Eigendegradation algorithm applied to visco-plastic weak layers

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1 **Abstract:** XXXXXX

4 **1. Introduction**

5 **2. Constitutive model**

6 *2.1. Rate dependent plasticity*

7   Along with the mathematical and numerical models described so far, it is essential
8 to choose an appropriate constitutive model for the materials. It has been studied
9 how clays suffer a viscous phenomena and delayed deformation due to creep behavior.
10 The concept of the viscoplastic model described in this section and used in the further
11 simulations is based on Perzyna's theory (1966-CITA). Perzyna's theory is a modification
12 of classical plasticity wherein viscous-like behavior is introduced by a time-rate flow
13 rule employing a plasticity yield function. Similar to the rate-independent theory the
14 strain rate is decomposed into an elastic and a viscoplastic strain rate:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{vp}} \tag{1}$$

15 The stress rate tensor $\dot{\sigma}$ is related to the elastic strain rate via a constitutive tensor
16 $\boldsymbol{D}_e$, which is constant in the case of linear elasticity and variable (stress dependent) in
17 the case of hypo or hyperelasticity:

$$\dot{\sigma} = \mathbf{D}_e(\dot{\varepsilon} - \dot{\varepsilon}^{vp}) \tag{2}$$

18   In the theory proposed by Perzyna (1966-CITA), the viscoplastic strain rate is
19 defined in a similar fashion as in the rate independent plasticity theory:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{vp}} = \gamma \langle \phi(f) \rangle \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{3}$$

20 where γ is the fluidity parameter (which is the reciprocal of viscosity). $\langle \phi(f) \rangle$ is the
21 viscous flow function, which represents the current magnitude of viscoplastic strain rate.
22 g denotes the viscoplastic potential function and f any valid plasticity function playing
23 the role of loading surface. A von-Mises yield surface, with a degradation curve for the
24 undrained shear strength, has been adopted for the failure criteria. Associative flow is
25 invoked by $f = g$. $\frac{\partial g}{\partial \sigma}$ represents the current direction of the viscoplastic strain rate. The
26 viscous flow function is defined by:

$$\langle \phi(f) \rangle = \left\{ \begin{array}{ll} \phi(f) & , f > 0 \\ 0 & , f \leq 0 \end{array} \right. , \tag{4}$$

where $<>$ denotes Macauley brackets. A choice for the function ϕ is:

$$\phi = \left(\frac{f - f_o}{f_o}\right)^\alpha \tag{5}$$

in which α is a material constant. Concerning algorithm aspects, in displacement-based finite element formulations, stress updates take place at the Gauss points for a known nodal displacement. We start from time t_n with the known converged state:

$$\left[\varepsilon_n, \varepsilon_n^{vp}, \sigma_n, \kappa_n\right] \tag{6}$$

(namely total strain, viscoplastic strain, stress and a scalar internal variable that characterizes the size of the loading surface for the purpose of introducing hardening or softening behavior) to calculate the corresponding values at time

$$t_{n+1} = t_n + \Delta t : \left[\varepsilon_{n+1}, \varepsilon_{n+1}^{vp}, \sigma_{n+1}, \kappa_{n+1}\right]. \tag{7}$$

In this incremental process, from Eqs. **??** and **??**:

$$\Delta\varepsilon = \varepsilon^e + \Delta\varepsilon^{vp} \tag{8}$$
$$\Delta\sigma = \mathbf{D}_e(\Delta\varepsilon - \Delta\varepsilon^{vp}) \tag{9}$$

Therefore the key feature of the stress updates is characterized by estimating the incremental viscoplastic strain $\Delta\varepsilon^{vp}$. Details of the numerical implementation can be found in the textbooks (Owen and Hinton 1986 CITA; De Souza Neto et al. 2008 - CITA). It has to be noticed also that for softening problems the viscoplasticity approach has a regularizing effect in the sense that the initial-value problem remains well-posed avoiding instability due to strain and strain-rate softening (Wang et al. 1997-CITA).

2.2. Eigenerosion and Eigensoftening algorithms

Within the context of OTM formulation, fracture can be modeled simply by eroding material points according to an energy-release criterion or stress criterion depending if eigenerosion [**? ? ? ?**] or eigensoftening [**? ?**] algorithm is adopted respectively. In both methodologies, when the material points are failed, they are neglected from the computation of stresses in the model, which approximates the presence of cracks. However, the way to reach the zero stiffness state is different for each model: meanwhile in the eigenerosion the failure is instantaneous, in the eigensoftening the material follows a softening curve which depends on the material. It needs to be pointed out that when a material point satisfies the erosion condition, its contribution to the internal force vector and to the material stiffness matrix is set to zero, but its contribution to the mass matrix is maintained. The mass of a material point is discarded only when an eroded material point is not connected to any nodes.

Next we compute the energy-release rate attendant to the failure of material point p, the starting point of the aforementioned methodologies:

$$G_{p,k+1} = \frac{C\epsilon}{m_{p,k+1}} \sum_{x_{q,k+1}\in B_\epsilon(x_{p,k+1})} m_q W_k(F_{q,k+1}),$$
$$m_{p,k+1} = \sum_{x_{q,k+1}\in B_\epsilon(x_{p,k+1})} m_q, \tag{10}$$

where $B_\epsilon(x_{p,k+1})$ is the sphere of radius ϵ centered at $x_{p,k+1}$ known as the ϵ-neighborhood of the material point, $m_{p,k+1}$ is the mass of the neighborhood at loading step $k+1$, $W_k(F_{q,k+1})$ is the current free-energy density per unit mass at the material point $x_{q,k+1}$ and C is a normalizing constant.
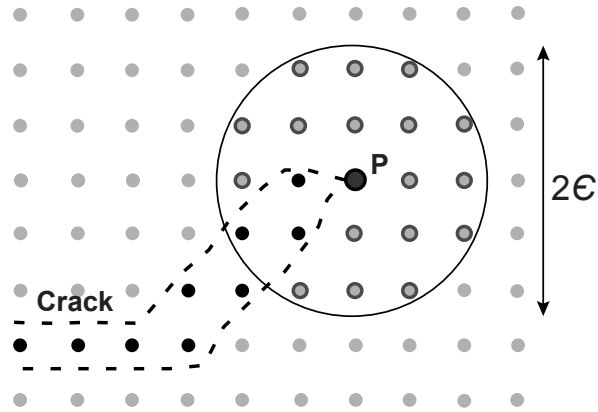
Figure 1. Scheme of a fractured layer (black dots) as set of failed material points, and of the ϵ-neighborhood (inside the circle) of the material point located at the crack tip (grey dots).

The material point is failed when $G_{p,k+1}$ surpasses a critical energy release rate that measures the material-specific energy, G_F, required to create a fracture surface of unit area. In calculations, the failed material-point set is updated at every time step according to this criterion. For linear elasticity, Schmidt *et al*. [?] have shown that this approximation converges to Griffith fracture in the limit of an infinitely fine discretization. Indeed, erosion schemes that estimate the energy-release rate based on the energy of a single material point suffer from mesh-dependency and may overestimate the toughness of the material.

A scheme of the configuration of the ϵ-neighborhood is plotted in Fig. 1.

On the other hand, the implementation of the eigensoftening algorithm consists in adopting a strength criterion for crack initiation and a softening law which is proper to the material under study before the formation of a stress-free crack. This second process tends to accumulate less energy until the crack. When the maximum tensile strength, f_t, is reached, a cohesive crack is formed with zero opening displacement. Once the opening displacement, w, reaches a critical vale, w_c, a stress-free crack is attained. The energy below the softening curve represents the static fracture energy per unit of area, G_F, which is sketched in Fig. 2. For the Eigensoftening calculation, Eq. (10) can be rewritten in terms of the principal stresses at time t_{k+1}, since this model employs the first principal stress as a failure criterion. Therefore, the variation of the averaged strain energy density in the ϵ-neighborhood of the material point $x_{p,k+1}$ can be expressed as,

$$\delta W_p^\epsilon = \frac{\partial G_p}{C\epsilon} = \frac{1}{m_p} \sum_{x_q \in B_\epsilon(x_p)} m_q \sigma_{q,1} \delta \varepsilon_q, \tag{11}$$

where $\sigma_{q,1}$ is the maximum principal stress at a neighboring material point $x_{q,k+1}$. Considering an effective strain ε_q at the material point $x_{q,k+1}$, such that the variation of the local strain energy can be obtained as $\delta W_q = \sigma_{q,1}\delta\varepsilon_q$, let assume the effective strain increment of each material point can be approximated by its counterpart in the neighborhood, being Eq. (11) simplified as:

$$\delta W_p^\epsilon = \frac{\delta \varepsilon_p}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \sigma_{q,1}. \tag{12}$$

Thus, the equivalent critical stress at the material point $x_{p,k+1}$ is defined as follows

$$\sigma_p^\epsilon = \frac{1}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \sigma_{q,1} \tag{13}$$
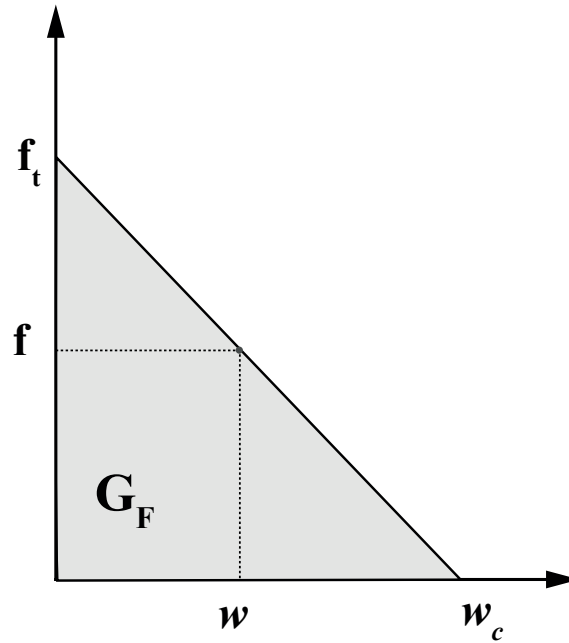
Figure 2. Scheme of a linear cohesive law, where the shade area is G_F, f_t is the tensile strength, and w_c is the critical opening displacement.

When $\sigma^\epsilon_{p,k+1}$ surpasses the tensile strength, f_t, the softening behavior is activated through the damage variable χ, which ranges between zero (an intact material) and one (completely failed material points). Of coarse, χ depends on the current and critical opening measures, w and w_c respectively. The later is a material parameter but the first one has to be measured in terms of the achieved strain and a length of affection called band width, h_ϵ, equivalent to the crack band model of Bažant [**?**]. It bears emphasis that a reference value for h^ϵ is between two and four times the maximum size of the aggregates for concrete according to Bažant [**?**]. Thus, this is a material parameter more than a numerical artifact. The relationship between strain and crack opening depends on the effective fracture strain, ε^ϵ_f, defined as the difference between the strain at crack initiation, $\varepsilon_1(x_{p,0})$, and the current strain, $\varepsilon_1(x_{p,k+1})$ for a material point p; and the band width as:

$$\varepsilon^\epsilon_f = \varepsilon_1(x_{p,k+1}) - \varepsilon_1(x_{p,0}) = \frac{w}{h^\epsilon} \tag{14}$$

2.3. Eigendegradation model

Following the work of Einav and Randolph (CITA), and the later implementations of .....(CITAS), the behavior of sensitive clays can be modeled by strain softening curves in order to reduce the strength of the material by a degradation obtained by accumulation of strain. Einav and Randolph assumed that the current shear strength depends on the accumulated absolute shear strain, ξ, which is taken as a state variable from which an isotropic strength reduction, $\delta(\xi)$, is calculated as

$$\delta(\xi) = s_u/s_{ui} = \delta_{\text{rem}} + (1 - \delta_{\text{rem}})e^{-3\xi/\xi_{95}} \tag{15}$$

where

$$\xi = \int_t |\dot{\gamma}_{\text{max}}| \, dt \tag{16}$$

and $|\dot{\gamma}_{\text{max}}|$ is the cumulative absolute shear strain, s_u and s_{ui} are the softened strength and initial strength, respectively, δ_{rem} is the fully remoulded strength ratio, and ξ_{95} is the cumulative shear strain required to cause 95% reduction (from peak to

remoulded). The assumption is that δ_{rem} may be taken as the inverse of the sensitivity of the soil, while an appropriate value for ξ_{95} must be deduced from laboratory test data, or by conducting cyclic penetration and extraction tests with T-bar or ball.

The calculation of the cumulative shear strain can be achieved by the eigendeformation technique, departing from Eq. (11), and considering, for the Eigendegradation calculation, that the stress remains constant in a neighborhood ϵ. Thus, Eq. (11) can be simplified as

$$\delta W_p^\epsilon = \frac{\delta \tau_p}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \gamma_q, \tag{17}$$

being $\delta \tau_p$ the increment of tangential stress of the neighborhood and γ_p^ϵ the current local shear strain, calculated as:

$$\gamma_p^\epsilon = \frac{1}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \gamma_q \tag{18}$$

Similarly, in the neighborhood ϵ, the non-local cumulative strain of a material point p is calculated, only when plasticity is activated, as follows:

$$\xi_p^\epsilon = \int_{t_{p0}}^{t_{k+1}} \left| \dot{\gamma}_p^\epsilon \right| \mathrm{d}t \tag{19}$$

being t_{k+1} referred to the current step and t_{p0} to the step when plasticity begins.

Considering only shear failure, yield shear stress τ is equivalent to the softened strength, s_u, and the residual yield shear stress, τ_{95}, can be reached by $\tau_{95} = \tau rem = s_{ui} \delta_{rem}$. Thus, in every state of degradation, the current yield shear stress, referred to the epsilon neighborhood, τ^ϵ, reads:

$$\tau^\epsilon = \tau_{95} + (\tau_i - \tau_{95})e^{-3\xi_p^\epsilon/\xi_{95}} \tag{20}$$

It is remarkable that, in laboratory, parameter ξ_{95} is not obtained. Instead, the displacement δ_{95} is achieves. In Fig. 3.A) the degradation of the strength in terms of the displacement is plotted. It can be seen how this law can be translated to the shear strain measurement (Fig. 3.B) by multiplying by ϵ, which, in this problem, is considered as the sliding length. Depending on the size of the soft layer, this parameter ϵ is obtained as the minimum length between the neighbor radius and the size of the soft layer (See Fig. 4) as follows:

$$2\epsilon = \min(h_s, 2C_\epsilon h) \tag{21}$$

2.4. Visco-Plastic Eigendegradation algorithm

Following, the pseudo-algorithm for the Eigendegradation model within a visco-plastic yield surface will be presented. It is worth mentioning that prior to the algorithm steps, we need to calculate the equivalent shear total strain of every material point as the norm of the deviatoric total strain tensor. Since large strain is considered, the strain tensor is obtained through the logarithm of the left Cauchy-Green strain tensor, $\mathbf{b}$:
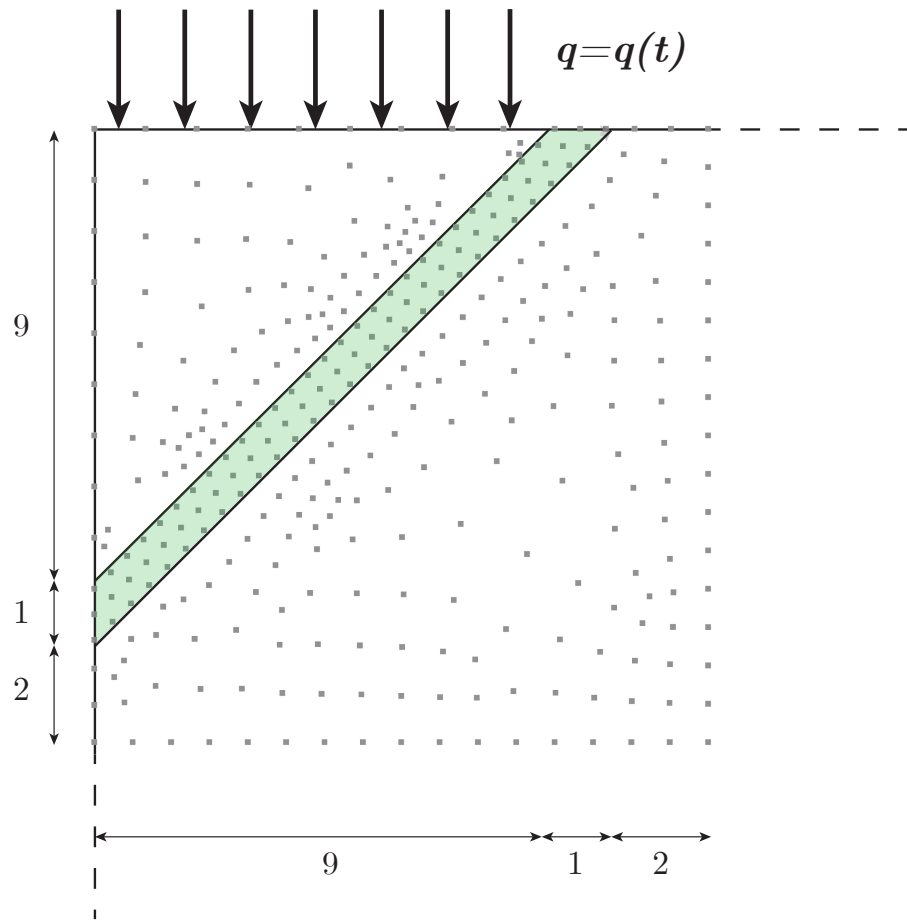
$$\varepsilon = \frac{1}{2} \log \boldsymbol{b} = \frac{1}{2} \log \boldsymbol{F}\boldsymbol{F}^T. \tag{22}$$
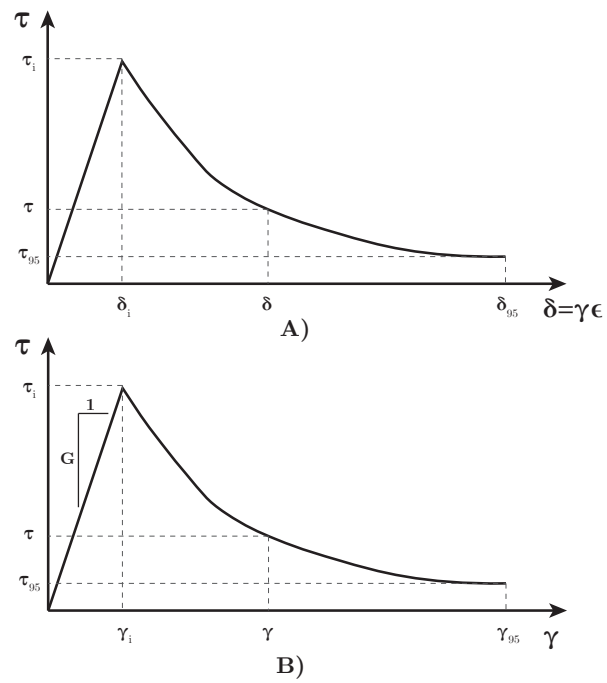
Figure 3. Degradation curve in terms of the displacement, A), and the shear strain, B).
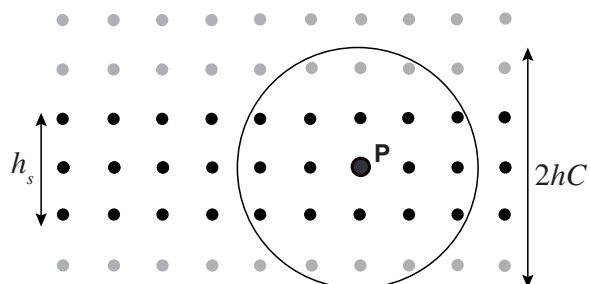
Figure 4. Scheme of the measurements of the soft layer (black dots) and the ϵ-neighborhood around material point P.

Figure 5. Geometry of the vertical cut analyzed through *Eigendegradation* and softening models and the location of the weak layer and the loaded zone. (Units in meters)

3. Computational tools

4. Applications

4.1. Shear test

*4.2. Strip footing load*

4.3. Vertical cut

This final application allows to understand the potentiality of the proposed methodology. A weak layer is supposed in a soil with a vertical cut on the left side. This weak layer is located forming a 45° angle, as it is sketched in Fig. 5. This layer, whose thick is 1 meter, will be considered plastic. Von Mises yield surface is employed, being its degradation modeled through both *Eigendegradation* and traditional softening in order to assess the performance of the first one. Out of the weak layer, the soil is considered elastic since its failure is far from the failure of the weak layer. The soil can be considered infinite on the right and on the bottom of the model.

The top left part is loaded by a surface load as it is shown in Fig. 5. This load is composed by two different waves, as it is sketched in Fig. 6.

Both waves follow the expression:

$$q_\alpha(t) = A_\alpha \cdot [1 - cos(\omega_\alpha t)]$$

where α varies for each of both loads. The bigger load is the one that provokes the triggering of the plastic mechanism. It can be considered as a abnormal scenario that
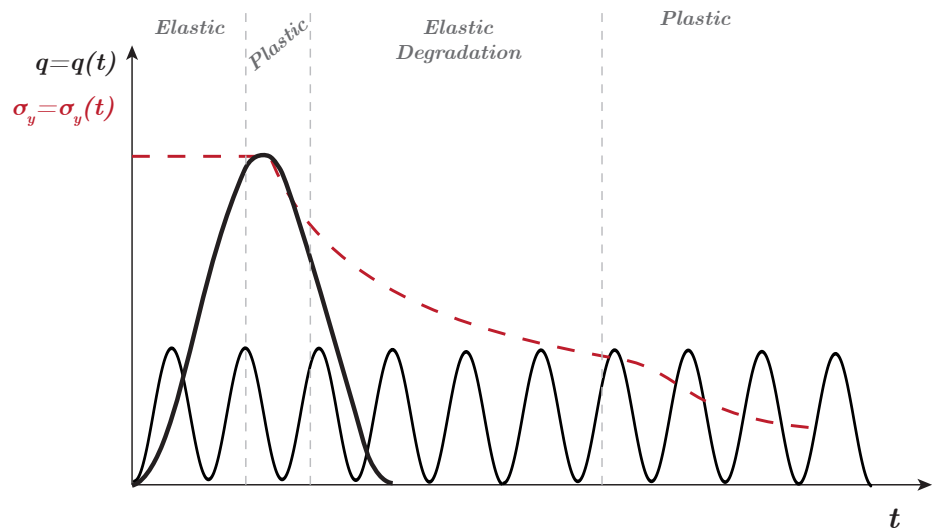
Figure 6. Scheme of the load and the yield strength along the time.

may lead to catastrophic consequences in the short or in the long time. Its parameters are

5. Conclusions

Abbreviations

The following abbreviations are used in this manuscript:

OTM	Optimal Transportation Meshfree
SPG	Smooth Particle Hydrodynamics
MPM	Material Point Method

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Algorithm 1 Visco-Plastic Eigendegradation algorithm

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1. Calculation of the small strain tensor

$$\varepsilon_{k+1}^{e\ trial} = \frac{1}{2} \log \mathbf{b}_{k+1}^{e\ trial}$$

2. Elastic Predictor: volumetric and deviatoric stress measurements

$$\text{Volumetric: } p_{k+1}^{trial} = K \left(\varepsilon_{vol}^{e} \right)_{k+1}^{trial}$$
$$\text{Deviatoric: } \mathbf{s}_{k+1}^{trial} = 2G \left(\varepsilon_{dev}^{e} \right)_{k+1}^{trial}$$

$$\text{being: } \boldsymbol{\infty}_{k+1}^{trial} = J^{-1} \boldsymbol{\infty}_{k+1}^{trial}$$

$$\text{and: } q_{k+1}^{trial} = \sqrt{\frac{3}{2}} \| \mathbf{s}_{k+1}^{trial} \|$$

3. Eigendegradation calculation:

if $t < t_{p0}$ **then**
$\sigma_y = \tau_i$
else

- $m_p = \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q$

- $\gamma_p^\epsilon = \frac{1}{m_p} \sum_{x_{q,k+1} \in B_\epsilon(x_{p,k+1})} m_q \gamma_q$

- $\xi_p^\epsilon = \sum_{k(t_{p0})}^{k+1} \left| \Delta \gamma^\epsilon(x_{p,k}) \right|$

- $\sigma_y = \tau_{95} + (\tau_i - \tau_{95}) e^{-3\xi_p^\epsilon / \xi_{95}}$

- Hardening modulus:

$$H = \frac{\partial \sigma_y}{\partial \bar{\varepsilon}^p} \simeq \frac{\partial \sigma_y}{\partial \xi_p^\epsilon} = -\frac{3(\tau_i - \tau_{95})}{\xi_{95}} e^{-3\xi_p^\epsilon / \xi_{95}}$$

end if

4. Yield condition: $\Delta \lambda = 0$

if $\phi = \frac{q_{k+1}^{trial} - \sigma_y}{\sigma_y} \leq 0$ **then** Elastic region:

$\boldsymbol{\infty}_{k+1} = \boldsymbol{\infty}_{k+1}^{trial}$

else Viscoplastic flow:

- **4.1** Derivative of the yield surface:

$$d = \frac{\partial \phi}{\partial \Delta \lambda} =$$
$$= -\frac{3G}{\sigma_y} - H \frac{q_{k+1}^{trial} - 3G\Delta\lambda}{\sigma_y^2} - \frac{1}{\alpha \gamma \Delta t} \left(\frac{\Delta \lambda}{\gamma \Delta t} \right)^{\frac{1}{\alpha} - 1}$$

- **4.2** Increment of plastic strain: $\Delta \lambda = \Delta \lambda - \frac{\phi}{d}$
- **4.3** Yield function:

$$\phi = \frac{q_{k+1}^{trial} - 3G\Delta\lambda}{\sigma_y} - \left[1 + \left(\frac{\Delta \lambda}{\gamma \Delta t} \right)^{1/\alpha} \right]$$

- **4.4** If $\phi < tolerance$ go to **4.5**, else go to **4.1**
- **4.5** Update

$$\bar{\varepsilon}_{k+1}^p = \bar{\varepsilon}_k^p + \Delta \gamma$$

$$\Delta \varepsilon_{k+1}^p = \frac{\Delta \gamma}{\| \mathbf{s}_{k+1}^{trial} \|} \mathbf{s}_{k+1}^{trial}$$

$$\left({}^{trial} \right) \left(1 - \frac{3G\Delta\gamma}{} \right) {}^{trial}$$