7 Matrices

LEARNING OBJECTIVES

Self-Directed Learning @SP

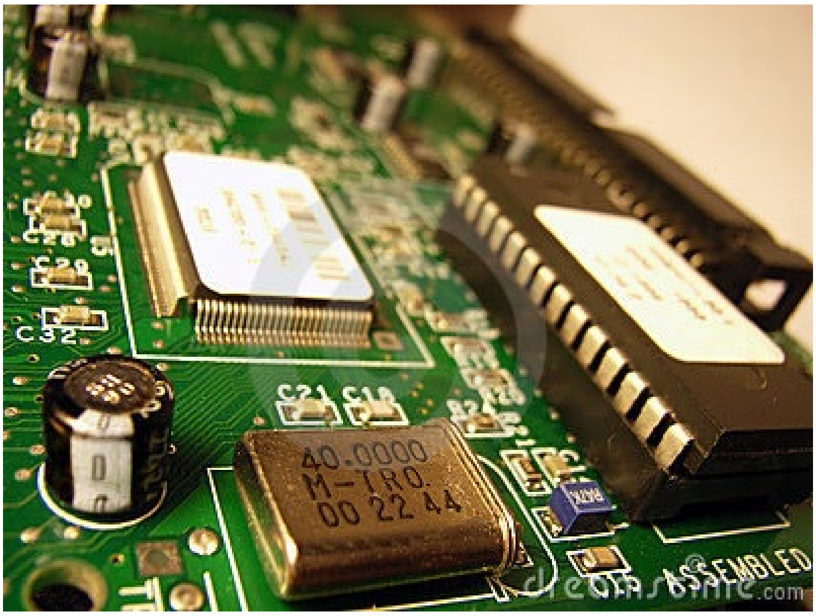
Plan Learning

Upon completion of Chapter 7, you should be able to:

1) Use matrix notation.
2) Add, subtract and multiply matrices.
3) Define identity matrix.
4) Define inverse matrix.
5) Express a pair of linear equations in matrix form and solve the equations by inverse matrix method.
6) Formulate a system of linear equations from a problem situation.
7) Use matrices to represent the basic transformations of reflection and scaling.

Relevant sections in e-book:

- 10.3 Operations on Matrices
- 10.4 Inverse Matrices and Matrix Equations



A computer motherboard contains thousands of logic circuits, millions of gates that perform calculations. These circuits can be represented by arrays of numbers known as matrices.

7.1 Matrix notation

7.1.1 Determine the Order of a Matrix

The **order of a matrix** is determined by the number of rows and number of columns. A matrix with m rows and n columns is an $m \times n$ matrix (read as "m by n" matrix).

VIDEO EXAMPLE 7.1

Determine the order of the matrix.

1. $\begin{bmatrix} 3 & 5 & -1 \\ \frac{1}{2} & \sqrt{3} & 1.7 \end{bmatrix}$

2. $\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$

3. $\begin{bmatrix} 2.4 & 6.9 \end{bmatrix}$

4. $\begin{bmatrix} -3.4 & 0 \\ 7.4 & -9 \end{bmatrix}$

- If a matrix has m rows and n columns, we say that the **dimensions or order** of the matrix is ______________.
- A matrix with only one column is called a ______________ matrix.
- A matrix with only one row is called a ______________ matrix.
- A matrix that has the same number of rows and columns is called a ______________ matrix.

7.1.2 Identify the elements of a matrix

A matrix can be represented generically as follows:

$$\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Using the double subscript notation, a_{43} represents the element in the 4th row, 3rd column. The notation a_{ij} represents the element in the ith row, jth column.

VIDEO EXAMPLE 7.2

Determine the value of the given element of the matrix.

1. a_{13}

2. a_{31}

$$\mathbf{A} = \begin{bmatrix} 3 & -6 & \frac{1}{3} \\ 2 & 4 & 0 \\ \sqrt{5} & 11 & 8.6 \\ \frac{1}{2} & 4 & 2 \end{bmatrix}$$

- We often name matrices with ______________ such as *A*, *B*, *C* and so on.

EXERCISE 7.1

The following is a matrix showing the average travelling time (in minutes) between 5 MRT stations along the East-West Line.

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Boon Lay} & \text{Jurong East} & \text{Outram Park} & \text{City Hall} & \text{Pasir Ris} \end{array} \\ \mathbf{D} = & \begin{pmatrix} 0 & 7 & 26 & 32 & 59 \\ 7 & 0 & 19 & 25 & 52 \\ 26 & 19 & 0 & 6 & 33 \\ 32 & 25 & 6 & 0 & 27 \\ 59 & 52 & 33 & 27 & 0 \end{pmatrix} \begin{array}{l} \text{Boon Lay} \\ \text{Jurong East} \\ \text{Outram Park} \\ \text{City Hall} \\ \text{Pasir Ris} \end{array} \end{array}$$

1. State the dimensions of **D**.
2. What is the average travelling time between Outram Park and Pasir Ris MRT stations?
3. Intepret the zeros in **D**.
4. Describe a special feature of **D**.

7.2 Addition and Subtraction of Matrices

VIDEO EXAMPLE 7.3

$$\mathbf{A}=\begin{bmatrix}6 & -1\\ 7 & \frac{1}{2}\\ 2 & \sqrt{2}\end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix}-9 & 2\\ 6.2 & 2\\ \frac{1}{3} & \sqrt{8}\end{bmatrix} \qquad \mathbf{C}=\begin{bmatrix}11 & 4\\ 1 & -\frac{1}{3}\\ 1 & 6\end{bmatrix} \qquad \mathbf{D}=\begin{bmatrix}2 & 3 & 8\\ -1 & 6 & \frac{1}{6}\end{bmatrix}$$

1. $\mathbf{C}-\mathbf{A}+\mathbf{B}$

2. $\mathbf{B}+\mathbf{D}$

EXERCISE 7.2

Consider the matrices

$$\mathbf{A}=\begin{pmatrix}2 & 3\\ -5 & 8\end{pmatrix}, \mathbf{B}=\begin{pmatrix}0 & 4\\ 7 & -9\end{pmatrix}, \mathbf{C}=\begin{pmatrix}9 & 2 & 1\\ -4 & 5 & -3\end{pmatrix}$$

(a) Find $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}+\mathbf{A}$.

(b) What is the relationship between $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}+\mathbf{A}$?

(c) Can you find the sum $\mathbf{A}+\mathbf{C}$? Why?

(d) Find a matrix $\mathbf{E}$ such that $\mathbf{A}+\mathbf{E}=\mathbf{A}$.

Matrix **E** is also known as a______________________ (also known as a null matrix). It is one in which all elements are zero, and usually denoted by the letter **O**.

$\mathbf{A}+$ ____ $=$ ____ $+\mathbf{A}=\mathbf{A}$

EXERCISE 7.3

The systolic blood pressure (in mm mercury) of a group of 4 patients with high blood pressure was measured just before and 24 hours after each of them has taken a pill of medicine. The data are recorded in the following matrices:

$$\begin{array}{cc} & \text{Blood Pressure} \\ \textit{Before}: \mathbf{B} = & \begin{pmatrix} 162 \\ 145 \\ 132 \\ 180 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Alif} \\ \text{Bala} \\ \text{Cai Feng} \\ \text{Doris} \end{array} \qquad \begin{array}{cc} & \text{Blood Pressure} \\ \textit{After}: \mathbf{A} = & \begin{pmatrix} 148 \\ 126 \\ 137 \\ 174 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Alif} \\ \text{Bala} \\ \text{Cai Feng} \\ \text{Doris} \end{array}$$

(a) Find $\mathbf{B} - \mathbf{A}$.

(b) Interpret the results in (a).

7.3 Multiplication of a Matrix by a scalar

The volume of sales of computer components in two shops in June is represented by the following matrix.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Monitor} & \text{Printer} & \text{Scanner} \end{array} \\ \mathbf{A} = & \begin{pmatrix} 320 & 358 & 107 \\ 216 & 195 & 54 \end{pmatrix} \end{array} \begin{array}{l} \text{Shop A} \\ \text{Shop B} \end{array}$$

Suppose that the sales of each component in both shops doubled in July. Then we can represent the volumes of sales in July by the matrix, called 2**A**, such that

$2\,\mathbf{A} =$

In general, a matrix can be multiplied by a real number (called a scalar). The process is known as scalar multiplication.

VIDEO EXAMPLE 7.4

Given $\mathbf{A}=\begin{bmatrix} 2 & 4 & -9 \\ 1 & \sqrt{3} & \frac{1}{2} \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} -1 & 0 & 4 \\ 2 & 9 & \frac{2}{3} \end{bmatrix}$, find $-2\mathbf{A}-7\mathbf{B}$.

EXERCISE 7.4

Let $\mathbf{A}=\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

(a) Find $7(\mathbf{A}+\mathbf{B})$.

(b) Find $7\mathbf{A}+7\mathbf{B}$.

(c) If k is a scalar, what is the relationship between $k(\mathbf{A}+\mathbf{B})$ and $k\mathbf{A}+k\mathbf{B}$?

7.4 Multiplication of Matrices

For Matrix Multiplication:

$$\underset{m \times n}{\mathbf{A}} \quad \times \quad \underset{n \times p}{\mathbf{B}} \quad = \quad \underset{m \times p}{\mathbf{AB}}$$

same

- Multiplying an $m \times n$ matrix by $n \times p$ matrix gives an__________ matrix.
- Given two matrices **A** and **B**, the matrix product **AB** is well defined provided the number of columns of **A** is ________ to the number of rows of **B**.
- The matrices **A** and **B** are said to be ___________ if the matrix product **AB** is well defined.
- In general, matrix multiplication is NOT commutative, that is **AB**____ **BA**.

VIDEO EXAMPLE 7.5

Given that $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ -6 & 3 \\ 1 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 & -1 \\ -2 & 0 & 10 \end{bmatrix}$, find the product $\mathbf{A} \bullet \mathbf{B}$.

EXERCISE 7.5

It is given that $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -4 \\ 0 & -8 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 6 \end{pmatrix}$. Find the following products where possible.

(a) **AB**

(b) **BA**

(c) **AC**

(d) **CA**

EXERCISE 7.6

It is given that $k = 2$, $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

Find the following products where possible.

(a) **AB**

(b) **AC**

(c) $k(\mathbf{AB})$

(d) $(k\mathbf{A})\mathbf{B}$

(e) $\mathbf{A}(k\mathbf{B})$

EXERCISE 7.7

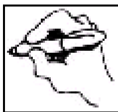
A café sells tea and coffee, each in small and large sizes. The cost of a small cup is \$1.20, while the cost of a large cup is \$1.50. In a day, the following number of cups of drink were sold.

	Small	Large
Coffee	200	100
Tea	100	50

Given that $\mathbf{A} = \begin{pmatrix} 200 & 100 \\ 100 & 50 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1.2 \\ 1.5 \end{pmatrix}$,

(a) find **AB**.

(b) Explain what each element of the matrix **AB** represents.



EXERCISE 7.8

The table below shows the number of visitors to an amusement park in one weekend. The entrance fees for an adult is \$25 and the entrance fees for a child is \$12.

	Saturday	Sunday
Number of Adults	1800	1200
Number of Children	2000	1000

(a) Write down two matrices such that the elements of the product will give the amount of entrance fees collected on Saturday and Sunday.

(b) Hence, calculate the total amount of entrance fees collected for that weekend.

7.5 Identity Matrix

The identity element when multiplying real numbers is 1 because $a \bullet 1 = a$ and $1 \bullet a = a$.

We now investigate a similar property when multiplying square matrices.

The **identity matrix** $\mathbf{I}_n$ is a $n \times n$ square matrix with ones along the main diagonal and zeros for all other elements. The main diagonal runs from the top left corner to the bottom right corner of the matrix.

Identity Matrices:

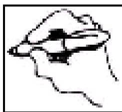
$$\mathbf{I}_2 = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad \mathbf{I}_3 = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

EXERCISE 7.9

Given $\mathbf{A} = \begin{bmatrix} -\frac{7}{8} & \sqrt{5} \\ 5.1 & 8 \end{bmatrix}$,

(a) Verify that $\mathbf{AI}_2 = \mathbf{A}$.

(b) Verify that $\mathbf{I}_2\mathbf{A} = \mathbf{A}$.



EXERCISE 7.10

Let $\mathbf{A} = \begin{pmatrix} 3 & 2 & -5 \\ 7 & -1 & 0 \\ 4 & 9 & 6 \end{pmatrix}$ and $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Compute $\mathbf{A}\mathbf{I}_3$ and $\mathbf{I}_3\mathbf{A}$.

(b) What is the relationship between $\mathbf{A}$, $\mathbf{A}\mathbf{I}_3$ and $\mathbf{I}_3\mathbf{A}$?

(c) Repeat the same for **AO** and **OA** where **O** is a zero matrix.

- For a $n \times n$ square matrix A,
  AI$_n$_____ **A** and **I$_n$A**_____ **A** (identity property of matrix multiplication)

- For zero matrices,
  AO_____**OA**______**O**.

7.6 Inverse Matrix

For a nonzero real number a, the multiplicative inverse of a is $\frac{1}{a}$ because $a \bullet \frac{1}{a} = 1$ and $\frac{1}{a} \bullet a = 1$.

The multiplicative inverse of a square matrix is defined in a similar fashion.

> Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{I}_n$ be the identity matrix of order n. If there exists an $n \times n$ matrix $\mathbf{A}^{-1}$ such that
>
> $$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n \quad \text{and} \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$
>
> then $\mathbf{A}^{-1}$ (read as "*A* inverse") is the **multiplicative inverse** of $\mathbf{A}$.

Note: The notation $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$, not the reciprocal of $\mathbf{A}$. That is, $\mathbf{A}^{-1} \neq \frac{1}{\mathbf{A}}$.

VIDEO EXAMPLE 7.6

Determine whether $\mathbf{A} = \begin{bmatrix} 10 & -3 \\ 4 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{5}{4} \end{bmatrix}$ are inverses.

$\mathbf{AB} =$

$\mathbf{BA} =$

Formula for the inverse of a 2×2 invertible matrix

> Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix, then the inverse $\mathbf{A}^{-1}$ is given by
>
> $$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
>
> The expression $ad - bc$ is called the determinant of $\mathbf{A}$.
>
> It is denoted by $|\mathbf{A}|$ and denoted as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
>
> When $ad - bc = 0$, then $\frac{1}{ad - bc} = \frac{1}{0}$, which is undefined.

Thus if $|\mathbf{A}| = 0,$ the matrix **A** does not have any inverse, i.e. **A** is **singular**

If **A** is an **invertible** matrix, then $|\mathbf{A}| \neq 0$.

VIDEO EXAMPLE 7.7

Find the inverse of matrix $\mathbf{A} = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix}$.

$$\mathbf{A} = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix}$$

EXERCISE 7.11

For matrices **A**, **B**, **C** and **D**, find the corresponding inverse matrix, if it exists.

$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -4 \\ 0 & -8 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 6 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 8 \\ -1 & -4 \end{pmatrix}$.

7.7 Solving Simultaneous Linear Equations By the Inverse Matrix Method

7.7.1 Equation in Matrix Form

Consider the simultaneous linear equations

$$4x+3y=6 \qquad (1)$$

$$2x-y=8 \qquad (2)$$

Let us learn to solve this by the **inverse matrix method**.

First, we represent these equations in the matrix form:

$$\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \qquad (3)$$

The matrix $\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$, whose elements are the coefficients of x and y, is called the **coefficient matrix**.

Let $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$,

$$\mathbf{AX} = \mathbf{B}$$

$$\mathbf{A}^{-1}\,\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{I}_2\,\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\mathbf{B}$$

Calculate the inverse of the coefficient matrix $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$.

$$\mathbf{A}^{-1} = \frac{1}{4(-1)-2(3)}\begin{pmatrix} -1 & -3 \\ -2 & 4 \end{pmatrix} = -\frac{1}{10}\begin{pmatrix} -1 & -3 \\ -2 & 4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{10}\begin{pmatrix} -1 & -3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 8 \end{pmatrix} = -\frac{1}{10}\begin{pmatrix} -30 \\ 20 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

The solution is: $x = 3$ and $y = -2$.

In general, for a pair of simultaneous linear equations

$$ax + by = h$$
$$cx + dy = k$$

We can rewrite them in the matrix form:

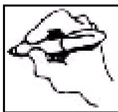
$$\mathbf{AX} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} h \\ k \end{pmatrix}$.

If $\mathbf{A}^{-1}$ exits, we have

$$\mathbf{A}^{-1}\,\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{I}_2\,\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Hence, the solution is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.



EXERCISE 7.12

Solve the following system of linear equations by the inverse matrix method.

$$x + y = 3$$

$$3x - 2y = 4$$

EXERCISE 7.13

The total capacity of 3 buckets and 4 flasks is 68 litres. The total capacity of 5 buckets and 6 flasks is 106 litres. Find the capacity of each bucket and each flask.

EXERCISE 7.14

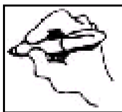
The total mass of 5 pieces of brick and 3 pieces of tile is 9 kg. The total mass of 8 pieces of brick and 6 pieces of tile is 15 kg. Find the mass of each piece of brick and tile.

7.8 Basic Transformations

A **transformation** on the 2-dimensional plane assigns to each point $\boldsymbol{u} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$ on the plane, a new location $\boldsymbol{u}' = \begin{pmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{pmatrix}$ on the plane by multiplying $\boldsymbol{u}$ with a matrix, $\boldsymbol{T}$ such that $\boldsymbol{u}' = \boldsymbol{T}\boldsymbol{u}$.

$\boldsymbol{T}$ is called a **transformation matrix**, and since it transforms or maps a point on the 2-dimensional plane to another point on the plane, $\boldsymbol{T}$ must be a 2 × 2 matrix. Depending on the entries of the transformation matrix, different transformations are effected. We shall explore the effects of two kinds of transformations: reflection and scaling.

Reflection



EXERCISE 7.15

We shall use a graphical object like a house as shown in Figure 6.1 to explore some transformations. Such a house can be represented by a matrix, H, where each of the columns contains the coordinates of each of the 5 corners a to e. Thus,

$$H = \begin{pmatrix} -1 & -3 & -3 & -2 & -1 \\ 1 & 1 & 3 & 4 & 3 \end{pmatrix}.$$

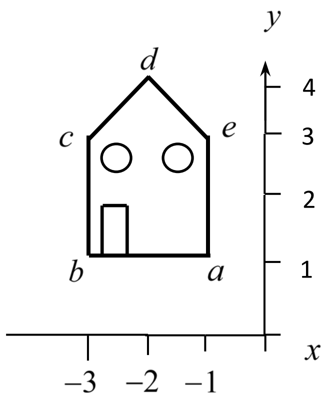


Figure 6.1 A house

Let us explore the geometrical effects of the following transformations matrices:

a) $\boldsymbol{T}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, b) $\boldsymbol{T}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, c) $\boldsymbol{T}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

a) To appreciate the effect of T_1, let us apply it to the house in Figure 6.1

Then the image

$$\boldsymbol{H}' = \boldsymbol{T}_1\boldsymbol{H}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -3 & -3 & -2 & -1 \\ 1 & 1 & 3 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 1 & 1 & 3 & 4 & 3 \end{pmatrix}$$

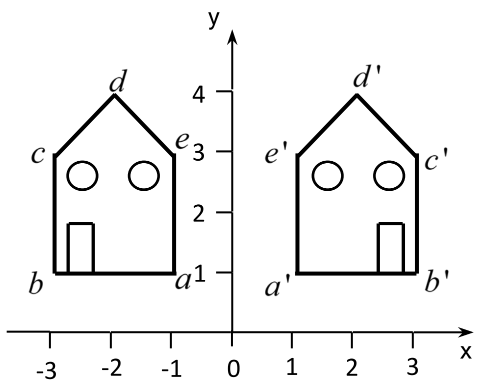


Figure 6.2 Reflection of a house about the y-axis

$\boldsymbol{T}_1$ change the sign of the x-coordinate of every point and leave the y-coordinate unchanged. So, for example, the point $\begin{pmatrix}-3\\1\end{pmatrix}$ is transformed into $\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$ and vice versa. In effect, the transformation matrix $\boldsymbol{T}_1$ gives a **reflection** that takes every point to its image on the opposite side of the y-axis, which is acting as a mirror.

b) To appreciate the effect of $\boldsymbol{T}_2$, let us apply it once again to the house in Figure 6.1.

The image of the house after the transformation is

$$\boldsymbol{H}'=\boldsymbol{T}_2\boldsymbol{H}$$
$$=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}-1&-3&-3&-2&-1\\1&1&3&4&3\end{pmatrix}$$
$$=\begin{pmatrix}-1&-3&-3&-2&-1\\-1&-1&-3&-4&-3\end{pmatrix}$$

and is as shown in Figure 6.3.

In effect, the transformation matrix $\boldsymbol{T}_2$ gives a **reflection** that takes every point to its image on the opposite side of the x-axis, which is acting as a mirror.

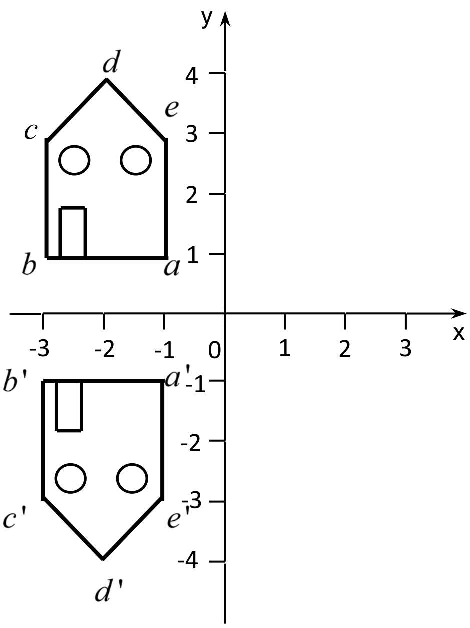


Figure 6.3 Reflection of a house about the x-axis

c) To understand the effect of $\boldsymbol{T}_3$, let us apply to the house in Figure 6.1.

The image of the house after the transformation is $\boldsymbol{H}'=\boldsymbol{T}_3\boldsymbol{H}$

$$=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}-1&-3&-3&-2&-1\\1&1&3&4&3\end{pmatrix}$$
$$=\begin{pmatrix}1&1&3&4&3\\-1&-3&-3&-2&-1\end{pmatrix}$$

and is as shown in Figure 6.4.

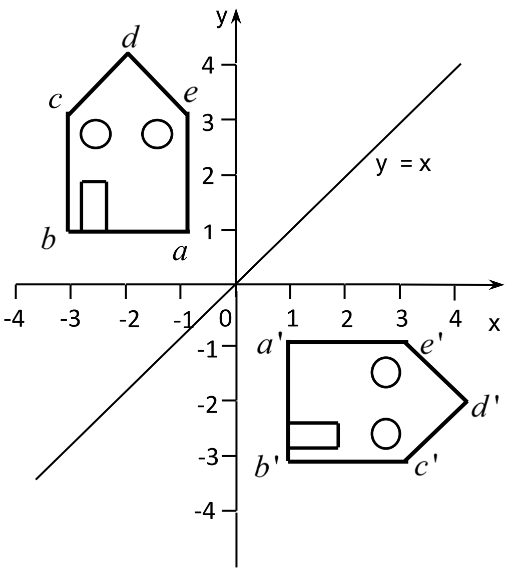


Figure 6.4 Reflection of a house about y = x

In effect, the transformation matrix $\boldsymbol{T}_3$ gives a **reflection** that takes every point to its image on the opposite side of the line y = x, which is acting as a mirror.

So the three transformation matrices, $\boldsymbol{T}_1=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$, $\boldsymbol{T}_2=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\boldsymbol{T}_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, each maps every point to its image on the opposite side of the y-axis, x-axis and the line y = x respectively.

EXERCISE 7.16

a) Taking the columns on the matrix $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ to represent the coordinates of four points of a square on the Cartesian axes, plot the figure in the space provided below.

b) Pre-multiply $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and plot transformed matrix in the same axes used in a). State what the matrix produced represents.

Scaling

EXERCISE 7.17

Let us next look at the geometrical effects of the following two transformation matrices which are diagonal matrices with positive diagonal elements:

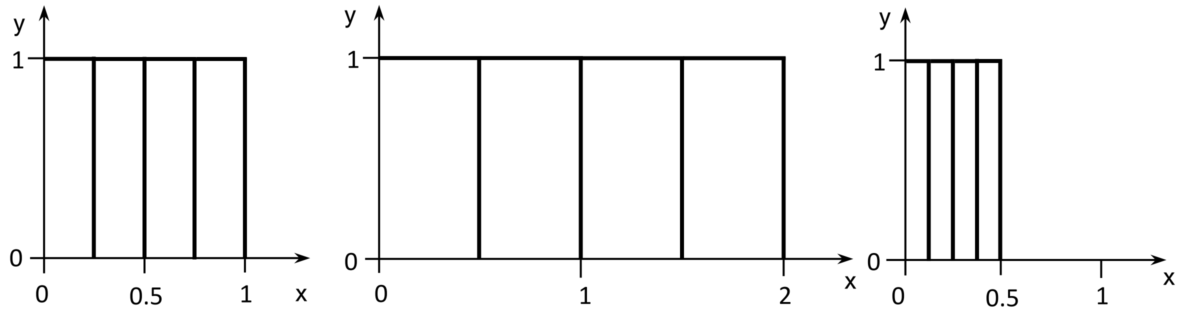
$$\boldsymbol{T}_4 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{T}_5 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The transformation effected by $\boldsymbol{T}_4 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ doubles the x-coordinate of every point and leaves the y-coordinate unchanged. Every point along the y-axis such as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is left unchanged, while every other point such as $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is moved in the x-direction e.g.:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

So effectively, $\boldsymbol{T}_4$ produces an **expansion** by a factor of 2 in the x-direction as can be seen in Figure 6.5 (b).

Similarly, $\mathrm{T}_5 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ half the x−coordinate of every point and leave the y-coordinate unchanged. So effectively, $\boldsymbol{T}_5$ produces a **contraction** by a factor of $\frac{1}{2}$ in the x−direction as can be seen in Figure 6.5 (c).



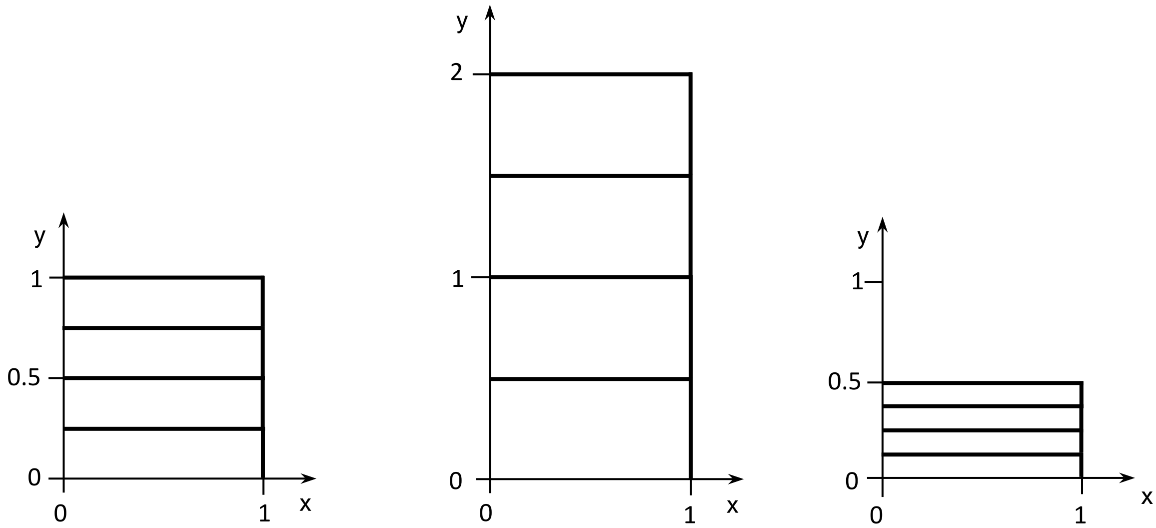
(a) Initial (b) Transform by $\boldsymbol{T}_4$ (c) Transform by $\boldsymbol{T}_5$

Figure 6.5 Scaling in the x-direction

With a similar argument, the following transformation matrices:

$$\boldsymbol{T}_6 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \boldsymbol{T}_7 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

can be shown to produce an **expansion** of a factor of 2 in the y-direction and a **contraction** of a factor of $\frac{1}{2}$ in the y-direction respectively as shown in Figure 6.6.



(a) Initial (b) Transform by $\boldsymbol{T}_6$ (c) Transform by $\boldsymbol{T}_7$

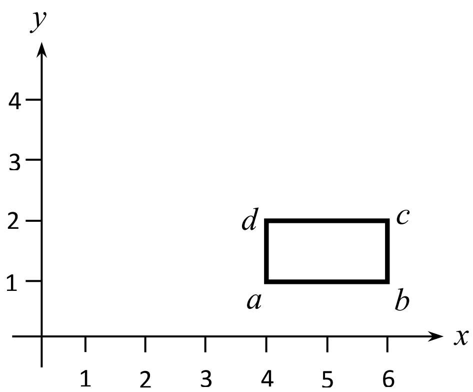
Figure 6.6 Scaling in the y-direction

In general the transformation matrix $\begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix}$ produces scaling relative to the origin in both the x and y-directions. The scaling factors in the x and y directions are k_x and k_y respectively. A factor with value **greater than 1**, will result in an **expansion**. On the other hand, a factor with a positive value **less than 1**, will result in a **contraction**.

EXERCISE 7.18

(a) Find the transformation matrix that will produce a contraction in the x-direction by a factor of $\frac{1}{2}$ and an expansion in the y-direction by a factor of 2.

(b) Using the above transformation matrix, find the image of the rectangle shown below and superimpose it on the same diagram. Besides the scaling effects, what can you observe about the position of the image? Explain your observation.



TUTORIAL CHAPTER 7

Multiple Choice Questions: Choose the best option.

1. The dimensions of the matrix $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$ are

   (a) 1×2 (b) 1×3

   (c) 2×1 (d) 3×1

2. If $|\mathbf{A}| = 0,$ then $\mathbf{A}$ is

   (a) a zero matrix.

   (b) a non-singular matrix.

   (c) a singular matrix.

   (d) 0.

3. If $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ then

   (a) $\mathbf{A} - \mathbf{B}$ cannot be evaluated

   (b) $\mathbf{A} - \mathbf{C}$ can be evaluated

   (c) $\mathbf{A} + \mathbf{B}$ can be evaluated

   (d) $\mathbf{A} + \mathbf{C}$ can be evaluated

4. Which of the following is always true?

   (a) $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

   (b) $\mathbf{AB} = \mathbf{BA}$

   (c) $\mathbf{A} - \mathbf{B} = \mathbf{B} - \mathbf{A}$

   (d) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

5. If $\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, which of the following cannot be evaluated?

   (a) $\mathbf{PQ}$

   (b) $\mathbf{PR}$

   (c) $\mathbf{QR}$

   (d) $\mathbf{R}^2$

Written solutions

1. Compute the following, where possible.

   (a) $\begin{pmatrix} 3 & -4 & 1 \\ 2 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 4 & 5 & -3 \\ 6 & -2 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 5 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -4 & 0 \\ 7 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 \\ -6 \\ 8 \end{pmatrix} \begin{pmatrix} 3 & 2 & -1 \end{pmatrix}$

2. Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 5 & 0 \\ 0 & 4 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -1 & 9 \\ 6 & -4 & 2 \\ 3 & 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 & 8 & 5 \\ 0 & -4 & 2 \\ 2 & 3 & 1 \end{pmatrix}$.

   Compute the following.

   (a) $2\mathbf{A} + 3\mathbf{B}$

   (b) $5\mathbf{B} - 4\mathbf{A} + \mathbf{C}$

   (c) $\mathbf{AB} + \mathbf{AC}$

   (d) $\mathbf{A}(\mathbf{BC})$

3. (a) Given that $\mathbf{R} = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$, $\mathbf{T} = \begin{pmatrix} 0 \\ 4 \\ a \end{pmatrix}$ and $\mathbf{RT} = \begin{pmatrix} 5 \end{pmatrix}$, find the value of a.

   (b) Given that $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & 0 \\ k & \frac{1}{3} \end{pmatrix}$, find the value of k which makes $\mathbf{AB}$ an identity matrix.

4. Given $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}$, find

   (a) $\mathbf{A} - 2\mathbf{B}$
   (b) $\mathbf{B}^{-1}$
   (c) $\mathbf{BA}$

5. A hardware company has two shops A and B. The following table shows the quantities of some tools in the two shops.

	Chiesel	Hammer	Saw
Shop A	18	22	15
Shop B	10	35	20

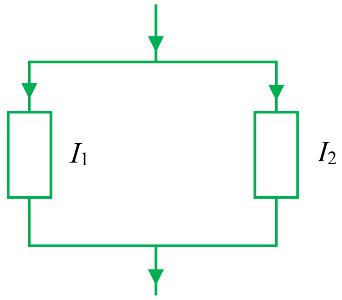
The selling prices of a chisel, a hammer and a saw are \$8, \$16 and \$30 respectively. Their cost prices are \$4, \$9 and \$20 respectively. The information is represented by the matrices $\mathbf{P}=\begin{pmatrix}18 & 22 & 15\\ 10 & 35 & 20\end{pmatrix}, \mathbf{S}=\begin{pmatrix}8\\16\\30\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}4\\9\\20\end{pmatrix}$.

(a) Find **PS** and interpret it.
(b) Find **PC** and interpret it.
(c) Find **P(S–C)** and interpret the values in the matrix
(d) If the two shops sold all their tools, what would be the total profit made by the company?

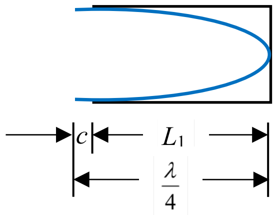
6. Under a Multi-Million Dollar Award Programme (MAP) devised by the Singapore National Olympic Council, an athlete who won an individual gold medal at the Asian Games will receive a cash payout of \$250 000. An individual silver medal will receive \$125 000 and an individual bronze medal will receive \$62 500.
In 2002 Asian Games at Busan, Singapore won 4 gold, 2 silver and 6 bronze individual medals. In the 2006 Asian Games at Doha, Singapore won 4 gold, 2 silver and 8 bronze individual medals.
(a) Write down two matrices which when multiplied, will give the total amount awarded to individual medal winners in the 2002 and 2006 Asian Games
(b) Hence, evaluate the matrix multiplication and state the total amount awarded to the individual medal winners in the 2002 and 2006 Asian Games respectively.

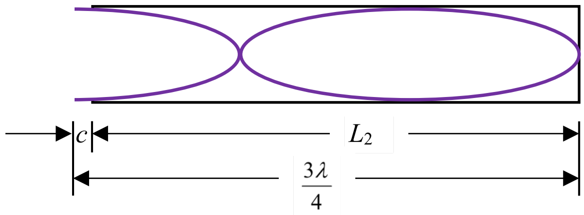
7. Solve the following pairs of simultaneous equations by the inverse matrix method.
(a) $x-y=11$ and $x+y=39$
(b) $19x-5y=13$ and $5x-2y=0$
(c) $x-y+6=0$ and $4x+3y+17=0$
(d) $7x+9-6(y-1)=24$ and $7y+10-6(x+3)+18=0$
(e) $\frac{x}{6}-\frac{5y}{3}=4$ and $\frac{x}{12}-\frac{10y}{3}=4$

8. In an electric circuit, the currents I_1 and I_2, in amperes, flowing through two wires satisfy the equations
$I_1+I_2=5$ and $3I_1+4I_2=18$
Find the values of I_1 and I_2.



9. In a resonance experiment, the wavelength, λ cm, of a sound and the end correction, c cm can be illustrated as follows:





where L_1 cm and L_2 cm are the lengths of air columns in a closed pipe.

(a) Form two linear equations relating λ and c,

(b) If $L_1 = 15$ and $L_2 = 47$, find the values of λ and c.

10. A customer purchases a blend of two coffees: Brazilian, costing \$3.50 per kg and Colombian, costing \$5.60 a kg. He buys 3 kg of the blend, which costs \$11.55. How many kilograms of each coffee went into the mixture?

11. The perimeter of a rectangle is 44 m. If the length of the rectangle is reduced by 3 m and the breadth is increased by the same amount, a square is formed instead. Find the area of the original rectangle.

12. *A two-digit number is equal to 7 times the sum of its digits. If the digits of the number are reversed, the new number is 18 less than the original number. Find the original two-digit number.

13. The equations of two lines L_1 and L_2 are

$L_1 : 2x + (k-3)y - 6 = 0$ and $L_2 : kx + 5y - 33 = 0$

(a) Find the y-intercept of L_2,

(b) Find the gradient of L_2 in terms of k.

(c) A line L_3 passes through the point $A(-2,4)$ and has the same y-intercept as L_2. Find the equation of L_3.

(d) When $k = 4$, using the inverse matrix method, find the coordinate of the point of intersection of L_1 and L_2.

(e) Find the possible values of k such that the lines L_1 and L_2 do not intersect.

14. *A developer buys 3 types of air-conditioners for a housing estate which has 4 different kinds of blocks. The table on the next page shows the number of different types of air-conditioners required per block of each kind.

	Double - System	Triple - System	Multi - System
Type *A* Block	0	3	2
Type *B* Block	3	2	1
Type *C* Block	2	1	0
Type *D* Block	1	1	0

The costs of the air-conditioners are \$2 000 for a Double-System, \$2 500 for a Triple-system and \$3 000 for a Multi-System. The information can be represented by the matrices $\mathbf{X} = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 2000 \\ 2500 \\ 3000 \end{pmatrix}$.

(a) Find **XY**, and explain what the numbers in your answer represent,

(b) There are 10 Type A blocks, 120 type B blocks, 180 Type C blocks and 90 Type D blocks in the estate. Write down a 1×4 matrix to show the number of blocks of each type. Hence, using the matrix multiplication, find the total cost of buying air-conditioners for the whole estate.

15. ** A bakery shop makes four types of bread, W, X, Y, Z. Each type of bread requires three main ingredients: flour, butter and sugar. The amount (in 100 grams) of each ingredient required for each loaf of bread and the cost per 100 grams, in dollars, are given in the following table.

Ingredients	Amount ($\times$100 grams)				Cost (\$) per 100 grams
	W	X	Y	Z	
Flour	1	2	0.5	0.75	0.20
Butter	0.5	0.75	0.4	0.25	0.80
Sugar	0.5	0.5	0.3	0.5	0.10

Everyday, the bakery makes 200 loaves of Bread W, 300 loaves of Bread X, 400 loaves of Bread Y and 100 loaves of Bread Z.

(a) Write down the matrices which when multiplied, will give the daily total cost of making these breads.

(b) Hence evaluate the daily total cost.

16. A triangle ΔABC with coordinates $A(-2, 0)$, $B(0, 5)$ and $C(-3, 2)$ is reflected in

(a) the x-axis,

(b) the y-axis and

(c) the line $y = x$.

Find the image of each of the reflections and sketch them.

17. An isosceles triangle with coordinates (1, 2), (1, 6) and (4, 4) is expanded by a factor of 2 in the x-direction and compressed by a factor of $\frac{1}{3}$ in the y-direction. Write down the transformation matrix and hence find the image. Sketch the triangle and the image in the same diagram.

Problem-solving Assignment 3

The goal of this series of problem-solving assignments is to develop problem-solving skills, not just to test your ability to get the answer. It's ok to try hard and not succeed at first (only your effort is evaluated), but <u>you must try</u>.

Question 1

A dietician prepares a special meal consisting of two types of meat *A* and *B*. Each kilogram of meat *A* consists of 110g of fat and 260g of protein. Each kilogram of meat *B* consists of 65g of fat and 190g of protein. If the meal should contain 158g of fat and 388 g of protein, how many kilograms of meat *A* and meat *B* should be used?

1. Understand the problem • State the given conditions and quantities. • Identify the unknown that you are asked to find. • If applicable, draw a diagram to describe the scenario.	

2. Devise a plan
• Break down the problem into smaller parts.
• Identify which are the relevant concepts that can be applied.
• The following are some strategies that may be useful:
    o Write an equation that describes the relationship between the unknown and given quantities.
    o Make a table.	

3. Implement the plan • Carry out the plan, showing each step clearly. • Any graph or diagram should be clearly labelled.	
4. Look back Substitute your answer back into the problem and check if it satisfies the given conditions.	

ANSWERS

Multiple Choice Questions:

1. B 2. C 3. C 4. D 5. C

Written solutions

1. (a) $\begin{pmatrix} 7 & 1 & -2 \\ 8 & 3 & 8 \end{pmatrix}$; (b) $\begin{pmatrix} 7 \\ 7 \\ -2 \end{pmatrix}$; (c) $\begin{pmatrix} 38 & 22 \end{pmatrix}$; (d) $\begin{pmatrix} 6 & 4 & -2 \\ -18 & -12 & 6 \\ 24 & 16 & -8 \end{pmatrix}$

2. (a) $\begin{pmatrix} 12 & -3 & 31 \\ 16 & -2 & 6 \\ 9 & 8 & 5 \end{pmatrix}$; (b) $\begin{pmatrix} -3 & 3 & 42 \\ 34 & -44 & 12 \\ 17 & -13 & 2 \end{pmatrix}$; (c) $\begin{pmatrix} 13 & 27 & 46 \\ 29 & -47 & 6 \\ 29 & -29 & 18 \end{pmatrix}$; (d) $\begin{pmatrix} 46 & 195 & 83 \\ -26 & 303 & 103 \\ -9 & 307 & 112 \end{pmatrix}$

3. (a) $a = -3$, (b) $k = \frac{1}{6}$

4. (a) $\begin{pmatrix} -7 & -8 \\ 8 & -1 \end{pmatrix}$; (b) $\begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$; (c) $\begin{pmatrix} 27 & -19 \\ -10 & 7 \end{pmatrix}$

5. (a) $\mathbf{PS} = \begin{pmatrix} 946 \\ 1240 \end{pmatrix}$; (b) $\mathbf{PC} = \begin{pmatrix} 570 \\ 755 \end{pmatrix}$; (c) $\begin{pmatrix} 376 \\ 485 \end{pmatrix}$; (d) \$861

6. (a) $\begin{pmatrix} 4 & 2 & 6 \\ 4 & 2 & 8 \end{pmatrix}$, $\begin{pmatrix} 250000 \\ 125000 \\ 62500 \end{pmatrix}$ OR $\begin{pmatrix} 250000 & 125000 & 62500 \end{pmatrix}$, $\begin{pmatrix} 4 & 4 \\ 2 & 2 \\ 6 & 8 \end{pmatrix}$

   (b) $\begin{pmatrix} 1625000 \\ 1750000 \end{pmatrix}$ OR $\begin{pmatrix} 1625000 & 1750000 \end{pmatrix}$

   In 2002, \$1 625 000 was awarded. In 2006, \$1 750 000 was awarded.

7. (a) $x = 25, y = 14$; (b) $x = 2, y = 5$; (c) $x = -5, y = 1$; (d) $x = \frac{3}{13}, y = \frac{-16}{13}$;

   (e) $x = 16, y = \frac{-4}{5}$;

8. $I_1 = 2\text{A}, I_2 = 3\text{A}$

9. (a) $L_1 + c = \frac{\lambda}{4}, L_2 + c = \frac{3\lambda}{4}$; (b) $\lambda = 64$ cm, $c = 1$ cm

10. 2.5 kg of Brazilian, 0.5 kg of Colombian

11. 112m^2

12. 42

13. (a) $\frac{33}{5}$; (b) $-\frac{k}{5}$; (c) $13x - 10y + 66 = 0$; (d) $\left(-\frac{1}{2}, 7\right)$; (e) -2 or 5

14. (a) $\begin{pmatrix} 13500 \\ 14000 \\ 6500 \\ 4500 \end{pmatrix}$, Total amount required for buying air-conditioners for each type block respectively; (b) $\begin{pmatrix} 10 & 120 & 180 & 90 \end{pmatrix}$, \$3 390 000

15. (a) $\begin{pmatrix}0.2 & 0.8 & 0.1\end{pmatrix}, \begin{pmatrix}1 & 2 & 0.5 & 0.75\\ 0.5 & 0.75 & 0.4 & 0.25\\ 0.5 & 0.5 & 0.3 & 0.5\end{pmatrix}, \begin{pmatrix}200\\300\\400\\100\end{pmatrix}$ OR $\begin{pmatrix}200 & 300 & 400 & 100\end{pmatrix}, \begin{pmatrix}1 & 0.5 & 0.5\\ 2 & 0.75 & 0.5\\ 0.5 & 0.4 & 0.3\\ 0.75 & 0.25 & 0.5\end{pmatrix}, \begin{pmatrix}0.2\\0.8\\0.1\end{pmatrix}$

(b) Daily Total Cost is \$665

19. $\boldsymbol{T} = \begin{pmatrix}2 & 0\\ 0 & \frac{1}{3}\end{pmatrix}$ Image $= \begin{pmatrix}2 & 2 & 8\\ \frac{2}{3} & 2 & \frac{4}{3}\end{pmatrix}$

Problem-solving Assignment 3:

1. 1.2 kg of A, 0.4 kg of B