

# THE FINITE ELEMENT METHOD.
## Truss elements

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Direct method. 1D formulation

Direct method: 2D and 3D structures

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## Contents

# Introduction



- The Finite Element Method is a numerical procedure for solving differential equations in an approximate way.
- The domain in which the problem is defined is divided in sub-domains named FINITE ELEMENTS
- The set of elements used for the discretization of the domain is named MESH
- It's possible to solve both boundary value problems (stationary problems) and initial value problems (transient problems)

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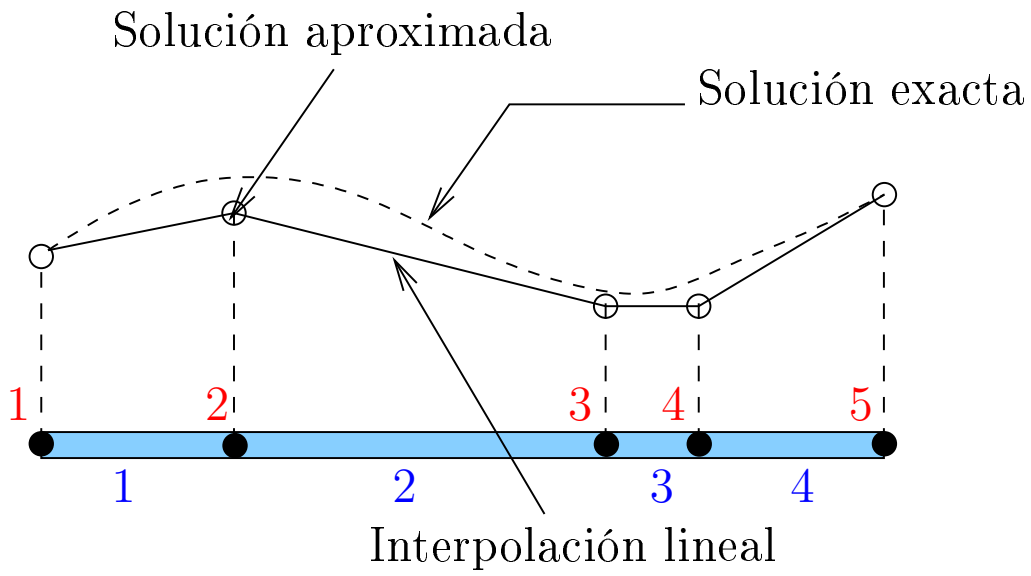
- The primary continuous variables is defined by its approximated values in some discrete points named NODES
- The values of this variable in an arbitrary point of a element is obtained interpolating the nodal values. Generally the interpolation functions are polynomials and they are named SHAPE FUNCTIONS
- The degrees of freedom are the primary variables defined at the nodes.

# Example: temperature distribution along a bar

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# Motivation

- The analysis of structures defined with articulated bars is based in a 1D formulation mainly. The elastic bar with axial loads is the basic model for this kind of computations, and it can be generalized for 2D and 3D problems.
- In this course, two formulations for the analysis of truss structures are presented and discussed:
  1. A direct method, based on concepts of the Mechanics of Materials, in order to formulate the equilibrium, compatibility and constitutive equations
  2. A weak formulation of the boundary value problem, as the basis of the finite element solution.

# Motivation



# Motivation

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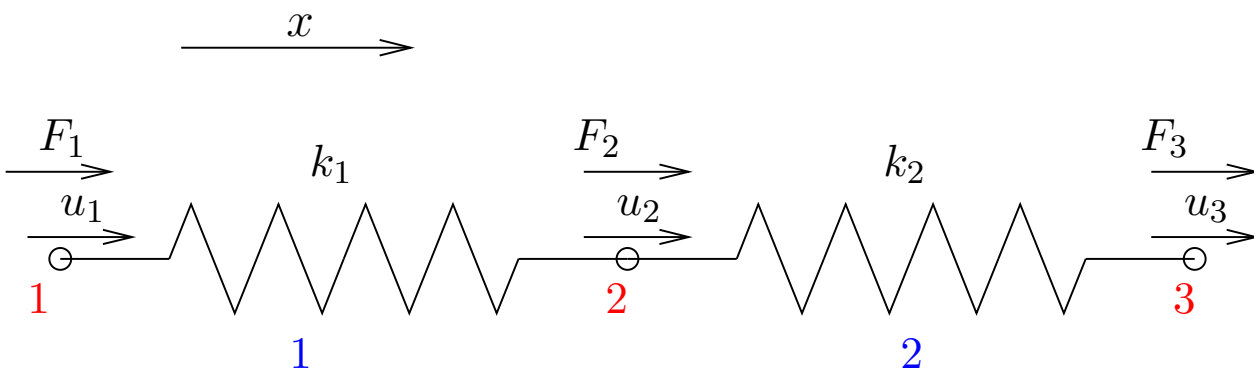
# General issues




- $F_i$ are the applied nodal forces
- $u_i$ are the nodal displacements
- $k_i$ are the stiffness constants of each bar

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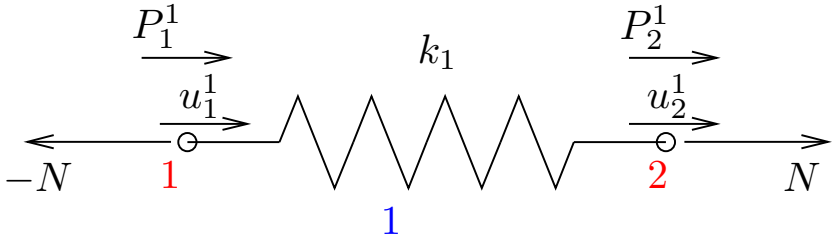
# General issues

- Taking into account only one bar:




- $P_i^e$ is the element nodal force corresponding to local node $i$ and element (bar) $e$
- $u_i^e$ are the displacements of the local node $i$ in element $e$
- $N$ y $-N$ are the internal forces

# Equations (element 1)



- Constitutive law (Hooke's law)

$$N = k(u_2 - u_1) \qquad k = \frac{EA}{L}$$

- Nodal equilibrium equations for element 1:

$$\begin{aligned} P_1^1 = -N \Rightarrow \quad & P_1^1 = -k_1(u_2^1 - u_1^1) \\ P_2^1 = N \ \Rightarrow \quad & P_2^1 = k_1(u_2^1 - u_1^1) \end{aligned}$$

$$\begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} P_1^1 \\ P_2^1 \end{Bmatrix}$$

# Elemental matrix equation



- For element 2

$$\begin{pmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} P_1^2 \\ P_2^2 \end{Bmatrix}$$

- The previous equations can be generalized for an element $e$, obtaining:

$$\boldsymbol{K}^e \boldsymbol{d}^e = \boldsymbol{P}^e$$

where:

$\boldsymbol{K}^e$ : Stiffness matrix of element $e$ (sym. and pos.def.)

$\boldsymbol{d}^e$ : Elemental vector of nodal displacements

$\boldsymbol{P}^e$ : Elemental force vector

# Compatibility and equilibrium. Assembly



- Taking into account the compatibility of displacements:

$$u_1 = u_1^1 \qquad u_2 = u_2^1 = u_1^2 \qquad u_3 = u_2^2$$

and the equilibrium of the bar:

$$F_1 = P_1^1 \qquad F_2 = P_2^1 + P_1^2 \qquad F_3 = P_2^2$$

the matrix equations expressed previously for elements 1 and 2, can be assembled in order to obtain the global equations of the complete structure:

$$\begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

# Support conditions



- $\det(\boldsymbol{K}) = 0 \Rightarrow$ there are rigid body movements because the structure is not well supported
- For avoiding this problem it's sufficient to impose, for example, $u_1 = 0$:

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$u_1 = 0 \qquad \text{(essential boundary conditio}$$

# Methodology

In the following, the 1D formulation is generalized for 2D and 3D structures.
To this end, the methodology for 2D/3D structures is the same that the one described previously:

1. Equilibrium equations in each element (constitutive properties and equilibrium conditions)
2. Compatibility (topology and connectivity)
3. Equilibrium of the complete structure
4. Boundary conditions
5. Solution of the full system of equations

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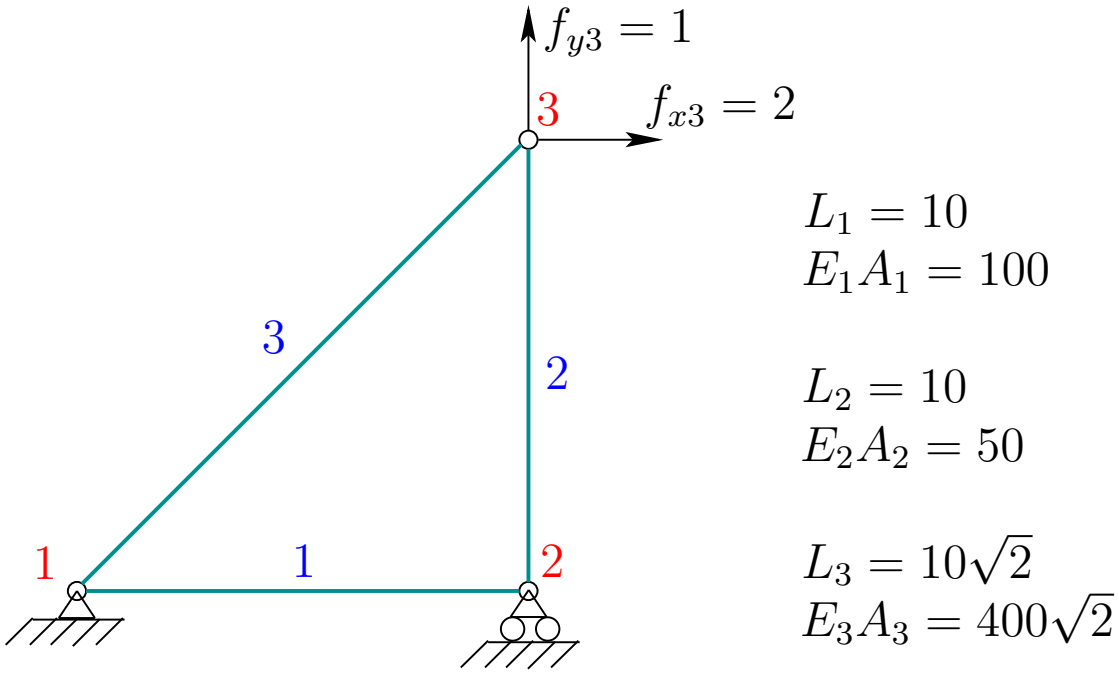
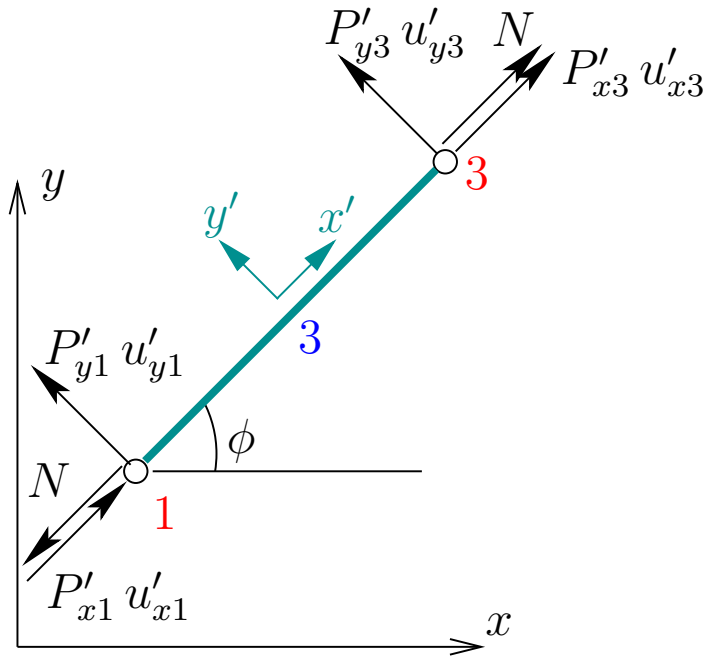
# Example

- Definition of the structure



# Example

- Equilibrium of element 3

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# Example

- Equilibrium of element 3 in local axes

$$\underbrace{\frac{E_3 A_3}{L_3}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{(\boldsymbol{K}^e)'} \begin{Bmatrix} u'_{x1} \\ u'_{y1} \\ u'_{x3} \\ u'_{y3} \end{Bmatrix} = \begin{Bmatrix} P'_{x1} \\ P'_{y1} \\ P'_{x3} \\ P'_{y3} \end{Bmatrix}$$

- Local - Global axes relations

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}}_{\boldsymbol{L}^T} \begin{Bmatrix} u'_x \\ u'_y \end{Bmatrix}; \qquad \boldsymbol{L}^T\boldsymbol{L} = \boldsymbol{1}$$

# Example



- Equilibrium of the element (global axes)

$$\boldsymbol{u}^e = \boldsymbol{L}^T (\boldsymbol{u}^e)'$$
$$\boldsymbol{P}^e = \boldsymbol{L}^T (\boldsymbol{P}^e)'$$

resulting:

$$\boldsymbol{K}^e = \boldsymbol{L}^T (\boldsymbol{K}^e)' \boldsymbol{L}$$
$$= \frac{E_3 A_3}{L_3} \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi & -\cos^2\phi & -\sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi & -\sin\phi\cos\phi & -\sin^2\phi \\ -\cos^2\phi & -\sin\phi\cos\phi & \cos^2\phi & \sin\phi\cos\phi \\ -\sin\phi\cos\phi & -\sin^2\phi & \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}$$

# Example



- Element stiffness matrix

$$\boldsymbol{K}^{e=1} = \begin{pmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \boldsymbol{K}^{e=2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{pmatrix}$$

$$\boldsymbol{K}^{e=3} = \begin{pmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{pmatrix}$$

# Example



- Assembly

$$\boldsymbol{K} = \begin{pmatrix} 10+20 & 20 & -10 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -20 & -20 & 0 & 0 & 20 & 20 \\ -20 & -20 & 0 & -5 & 20 & 5+20 \end{pmatrix}$$

# Example



- Boundary conditions

$$u_{x1} = 0 \qquad u_{y1} = 0 \qquad u_{y2} = 0$$

$$\boldsymbol{K} = \begin{pmatrix} 30 & 20 & -10 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -20 & -20 & 0 & 0 & 20 & 20 \\ -20 & -20 & 0 & -5 & 20 & 25 \end{pmatrix}$$

# Solution of the equations system

- The least efficient method use the inverse of the stiffness matrix:

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f} \Rightarrow \qquad \boldsymbol{d} = \boldsymbol{K}^{-1}\boldsymbol{f}$$

- Methodology: Gauss elimination

$$\begin{pmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \vdots & \vdots & \dots & \vdots \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{pmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix}$$

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# Direct method: 2D and 3D structures

- With elemental operations of the pivots, we obtain the system:

$$\begin{pmatrix} k'_{11} & k'_{12} & \dots & k'_{1n} \\ 0 & k'_{22} & \dots & k'_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & k'_{nn} \end{pmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix}$$

The solution is:

$$d_n = \frac{f_n}{k'_{nn}}; \quad d_{n-1} = \frac{f_{n-1} - k'_{n-1\,n} d_n}{k'_{n-1\,n-1}}; \quad \dots \quad d_1 = \frac{f_1 - \sum_{i=2}^{n} k'_{1i} d_i}{k'_{11}}$$

- Example: Solution

$$u_{x2} = 0.0 \qquad u_{x3} = 0.3 \qquad u_{y3} = -0.2$$

# Boundary value problem



From the continuum mechanics, the equations defining the behavior of a hinged bar are:

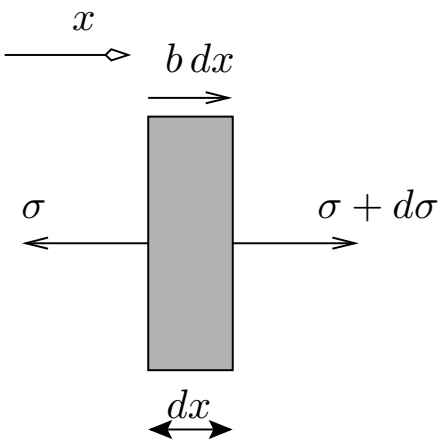
$$A\frac{d\sigma}{dx} + b = 0$$

$$\varepsilon = \frac{du}{dx}$$

$$\sigma = E\varepsilon$$

$$\sigma(0) = -\overline{t}$$

$$u(L) = \overline{u}$$




The *natural boundary conditions* are expressed in term of the derivatives of $u$. The *essential boundary conditions* are expressed directly in terms of $u$.

# Strong formulation



Let be $\overline{\Omega} = \Omega \cup \partial\Omega$ the part of the local axis $x$ occupied by the hinged bar ($\Omega = (0, L)$, $\overline{\Omega} = [0, L]$ and $\partial\Omega$ the boundary of this bar: $x = 0$ y $x = L$). The strong formulation of the boundary value problem is established in the following terms:
Known $b : \overline{\Omega} \to \mathbb{R}$ and the constants $\overline{u} \in \mathbb{R}$, $\overline{t} \in \mathbb{R}$, find the displacement field $u \in \mathbb{R}$, verifying:

$$EA\frac{d^2u}{dx^2} + b = 0 \text{ in } \Omega$$

$$u(L) = \overline{u}$$

$$E\frac{du}{dx}\bigg|_{x=0} = -\overline{t}$$

# Weak formulation



Known $b : \overline{\Omega} \to \mathbb{R}$ and the constants $\overline{u} \in \mathbb{R}$ $\overline{t} \in \mathbb{R}$, find the displacement field $u \in \mathcal{U} \mid \forall \delta u \in \mathcal{V}$ verifying:

$$\int_0^L E\,A \frac{d\delta u}{dx}\frac{du}{dx}\,dx = \int_0^L \delta u b\,dx + \delta u(0)\overline{t}$$

where:

$$\mathcal{V} = \left\{\delta u \mid \delta u \in H^1,\ \delta u(L) = 0\right\};\quad \mathcal{U} = \left\{u \mid u \in H^1,\ u(L) = \overline{u}\right\}$$

being $H^1$ the order 1 degree 2 Sobolev space:

$$H^1 = \left\{ u(x) : \Omega \to \mathbb{R} \quad \middle| \quad \int_0^L \left(\frac{du}{dx}\right)^2 dx < \infty \right\}$$

# Galerkin method



- The Galerkin formulation is the point of departure for the finite element method: it allows to obtain an approximate solution of the weak formulation:
- At first, it's necessary to construct the functional subspaces $\mathcal{V}^h$ y $\mathcal{U}^h$, which are a finite dimension approximation of the functional subspaces $\mathcal{V}$ y $\mathcal{U}$:

$$\mathcal{V}^h \subset \mathcal{V} \qquad (\delta u^h \in \nu^h \Rightarrow \delta u^h \in \nu)$$
$$\mathcal{U}^h \subset \mathcal{U} \qquad (u^h \in \delta^h \Rightarrow u^h \in \delta)$$

- The Galerkin method establishes that the functions $u^h \in \mathcal{U}^h$ are constructed using the functions $v^h \in \mathcal{V}^h$ via:

$$u^h = v^h + \overline{u}^h$$

  where $\overline{u}^h$ is a given function verifying $\overline{u}^h(L) = \overline{u}$
- **The key idea is that the functional subspaces $\mathcal{V}^h$ y $\mathcal{U}^h$ contain the same functions with the exception of $\overline{u}^h$**

# Galerkin method



The Galerkin approximation of the boundary value problem is obtained expressing the weak formulation in terms of the finite dimension subspaces $\nu^h$ y $\delta^h$:

$$\int_0^L E\,A\frac{d\delta u^h}{dx}\frac{d(v^h+\overline{u}^h)}{dx}\,dx=\int_0^L \delta u^h b\,dx+\delta u^h(0)\overline{t}$$

obtaining:

$$\int_0^L E\,A\frac{d\delta u^h}{dx}\frac{dv^h}{dx}\,dx=\int_0^L \delta u^h b\,dx+\delta u^h(0)\overline{t}-\int_0^L E\,A\frac{d\delta u^h}{dx}\frac{d\overline{u}^h}{dx}\,dx$$

# Galerkin formulation



- Known $b:\overline{\Omega}\to\mathbb{R}$ and the constants $\overline{u}\in\mathbb{R}$ $\overline{t}\in\mathbb{R}$, find the displacement field $u^h=v^h+\overline{u}^h\in\mathcal{U}^h$, with $v^h\in\mathcal{V}^h$, such that $\forall\delta u^h\in\mathcal{V}^h$ must be verified:

$$\int_0^L E\,A\frac{d\delta u^h}{dx}\frac{dv^h}{dx}\,dx=\int_0^L \delta u^h b\,dx+\delta u^h(0)\overline{t}-\int_0^L E\,A\frac{d\delta u^h}{dx}\frac{d\overline{u}^h}{dx}\,dx \quad (1)$$

- The functions $v^h$, $\delta u^h$ are in the same subspace $\mathcal{V}^h$. There are other methods, named "*Petrov-Galerkin methods*", such that the functions $v^h$ are not in the subspace $\mathcal{V}^h$. These methods are mainly used in Computational Fluid Mechanics

# Matrix formulation



From the expressions:

$$\delta u^h = \sum_{A=1}^{n} c_A N_A \qquad u^h = v^h + \overline{u}^h = \sum_{A=1}^{n} d_A N_A + \overline{u} N_{n+1}$$

the Galerkin formulation results in:

$$\int_0^L E\,A \left( \sum_{A=1}^{n} c_A \frac{dN_A}{dx} \right) \left( \sum_{B=1}^{n} d_B \frac{dN_B}{dx} \right) dx = \int_0^L \left( \sum_{A=1}^{n} c_A N_A \right) b\,dx + \left( \sum_{A=1}^{n} c_A N_A(0) \right) \overline{t} - \int_0^L E\,A \left( \sum_{A=1}^{n} c_A \frac{dN_A}{dx} \right) \overline{u} \frac{dN_{n+1}}{dx}\,dx$$

# Matrix formulation



Considering that the coefficients $c_A$ have an arbitrary value, the following result is obtained:

$$\sum_{A=1}^{n} c_A G_A = 0$$

being:

$$G_A = \sum_{B=1}^{n} \left( \int_0^L E\,A \frac{dN_A}{dx} \frac{dN_B}{dx}\,dx \right) d_B - \int_0^L N_A b\,dx - N_A(0)\overline{t} + \int_0^L E\,A \frac{dN_A}{dx} \overline{u} \frac{dN_{n+1}}{dx}\,dx$$

# Matrix formulation



Finally, we obtain:

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f}$$

where:

$$K_{AB} = \int_0^L E\,A \frac{dN_A}{dx}\frac{dN_B}{dx}\,dx$$

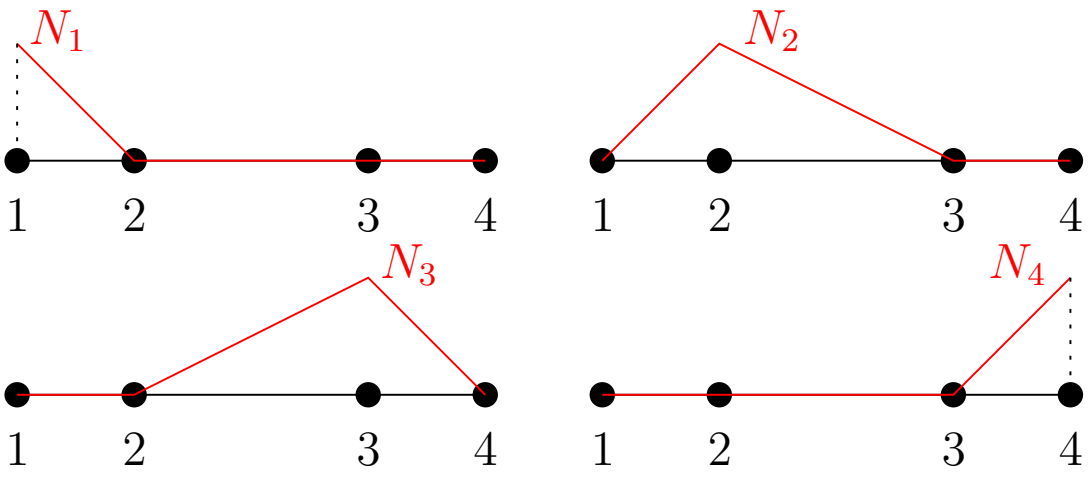
$$f_A = \int_0^L N_A b\,dx + N_A(0)\bar{t} - \int_0^L E\,A \frac{dN_A}{dx}\bar{u}\frac{dN_{n+1}}{dx}\,dx$$

# 1D Elements



- Linear interpolation functions (Shape Functions)

$$N_A(x) = \begin{cases} \frac{x - x_{A-1}}{h_{A-1}} & x_{A-1} < x < x_A \\ \frac{x_{A+1} - x}{h_A} & x_A < x < x_{A+1} \end{cases}$$



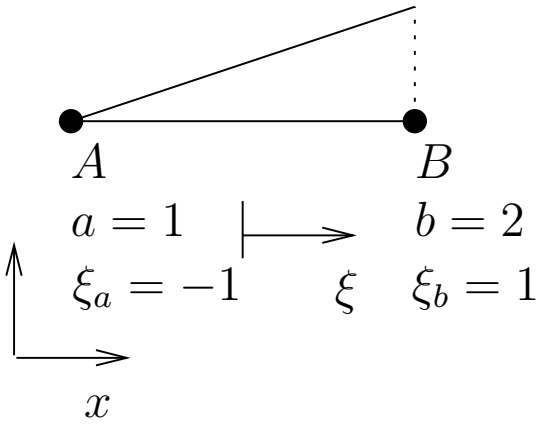
# 1D Elements



- Global and local references

$$N_B(x) = \frac{x - x_A}{x_B - x_A} = \frac{x - x_A}{h^e} \tag{2}$$

$$N_b(\xi) = \frac{1 + \xi_b \xi}{2} \tag{3}$$




# 1D Elements



- Interpolation of the displacement field

$$u^h(x) = u_A N_A(x) + u_B N_B(x) \tag{4}$$

$$u^h(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) \tag{5}$$

- Relation between local and global coordinates

$$\xi(x) = \frac{2x - x_A - x_B}{h^e} \tag{6}$$

$$x^e(\xi) = \frac{h^e \xi + x_A + x_B}{2} = \sum_{a=1}^{2} N_a(\xi) x_a^e$$

$$= \frac{1}{2}\left((1 - \xi)x_A + (1 + \xi)x_B\right) \tag{7}$$

# 1D Elements

- Integrals with local coordinates

$$K^e_{ab} = \int_{-1}^{+1} N_{a,\xi}(\xi) EA N_{b,\xi}(\xi) \frac{1}{\frac{\partial x}{\partial \xi}} d\xi \tag{8}$$

substituting:

$$N_{a,\xi} = \frac{\partial}{\partial \xi}\left[\frac{1}{2}(1+\xi_a\xi)\right] = \frac{1}{2}\xi_a \tag{9}$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\left[\frac{h^e\xi + x_A + x_B}{2}\right] = \frac{h^e}{2} \tag{10}$$

in (8), results:

$$\mathbf{K}^e = \frac{EA}{h^e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{11}$$

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# Additional material ...

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