Structural Models: Beam Elements

Finite Element Methods

José M. Goicolea

Grupo de Mecánica Computacional, Escuela de Ingenieros de Caminos,
Universidad Politécnica de Madrid

19 October 2023

Outline

Índice

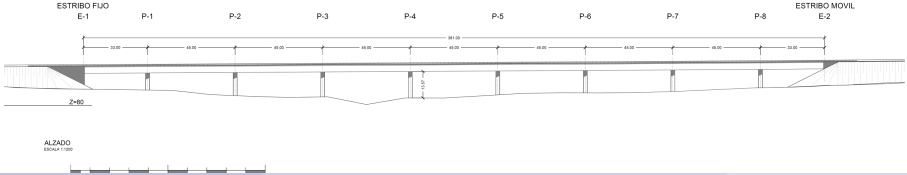
Las Piedras. $L=19\times 63.5=1206.5$ m; $H=92$ m

Córdoba–Málaga HS line

Contreras viaducts; Arch L=261 m

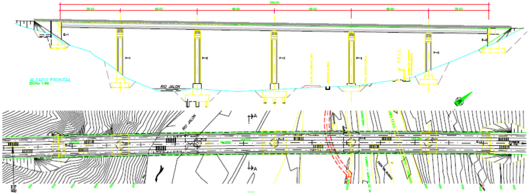
Madrid–Valencia HS line

La Marota. L=381 m; Córdoba–Málaga HS line

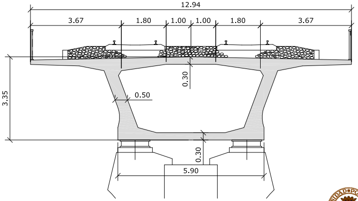


Villanueva del Jalón Bridge

Madrid-Barcelona HS line

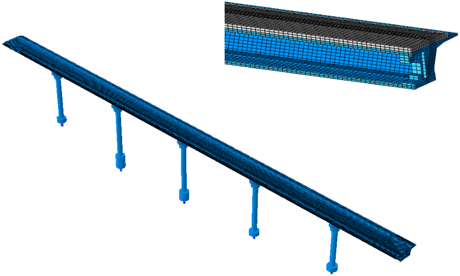


- Madrid-Barcelona High Speed Railway
- Prestressed concrete
- Total length: 250 m
- 6 spans (35 + 4 x 45 + 35)

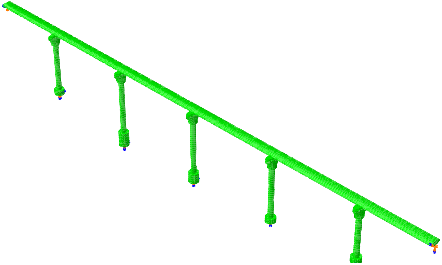


Villanueva del Jalón: types of FE models

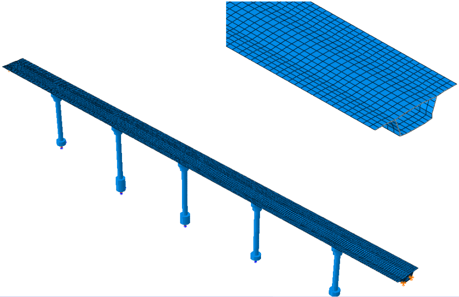
Continuum elements



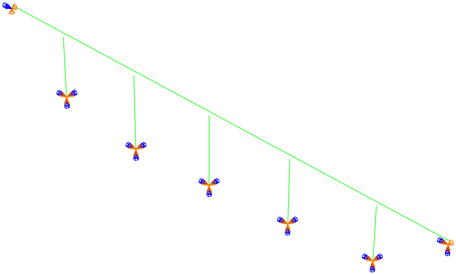
Beam elements (ornate)



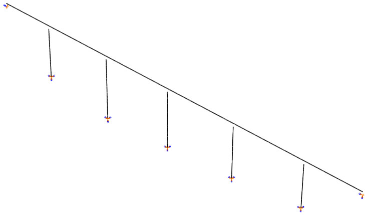
Shell elements



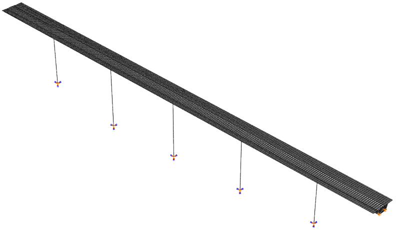
Beam elements (plain)



Villanueva del Jalón: Numerical models



FEM (Beams)



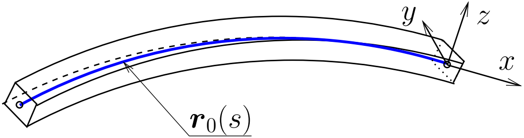
FEM (Shells)

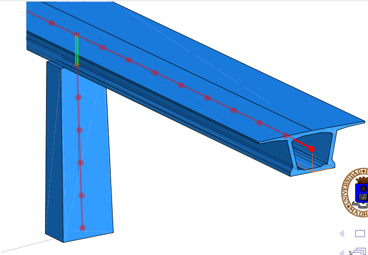
	Frequency [Hz]		
	1st lateral	1st vertical	1st torsion
Beam elements	0.623	**3.228**	3.498
Shell elements	**0.627**	3.059	**5.823**

Índice

Assumptions:

- Prismatic slender rod, with one dimension (longitudinal) much greater than the other two (transverse) dimensions
- Base curve (*"directriz"*) $\boldsymbol{r}_0(s)$, with $s =$ arc-length;
- Normal cross section at each point, $\perp \boldsymbol{r}_0(s)$.
  Rigid (plane and undeformable)
- Local axes: x tangent to base curve; (y, z) transverse (normal to x).
- 2D beams: loads (q_x, q_z) and deflection (u, w) within xz plane

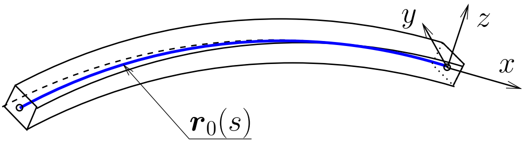


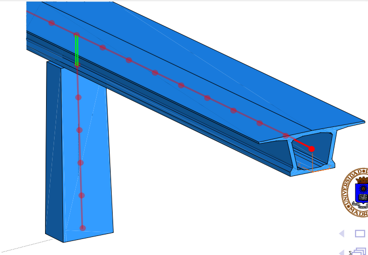


Beams: General Assumptions

Assumptions:

- Prismatic slender rod, with one dimension (longitudinal) much greater than the other two (transverse) dimensions
- Base curve (*"directriz"*) $\boldsymbol{r}_0(s)$, with $s =$ arc-length;
- Normal cross section at each point, $\perp \boldsymbol{r}_0(s)$.
  Rigid (plane and undeformable)
- Local axes: x tangent to base curve; (y, z) transverse (normal to x).
- 2D beams: loads (q_x, q_z) and deflection (u, w) within xz plane

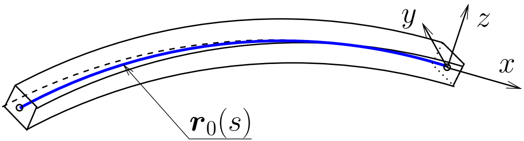


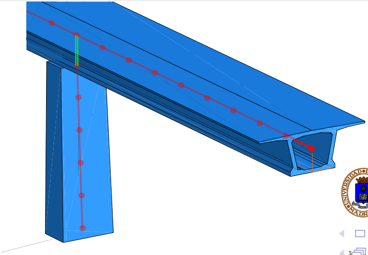


Beams: General Assumptions

Assumptions:

- Prismatic slender rod, with one dimension (longitudinal) much greater than the other two (transverse) dimensions
- Base curve (*"directriz"*) $\boldsymbol{r}_0(s)$, with $s =$ arc-length;
- Normal cross section at each point, $\perp \boldsymbol{r}_0(s)$.
  Rigid (plane and undeformable)
- Local axes: x tangent to base curve; (y, z) transverse (normal to x).
- 2D beams: loads (q_x, q_z) and deflection (u, w) within xz plane

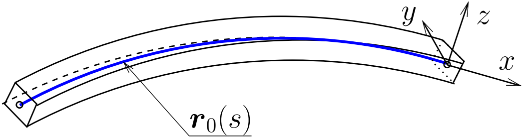


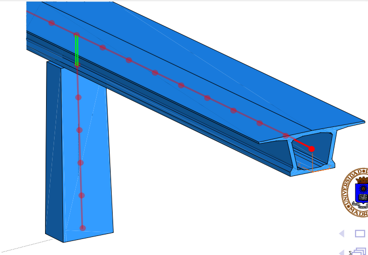


Beams: General Assumptions

Assumptions:

- Prismatic slender rod, with one dimension (longitudinal) much greater than the other two (transverse) dimensions
- Base curve (*"directriz"*) $\boldsymbol{r}_0(s)$, with $s =$ arc-length;
- Normal cross section at each point, $\perp \boldsymbol{r}_0(s)$.
  Rigid (plane and undeformable)
- Local axes: x tangent to base curve; (y, z) transverse (normal to x).
- 2D beams: loads (q_x, q_z) and deflection (u, w) within xz plane

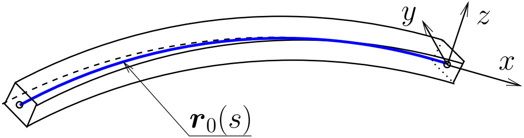


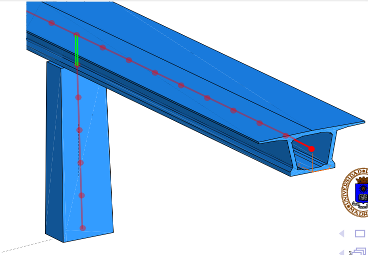


Beams: General Assumptions

Assumptions:

- Prismatic slender rod, with one dimension (longitudinal) much greater than the other two (transverse) dimensions
- Base curve (*"directriz"*) $\boldsymbol{r}_0(s)$, with s = arc-length;
- Normal cross section at each point, $\perp \boldsymbol{r}_0(s)$.
  Rigid (plane and undeformable)
- Local axes: x tangent to base curve; (y, z) transverse (normal to x).
- 2D beams: loads (q_x, q_z) and deflection (u, w) within xz plane





Axial resultant:

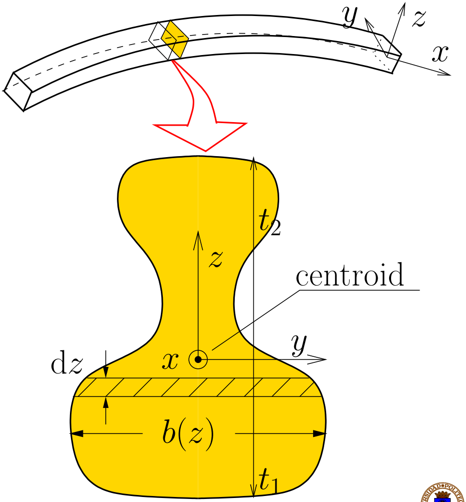
$$N \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \sigma_{xx} b(z)\,\mathrm{d}z \qquad (1)$$

Shear resultant:

$$V \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \sigma_{xz} b(z)\,\mathrm{d}z \qquad (2)$$

Bending moment:

$$M \stackrel{\text{def}}{=} -\int_{t_1}^{t_2} z\sigma_{xx} b(z)\,\mathrm{d}z \qquad (3)$$



2D Beams: Section Resultants

Axial resultant:

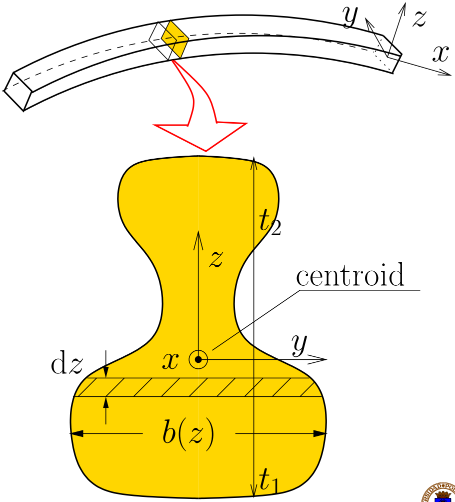
$$N \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \sigma_{xx} b(z)\,\mathrm{d}z \qquad (1)$$

Shear resultant:

$$V \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \sigma_{xz} b(z)\,\mathrm{d}z \qquad (2)$$

Bending moment:

$$M \stackrel{\text{def}}{=} -\int_{t_1}^{t_2} z\sigma_{xx} b(z)\,\mathrm{d}z \qquad (3)$$



2D Beams: Section Resultants

Axial resultant:

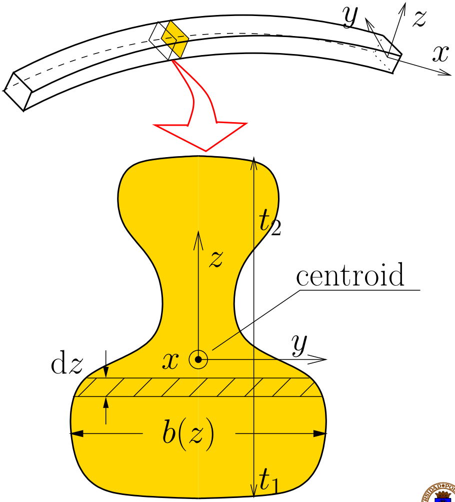
$$N \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \sigma_{xx} b(z)\,\mathrm{d}z \qquad (1)$$

Shear resultant:

$$V \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \sigma_{xz} b(z)\,\mathrm{d}z \qquad (2)$$

Bending moment:

$$M \stackrel{\text{def}}{=} -\int_{t_1}^{t_2} z\sigma_{xx} b(z)\,\mathrm{d}z \qquad (3)$$



2D Beams: Equilibrium equations

Axial equilibrium

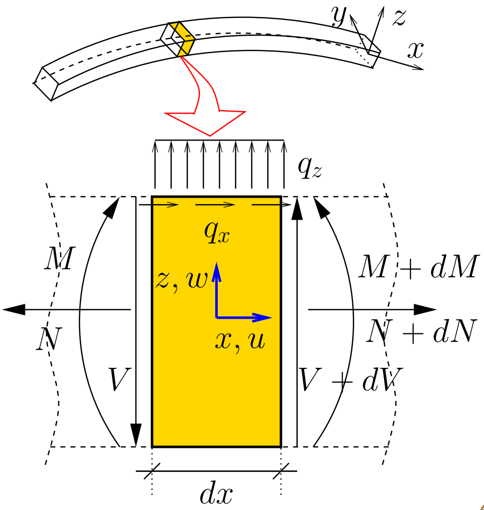
$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0; \qquad (4)$$

Shear equilibrium

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0; \qquad (5)$$

Bending moment equilibrium

$$\frac{\mathrm{d}M}{\mathrm{d}x} + V = 0 \qquad (6)$$



Axial equilibrium

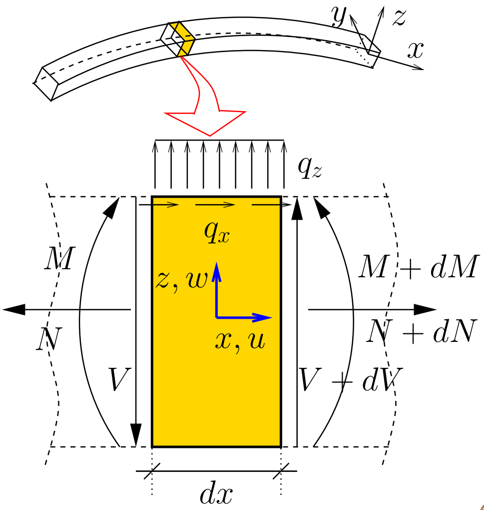
$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0; \qquad (4)$$

Shear equilibrium

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0; \qquad (5)$$

Bending moment equilibrium

$$\frac{\mathrm{d}M}{\mathrm{d}x} + V = 0 \qquad (6)$$



Axial equilibrium

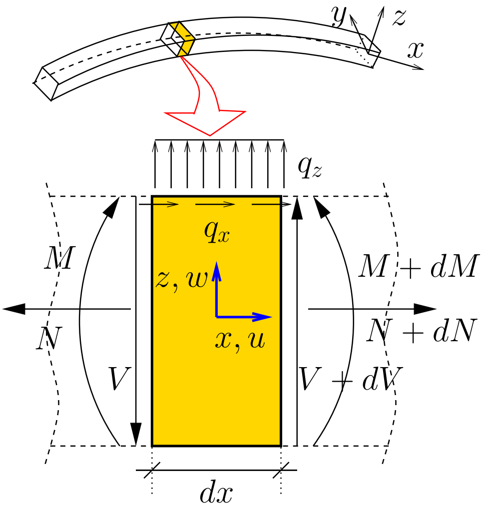
$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0; \qquad (4)$$

Shear equilibrium

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0; \qquad (5)$$

Bending moment equilibrium

$$\frac{\mathrm{d}M}{\mathrm{d}x} + V = 0 \qquad (6)$$

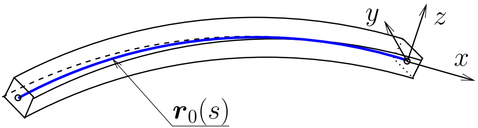


♣ Universal validity, even with material or geometric nonlinearities

Índice

Bernoulli-Euler Assumptions

1. Prismatic rod, with base curve *("directriz")* $\boldsymbol{r}_0(s)$ (line of centroids)



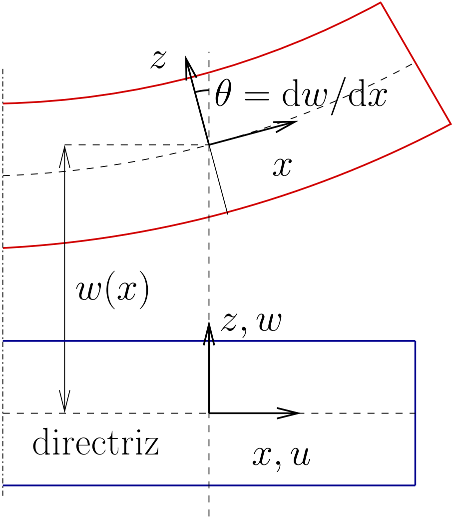
2. After deformation, plane cross sections (yz) remain plane and normal to the base curve
3. (2D) Displacements normal to the beam ($w \stackrel{\text{def}}{=} u_z$) are uniform along the cross section and equal to those of the base curve:

$$\boxed{w(x, z) = w(x).} \tag{7}$$

4. (2D) Plane stress:

$$\boxed{\sigma_{zz} = 0.} \tag{8}$$

Displacements (2D)



- Rotation of cross section: θ
- Rotation of tangent to base curve: $\dfrac{\mathrm{d}w}{\mathrm{d}x} = w'$
- Cross section normal to base curve:

$$\boxed{\theta = \frac{\mathrm{d}w}{\mathrm{d}x}} \tag{9}$$

- Longitudinal displacements:

$$\begin{aligned} u(x,z) &= u_0(x) - z\theta \\ &= u_0(x) - z\frac{\mathrm{d}w}{\mathrm{d}x} \end{aligned} \tag{10}$$

Strains

From the above displacement equations,

$$(??) \longrightarrow \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}; \tag{11}$$

$$(??) \longrightarrow \qquad \gamma_{xz} \overset{\text{def}}{=} 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \tag{12}$$

$$\gamma_{yz} = \gamma_{xy} = 0 \tag{13}$$

$$(??) \longrightarrow \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0; \tag{14}$$

Implications

- No shear deformation: $\boxed{\gamma_{xz} = 0}$
- Assumption valid for slender (thin) beams: $\lambda = \frac{l}{t} \gtrapprox 20.$

Strains

From the above displacement equations,

$$\text{(??)} \longrightarrow \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}; \tag{11}$$

$$\text{(??)} \longrightarrow \qquad \gamma_{xz} \overset{\text{def}}{=} 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \tag{12}$$

$$\gamma_{yz} = \gamma_{xy} = 0 \tag{13}$$

$$\text{(??)} \longrightarrow \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0; \tag{14}$$

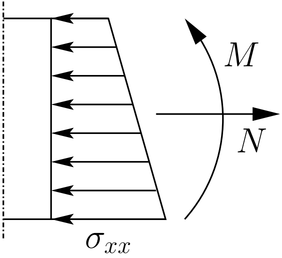
Implications

- No shear deformation: $\boxed{\gamma_{xz} = 0}$
- Assumption valid for slender (thin) beams: $\lambda = \dfrac{l}{t} \gtrapprox 20.$

Constitutive Relations: Moment

- Moment resultant:

$$M = -\int_{t_1}^{t_2} z\sigma_{xx} b(z)\,\mathrm{d}z \qquad (15)$$



- from (??):

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) \qquad (16)$$

- Substituting (??) in (??):

$$M = \cancel{-E\int_{t_1}^{t_2} z\frac{\mathrm{d}u_0}{\mathrm{d}x} b(z)\,\mathrm{d}z} + E\int_{t_1}^{t_2} z^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} b(z)\,\mathrm{d}z = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \qquad (17)$$

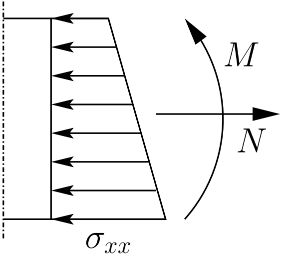
- Curvature (for $\mathrm{d}w/\mathrm{d}x \ll 1$)

$$\kappa = \frac{\mathrm{d}^2 w/\mathrm{d}x^2}{[1 + (\mathrm{d}w/\mathrm{d}x)^2]^{3/2}} \approx \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \implies \boxed{M = EI\kappa} \qquad (18)$$

Constitutive Relations: Moment

- Moment resultant:

$$M = -\int_{t_1}^{t_2} z\sigma_{xx} b(z)\,\mathrm{d}z \qquad (15)$$



- from (??):

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) \qquad (16)$$

- Substituting (??) in (??):

$$M = -E\cancel{\int_{t_1}^{t_2} z\frac{\mathrm{d}u_0}{\mathrm{d}x} b(z)\,\mathrm{d}z} + E\int_{t_1}^{t_2} z^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} b(z)\,\mathrm{d}z = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \qquad (17)$$

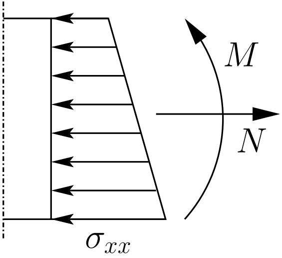
- Curvature (for $\mathrm{d}w/\mathrm{d}x \ll 1$)

$$\kappa = \frac{\mathrm{d}^2 w/\mathrm{d}x^2}{[1 + (\mathrm{d}w/\mathrm{d}x)^2]^{3/2}} \approx \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \implies \boxed{M = EI\kappa} \qquad (18)$$

Constitutive Relations: Moment

- Moment resultant:

$$M = -\int_{t_1}^{t_2} z\sigma_{xx} b(z)\,\mathrm{d}z \qquad (15)$$



- from (??):

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) \qquad (16)$$

- Substituting (??) in (??):

$$M = \cancel{-E\int_{t_1}^{t_2} z\frac{\mathrm{d}u_0}{\mathrm{d}x} b(z)\,\mathrm{d}z} + E\int_{t_1}^{t_2} z^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} b(z)\,\mathrm{d}z = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \qquad (17)$$

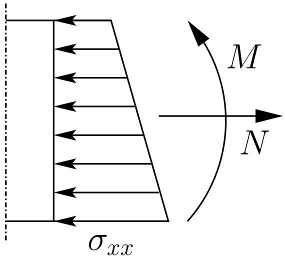
- Curvature (for $\mathrm{d}w/\mathrm{d}x \ll 1$)

$$\kappa = \frac{\mathrm{d}^2 w/\mathrm{d}x^2}{[1+(\mathrm{d}w/\mathrm{d}x)^2]^{3/2}} \approx \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \implies \boxed{M = EI\kappa} \qquad (18)$$

Constitutive Relations: Moment

- Moment resultant:

$$M = -\int_{t_1}^{t_2} z\sigma_{xx} b(z)\,\mathrm{d}z \qquad (15)$$



- from (??):

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) \qquad (16)$$

- Substituting (??) in (??):

$$M = \cancel{-E\int_{t_1}^{t_2} z\frac{\mathrm{d}u_0}{\mathrm{d}x} b(z)\,\mathrm{d}z} + E\int_{t_1}^{t_2} z^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} b(z)\,\mathrm{d}z = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \qquad (17)$$

- Curvature (for $\mathrm{d}w/\mathrm{d}x \ll 1$)

$$\kappa = \frac{\mathrm{d}^2 w/\mathrm{d}x^2}{[1+(\mathrm{d}w/\mathrm{d}x)^2]^{3/2}} \approx \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \quad \Longrightarrow \quad \boxed{M = EI\kappa} \qquad (18)$$

Constitutive Relations: Axial & Shear Resultants

- Integrating the normal stresses

$$N = \int_{t_1}^{t_2} \sigma_{xx} b(z)\,\mathrm{d}z = EA\frac{\mathrm{d}u_0}{\mathrm{d}x} \tag{19}$$

$$V = \int_{t_1}^{t_2} \sigma_{xz} b(z)\,\mathrm{d}z; \tag{20}$$

- Contradiction: from (??),

$$\gamma_{xz} = 0 \quad \Rightarrow \quad \sigma_{xz} = G\gamma_{xz} = 0 \quad \Rightarrow \quad \boxed{V = 0\ !} \tag{21}$$

However, we know $V \neq 0$.

- In practice, shear V is evaluated from the equilibrium equations (??):

$$\boxed{V = -\mathrm{d}M/\mathrm{d}x}$$

Strong Formulation: Transverse Deflection

- Strong Formulation / Transverse Deflection (??), (??):

$$\left.\begin{aligned} \frac{\mathrm{d}M}{\mathrm{d}x} + V = 0 \\ \frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0 \end{aligned}\right\} \quad \Rightarrow \quad \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = q_z \tag{22}$$

$$M = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \quad \Rightarrow \quad \boxed{EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q_z} \tag{23}$$

(assuming EI constant)

- Differential equation (4th order) of the *elastica*, as function of $w(x)$

Strong Formulation: Axial Displacement

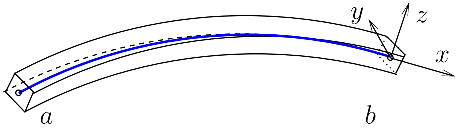
- Axial resultants / displacements $u = u_0$ (??):

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0\,; \quad N = EA\frac{\mathrm{d}u_0}{\mathrm{d}x} \tag{24}$$

$$\boxed{EA\frac{\mathrm{d}^2u_0}{\mathrm{d}x^2} + q_x = 0} \tag{25}$$

- We assume axial resultants N uncoupled from moment and shear resultants (M, V). In a first stage we will not consider them for simplicity: *beam* elements.
- Elements of frame type (aka *beam-column*): include simultaneously axial and transverse deformation $(\to M, V, N)$

Weak Formulation



- Weight functions *(virtual displacements)* $\bar{w}$, of arbitrary value. From (**??**), integrating on body $[a,b]$:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = q_z \quad \Rightarrow \quad \int_a^b \bar{w}\frac{\mathrm{d}^2 M}{\mathrm{d}x^2}\,\mathrm{d}x = \int_a^b \bar{w} q_z\,\mathrm{d}x \quad \forall \bar{w} \tag{26}$$

- Integrating by parts (twice):

$$-\int_a^b \frac{\mathrm{d}\bar{w}}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x + \left[\bar{w}\frac{\mathrm{d}M}{\mathrm{d}x}\right]_a^b = \int_a^b \bar{w} q_z\,\mathrm{d}x$$

$$\boxed{\int_a^b \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x = \int_a^b \bar{w} q_z\,\mathrm{d}x + \left[\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} M\right]_a^b + \left[\bar{w} V\right]_a^b} \tag{27}$$

- M_a, V_a, M_b, V_b: *natural* boundary conditions.

Components of Virtual Work

- Virtual work from bending (*internal*)

$$\boxed{\int_a^b \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x}$$

- Virtual work from applied loads

$$\boxed{\int_a^b \bar{w} q_z \,\mathrm{d}x}$$

- Virtual work from boundary loads (natural BC's)

$$\boxed{\left[\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} M\right]_a^b + \left[\bar{w} V\right]_a^b}$$

Components of Virtual Work

- Virtual work from bending (*internal*)

$$\boxed{\int_a^b \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x}$$

- Virtual work from applied loads

$$\boxed{\int_a^b \bar{w} q_z\,\mathrm{d}x}$$

- Virtual work from boundary loads (natural BC's)

$$\boxed{\left[\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} M\right]_a^b + \left[\bar{w} V\right]_a^b}$$

Components of Virtual Work

- Virtual work from bending (*internal*)

$$\boxed{\int_a^b \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x}$$

- Virtual work from applied loads

$$\boxed{\int_a^b \bar{w} q_z \,\mathrm{d}x}$$

- Virtual work from boundary loads (natural BC's)

$$\boxed{\left[\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} M\right]_a^b + [\bar{w} V]_a^b}$$

Índice

Finite Element Approximation

- Interpolation functions for displacements, $N_i(x)$:

$$w(x) \approx w_h(x) = \sum_{i=1}^{n} N_i(x) a_i = \{\mathbf{N}\}^{\mathsf{T}}\{\mathbf{a}\}$$

$$= \begin{bmatrix} N_1(x) & N_2(x) & \dots & N_n(x) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}$$

- Interpolation of *strains*: $[\mathbf{B}]$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \approx [\mathbf{B}]\{\mathbf{a}\}; \tag{28}$$

$$[\mathbf{B}] = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\{\mathbf{N}\}^{\mathsf{T}} = \begin{bmatrix} \frac{\mathrm{d}^2 N_1}{\mathrm{d}x^2} & \frac{\mathrm{d}^2 N_2}{\mathrm{d}x^2} & \dots \frac{\mathrm{d}^2 N_n}{\mathrm{d}x^2} \end{bmatrix} \tag{29}$$

Same interpolation for $\bar{w}$ as for w

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} \approx [\mathbf{B}]\{\bar{\mathbf{a}}\} = \{\bar{\mathbf{a}}\}^\mathsf{T}[\mathbf{B}]^\mathsf{T} \tag{30}$$

$$\int_a^b \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2}\,\mathrm{d}x \approx \int_a^b \{\bar{\mathbf{a}}\}^\mathsf{T}\left([\mathbf{B}]^\mathsf{T} EI [\mathbf{B}]\right)\{\mathbf{a}\}\,\mathrm{d}x \tag{31}$$

$$\int_a^b \bar{w} q\,\mathrm{d}x \approx \int_a^b \{\bar{\mathbf{a}}\}^\mathsf{T}\{\mathbf{N}\} q\,\mathrm{d}x = \{\bar{\mathbf{a}}\}^\mathsf{T}\{\mathbf{f}_{\mathsf{dist}}\} \tag{32}$$

$$\{\bar{\mathbf{a}}\}^\mathsf{T} \underbrace{\left[\boxed{\left(\int_a^b \underbrace{[\mathbf{B}]^\mathsf{T}}_{(n\times 1)} EI \underbrace{[\mathbf{B}]}_{(1\times n)}\,\mathrm{d}x\right)}\right]}_{[\mathbf{K}]\ (n\times n)} \{\mathbf{a}\} = \{\bar{\mathbf{a}}\}^\mathsf{T}\Big[\{\mathbf{f}_{\mathsf{dist}}\} + \{\mathbf{f}_{\mathsf{boun}}\}\Big] \tag{33}$$

Matrix Formulation

- Taking arbitrary $\{\bar{\mathbf{a}}\}^{\mathsf{T}}$; and calling $\{\mathbf{f}\} = \{\mathbf{f}_{\text{dist}}\} + \{\mathbf{f}_{\text{boun}}\}$:

$$\boxed{[\mathbf{K}]\{\mathbf{a}\} = \{\mathbf{f}\}}$$

  with
  - $w, \mathrm{d}w/\mathrm{d}x$: kinematic or *essential* boundary conditions
  - M, V: static or *natural* boundary conditions $(\to \{\mathbf{f}\}_{\text{boun}})$
- Integration is performed locally for each subdomain Ω^e (*element* e):

$$[\mathbf{K}^e]\{\mathbf{a}^e\} = \{\mathbf{f}^e\}; \tag{34}$$

$$\boxed{[\mathbf{K}] = \overset{\text{numel}}{\mathsf{A}}\,[\mathbf{K}^e]; \quad \{\mathbf{f}\} = \overset{\text{numel}}{\mathsf{A}}\,\{\mathbf{f}^e\}.} \tag{35}$$

- $\overset{\text{numel}}{\mathsf{A}}$: Assembly operator

Requirements for Convergence

Completeness

- $w(x)$ capable to represent an arbitrary rigid body displacement $\left(w^0, \theta^0 = \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^0\right)$
- $w(x)$ capable to represent deformations with arbitrary constant curvature, $\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^0 = \kappa^0$. $\quad \rightarrow \quad w(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$

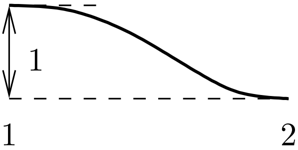
Compatibility

- Variational index m: order of derivatives in weak formulation $(\mathrm{d}^2 w/\mathrm{d}x^2 \ \rightarrow \ m = 2)$
- $w(x)$ must be continuous at least of order $(m-1) = 1$:
  $\rightarrow \ w(x) \in C^1$
  4 conditions: $w_a, \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)_a, w_b, \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)_b$.

$$\boxed{w(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3} \tag{36}$$

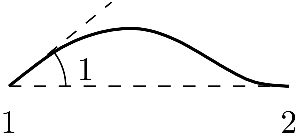
Element with Cubic (Hermitic) Interpolation

$$N_1(x) = 1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}$$



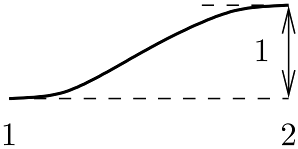
$$a_1 = w_1$$

$$N_2(x) = x\left(1 - 2\frac{x}{l} + \frac{x^2}{l^2}\right)$$



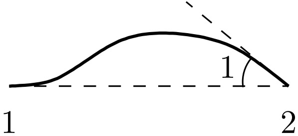
$$a_2 = \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_1$$

$$N_3(x) = \frac{x^2}{l^2}\left(3 - 2\frac{x}{l}\right)$$



$$a_3 = w_2$$

$$N_4(x) = \frac{x^2}{l}\left(\frac{x}{l} - 1\right)$$



$$a_4 = \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_2$$

Stiffness Matrix

Integrating the individual matrix components in (??)

$$K_{11}^e = EI \int_0^l B_1^2 \, \mathrm{d}x; \quad B_1 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} N_1(x) = -\frac{6}{l^2} + \frac{12x}{l^3}; \tag{37}$$

$$K_{11}^e = \frac{12EI}{l^3}; \tag{38}$$

$$K_{12}^e = \frac{6EI}{l^2}; \quad K_{13}^e = -\frac{12EI}{l^3}; \ldots \tag{39}$$

$$\boxed{[\mathbf{K}^e] = \frac{12EI}{l^3} \begin{pmatrix} 1 & l/2 & -1 & l/2 \\ l/2 & l^2/3 & -l/2 & l^2/6 \\ -1 & -l/2 & 1 & -l/2 \\ l/2 & l^2/6 & -l/2 & l^2/3 \end{pmatrix}} \tag{40}$$

Consistent Load Vectors

- Integrating the individual terms in (??), (??) for each element e, assuming constant distributed load q:

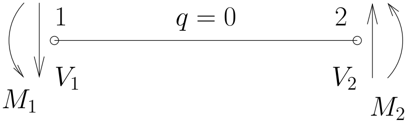
$$\{\mathbf{f}^e\} = \underbrace{\begin{Bmatrix} ql/2 \\ ql^2/12 \\ ql/2 \\ -ql^2/12 \end{Bmatrix}}_{\{\mathbf{f}_{\text{dist}}\}} + \underbrace{\begin{Bmatrix} -V_1 \\ -M_1 \\ V_2 \\ M_2 \end{Bmatrix}}_{\{\mathbf{f}_{\text{boun}}\}} \tag{41}$$

- $\{\mathbf{f}_{\text{dist}}\}$ includes the distributed transverse loads q and the fixed-end moments (*"momentos de empotramiento perfecto"*)
- $\{\mathbf{f}_{\text{boun}}\}$ corresponds to the loads at the ends (boundaries) of the element

Direct Matrix Formulation (w/o Finite Elements)

- General solution in beam 1–2 without distributed loads $(q = 0)$:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0$$



- Boundary conditions: $w_1, \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_1, w_2, \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_2$

$$w(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \tag{42}$$

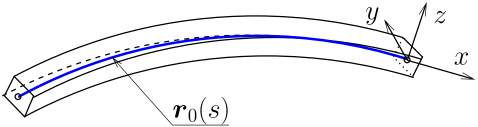
4 conditions $\Rightarrow$ 4 parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$
Exact solution with FE interpolation!

- Direct formulation of matrix structural equations
- Warning: distributed loads $(q \neq 0)$ require correction (fixed-end moments)

Índice

Timoshenko Beam Model: Assumptions

1. Prismatic rod, with base curve *("directriz")* $\boldsymbol{r}_0(s)$ (line of centroids)



2. After deformation, plane cross sections (yz) remain plane but not necessarily normal to the base curve
3. (2D) Displacements normal to the beam (z) are uniform along the cross section and equal to those of the base curve:

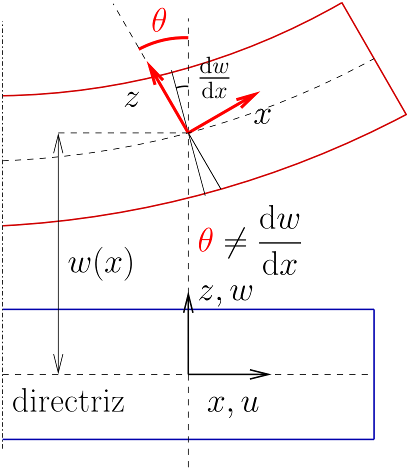
$$\boxed{w(x, z) = w(x).} \tag{43}$$

4. (2D) Plane stress:

$$\boxed{\sigma_{zz} = 0.} \tag{44}$$

Timoshenko Beams: Displacements

- θ: rotation of cross section; x: normal to cross section
- $\mathrm{d}w/\mathrm{d}x = w'$: rotation of base curve tangent (1st order)



- Cross section not normal to base curve:

$$\boxed{\theta \neq \frac{\mathrm{d}w}{\mathrm{d}x}} \tag{45}$$

- Longitudinal displacements:

$$u(x,z) = u_0(x) - z\theta \tag{46}$$

Timoshenko Beams: Strains

From the above displacement equations,

$$(??): \rightarrow \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{47}$$

$$(??): \rightarrow \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0; \tag{48}$$

$$(??): \rightarrow \qquad \gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\theta + \frac{\mathrm{d}w}{\mathrm{d}x} \neq 0 \tag{49}$$

$$\gamma_{yz} = \gamma_{xy} = 0 \tag{50}$$

Implications

- *Shear deformation* $\gamma_{xz} \neq 0$ is considered
- Shear deformation is constant across cross section (Navier hypothesis)
- Assumption valid for not so slender beams: $\lambda = \dfrac{l}{t} \gtrapprox 8$.

Constitutive relations

- Stresses

$$\sigma_{xx} = E\varepsilon_{xx} = E\frac{\mathrm{d}u_0}{\mathrm{d}x} - Ez\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{51}$$

$$\sigma_{xz} = \tau = G\gamma_{xz} = G\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right) \tag{52}$$

- Resultants

$$M = \int_{t_1}^{t_2} -Ez^2\frac{\mathrm{d}\theta}{\mathrm{d}x}b(z)\,\mathrm{d}z = EI\frac{\mathrm{d}\theta}{\mathrm{d}x} = EI\kappa \tag{53}$$

$$V = \int_{t_1}^{t_2} G\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right)b(z)\,\mathrm{d}z = GA\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right) = GA\gamma_{xz} \tag{54}$$

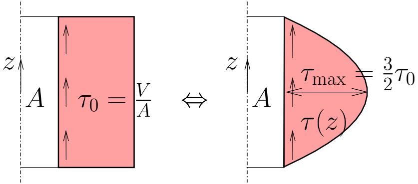
$$N = \int_{t_1}^{t_2} E\left(\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)b(z)\,\mathrm{d}z = EA\frac{\mathrm{d}u_0}{\mathrm{d}x} \tag{55}$$

Shear: stress distribution in cross section

- Following (??), shear strain is assumed constant throughout the cross section, which implies a similar constant value for the shear stress

$$\gamma_{xz} = \gamma_0 \text{ (const.)} \quad \Rightarrow \quad \tau_0 = G\gamma_0 = \frac{V}{A} \text{ (const.)} \qquad (56)$$

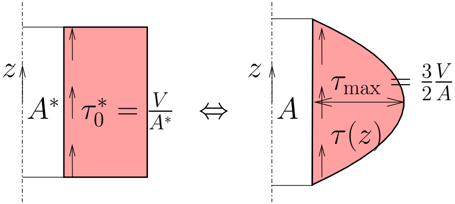
- However, basic equilibrium requires $\tau = 0$ at the top and bottom boundaries. Assuming for simplicity a rectangular cross section (width b, depth h), the correct distribution of shear stress $\tau(z)$ consistent with the continuum mechanics equilibrium equations is parabolic.
- Equating the shear resultant for both cases, $V = \int_A \tau(z)\,\mathrm{d}A$:



- *Note*: For a parabolic distribution, elasticity equations imply non-uniform shear strain and warping of the cross section.

Shear: reduced shear area

- In order to ensure correct shear deformation of the Timoshenko beam model, we now equate the strain energy for both cases, assuming an equivalent shear area A^*



$$\frac{1}{2GA^*}V^2 = \frac{1}{2}\int_A \frac{1}{G}\tau^2(z)\,\mathrm{d}A \quad \Rightarrow \quad A^* = \alpha A.$$

- The shear correction factor α depends on the shape of the section.
- For the rectangular section, with $\mathrm{d}A = b\,\mathrm{d}z$, the above equation yields

$$\boxed{A^* = \frac{5}{6}A \;\Rightarrow\; \alpha = \frac{5}{6}}$$

Strong Formulation

- From equilibrium equations

$$\text{(??)}: \quad \frac{\mathrm{d}M}{\mathrm{d}x} + V = 0 \quad \Rightarrow \quad EI\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + V = 0 \qquad (57)$$

$$\text{(??)}: \quad \frac{\mathrm{d}V}{\mathrm{d}x} + q = 0 \quad \Rightarrow \quad GA^*\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - \frac{\mathrm{d}\theta}{\mathrm{d}x}\right) + q = 0 \qquad (58)$$

- These include 2nd order derivatives of w, θ.

Weak Formulation (Moments)

Taking in 1st place the moment equation (??):

$$\int_1^2 \bar{\theta} \left(EI \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} \right) \mathrm{d}x + \int_1^2 \bar{\theta} \overbrace{V}^{GA^* \gamma_{xz}} \mathrm{d}x = 0 \quad \forall \bar{\theta} \tag{59}$$

Integrating by parts,

$$-\int_1^2 \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}x} EI \frac{\mathrm{d}\theta}{\mathrm{d}x} \mathrm{d}x + \left[\bar{\theta}\, EI \frac{\mathrm{d}\theta}{\mathrm{d}x} \right]_1^2 + \int_1^2 \bar{\theta} GA^* \left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta \right) \mathrm{d}x = 0 \quad \forall \bar{\theta} \tag{60}$$

Weak Formulation (Shear)

Doing the same for the shear equation (**??**):

$$\int_1^2 \bar{w}\, GA^* \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{\mathrm{d}\theta}{\mathrm{d}x} \right) \mathrm{d}x + \int_1^2 q\bar{w}\, \mathrm{d}x = 0 \quad \forall \bar{w} \tag{61}$$

Integrating by parts:

$$-\int_1^2 \frac{\mathrm{d}\bar{w}}{\mathrm{d}x} GA^* \left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta \right) \mathrm{d}x + \left[\bar{w}\, GA^* \left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta \right) \right]_1^2 + \int_1^2 \bar{w} q\, \mathrm{d}x = 0 \quad \forall \bar{w} \tag{62}$$

Weak Formulation (Joint)

Adding (??) & (??):

$$\int_1^2 \underbrace{\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} - \bar{\theta}\right)}_{\bar{\gamma}} GA^* \underbrace{\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right)}_{\gamma} \mathrm{d}x + \int_1^2 \underbrace{\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}x}}_{\bar{\kappa}} EI \underbrace{\frac{\mathrm{d}\theta}{\mathrm{d}x}}_{\kappa} \mathrm{d}x$$

$$= \left[\bar{w}\, \underbrace{GA^* \left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right)}_{V_i}\right]_1^2 + \left[\bar{\theta}\, \underbrace{EI \frac{\mathrm{d}\theta}{\mathrm{d}x}}_{M_i}\right]_1^2 + \int_1^2 \bar{w} q \, \mathrm{d}x \quad \forall (\bar{w}, \bar{\theta}) \tag{63}$$

Convergence Requirements

- The weak formulation includes first order derivatives of $w, \bar{w}, \theta, \bar{\theta}$
- $\Rightarrow$ FEM requires only C^0 approximation

Índice

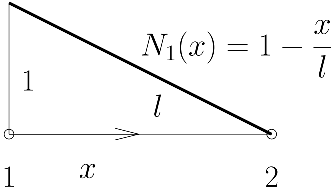
Finite Element Approximation (Galerkin)

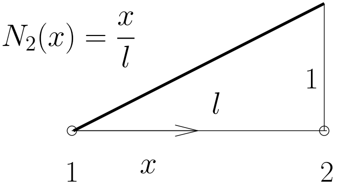
Weak formulation (from (??))

$$\delta W = \int_1^2 \bar{\gamma}\, GA^*\, \gamma \,\mathrm{d}x + \int_1^2 \bar{\kappa}\, EI\, \kappa \,\mathrm{d}x - \int_1^2 \bar{w}\, q \,\mathrm{d}x - [\bar{w}\, V]_1^2 - \left[\bar{\theta}\, M\right]_1^2 = 0 \quad \forall(\bar{\gamma}, \bar{\kappa})$$

Linear interpolation functions (C^0 continuity):

$$w^h(x) = w_1 N_1(x) + w_2 N_2(x);$$
$$\theta^h(x) = \theta_1 N_1(x) + \theta_2 N_2(x);$$





Interpolation of Strains

$$\kappa^h = \theta_1 \frac{\mathrm{d}N_1}{\mathrm{d}x} + \theta_2 \frac{\mathrm{d}N_2}{\mathrm{d}x} = \underbrace{\begin{bmatrix} 0 & -1/l & 0 & 1/l \end{bmatrix}}_{[\mathbf{B}_f^e]} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \quad \text{(cte.)}$$

$$\gamma^h = \left(\frac{\mathrm{d}w^h}{\mathrm{d}x} - \theta^h \right) = w_1 \frac{\mathrm{d}N_1}{\mathrm{d}x} - \theta_1 N_1 + w_2 \frac{\mathrm{d}N_2}{\mathrm{d}x} - \theta_2 N_2$$

$$= \underbrace{\begin{bmatrix} -1/l & -1 + x/l & 1/l & -x/l \end{bmatrix}}_{[\mathbf{B}_c^e]} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \quad \text{(linear)}$$

Element Matrices (1)

$$\delta W^{h,e} = \{\bar{\mathbf{a}}^e\}^\mathsf{T} \left[\left([\mathbf{K}_f^e] + [\mathbf{K}_c^e] \right) \{\mathbf{a}^e\} - \{\mathbf{f}_\mathsf{int}^e\} - \{\mathbf{f}_\mathsf{ext}^e\} \right]$$

$$\boxed{[\mathbf{K}_f^e] = \int_1^2 [\mathbf{B}_f^e]^\mathsf{T} EI [\mathbf{B}_f^e] \,\mathrm{d}x \,;} \qquad \boxed{[\mathbf{K}_c^e] = \int_1^2 [\mathbf{B}_c^e]^\mathsf{T} GA^* [\mathbf{B}_c^e] \,\mathrm{d}x}$$

$$\{\mathbf{f}_\mathsf{dist}^e\} = \int_1^2 \{\mathbf{N}^e\} q(x) \,\mathrm{d}x; \qquad \{\mathbf{f}_\mathsf{boun}^e\} = \begin{Bmatrix} -V_1 \\ -M_1 \\ V_2 \\ M_2 \end{Bmatrix}$$

Element Matrices (2)

Exact closed form integration:

$$[\mathbf{K}_f^e] = \left(\frac{EI}{l}\right)^e \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

(constant: 1 Gauss point)

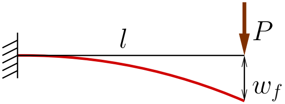
$$[\mathbf{K}_c^e] = \left(\frac{GA^*}{l}\right)^e \begin{pmatrix} 1 & l/2 & -1 & l/2 \\ l/2 & l^2/3 & -l/2 & l^2/6 \\ -1 & -l/2 & 1 & -l/2 \\ l/2 & l^2/6 & -l/2 & l^2/3 \end{pmatrix}$$

(quadratic: 2 Gauss points)

Índice

Deformation of Cantilever

- Bernoulli beam (EI): $w_f = P\dfrac{l^3}{3EI}$



- Shear beam

  (idealisation) (GA^*): $w_c = P\dfrac{l}{GA^*}$

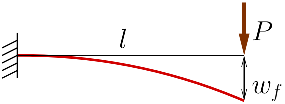
- Timoshenko beam (EI, GA^*):

$$w_t = P\left(\frac{l^3}{3EI} + \frac{l}{GA^*}\right)$$

(more flexible than Bernoulli !)

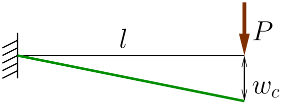
Deformation of Cantilever

- Bernoulli beam (EI): $w_f = P\dfrac{l^3}{3EI}$



- Shear beam

  (idealisation) (GA^*): $w_c = P\dfrac{l}{GA^*}$



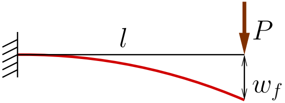
- Timoshenko beam (EI, GA^*):

$$w_t = P\left(\frac{l^3}{3EI} + \frac{l}{GA^*}\right)$$

(more flexible than Bernoulli !)

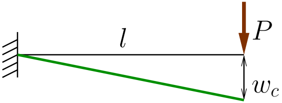
Deformation of Cantilever

- Bernoulli beam (EI): $w_f = P\dfrac{l^3}{3EI}$



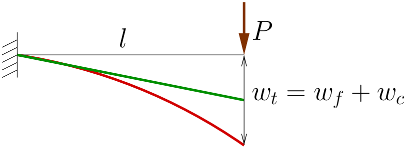
- Shear beam

  (idealisation) (GA^*): $w_c = P\dfrac{l}{GA^*}$



- Timoshenko beam (EI, GA^*):

$$w_t = P\left(\frac{l^3}{3EI} + \frac{l}{GA^*}\right)$$



(more flexible than Bernoulli !)

Comparison of models – Shear deflection

- Assuming rectangular cross section

$$I = \frac{1}{12} b t^3; \qquad A^* = \frac{5}{6} b t$$

- The bending and shear stiffness coefficients are:

$$K_f = \frac{3EI}{l^3}; \qquad K_c = \frac{GA^*}{l}$$

- The ratio K_f/K_c gives the relative value of shear deflection over bending deflection; it quantifies the relevance of Timoshenko beam:

$$\frac{K_f}{K_c} = \frac{3}{5}(1+\nu)\frac{1}{\lambda^2} \qquad \left(\lambda = \frac{l}{t},\ \text{slenderness}\right)$$

- Example: assuming $\nu = 0.25$,

λ	K_f/K_c
10	$0.75\,\%$
3	$8.33\,\%$

Comparison of models – Shear deflection

- Assuming rectangular cross section

$$I = \frac{1}{12} b t^3; \qquad A^* = \frac{5}{6} b t$$

- The bending and shear stiffness coefficients are:

$$K_f = \frac{3EI}{l^3}; \qquad K_c = \frac{GA^*}{l}$$

- The ratio K_f/K_c gives the relative value of shear deflection over bending deflection; it quantifies the relevance of Timoshenko beam:

$$\frac{K_f}{K_c} = \frac{3}{5}(1+\nu)\frac{1}{\lambda^2} \qquad \left(\lambda = \frac{l}{t}, \text{ slenderness}\right)$$

- Example: assuming $\nu = 0.25$,

λ	K_f/K_c
10	0.75 %
3	8.33 %

Comparison of models – Shear deflection

- Assuming rectangular cross section

$$I = \frac{1}{12} b t^3; \qquad A^* = \frac{5}{6} b t$$

- The bending and shear stiffness coefficients are:

$$K_f = \frac{3EI}{l^3}; \qquad K_c = \frac{GA^*}{l}$$

- The ratio K_f/K_c gives the relative value of shear deflection over bending deflection; it quantifies the relevance of Timoshenko beam:

$$\frac{K_f}{K_c} = \frac{3}{5}(1+\nu)\frac{1}{\lambda^2} \qquad \left(\lambda = \frac{l}{t}, \text{ slenderness}\right)$$

- Example: assuming $\nu = 0.25$,

λ	K_f/K_c
10	$0.75\,\%$
3	$8.33\,\%$

Comparison of models – Shear deflection

- Assuming rectangular cross section

$$I = \frac{1}{12}bt^3; \qquad A^* = \frac{5}{6}bt$$

- The bending and shear stiffness coefficients are:

$$K_f = \frac{3EI}{l^3}; \qquad K_c = \frac{GA^*}{l}$$

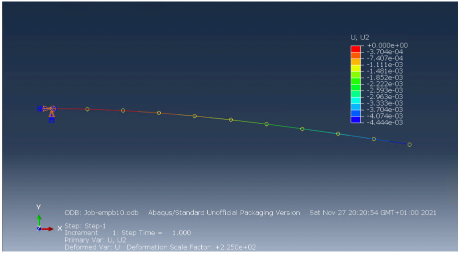
- The ratio K_f/K_c gives the relative value of shear deflection over bending deflection; it quantifies the relevance of Timoshenko beam:

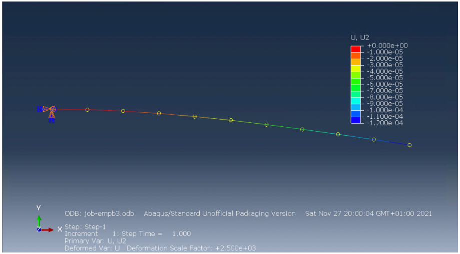
$$\frac{K_f}{K_c} = \frac{3}{5}(1+\nu)\frac{1}{\lambda^2} \qquad \left(\lambda = \frac{l}{t}, \text{ slenderness}\right)$$

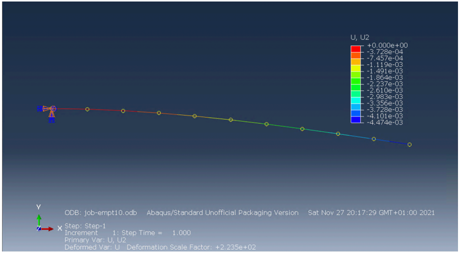
- Example: assuming $\nu = 0.25$,

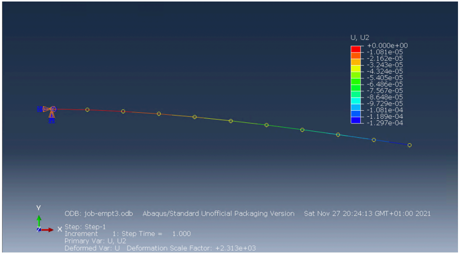
λ	K_f/K_c
10	0.75 %
3	8.33 %

Example of cantilever beam FE models









$$\lambda = \frac{l}{t} = 10 \;\Rightarrow\; \frac{u_t - u_b}{u_b} = 0.68\,\%$$

acceptable error

$$\lambda = \frac{l}{t} = 3 \;\Rightarrow\; \frac{u_t - u_b}{u_b} = 8.08\,\%$$

excessive error!

Índice

Locking of (Timoshenko) Slender Beam

- In principle, it would be desirable to use the more general Timoshenko model for all cases, including slender beams.
- In the limit for very slender beams:

$$\lambda \to \infty \quad \Rightarrow \quad \frac{K_f}{K_c} \to 0,\ \theta = \frac{\mathrm{d}w}{\mathrm{d}x}$$

- Considering the FE approximation in a cantilever, with linear interpolation functions for w and θ around node 1
- Starting from the fixed end, the interpolation implies the propagation to the adjoining nodes:

$$\theta_0 = 0 \ \Rightarrow \ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_0 = 0 \ \overset{(w \text{ linear in elem.})}{\Longrightarrow} \ w_1 = 0,\ \theta_1 = 0 \ \Rightarrow \ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_1 = 0, \ldots$$

$\boxed{\text{locking!}}$

Locking of (Timoshenko) Slender Beam

- In principle, it would be desirable to use the more general Timoshenko model for all cases, including slender beams.
- In the limit for very slender beams:

$$\lambda \to \infty \quad \Rightarrow \quad \frac{K_f}{K_c} \to 0,\ \theta = \frac{\mathrm{d}w}{\mathrm{d}x}$$

- Considering the FE approximation in a cantilever, with linear interpolation functions for w and θ around node 1
- Starting from the fixed end, the interpolation implies the propagation to the adjoining nodes:

$$\theta_0 = 0 \ \Rightarrow \ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_0 = 0 \ \overset{(w \text{ linear in elem.})}{\Rightarrow} \ w_1 = 0,\ \theta_1 = 0 \ \Rightarrow \ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_1 = 0, \ldots$$

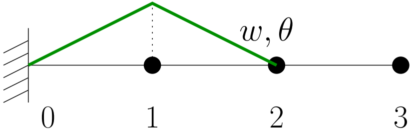
$\boxed{\text{locking!}}$

Locking of (Timoshenko) Slender Beam

- In principle, it would be desirable to use the more general Timoshenko model for all cases, including slender beams.
- In the limit for very slender beams:

$$\lambda \to \infty \quad \Rightarrow \quad \frac{K_f}{K_c} \to 0,\ \theta = \frac{\mathrm{d}w}{\mathrm{d}x}$$

- Considering the FE approximation in a cantilever, with linear interpolation functions for w and θ around node 1



- Starting from the fixed end, the interpolation implies the propagation to the adjoining nodes:

$$\theta_0 = 0 \ \Rightarrow \ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_0 = 0 \ \xrightarrow{(w \text{ linear in elem.})} \ w_1 = 0,\ \theta_1 = 0 \ \Rightarrow \ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_1 = 0, \dots$$

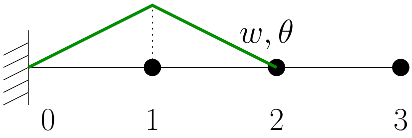
locking!

Locking of (Timoshenko) Slender Beam

- In principle, it would be desirable to use the more general Timoshenko model for all cases, including slender beams.
- In the limit for very slender beams:

$$\lambda \to \infty \quad \Rightarrow \quad \frac{K_f}{K_c} \to 0, \; \theta = \frac{\mathrm{d}w}{\mathrm{d}x}$$

- Considering the FE approximation in a cantilever, with linear interpolation functions for w and θ around node 1



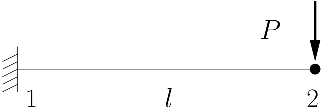
- Starting from the fixed end, the interpolation implies the propagation to the adjoining nodes:

$$\theta_0 = 0 \;\Rightarrow\; \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_0 = 0 \overset{(w \text{ linear in elem.})}{\Rightarrow} w_1 = 0, \; \theta_1 = 0 \;\Rightarrow\; \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_1 = 0, \ldots$$

$\boxed{\text{locking!}}$

Example: Cantilever with 1 element (1)

♣ 1 element Timoshenko beam



$$\begin{pmatrix} \frac{GA^*}{l} & \frac{GA^*}{2} & -\frac{GA^*}{l} & \frac{GA^*}{2} \\ & \frac{GA^*}{3}l + \frac{EI}{l} & -\frac{GA^*}{2} & \frac{GA^*}{6}l - \frac{EI}{l} \\ & & \frac{GA^*}{l} & -\frac{GA^*}{2} \\ & & & \frac{GA^*}{3}l + \frac{EI}{l} \end{pmatrix} \begin{Bmatrix} 0 \\ 0 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} V_1 \\ M_1 \\ P \\ 0 \end{Bmatrix}$$

♣ Eliminating the equations for (V_1, M_1):

$$\begin{pmatrix} \frac{GA^*}{l} & -\frac{GA^*}{2} \\ -\frac{GA^*}{2} & \frac{GA^*}{3}l + \frac{EI}{l} \end{pmatrix} \begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Example: Cantilever with 1 element (2)

♣ Inverting:

$$\begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \frac{\mu}{1+\mu} \begin{pmatrix} \frac{l}{GA^*} + \frac{l^3}{3EI} & \frac{l^2}{2EI} \\ \frac{l^2}{2EI} & \frac{l}{2EI} \end{pmatrix} \begin{Bmatrix} P \\ 0 \end{Bmatrix} \qquad \left(\mu = \frac{12EI}{GA^* l^2} \right)$$

$$w_2 = \frac{\mu}{1+\mu} P \left(\frac{l}{GA^*} + \frac{l^3}{3EI} \right)$$

♣ Identical to exact solution (with bending and shear), except for the factor $\boxed{\frac{\mu}{1+\mu}}$.

♣ Value for an infinitely slender rectangular section beam $(b \times t)$ & $\nu = 1/4$:

$$\frac{\mu}{1+\mu} = \frac{3}{3+\lambda^2}; \quad \lim_{\lambda\to\infty} \frac{\mu}{1+\mu} = 0 \ \Rightarrow \ \boxed{w_2 = 0}$$

Locking!

Reduced Integration

- In principle it would be necessary to integrate at one point the bending terms and at two points the shear terms
- It is possible to modify these integration rules so as to relax the locking exhibited by these elements
- Particularising $[\mathbf{K}_c]$ at the center of the element, and integrating with only one point:

$$
\begin{aligned}
[\mathbf{K}_c^e] &= \int_1^2 \left[\mathbf{B}_c^e\right]^{\mathsf{T}}_{x=l/2} GA^* \left[\mathbf{B}_c^e\right]_{x=l/2} \mathrm{d}x \\
&= \left(\frac{GA^*}{l}\right)^e \begin{pmatrix} 1 & l/2 & -1 & l/2 \\ l/2 & l^2/4 & -l/2 & l^2/4 \\ -1 & -l/2 & 1 & -l/2 \\ l/2 & l^2/4 & -l/2 & l^2/4 \end{pmatrix}
\end{aligned}
$$

Cantilever, 1 element Reduced Integration

- Matrix equation:

$$\begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{pmatrix} \frac{l}{GA^*} + \frac{l^3}{4EI} & \frac{l^2}{2EI} \\ \frac{l^2}{2EI} & \frac{l}{2EI} \end{pmatrix} \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

- For $\lambda \to \infty$:

$$\frac{w^{\mathsf{FE}}}{w^{\mathsf{exact}}} \to \frac{l^3/(4EI)}{l^3/(3EI)} = \frac{3}{4}$$

Locking-free solution!
(although slightly stiffer than exact solution)

- The error vanishes for a sufficiently refined mesh:
with only 2 elements, $w^{\mathsf{FE}}/w^{\mathsf{exact}} = 0.938$ (for $\lambda \to \infty$).

Cantilever, 1 element Reduced Integration

- Matrix equation:

$$\begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{pmatrix} \frac{l}{GA^*} + \frac{l^3}{4EI} & \frac{l^2}{2EI} \\ \frac{l^2}{2EI} & \frac{l}{2EI} \end{pmatrix} \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

- For $\lambda \to \infty$:

$$\frac{w^{\mathsf{FE}}}{w^{\mathsf{exact}}} \to \frac{l^3/(4EI)}{l^3/(3EI)} = \frac{3}{4}$$

Locking-free solution!
(although slightly stiffer than exact solution)

- The error vanishes for a sufficiently refined mesh: with only 2 elements, $w^{\mathsf{FE}}/w^{\mathsf{exact}} = 0.938$ (for $\lambda \to \infty$).

Cantilever, 1 element Reduced Integration

- Matrix equation:

$$\begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{pmatrix} \frac{l}{GA^*} + \frac{l^3}{4EI} & \frac{l^2}{2EI} \\ \frac{l^2}{2EI} & \frac{l}{2EI} \end{pmatrix} \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

- For $\lambda \to \infty$:

$$\frac{w^{\mathsf{FE}}}{w^{\mathsf{exact}}} \to \frac{l^3/(4EI)}{l^3/(3EI)} = \frac{3}{4}$$

  Locking-free solution!
  (although slightly stiffer than exact solution)

- The error vanishes for a sufficiently refined mesh: with only 2 elements, $w^{\mathsf{FE}}/w^{\mathsf{exact}} = 0.938$ (for $\lambda \to \infty$).

Assumed Strains (1)

- Strain field (Timoshenko, linear interpolation):

$$\gamma = \frac{1}{l}(w_2 - w_1) + \theta_1 \left(1 - \frac{x}{l}\right) + \theta_2 \left(-\frac{x}{l}\right)$$

- Using isoparametric coordinates:

$x = 0$	$x = l/2$	$x = l$
$\xi = -1$	$\xi = 0$	$\xi = +1$

$$\gamma = \underbrace{\frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2)}_{\alpha_1} + \underbrace{\frac{1}{2}(\theta_1 - \theta_2)}_{\alpha_2} \xi = \alpha_1 + \alpha_2 \xi$$

- Very slender beams:

$$\alpha_1 \to 0 \;\Rightarrow\; \alpha_2 \to 0 \;\Rightarrow\; \theta_1 = \theta_2$$

♣ Solution: independent ("assumed") field, $\gamma(x) = [\mathbf{N}_\gamma]\{\boldsymbol{\gamma}\}$

Assumed Strains (2)

- Mixed method: independent interpolation of deflections (w), rotations (θ) and shear strains (γ)
- Simplest case: prescribe $\gamma = \text{const.}$:

$$\gamma(\xi) = [\widehat{\mathbf{B}}_c]\{\mathbf{a}\} = \alpha_1 = \frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2)$$

$$= \begin{bmatrix} -\frac{1}{l} & -\frac{1}{2} & \frac{1}{l} & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

- In this case, identical result to reduced integration

Assumed Strains (2)

- Mixed method: independent interpolation of deflections (w), rotations (θ) and shear strains (γ)
- Simplest case: prescribe $\gamma =$ const.:

$$\gamma(\xi) = [\widehat{\mathbf{B}}_c]\{\mathbf{a}\} = \alpha_1 = \frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2)$$

$$= \begin{bmatrix} -\frac{1}{l} & -\frac{1}{2} & \frac{1}{l} & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

- In this case, identical result to reduced integration

Assumed Strains (2)

- Mixed method: independent interpolation of deflections (w), rotations (θ) and shear strains (γ)
- Simplest case: prescribe $\gamma = \text{const.}$:

$$\gamma(\xi) = [\widehat{\mathbf{B}}_c]\{\mathbf{a}\} = \alpha_1 = \frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2)$$

$$= \begin{bmatrix} -\frac{1}{l} & -\frac{1}{2} & \frac{1}{l} & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

- In this case, identical result to reduced integration

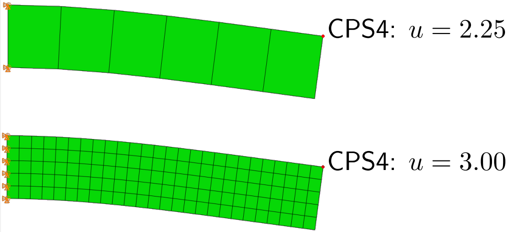
Índice

Continuum elements? – 1

Remarks

- A bilinear continuum element when subjected to pure bending is constrained to deform by shear and behaves as too stiff. The solution may be very poor and requires very fine meshes for an acceptable precision.
- The "incompatible modes" allow a dramatic improvement
- Quadratic elements also improve results, with additional dof's

ABAQUS Continuum Plane Strain (CPS):
CPS4: bilinear;

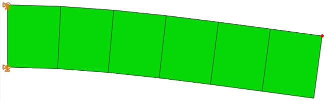


Continuum elements? – 1

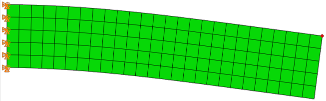
Remarks

- A bilinear continuum element when subjected to pure bending is constrained to deform by shear and behaves as too stiff. The solution may be very poor and requires very fine meshes for an acceptable precision.
- The "incompatible modes" allow a dramatic improvement
- Quadratic elements also improve results, with additional dof's

ABAQUS Continuum Plane Strain (CPS):
CPS4: bilinear;
CPS4I: incompatible modes;



CPS4: $u = 2.25$
CPS4I: $u = 3.04$



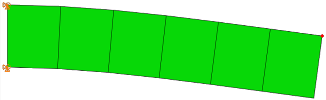
CPS4: $u = 3.00$
CPS4I: $u = 3.06$

Continuum elements? – 1

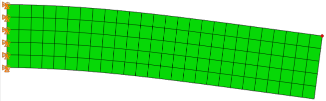
Remarks

- A bilinear continuum element when subjected to pure bending is constrained to deform by shear and behaves as too stiff. The solution may be very poor and requires very fine meshes for an acceptable precision.
- The "incompatible modes" allow a dramatic improvement
- Quadratic elements also improve results, with additional dof's

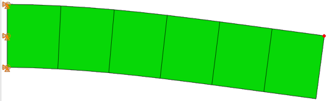
ABAQUS Continuum Plane Strain (CPS):
CPS4: bilinear; CPS8: quadratic;
CPS4I: incompatible modes;



CPS4: $u = 2.25$
CPS4I: $u = 3.04$



CPS4: $u = 3.00$
CPS4I: $u = 3.06$
CPS8: $u = 3.06$



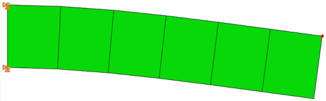
CPS8: $u = 3.05$

Continuum elements? – 1

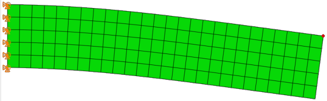
Remarks

- A bilinear continuum element when subjected to pure bending is constrained to deform by shear and behaves as too stiff. The solution may be very poor and requires very fine meshes for an acceptable precision.
- The "incompatible modes" allow a dramatic improvement
- Quadratic elements also improve results, with additional dof's

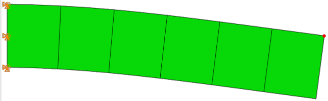
ABAQUS Continuum Plane Strain (CPS):
CPS4: bilinear; CPS8: quadratic;
CPS4I: incompatible modes;
CPS4R: reduced integration



CPS4: $u = 2.25$
CPS4I: $u = 3.04$
CPS4R: $u = 1207$



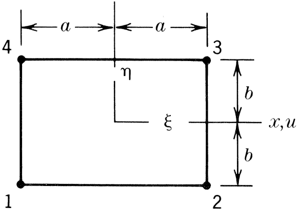
CPS4: $u = 3.00$
CPS4I: $u = 3.06$
CPS8: $u = 3.06$



CPS8: $u = 3.05$

Continuum elements? – 2

Consider a bilinear quad, under pure bending → prescribed displacement field $\bar{u}$ (extension in top fibre and compression in bottom fibre):

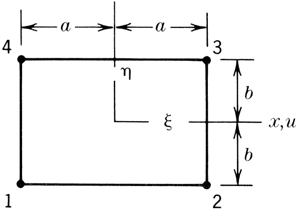


Bilinear parent element

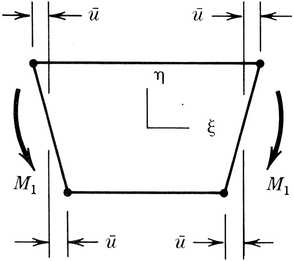
The constraints imposed by the shape functions for the bilinear quad require a greater applied moment than the real one (elastic solid):

$$\boxed{M_1 > M_2}$$

Consider a bilinear quad, under pure bending → prescribed displacement field $\bar{u}$ (extension in top fibre and compression in bottom fibre):



Bilinear parent element



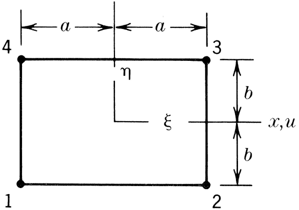
Bilinear quad, required bending moment M_1.

The constraints imposed by the shape functions for the bilinear quad require a greater applied moment than the real one (elastic solid):

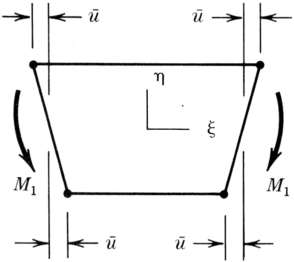
$$\boxed{M_1 > M_2}$$

Continuum elements? – 2

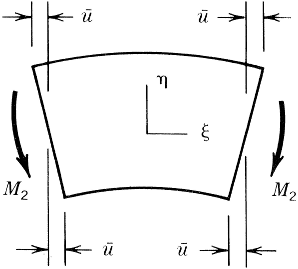
Consider a bilinear quad, under pure bending → prescribed displacement field $\bar{u}$ (extension in top fibre and compression in bottom fibre):



Bilinear parent element



Bilinear quad, required bending moment M_1.



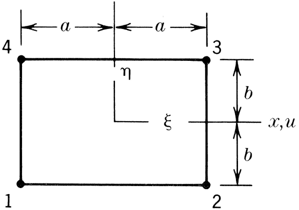
Elastic solid, required bending moment M_2.

The constraints imposed by the shape functions for the bilinear quad require a greater applied moment than the real one (elastic solid):

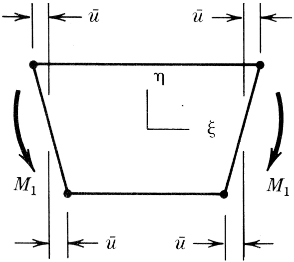
$$\boxed{M_1 > M_2}$$

Continuum elements? – 2

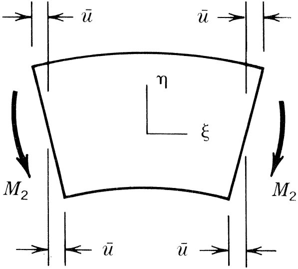
Consider a bilinear quad, under pure bending → prescribed displacement field $\bar{u}$ (extension in top fibre and compression in bottom fibre):



Bilinear parent element



Bilinear quad, required bending moment M_1.



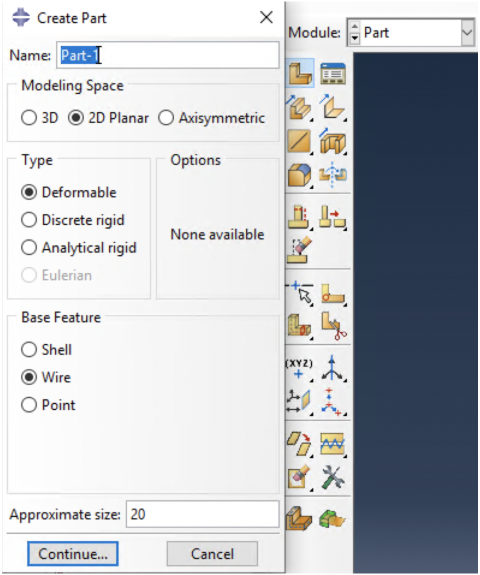
Elastic solid, required bending moment M_2.

The constraints imposed by the shape functions for the bilinear quad require a greater applied moment than the real one (elastic solid):

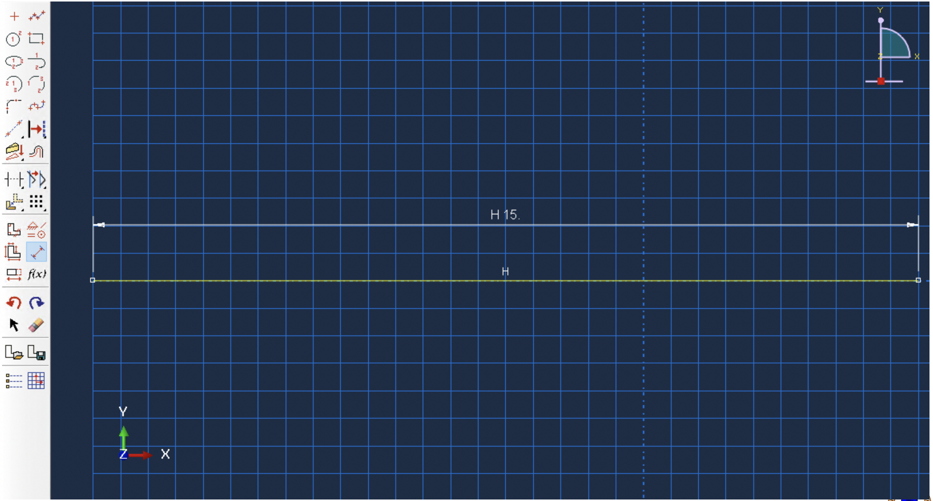
$$\boxed{M_1 > M_2}$$

Índice

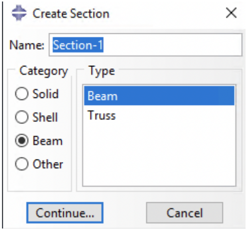
Part definition (1)

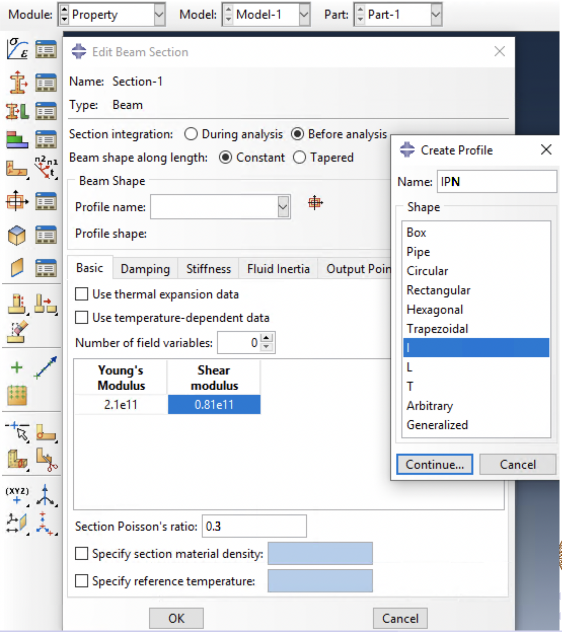


Part module (2)

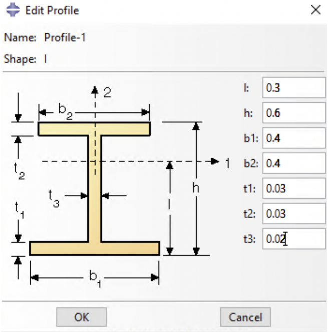


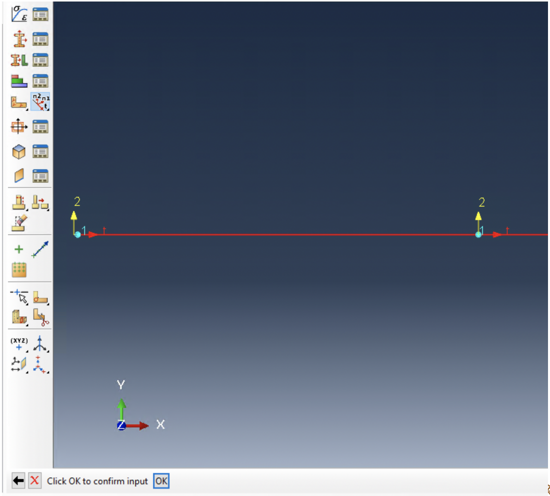
Property module



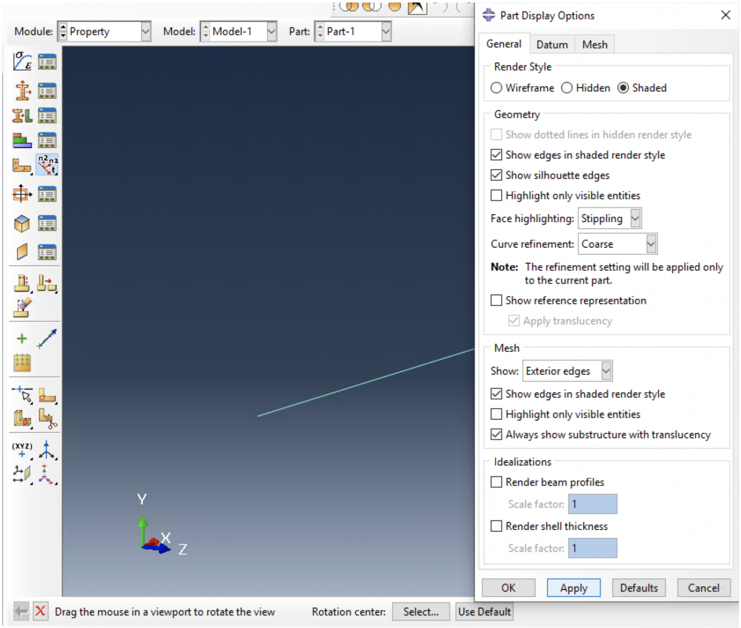


Property module – beam section

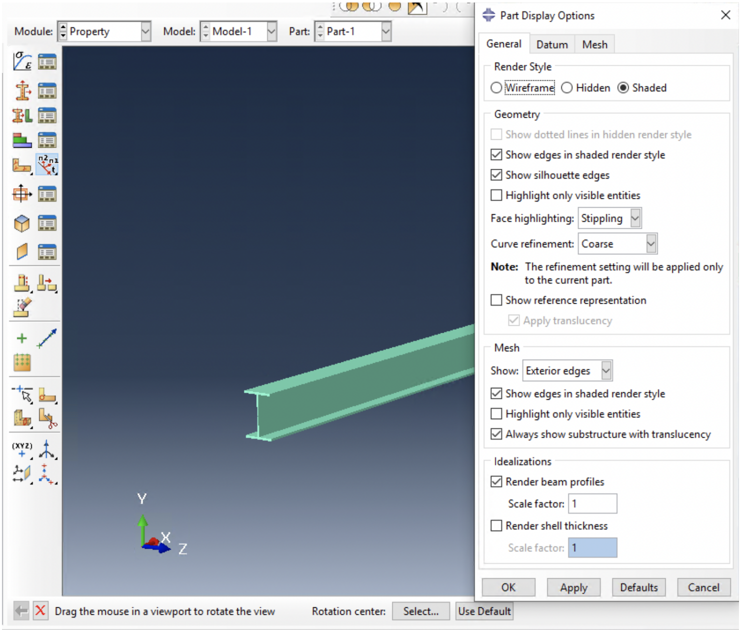




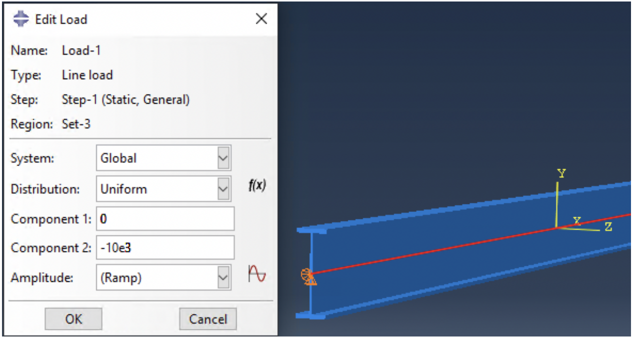
Property module – profile and orientation

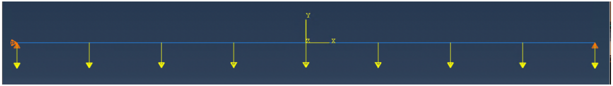


Property module – display options

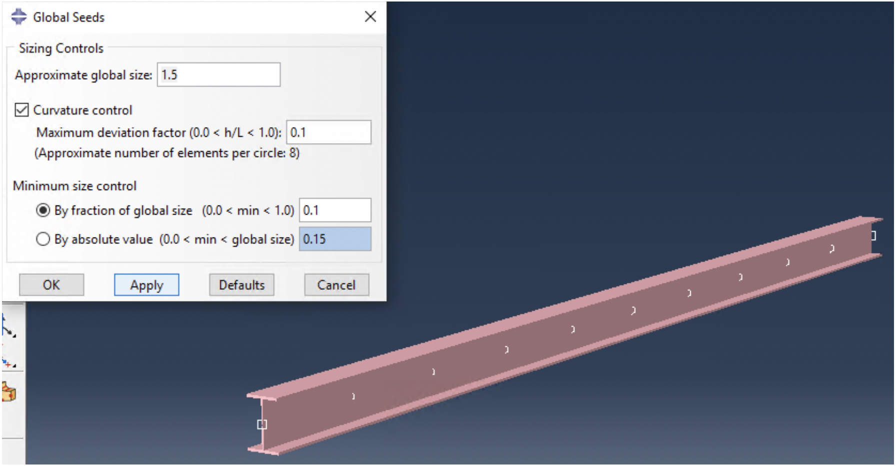


Load module – line load

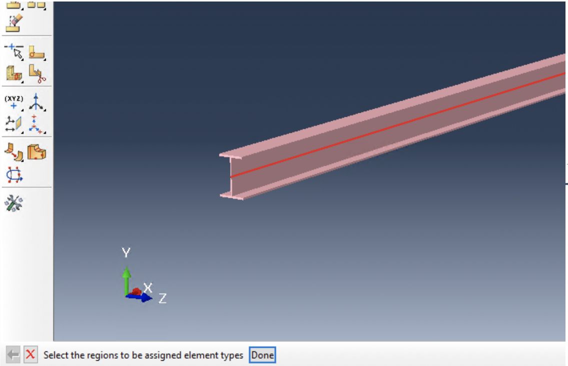


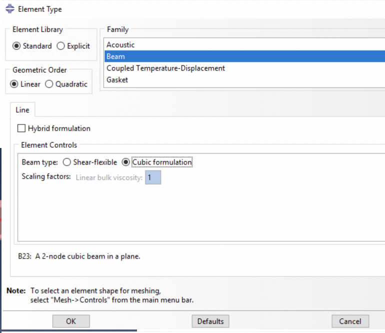


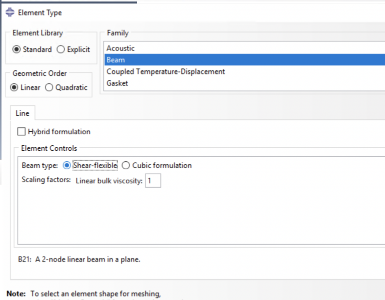
Mesh module – seeds



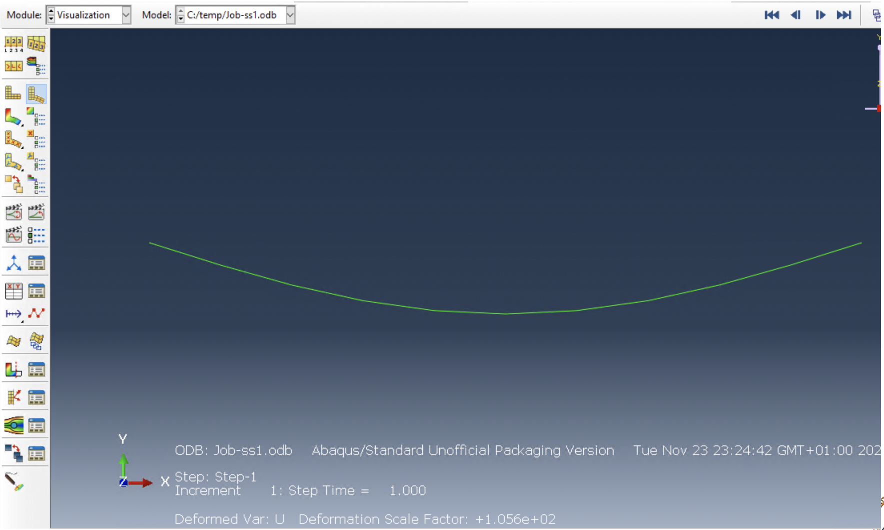
Mesh module – beam element type



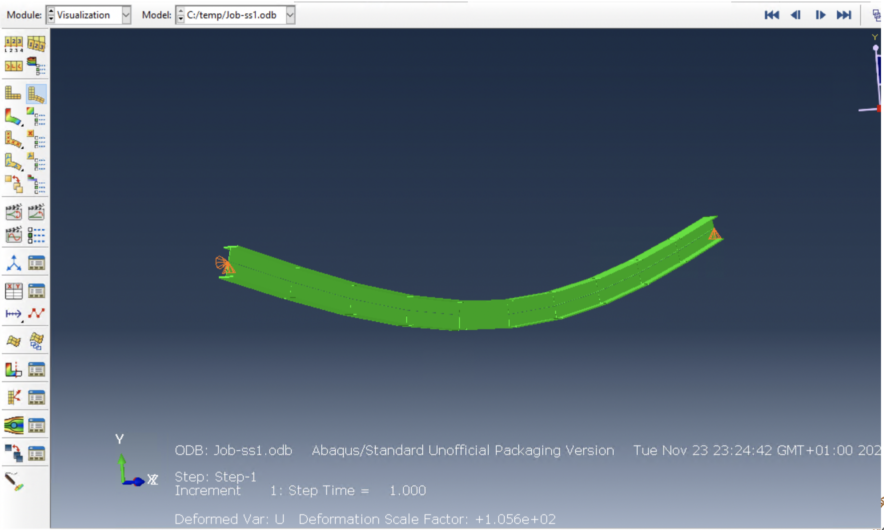




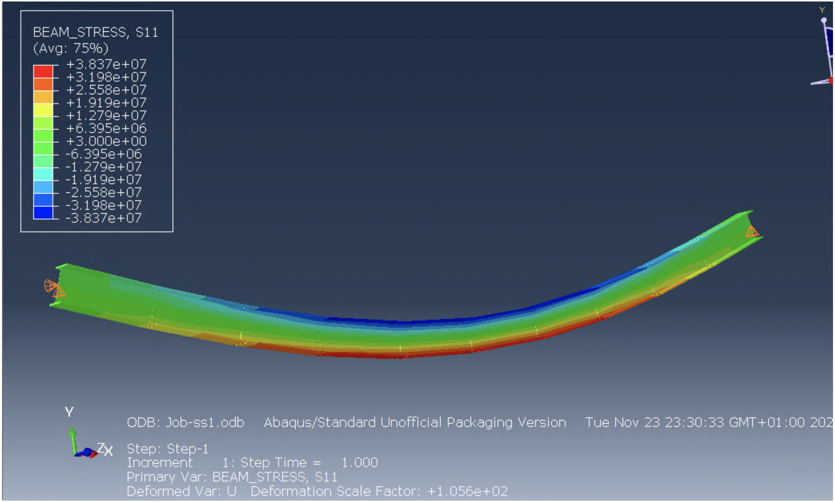
Results


Module: Visualization
Model: C:/temp/Job-ss1.odb
Y
X
ODB: Job-ss1.odb   Abaqus/Standard Unofficial Packaging Version   Tue Nov 23 23:24:42 GMT+01:00 202
Step: Step-1
Increment   1: Step Time =   1.000
Deformed Var: U   Deformation Scale Factor: +1.056e+02

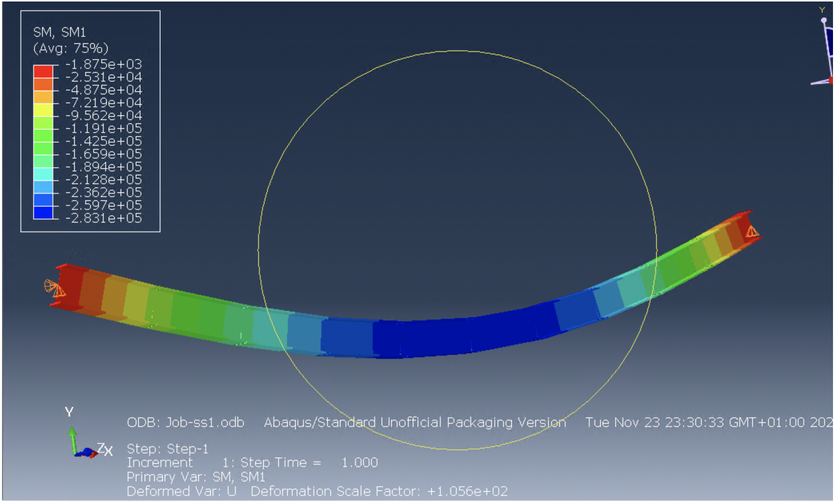
Results


Module: Visualization
Model: C:/temp/Job-ss1.odb
ODB: Job-ss1.odb Abaqus/Standard Unofficial Packaging Version Tue Nov 23 23:24:42 GMT+01:00 202
Step: Step-1
Increment 1: Step Time = 1.000
Deformed Var: U Deformation Scale Factor: +1.056e+02

Results


BEAM_STRESS, S11
(Avg: 75%)
+3.837e+07
+3.198e+07
+2.558e+07
+1.919e+07
+1.279e+07
+6.395e+06
+3.000e+00
-6.395e+06
-1.279e+07
-1.919e+07
-2.558e+07
-3.198e+07
-3.837e+07
Y
Z X
ODB: Job-ss1.odb Abaqus/Standard Unofficial Packaging Version Tue Nov 23 23:30:33 GMT+01:00 202
Step: Step-1
Increment 1: Step Time = 1.000
Primary Var: BEAM_STRESS, S11
Deformed Var: U Deformation Scale Factor: +1.056e+02

Results


SM, SM1
(Avg: 75%)
-1.875e+03
-2.531e+04
-4.875e+04
-7.219e+04
-9.562e+04
-1.191e+05
-1.425e+05
-1.659e+05
-1.894e+05
-2.128e+05
-2.362e+05
-2.597e+05
-2.831e+05
Y
Z X
ODB: Job-ss1.odb   Abaqus/Standard Unofficial Packaging Version   Tue Nov 23 23:30:33 GMT+01:00 202
Step: Step-1
Increment 1: Step Time = 1.000
Primary Var: SM, SM1
Deformed Var: U   Deformation Scale Factor: +1.056e+02