viscoplastic strain, stress and a scalar internal variable that characterizes the size of the loading surface for the purpose of introducing hardening or softening behavior) to calculate the corresponding values at time

$$t_{n+1} = t_n + \Delta t : \left[\varepsilon_{n+1}, \varepsilon_{n+1}^{vp}, \sigma_{n+1}, \kappa_{n+1} \right]. \quad (7)$$

Indeed, this process has been carried out in an incremental way, being calculated:

$$\Delta \varepsilon = \varepsilon^e + \Delta \varepsilon^{vp} \quad (8)$$

$$\Delta \sigma = \mathbf{D}_e (\Delta \varepsilon - \Delta \varepsilon^{vp}) \quad (9)$$

Therefore, the key feature of the stress updates is characterized by estimating the incremental viscoplastic strain $\Delta \varepsilon^{vp}$. Details of the numerical implementation can be found in the textbooks (Owen and Hinton, 1986 [39]; De Souza Neto *et al.*, 2008 [38]). It has to be noticed also that for softening problems the viscoplasticity approach has a regularizing effect in the sense that the initial-value problem remains well-posed avoiding instability due to strain and strain-rate softening (Wang *et al.* [24]).

2.2. Eigenerosion and Eigensoftening algorithms

Within the context of OTM formulation, fracture can be modeled simply by eroding material points according to an energy-release criterion or stress criterion depending on whether the eigenerosion [40–43] or the eigensoftening [36,37,44] algorithm is adopted respectively. In both methodologies, when the material points are failed, they are neglected from the computation of stresses in the model, which approximates the presence of cracks. However, the way to reach the zero stiffness state is different for each model: meanwhile in the eigenerosion the failure is instantaneous, in the eigensoftening the material follows a softening curve which depends on the material. It needs to be pointed out that when a material point satisfies the erosion condition, its contribution to the internal force vector and to the material stiffness matrix is set to zero, but its contribution to the mass matrix is maintained. The mass of a material point is discarded only when an eroded material point is not connected to any nodes.

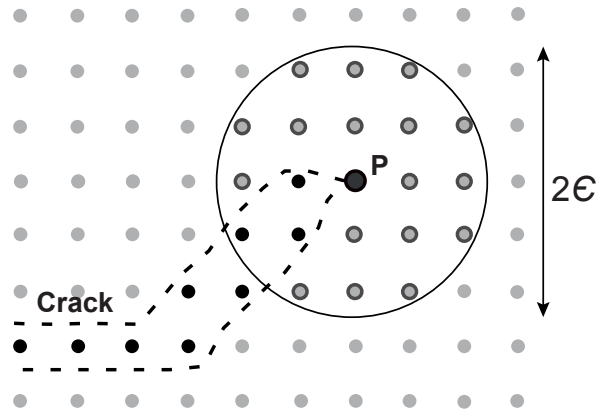


Figure 1. Scheme of a fractured layer (black dots) as set of failed material points, and of the ϵ -neighborhood (inside the circle) of the material point located at the crack tip (gray dots).

120 Next we compute the energy-release rate attendant to the failure of the material
121 point p , the starting point of the aforementioned methodologies:

$$\begin{aligned} G_{p,k+1} &= \frac{C\epsilon}{m_{p,k+1}} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q W_k(F_{q,k+1}), \\ m_{p,k+1} &= \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q, \end{aligned} \quad (10)$$

122 where $B_\epsilon(x_{p,k+1})$ is the sphere of radius ϵ centered at $x_{p,k+1}$ known as the ϵ -neighborhood
123 of the material point, $m_{p,k+1}$ is the mass of the neighborhood at loading step $k+1$,
124 $W_k(F_{q,k+1})$ is the current free-energy density per unit mass at the material point $x_{q,k+1}$
125 and C is a normalizing constant, which ranges from 1 to 2, in order to extend the influence
126 from the first line of neighbors to the second [40,41].

127 The material point is failed when $G_{p,k+1}$ surpasses a critical energy release rate
128 that measures the material-specific energy, G_F , required to create a fracture surface of
129 unit area. In calculations, the failed material-point set is updated at every time step
130 according to this criterion. For linear elasticity, Schmidt *et al.* [40] have shown that this
131 approximation converges to Griffith fracture in the limit of an infinitely fine discretization.
132 Indeed, erosion schemes that estimate the energy-release rate based on the energy of a
133 single material point suffer from mesh-dependency and may overestimate the toughness
134 of the material.

135 A scheme of the configuration of the ϵ -neighborhood is plotted in Fig. 1.

On the other hand, the implementation of the eigensoftening algorithm consists in adopting a strength criterion for crack initiation and a softening law which is proper to the material under study before the formation of a stress-free crack. This second process tends to accumulate less energy until the crack. When the maximum tensile strength, f_t , is reached, a cohesive crack is formed with zero opening displacement. Once the opening displacement, w , reaches a critical value, w_c , a stress-free crack is attained. The energy below the softening curve represents the static fracture energy per unit of area, G_F , which is sketched in Fig. 2. For the Eigensoftening calculation, Eq. (10) can be rewritten in terms of the principal stresses at time t_{k+1} , since this model employs the first principal stress as a failure criterion. Therefore, the variation of the averaged strain energy density in the ϵ -neighborhood of the material point $x_{p,k+1}$ can be expressed as,

$$\delta W_p^\epsilon = \frac{\partial G_p}{C\epsilon} = \frac{1}{m_p} \sum_{x_q \in B_\epsilon(x_p)} m_q \sigma_{q,1} \delta \epsilon_q, \quad (11)$$

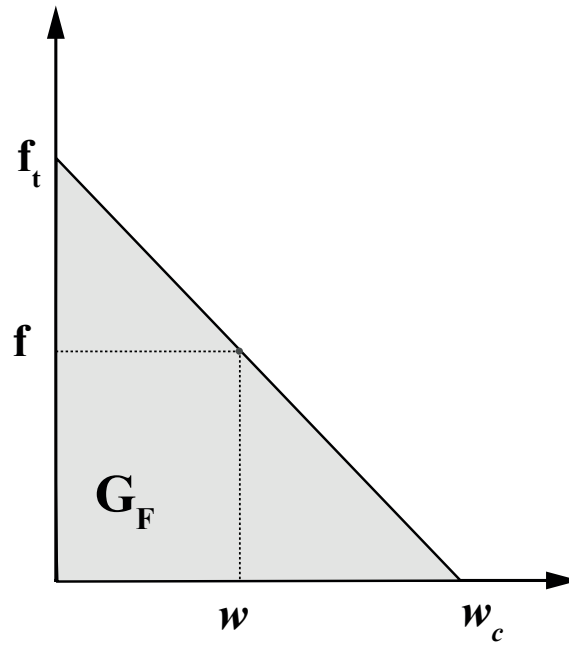


Figure 2. Scheme of a linear cohesive law, where the shade area is G_F , f_t is the tensile strength, and w_c is the critical opening displacement.

where $\sigma_{q,1}$ is the maximum principal stress at a neighboring material point $x_{q,k+1}$. Considering an effective strain ε_q at the material point $x_{q,k+1}$, such that the variation of the local strain energy can be obtained as $\delta W_q = \sigma_{q,1} \delta \varepsilon_q$, let assume the effective strain increment of each material point can be approximated by its counterpart in the neighborhood, being Eq. (11) simplified as:

$$\delta W_p^e = \frac{\delta \varepsilon_p}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \sigma_{q,1}. \quad (12)$$

Thus, the equivalent critical stress at the material point $x_{p,k+1}$ is defined as follows

$$\sigma_p^e = \frac{1}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \sigma_{q,1} \quad (13)$$

136 When $\sigma_{p,k+1}^e$ surpasses the tensile strength, f_t , the softening behavior is activated
 137 through the damage variable χ , which ranges between zero (an intact material) and one
 138 (completely failed material points). Of course, χ depends on the current and critical
 139 opening measures, w and w_c respectively. The latter is a material parameter, but the first
 140 one has to be measured in terms of the achieved strain and a length of affection called
 141 band width, h_ϵ , equivalent to the crack band model of Bažant [45]. It bears emphasis
 142 that a reference value for h^ϵ is between two and four times the maximum size of the
 143 aggregates for concrete according to Bažant [46]. Thus, this is a material parameter more
 144 than a numerical artifact. The relationship between strain and crack opening depends
 145 on the effective fracture strain, ε_f^ϵ , defined as the difference between the strain at crack
 146 initiation, $\varepsilon_1(x_{p,0})$, and the current strain, $\varepsilon_1(x_{p,k+1})$ for a material point p ; and the band
 147 width as:

$$\varepsilon_f^\epsilon = \varepsilon_1(x_{p,k+1}) - \varepsilon_1(x_{p,0}) = \frac{w}{h^\epsilon} \quad (14)$$

2.3. Eigendegradation model

Following the work of Einav and Randolph [47], and the later implementations of Zhang *et al.* [47] (similar also of some other implementations [48,49]), the behavior of sensitive clays can be modeled by strain softening curves in order to reduce the strength of the material by a degradation obtained by accumulation of strain. Einav and Randolph assumed that the current shear strength depends on the accumulated absolute shear strain, ζ , which is taken as a state variable from which an isotropic strength reduction, $\delta(\zeta)$, is calculated as

$$\delta(\zeta) = s_u / s_{ui} = \delta_{rem} + (1 - \delta_{rem})e^{-3\zeta/\zeta_{95}} \quad (15)$$

where

$$\zeta = \int_t |\dot{\gamma}_{max}| dt \quad (16)$$

and $|\dot{\gamma}_{max}|$ is the cumulative absolute shear strain, s_u and s_{ui} are the softened strength and initial strength, respectively, δ_{rem} is the fully remolded strength ratio, and ζ_{95} is the cumulative shear strain required to cause 95% reduction (from peak to remolded). The assumption is that δ_{rem} may be taken as the inverse of the sensitivity of the soil, while an appropriate value for ζ_{95} must be deduced from laboratory test data, or by conducting cyclic penetration and extraction tests with T-bar or ball.

The calculation of the cumulative shear strain can be achieved by the eigendeformation technique, departing from Eq. (11), and considering, for the Eigendegradation calculation, that the stress remains constant in a neighborhood ϵ . Thus, Eq. (11) can be simplified as

$$\delta W_p^\epsilon = \frac{\delta \tau_p}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \gamma_q, \quad (17)$$

being $\delta \tau_p$ the increment of tangential stress of the neighborhood and γ_p^ϵ the current local shear strain, calculated as:

$$\gamma_p^\epsilon = \frac{1}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \gamma_q \quad (18)$$

Similarly, in the neighborhood ϵ , the non-local cumulative strain of a material point p is calculated, only when plasticity is activated, as follows:

$$\zeta_p^\epsilon = \int_{t_{p0}}^{t_{k+1}} |\dot{\gamma}_p^\epsilon| dt \quad (19)$$

being t_{k+1} referred to the current step and t_{p0} to the step when plasticity begins.

Considering only shear failure, yield shear stress τ is equivalent to the softened strength, s_u , and the residual yield shear stress, τ_{95} , can be reached by $\tau_{95} = \tau_{rem} = s_{ui} \delta_{rem}$. Thus, in every state of degradation, the current yield shear stress, referred to the epsilon neighborhood, τ^ϵ , reads:

$$\tau^\epsilon = \tau_{95} + (\tau_i - \tau_{95})e^{-3\zeta_p^\epsilon/\zeta_{95}} \quad (20)$$

It is remarkable that, in laboratory, parameter ζ_{95} is not obtained. Instead, the displacement δ_{95} is achieved. In Fig. 3.A) the degradation of the strength in terms of the displacement is plotted. It can be seen how this law can be translated to the shear strain measurement (Fig. 3.B) by multiplying by ϵ , which, in this problem, is considered as the sliding length. Depending on the size of the soft layer, this parameter ϵ is obtained as the minimum length between the neighbor radius and the size of the soft layer (See Fig. 4) as follows:

$$2\epsilon = \min(h_s, 2C_\epsilon h) \quad (21)$$

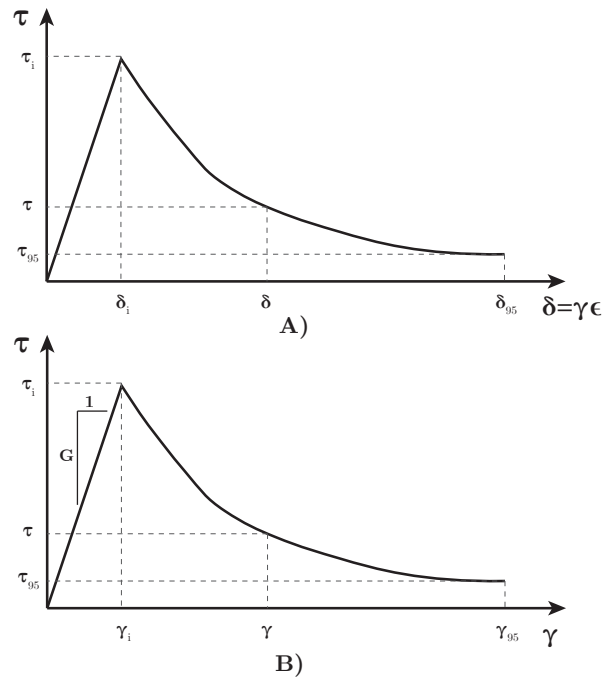


Figure 3. Degradation curve in terms of the displacement, A), and the shear strain, B).

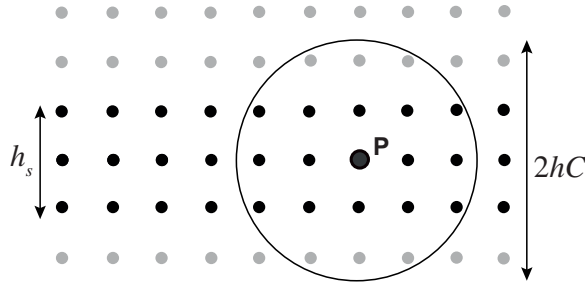


Figure 4. Scheme of the measurements of the soft layer (black dots) and the ϵ -neighborhood around material point P.

173

174 2.4. Visco-Plastic Eigendegradation algorithm

175 Following, the pseudo-algorithm for the Eigendegradation model within a visco-
 176 plastic yield surface will be presented. It is worth mentioning that, prior to the algorithm
 177 steps, we need to calculate the equivalent shear total strain of every material point as
 178 the norm of the deviatoric total strain tensor. Since large strain is considered, the strain
 179 tensor is obtained through the logarithm of the left Cauchy-Green strain tensor, \mathbf{b} :

$$\epsilon = \frac{1}{2} \log \mathbf{b} = \frac{1}{2} \log \mathbf{F}\mathbf{F}^T. \quad (22)$$

180

The aforementioned procedure can be followed in Algorithm 1.

181 3. Time and spatial discretization

182 Following, the rest of computational tools that were employed in the present
 183 research are highlighted. First, the spatial discretization is shown, while the time dis-
 184 cretization is mentioned in the final subsection.

Algorithm 1 Visco-Plastic Eigendegradation algorithm**1. Calculation of the small strain tensor**

$$\boldsymbol{\varepsilon}_{k+1}^{e\ trial} = 1/2 \log \mathbf{b}_{k+1}^{e\ trial}$$

2. Elastic Predictor: volumetric and deviatoric stress measurements

$$\text{Volumetric: } p_{k+1}^{trial} = K(\boldsymbol{\varepsilon}_{vol}^e)_{k+1}^{trial}$$

$$\text{Deviatoric: } \mathbf{s}_{k+1}^{trial} = 2G(\boldsymbol{\varepsilon}_{dev}^e)_{k+1}^{trial}$$

$$\text{being: } \boldsymbol{\sigma}_{k+1}^{trial} = J^{-1} \boldsymbol{\tau}_{k+1}^{trial}$$

$$\text{and: } q_{k+1}^{trial} = \sqrt{\frac{3}{2}} \|\mathbf{s}_{k+1}^{trial}\|$$

3. Eigendegradation calculation:

if $t < t_{p0}$ **then** $\sigma_y = \tau_i$

else

- $m_p = \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q$
- $\gamma_p^e = \frac{1}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \gamma_q$
- $\xi_p^e = \sum_{k(t_{p0})}^{k+1} |\Delta \gamma^e(x_{p,k})|$
- $\sigma_y = \tau_{95} + (\tau_i - \tau_{95}) e^{-3\xi_p^e / \xi_{95}}$
- Hardening modulus: $H = \frac{\partial \sigma_y}{\partial \bar{\varepsilon}^p} \simeq \frac{\partial \sigma_y}{\partial \xi_p^e} = -\frac{3(\tau_i - \tau_{95})}{\xi_{95}} e^{-3\xi_p^e / \xi_{95}}$

end if

4. Yield condition: $\Delta\lambda = 0$

if $\phi = q_{k+1}^{trial} - \sigma_y \leq 0$ **then** Elastic region: $\sigma_{k+1} = \boldsymbol{\sigma}_{k+1}^{trial}$

else Viscoplastic flow:

- **4.1** Derivative of the yield surface:

$$d = \frac{\partial \phi}{\partial \Delta\lambda} = - \left(3G + \alpha \frac{q_{k+1}^{trial} - 3G\Delta\lambda}{\Delta\lambda + \gamma\Delta t} \right) \left[\frac{\gamma\Delta t}{\Delta\lambda + \gamma\Delta t} \right]^\alpha - H$$

- **4.2** Increment of plastic strain: $\Delta\lambda = \Delta\lambda - \frac{\phi}{d}$
- **4.3** Yield function: $\phi = \left(q_{k+1}^{trial} - 3G\Delta\lambda \right) \left[\frac{\gamma\Delta t}{\Delta\lambda + \gamma\Delta t} \right]^\alpha$
- **4.4** If $\phi < tolerance$ go to **4.5**, else go to **4.1**
- **4.5** Update

$$\bar{\varepsilon}_{k+1}^p = \bar{\varepsilon}_k^p + \Delta\gamma$$

$$\Delta \boldsymbol{\varepsilon}_{k+1}^p = \frac{\Delta\gamma}{\|\mathbf{s}_{k+1}^{trial}\|} \mathbf{s}_{k+1}^{trial}$$

$$\boldsymbol{\sigma}_{k+1} = \left(p_{k+1}^{trial} \right) \mathbf{I} + \left(1 - \frac{3G\Delta\gamma}{q_{k+1}^{trial}} \right) \mathbf{s}_{k+1}^{trial}$$

end if

5. Update elastic left Cauchy-Green Tensor

$$\boldsymbol{\varepsilon}_{k+1}^e = \boldsymbol{\varepsilon}_{k+1}^{e\ trial} - \Delta \boldsymbol{\varepsilon}_{k+1}^p$$

$$\mathbf{b}_{k+1}^e = \exp(2\boldsymbol{\varepsilon}_{k+1}^e)$$

185 3.1. Spatial discretization

The dynamic problem of a dry soil (mono-phase material) is studied in this research, being the time an important issue of the following analyses. The governing equation of the dynamic problem can be defined by the linear momentum balance equation:

$$\operatorname{div} \sigma - \rho a + \rho g = 0. \quad (23)$$

186 The derivation of the weak form of the problem needs the multiplication by the test
187 function δu , the virtual displacement, and the integration over the domain. After the
188 application of the Green's Theorem, Equation (23) yields:

$$-\int_{\Omega} \sigma : \operatorname{grad} \delta u \, d\Omega + \int_{\Gamma_t} \delta u \cdot \bar{t} \, d\gamma - \int_{\Omega} \delta u \cdot \rho a \, d\Omega + \int_{\Omega} \delta u \cdot \rho g \, d\Omega = 0 \quad (24)$$

where Ω represents the volume of the body and Γ the boundary where tractions are applied. The first term of the equation is defined as the internal forces, meanwhile the second and forth conform the external forces. The next step is the interpolation through the Optimal Transportation Meshfree [50–52], a meshfree method that has been demonstrated to perform reasonably well in geotechnical problems [53,54]. It is based in the conjunction of material points and nodes. As mentioned before, the shape functions are based on the work of Arroyo and Ortiz [55], who defined the Local Max-Ent shape function (LME) of the material point (x) with respect to the neighborhood (x_a) as follows:

$$N_a(\mathbf{x}) = \frac{\exp[-\beta_{LME} |\mathbf{x} - \mathbf{x}_a|^2 + \lambda^* \cdot (\mathbf{x} - \mathbf{x}_a)]}{Z(\mathbf{x}, \lambda^*(\mathbf{x}))}, \quad (25)$$

where the computation is done along a neighborhood N_b and

$$Z(\mathbf{x}, \lambda) = \sum_{a=1}^{N_b} \exp[-\beta_{LME} |\mathbf{x} - \mathbf{x}_a|^2 + \lambda \cdot (\mathbf{x} - \mathbf{x}_a)]. \quad (26)$$

189 The first derivatives of the shape function can be obtained from the own shape function
190 and its Hessian matrix \mathbf{J} by employing the following expression:

$$\nabla N_a^* = -N_a^* (\mathbf{J}^*)^{-1} (\mathbf{x} - \mathbf{x}_a), \quad (27)$$

The parameter β_{LME} defines the shape of the neighborhood, and it is related with the discretization size (or nodal spacing), h , and the constant, γ_{LME} , which controls the locality of the shape functions, as follows,

$$\beta = \frac{\gamma_{LME}}{h^2}. \quad (28)$$

191 It bears emphasis that $\lambda^*(\mathbf{x})$ comes from the minimization of the function $g(\lambda) =$
192 $\log Z(\mathbf{x}, \lambda)$ to guarantee the maximum entropy.

193

By employing the outlined shape functions and applying Galerkin procedure to the weak form, u can be interpolated by employing:

$$u \approx u^h = N_u \cdot \tilde{u} \quad (29)$$

194 where \square^h represents the OTM approximation of the field \square and $\tilde{\square}$ the nodal values. $N_u =$
195 $[N_1 I, N_2 I, \dots, N_m I]$ represents the shape function and m is the number of nodes. Since
196 the shape functions are defined in the reference configuration and time independent,
197 $N = N(X)$, the following property holds: $\dot{\square}^h = N \cdot \dot{\tilde{\square}}$.

Table 1: The α -parameters of the Newmark scheme.

$$\alpha_1 = \frac{1}{\beta \Delta t^2} \quad \alpha_2 = \frac{1}{\beta \Delta t} \quad \alpha_3 = \frac{1}{2\beta} - 1$$

3.2. Time discretization

In this work, an implicit scheme has been proposed, since several applications cover a wide range of loading rates; from slow scenarios to quick phenomena. For the first ones, an explicit scheme would provide long computation time. Thus, the Newmark Implicit Scheme has been employed, with the parameters $\gamma = 0.6$ and $\beta = 0.325$ that are known to be suitable for dynamic problems [56]. To construct this scheme, Eq. (24) is reformulated as a system of equations, read as

$$\mathbf{R}_{k+1} + \mathbf{M} \ddot{\mathbf{u}}_{k+1} = \mathbf{P}_{k+1}, \quad (30)$$

where \mathbf{R} and \mathbf{M} respectively denote the internal forces vector and mass matrix, whereas \mathbf{P} is the external forces vector, which contains both gravity acceleration and external nodal forces. $k + 1$ represents the current step. Eq. (30) can be re-written with the Newmark scheme as:

$$\mathbf{G}_{k+1} = \mathbf{M}[\alpha_1 \Delta \mathbf{u}_{k+1} - \alpha_2 \dot{\mathbf{u}}_k - \alpha_3 \ddot{\mathbf{u}}_k] + \mathbf{R}_{k+1} - \mathbf{P}_{k+1} = \mathbf{0}, \quad (31)$$

where the α -parameters are listed in Table 1 according to Wriggers [57]. These coefficients can be easily extended to any other time integration schemes.

Solving the above non-linear equations with a Newton-Raphson method, the resulting iterative scheme, taking into account the matrices that are involved in our problem, can be written as:

$$\begin{aligned} [\alpha_1 \mathbf{M} + \mathbf{K}_{k+1}^i] \Delta \mathbf{u}_{k+1}^{i+1} &= [\mathbf{K}_*]_{k+1}^i \Delta \mathbf{u}_{k+1}^{i+1} = -\mathbf{G}(\mathbf{u}_{k+1}^i), \\ \text{where } \mathbf{u}_{k+1}^{i+1} &= \mathbf{u}_{k+1}^i + \Delta \mathbf{u}_{k+1}^{i+1}. \end{aligned} \quad (32)$$

where \mathbf{K} is the tangential stiffness matrix:

$$\mathbf{K}(\mathbf{u}_{k+1}^i) = \mathbf{K}_{k+1}^i = \left. \frac{\partial \mathbf{R}}{\partial \mathbf{u}} \right|_{\mathbf{u}_{k+1}^i}. \quad (33)$$

and i depicts the iteration index. The iteration finishes when \mathbf{G}_{k+1}^i is smaller than a given tolerance.

4. Applications

Three tests have been studied in this research. The two first applications are devoted to show the performance of the two main principal properties of the proposed constitutive model: the degradation (Shear test, Sec. 4.1) and the viscous behavior (Strip footing load, Sec. 4.2). The last example shows the suitability of the model when the triggering and propagation of a slope due to cyclic loading is sought.

4.1. Shear test

In the first example, the behavior of a weak layer under a shear test is analyzed. In Fig. 5 an embankment of 10 meters of depth is presented. A weak layer in the bottom of the domain of 0.5 meters appears. The proposed shear test is similar to the one proposed by Zhang *et al.* [47]. Although different constitutive laws are employed (local and non-local), since both are based on similar degradation processes, it is expected to obtain comparable results.

The original embankment's length is 700 m. Since the softened zone only extends 90 m. and considering infinite conditions at both sides of this softened zone, only this 90 m.

Table 2: Parameters for the shear degradation analysis.

| | |
|---------------------------------------------------------------|-----------------------|
| Softened/ modeled length $L = l_0$ | 90 m |
| Overall height, H | 10 m |
| Height of sliding material, h | 7.2 m |
| Shear band thickness, s | 0.5 m |
| Submerged density of the soil, ρ | 600 kg/m ³ |
| Poisson's ratio, ν | 0.495 |
| Young's modulus, E | 1.98 MPa |
| Peak shear strength, $\tau_p = \tau_i$ | 10 kPa |
| Residual (95%) shear strength, $\tau_{95} = \tau_r$ | 1.25 kPa |
| Plastic shear strain to 95% reduction in strength, γ_p | 0.6 |
| Neighborhood parameter, C_e | 1.5 |

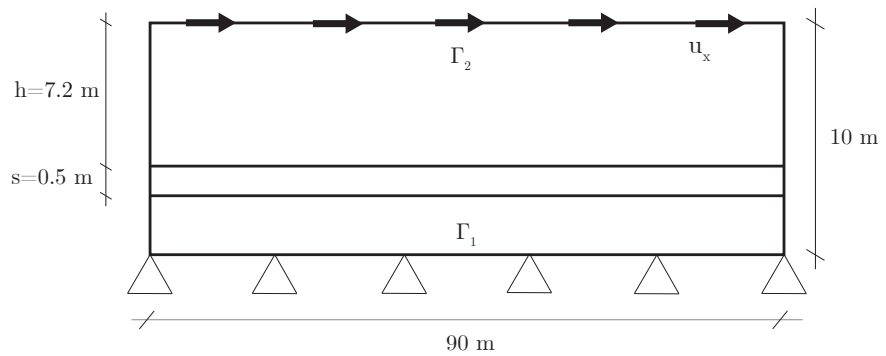


Figure 5. Geometry and loading conditions of the shear degradation problem.

are modeled in order to save computational time. Unlike the original example, gravity conditions are neglected, considering the failure of the embankment at the final of the residual strength. Thus, the parameters needed in this example are shown in Table 2. It is important to point out that, in this research, the proposed non-local degradation model has been employed with a neighborhood parameter, C_{ϵ} , of 1.5. This is an important difference with the model proposed for the original example, which evaluates the degradation locally in an Arbitrary Eulerian-Lagrangian (ALE) configuration.

In addition, the geometry and boundary conditions can be seen in Fig. 5. About the latest, two boundary conditions have been considered. In the first one, Γ_1 in Fig. 5, both vertical and horizontal displacements are constrained. In Γ_2 a horizontal displacement of 10 meters is imposed gradually from 0 to 1000 s.

The stress behavior is analyzed in Fig. 6. In order to assess the performance of the proposed algorithm, similar to the figure proposed by Zhang *et al.*[47], in Fig. 6 a dimensionless measurement of the shear stress is plotted. It is calculated as a relative increment from the residual shear strength τ_{95} and divided by the maximum increment, measured from the initial (or peak) strength to the residual one, $\tau_p - \tau_{95}$. On the other hand, in the abscissa, the dimensionless distance from the beginning of the degradation is plotted. Thus, the degradation starts from 0 until reaching the τ_{95} at distance 1.

Results obtained in this research and the ones obtained by Zhang *et al.*[47] are not coincident since they are different approaches; however, the overall trend, mainly after the beginning of the degradation, is similar to the reference research. This application allows us to assess the performance of a degraded layer of soft clay, which has been proved to perform similar to other validated studies.

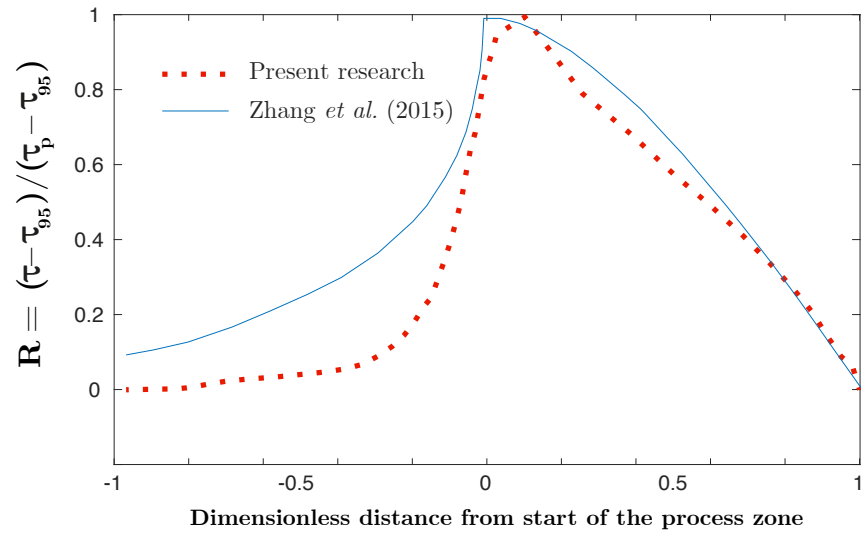


Figure 6. Geometry and loading conditions of the shear degradation problem.

248 4.2. Strip footing load

Following, the behavior of the soil under different loading rates of a strip footing is analyzed. Similarly, the viscous properties of the soil are varied in order to extend the analysis to the behavior of the soil. This classic problem has been extensively used to verify the solutions provided by numerical models under viscoplastic conditions. The two main features to assess are the mechanism of failure and the behavior of the reaction forces at different viscoplastic scenarios. These results have been previously presented in the work of Pastor and coworkers [35,58], where the loading is applied as an incremental velocity downwards at the base of the strip footing. In the present research, the loading is applied as a negative displacement according to the following expression:

$$u_y = u_f \left(t/t_f \right)^2,$$

249 where $u_f = 0.04$ m. and $t_f = 4$ s. The geometry and soil parameters can be seen in
 250 Fig.7. Parameters of the Eigendegradation algorithm are also depicted. Opposite to the
 251 previous application, in this case there is no weak layer, acting the whole domain as a
 252 viscoplastic degradable soil.

253 The first calculation is made by activating the degradation part. This degradation
 254 acts as a softening of the material following the proposed exponential expression. It is
 255 known how the softening of the material boosts the formation of shear bands. In Fig.
 256 8 the mechanism of failure is depicted. In [58], there is a study of the influence of the
 257 discretization size and the parameters of the meshfree model. Optimal options achieved
 258 in the aforementioned study have been activated here.

259 In order to verify the performance of the full model, different viscous parameters
 260 have been employed in the calculation of the failure load of the strip footing. Establishing
 261 the sensitivity parameter α as 1.0, the viscous parameter has been varied with different
 262 values (see Fig. 7). The obtained results have been depicted in Fig.9 for different values
 263 of μ . The smaller the value of the μ parameter (*i.e.* large values of γ), the higher the final
 264 loading is obtained from the footing loading, as expected from the viscous model. In
 265 addition, in dashed line, the reference value obtained by Navas *et al.* [58] is depicted.
 266 This line can be considered as the value of a pseudo-static load is applied, without any
 267 rigidization due to the viscous behavior. This value is very close to the one obtained
 268 with $\mu = 200$ s.

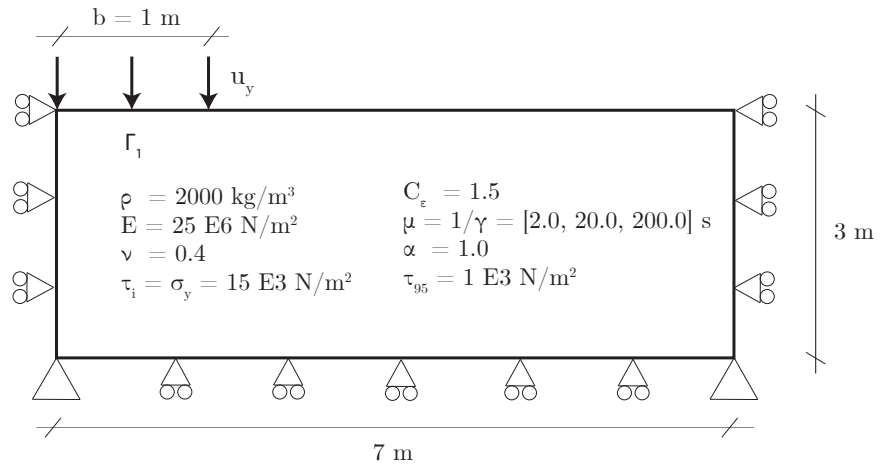


Figure 7. Geometry, material parameters and loading conditions of the strip footing problem.

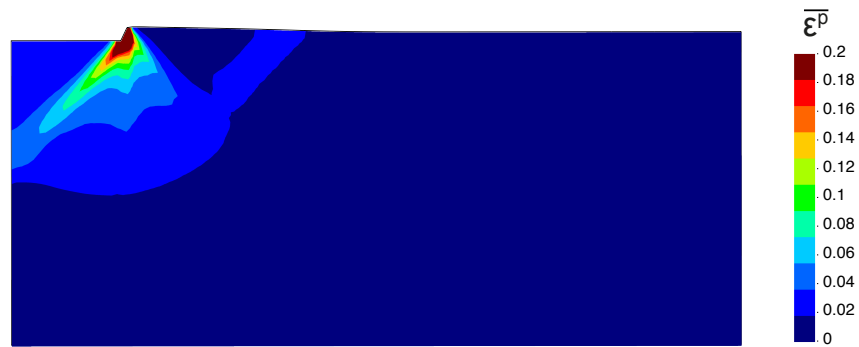


Figure 8. Equivalent plastic deformation of the footing problem using von Mises law with softening through degradation at the final of the simulation.

269 Similarly, depending on the loading rate, the material can get stiffer and provide
 270 bigger response of the reaction forces. Thus, for $\mu = 200$ s., 3 different loading rates
 271 have been tested. The obtained results have been depicted in Fig.10 for different values
 272 of t_f , being this parameter the final time of application of the imposed displacement
 273 ($u_y = u_f \left(t/t_f \right)^2$). The quicker is the application of the displacement, the higher the final
 274 loading is obtained from the footing loading, as expected from the viscous model. In
 275 addition, in dashed line, the reference pseudo-static value, obtained by Navas *et al.* [58],
 276 is depicted also in this figure, being close to the slowest case.

277 4.3. Vertical cut

278 This final application allows understanding the potentiality of the proposed method-
 279 ology. A weak layer is supposed in a soil with a vertical cut on the left side. This weak
 280 layer is located forming a 45° angle, as it is sketched in Fig. 11. This layer, whose thick-
 281 is 1 meter, will be considered plastic. Von Mises yield surface is employed, being its
 282 degradation modeled through both *Eigendegradation* and traditional softening, in order to
 283 assess the performance of the first one against the latter. Out of the weak layer, the soil is
 284 considered elastic since its failure is far from the failure of the weak layer. Parameters of
 285 both models are presented in the right part of Fig. 11. In the traditional softening model,

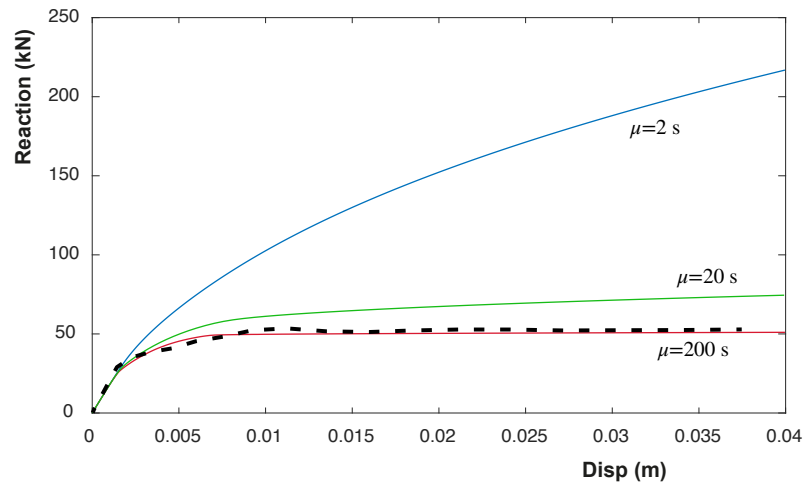


Figure 9. Obtained reaction with different μ values for the problem of the strip footing using viscoplastic von Mises law. The dashed line represents the pseudo-static behavior.

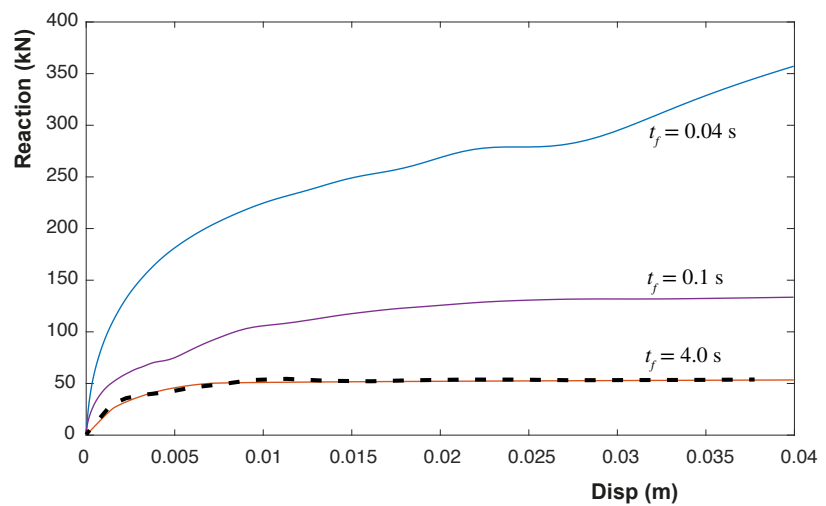


Figure 10. Obtained reaction with different loading rates for the problem of the strip footing using viscoplastic von Mises law, being t_f the final time of the application of the load. The dashed line represents the pseudo-static behavior.

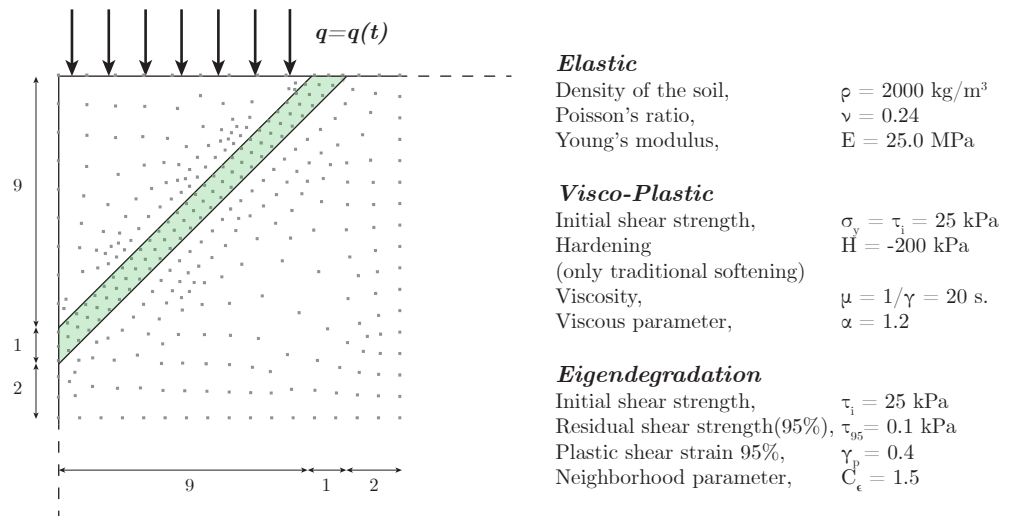


Figure 11. Left: Geometry of the vertical cut analyzed through *Eigendegradation* and softening models and the location of the weak layer and the loaded zone. (Units in meters). Right: parameters of the employed models.

no Eigendegradation parameters are needed. Instead, a negative hardening of 200 kPa is employed. This parameter is not employed in the Eigendegradation simulation.

The soil can be considered infinite on the right and on the bottom of the model; thus, any movement in these directions is prevented. The 12 first meters are modeled for the sake of simplicity.

The top left part is loaded by a surface load, as it is shown in Fig. 11. This load is composed by two different waves, as it is sketched in Fig. 12.

Both waves follow the expression:

$$q_\alpha(t) = A_\alpha \cdot [1 - \cos(\omega_\alpha t)]$$

where α varies for each of both loads. The bigger load is the one that provokes the triggering of the plastic mechanism. It can be considered as an abnormal scenario that may lead to catastrophic consequences in the short or in the long time. Their parameters are $A_1 = 11 \text{ kPa}$ and $\omega_1 = 0.1\pi \text{ rad/s}$. This load is applied only in the first 20 seconds of simulation. The secondary load is of a lower magnitude. It could be considered as a usual load that the soil suffers permanently and is not capable to provoke the breakage of the slope by itself. The amplitude of this load is half of the first one, $A_2 = 5.5 \text{ kPa}$ and the frequency is much higher, $\omega_2 = 3\pi \text{ rad/s}$. This load is maintained through the 80 s. of the simulation.

Following, in Fig. 13, the evolution of both shear strength and the equivalent plastic strain are depicted along the time for both Eigendegradation and softening models. On the left, the shear strength is plotted. The first part of the figure is leaded by the first load: the material reach the yield stress close to 10 s. and start to decrease the strength till 15 s. Both materials, until this point, behave similarly. Observing the right figure, we can see how both models obtain plastic strain until 15 s. as well. Obviously, the amount of shear strain is different since one law is logarithmic (eigendegradation) and the other one is linear. After this point, the eigendegradation model accumulates shear strain (elastic in this case) that makes the shear strength to decrease. This elastic shear strain comes from the second law (the one with small amplitude). We know that is elastic strain since, in the right figure, no accumulated plastic strain is obtained from 15 s. to 50 s. From this point on, the equivalent plastic strain increases drastically. It is translated in an increment of the descent of the yield stress. However, since this accumulated strain provoked by the

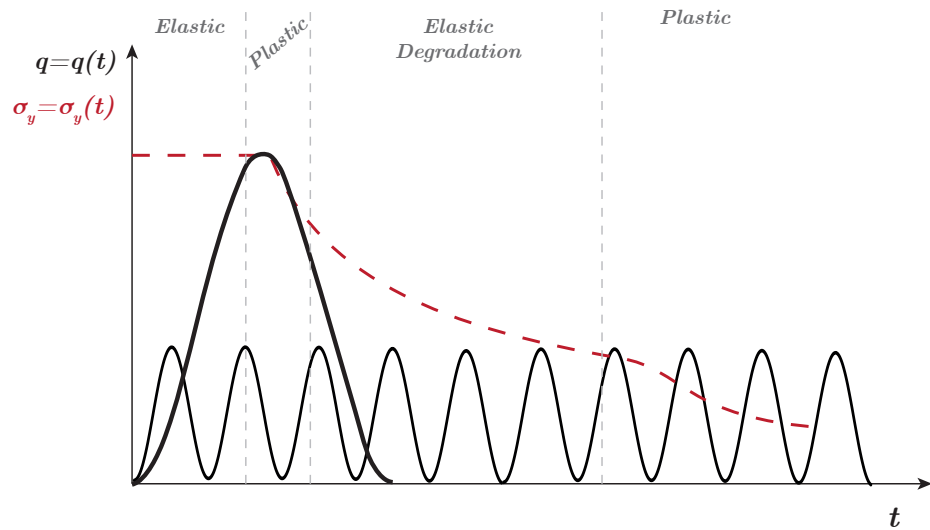


Figure 12. Scheme of the load and the yield strength along the time.

second load is elastic, no variation of the shear strength with the traditional softening model is observed.

Another observation that arises from Fig. 13 is the capability of the model of being sensitive to both loading and unloading conditions of the load, *i.e.*, any variation of the strain, positive or negative, makes the material to degrade and lose shear strength. It is seen in the slope of the yield stress curve from 15 to 50 s., that remains constant along the whole loading cycle, equal in the loading or unloading branch.

Finally, in Fig. 14, the distribution of the equivalent plastic strain in the deformed model at 4 different times is depicted. The chosen moments were: i) the peak of the first load (around 15 s.), ii) the beginning of the secondary plastification of the material (50 s.), iii) moments before the failure of the slope (75 s.) and iv) moments after the triggering of the slope.

5. Conclusions

A visco-plastic degradation model has been proposed in order to simulate the behavior of layers of soft clays. The non-local performance of the proposed algorithm has been carried out following concepts of non-local meshfree models capable to reproduce both brittle (Eigenerosion) and quasi-brittle (Eigensoftening) behaviors. Thus, the proposed algorithm takes the name of *Eigendegradation*.

Two main properties defined the proposed constitutive model: softening due to degradation and visco-plastic behavior. Thus, two different examples are provided in order to validate both properties.

The first example reproduces a shear test of a soil with a thin soft layer. This test shows the performance of the degradation behavior from the beginning of the reduction of the shear strength to the final of this reduction, obtaining a residual strength. This result has been compared to other similar work of degradation of clays, obtaining very close curves, what let us think the behavior of the proposed algorithm when modeling this material is correct.

The second example studies the behavior of a soil loaded by a strip footing. In this case, the viscoplastic concept is the one to be assessed. Since the von Mises yield surface is employed, the traditional Prandtl mechanism is obtained. Moreover, a hardening behavior is observed either when the viscosity becomes higher or when the load is applied quicker. This example allows us to validate the viscous behavior, as well as to

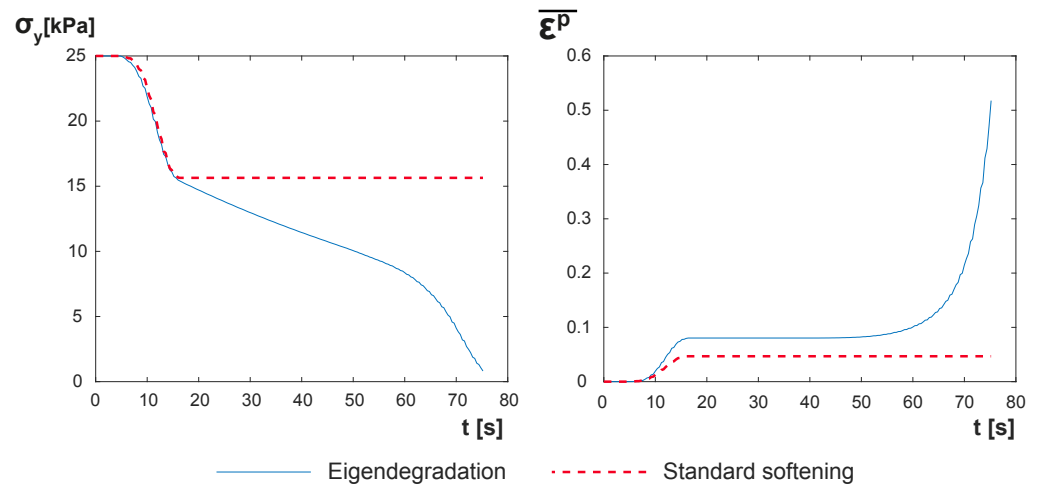


Figure 13. Evolution of the shear strength and the equivalent plastic strain along the time for both Eigendegradation and softening models.

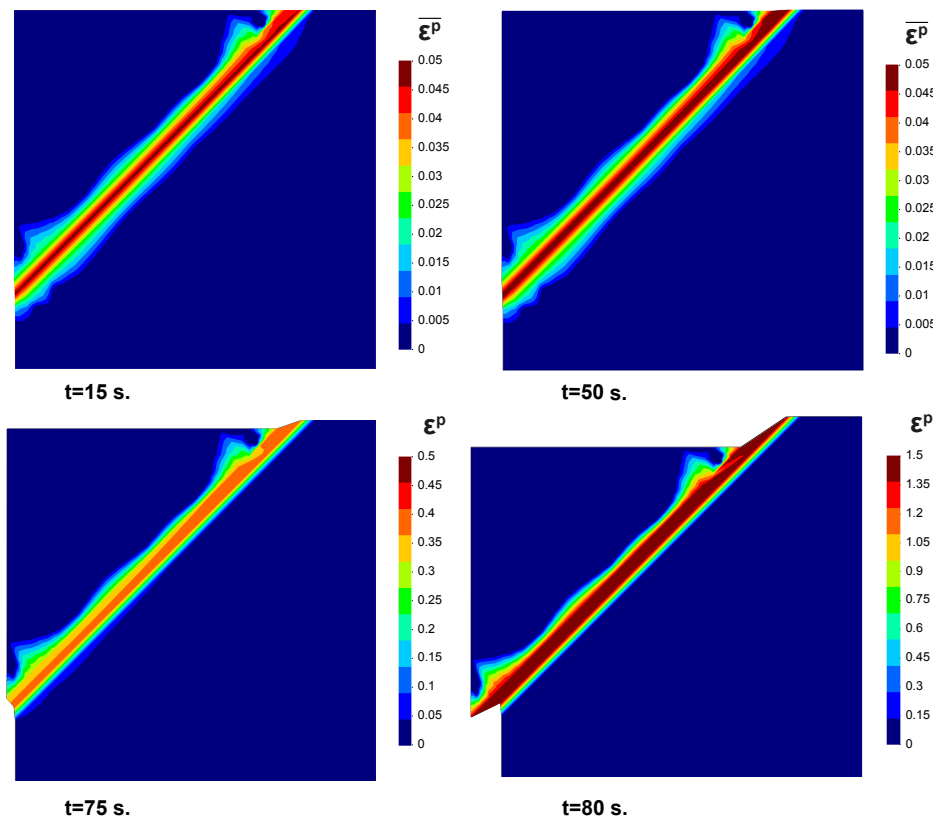


Figure 14. Distribution of the equivalent plastic strain in the deformed model at 4 different times.

verify that the degradation acts as a softening of the material that helps to define clearer the shear bands formed in the plastic mechanism.

Finally, a vertical cut of soil is modeled. This soil contains a 45° soft layer. A compound load is applied, being the first part what makes the beginning of the degradation of the material and the last part the one that makes the whole degradation of the shear strength and the final failure of the slope. In comparison with a traditional softening law, the proposed one is capable to produce degradation with elastic accumulated strain, being able the accumulation by both loading or unloading conditions. This behavior is seen in many failures of these kinds of soft layers, where a load much lower than the critical one but held in the time produces the fatigue of the material and the final failure of the soil structure.

The present algorithm has performed successfully with the proposed applications. Further research can be made in order to be able to reproduce a wide range of problems. First of all, this model should be calibrated against experimental tests such as the shear test. Moreover, only the von Mises yield surface has been validated in this manuscript. Although it is able to reproduce the undrained conditions of the soil, more sophisticated yield surfaces, such as the Cam-Clay one, would improve the type of problems to be modeled. The soil should also be modeled as a biphasic material, including the water in the formulation of the problem. Finally, some other spatial discretization, such as FEM or MPM, could be employed to assess the performance of this algorithm.

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Abbreviations

The following abbreviations are used in this manuscript:

| | |
|-----|---------------------------------|
| FEM | Finite Element Method |
| OTM | Optimal Transportation Meshfree |
| SPH | Smooth Particle Hydrodynamics |
| MPM | Material Point Method |
| ALE | Arbitrary Eulerian-Lagrangian |

References

- Drucker, D.C.; Prager, W. Soil mechanics and plastic analysis for limit design. *Quarterly of Applied Mathematics* **1952**, *10*(2), 157–165.
- Darve, F.; Laouafa, F. Instabilities in granular materials and application to landslides. *Mechanics of Cohesive-frictional Materials* **2000**, *5*, 627–652.
- Pastor, M.; Manzanal, D.; Merodo, J.A.F.; Mira, P.; Blanc, T.; Drempeit, V.; Pastor, M.J.; Haddad, B.; Sánchez, M. From solids to fluidized soils: diffuse failure mechanisms in geostructures with applications to fast catastrophic landslides. *Granular Matter* **2009**, *12*, 211–228. doi:10.1007/s10035-009-0152-4.
- Ledesma, O.; Manzanal, D.; Sfriso, A. Formulation and numerical implementation of a state parameter-based generalized plasticity model for mine tailings. *Computers and Geotechnics* **2021**, *135*, 104158. doi:10.1016/j.compgeo.2021.104158.
- Ledesma, O.; Sfriso, A.; Manzanal, D. Procedure for assessing the liquefaction vulnerability of tailings dams. *Computers and Geotechnics* **2022**, *144*, 104632. doi:10.1016/j.compgeo.2022.104632.

- 395 6. Laouafa, F.; Darve, F. Modelling of slope failure by a material instability mechanism. *Com-*
396 *puters and Geotechnics* **2002**, *29*, 301–325. doi:10.1016/s0266-352x(01)00030-1.
- 397 7. Fernández Merodo, J.A.; Pastor, M.; Mira, P.; Tonni, L.; Herreros, M.I.; Gonzalez, E.; Ta-
398 magnini, R.; Merodo, J.A.F.; Pastor, M.; Mira, P.; Tonni, L.; Herreros, M.I.; Gonzalez, E.;
399 Tamagnini, R. Modelling of diffuse failure mechanisms of catastrophic landslides. *Computer*
400 *Methods in Applied Mechanics and Engineering* **2004**, *193*, 2911–2939.
- 401 8. López-Querol, S.; Blázquez, R. Liquefaction and cyclic mobility model in saturated granular
402 media. *International Journal for Numerical and Analytical Methods in Geomechanics* **2006**, *30*, 413–
403 439. doi:10.1002/nag.488.
- 404 9. Manzanal, D.; Bertelli, S.; Lopez-Querol, S.; Rossetto, T.; Mira, P. Influence of fines content on
405 liquefaction from a critical state framework: the Christchurch earthquake case study. *Bulletin*
406 *of Engineering Geology and the Environment* **2021**, *80*, 4871–4889. doi:10.1007/s10064-021-02217-
407 2.
- 408 10. Manzanal, D.; Dremptic, V.; Haddad, B.; Pastor, M.; Stickle, M.M.; Mira, P. Application of a
409 New Rheological Model to Rock Avalanches: An SPH Approach. *Rock Mechanics and Rock*
410 *Engineering* **2016**, *49*, 2353–2372. doi:10.1007/s00603-015-0909-5.
- 411 11. Dutto, P.; Stickle, M.; Pastor, M.; Manzanal, D.; Yague, A.; Tayyebi, S.M.; Lin, C.; Elizalde,
412 M. Modelling of Fluidised Geomaterials: The Case of the Aberfan and the Gypsum Tailings
413 Impoundment Flowslides. *Materials* **2017**, *10*, 562. doi:10.3390/ma10050562.
- 414 12. Longo, A.; Pastor, M.; Sanavia, L.; Manzanal, D.; Stickle, M.M.; Lin, C.; Yague, A.; Tayyebi,
415 S.M. A depth average SPH model including $\mu(I)$ rheology and crushing for rock avalanches.
416 *International Journal for Numerical and Analytical Methods in Geomechanics* **2019**, *43*, 833–857.
417 doi:10.1002/nag.2912.
- 418 13. Lin, C.; Pastor, M.; Yague, A.; Tayyebi, S.M.; Stickle, M.M.; Manzanal, D.; Li, T.; Liu, X. A
419 depth-integrated SPH model for debris floods: application to Lo Wai (Hong Kong) debris
420 flood of August 2005. *Géotechnique* **2019**, *69*, 1035–1055. doi:10.1680/jgeot.17.p.267.
- 421 14. Pastor, M.; Yague, A.; Stickle, M.; Manzanal, D.; Mira, P. A two-phase SPH model for debris
422 flow propagation. *International Journal for Numerical and Analytical Methods in Geomechanics*
423 **2017**, *42*, 418–448. doi:10.1002/nag.2748.
- 424 15. Pastor, M.; Tayyebi, S.M.; Stickle, M.M.; Yague, A.; Molinos, M.; Navas, P.; Manzanal, D. A
425 depth integrated, coupled, two-phase model for debris flow propagation. *Acta Geotechnica*
426 **2021**, *Online*. doi:10.1007/s11440-020-01114-4.
- 427 16. Zabala, F.; Alonso, E.E. Progressive failure of Aznalcóllar dam using the material point
428 method. *Géotechnique* **2011**, pp. 795–808. doi:10.1680/geot.9.P.134.
- 429 17. Yerro, A.; Alonso, E.E.; Pinyol, N.M. The material point method for unsaturated soils.
430 *Geotechnique* **2015**, *65*, 201–217. doi:10.1680/geot.14.P.163.
- 431 18. Yerro, A.; Alonso, E.E.; Pinyol, N.M. Run-out of landslides in brittle soils. *Computers and*
432 *Geotechnics* **2016**, *80*, 427–439.
- 433 19. Feng, K.; Wang, G.; Huang, D.; Jin, F. Material point method for large-deformation modeling
434 of coseismic landslide and liquefaction-induced dam failure. *Soil Dynamics and Earthquake*
435 *Engineering* **2021**, *150*, 106907. doi:https://doi.org/10.1016/j.soildyn.2021.106907.
- 436 20. Sizkow, S.F.; El Shamy, U. SPH-DEM simulations of saturated granular soils liquefaction
437 incorporating particles of irregular shape. *Computers and Geotechnics* **2021**, *134*, 104060. doi:
438 https://doi.org/10.1016/j.compgeo.2021.104060.
- 439 21. Rice, J.R. The Initiation and growth of shear band. In *Plasticity and soil mechanics* (ed. A. C.
440 Palmer). Cambridge: Cambridge University Engineering Department., 1973, pp. 263–274.
- 441 22. Desrues, J. La localisation de la déformation dans les milieux granulaires,. PhD thesis, 1984.
- 442 23. Sulem, J.; Vardoulakis, I. *Bifurcation Analysis in Geomechanics*; CRC Press, 1995. doi:
443 10.1201/9781482269383.
- 444 24. Wang, W.; Sluys, L.; De Borst, R. Viscoplasticity for instabilities due to strain softening
445 and strain-rate softening. *International Journal for Numerical Methods in Engineering* **1997**,
446 *40*, 3839–3864.
- 447 25. Gutiérrez, M.A.; De Borst, R. Numerical analysis of localization using a viscoplastic regular-
448 ization: influence of stochastic material defects. *International Journal for Numerical Methods in*
449 *Engineering* **1999**, *44*, 1823–1841.
- 450 26. Bjerrum, L. Engineering Geology of Norwegian Normally-Consolidated Marine
451 Clays as Related to Settlements of Buildings. *Géotechnique* **1967**, *17*, 83–118. doi:
452 10.1680/geot.1967.17.2.83.

- 453 27. Kim, Y.T.; Leroueil, S. Modeling the viscoplastic behaviour of clays during consolidation:
454 application to Berthierville clay in both laboratory and field conditions. *Canadian Geotechnical*
455 *Journal* **2001**, *38*, 484–497.
- 456 28. In *Creep of Soils and Related Phenomena*; Developments in Geotechnical Engineering, Elsevier,
457 1992; pp. 3–422.
- 458 29. Manzanal, D.; Pastor, M.; Javanmardi, Y.; Fernández Merodo, J.A.; Mira, P.; Martín Stickle,
459 M.; Yagüe, A. *A reference state curve to define the state of soils over a wide range of pressures and*
460 *densities*, coupled an ed.; 2015.
- 461 30. Adachi, T.; Okano, M. A Constitutive Equation for Normally Consolidated Clay. *Soils and*
462 *Foundations* **1974**, *14*, 69–73. doi:10.3208/sandf1972.14.4_55.
- 463 31. Heeres, O.M.; Suiker, A.S.; de Borst, R. A comparison between the Perzyna viscoplastic
464 model and the Consistency viscoplastic model. *European Journal of Mechanics - A/Solids* **2002**,
465 *21*, 1–12. doi:https://doi.org/10.1016/S0997-7538(01)01188-3.
- 466 32. Perzyna, P. Fundamental Problems in Viscoplasticity. *Advances in Applied Mechanics* **1966**,
467 *9*, 243–377.
- 468 33. Naghdi, P.M.; Murch, S.A. On the Mechanical Behavior of Viscoelastic/Plastic Solids. *Journal*
469 *of Applied Mechanics* **1963**, *30*, 321–328. doi:10.1115/1.3636556.
- 470 34. Nova. A viscoplastic constitutive model for normally consolidated clays. *International union*
471 *of theoretical and applied mechanics conference on deformation and failure of granular materials*. Delft,
472 *Netherlands*. **1982**, pp. 287–295.
- 473 35. Blanc, T.; Pastor, M. A stablized {Runge-Kutta, Taylor} smoothed particle hydrodynamics
474 algorithm for large deformation problems in dynamics. *International Journal for Numerical*
475 *Methods in Engineering* **2012**, *91*, 1427–1458. doi:10.1002/nme.
- 476 36. Navas, P.; Yu, R.C.; Li, B.; Ruiz, G. Modeling the dynamic fracture in concrete: an eigensoft-
477 ening meshfree approach. *International Journal of Impact Engineering* **2018**, *113*, 9–20. doi:
478 10.1016/j.ijimpeng.2017.11.004.
- 479 37. Molinos, M.; Navas, P.; Manzanal, D.; Pastor, M. Local Maximum Entropy Material Point
480 Method applied to quasi-brittle fracture. *Engineering Fracture Mechanics* **2021**, *241*, 107394.
481 doi:10.1016/j.engfracmech.2020.107394.
- 482 38. de Souza Neto, E.A.; Pires, F.M.; Owen, D.R.J.; Andrade Pires, F.M. F-bar-based linear
483 triangles and tetrahedra for finite strain analysis of nearly incompressible solids. {Part I:}
484 formulation and benchmarking. *International Journal for Numerical Methods in Engineering*
485 **2005**, *62*, 353–383. doi:10.1002/nme.1187.
- 486 39. Owen, D.R.J.; Hinton, E. *Finite elements in plasticity—theory and practice*; Pineridge Press,
487 Swansea, 1981; p. 603. doi:10.1002/nme.1620170712.
- 488 40. Schmidt, B.; Fraternali, F.; Ortiz, M. Eigenfracture: an eigendeformation approach to varia-
489 tional fracture. *SIAM J. Multiscale Model. Simul.* **2009**, *7*, 1237–1266. doi:10.1137/080712568.
- 490 41. Pandolfi, A.; Ortiz, M. An eigenerosion approach to brittle fracture. *International Journal for*
491 *Numerical Methods in Engineering* **2012**, *92*, 694–714. doi:10.1002/nme.
- 492 42. Li, B.; Kadane, A.; Ravichandran, G.; Ortiz, M.; Kidane, A. Verification and validation of
493 the optimal-transportation meshfree (OTM) simulation of terminal ballistics. *International*
494 *Journal of Impact Engineering* **2012**, *42*, 25–36. doi:10.1016/j.ijimpeng.2011.11.003.
- 495 43. Pandolfi, A.; Li, B.; Ortiz, M. Modeling fracture by material-point erosion. *International*
496 *Journal of fracture* **2013**, *184*, 3–16. doi:10.1007/s10704-012-9788-x.
- 497 44. Yu, R.C.; Navas, P.; Ruiz, G. Meshfree modeling of the dynamic mixed-mode fracture in
498 FRC through an eigensoftening approach. *Engineering Structures* **2018**, *172*, 94–104. doi:
499 10.1016/j.engstruct.2018.06.010.
- 500 45. Bažant, Z.P.; Oh, B.H. Crack band theory for fracture in concrete. *Materials and Structures*
501 **1983**, *16*, 155–177.
- 502 46. Bažant, Z.P.; Planas, J. *Fracture and Size Effect in Concrete and Other Quasibrittle Materials*; New
503 directions in Civil Engineering, CRC Press, Boca Raton, Florida, USA, 2019; pp. 1–170. doi:
504 10.1201/9780203756799.
- 505 47. Zhang, W.; Wang, D.; Randolph, M.; Puzrin, A., M. Catastrophic failure in planar landslides
506 with a fully softened weak zone. *Géotechnique* **2015**, *65*, 755–769. doi:10.1680/geot14.P218.
- 507 48. Wang, L.; Zhang, X.; Tinti, S. Large deformation dynamic analysis of progressive failure in
508 layered clayey slopes under seismic loading using the particle finite element method. *Acta*
509 *Geotechnica* **2021**, *16*, 2435–2448. doi:10.1007/s11440-021-01142-8.

- 510 49. Singh, V.; Stanier, S.; Bienen, B.; Randolph, M.F. Modelling the behaviour of sensitive clays
511 experiencing large deformations using non-local regularisation techniques. *Computers and*
512 *Geotechnics* **2021**, *133*, 104025. doi:10.1016/j.compgeo.2021.104025.
- 513 50. Li, B.; Habbal, F.; Ortiz, M. Optimal transportation meshfree approximation schemes for
514 fluid and plastic flows. *International Journal for Numerical Methods in Engineering* **2010**,
515 *83*, 1541–1579. doi:10.1002/nme.
- 516 51. Li, B.; Stalzer, M.; Ortiz, M. A massively parallel implementation of the Optimal Transporta-
517 tion Meshfree (pOTM) method for explicit solid dynamics. *International Journal for Numerical*
518 *Methods in Engineering* **2014**, *100*, 40–61.
- 519 52. Huang, D.; Weißenfels, C.; Wriggers, P. Modelling of serrated chip formation processes using
520 the stabilized optimal transportation meshfree method. *International Journal of Mechanical*
521 *Sciences* **2019**, *155*, 323–333. doi:10.1016/j.ijmecsci.2019.03.005.
- 522 53. Navas, P.; Manzanal, D.; Martín Stickle, M.; Pastor, M.; Molinos, M. Meshfree modeling of
523 cyclic behavior of sands within large strain Generalized Plasticity Framework. *Computers*
524 *and Geotechnics* **2020**, *122*, 103538. doi:10.1016/j.compgeo.2020.103538.
- 525 54. Navas, P.; Pastor, M.; Yagüe, A.; Stickle, M.M.; Manzanal, D.; Molinos, M. Fluid stabilization
526 of the u-w Biot's formulation at large strain. *International Journal for Numerical and Analytical*
527 *Methods in Geomechanics* **2021**, *45*, 336–352. doi:https://doi.org/10.1002/nag.3158.
- 528 55. Arroyo, M.; Ortiz, M. Local maximum-entropy approximation schemes: a seamless bridge
529 between finite elements and meshfree methods. *International Journal for Numerical Methods in*
530 *Engineering* **2006**, *65*, 2167–2202. doi:10.1002/nme.1534.
- 531 56. Kontoe, S. Developement of time integration schemes and advanced boundary conditions
532 for dynamic geotechnical analysis. PhD thesis, D. thesis, 2006.
- 533 57. Wriggers, P. *Nonlinear Finite Element Methods*; Vol. 2008, Springer, 2008.
- 534 58. Navas, P.; López-Querol, S.; Yu, R.C.; Pastor, M. Optimal transportation meshfree method in
535 geotechnical engineering problems under large deformation regime. *International Journal for*
536 *Numerical Methods in Engineering* **2018**, *115*, 1217–1240. doi:10.1002/nme.5841.