4 Understand Functions and Their Graphs

LEARNING OBJECTIVES

Self-Directed Learning @SP

Plan Learning

Upon completion of Chapter 4, you should be able to:

1) Define a relation.
2) Define a function.
3) Use function notation.
4) State the domain and range of a function.
5) Sketch graphs of absolute value functions.
6) Sketch graphs of functions of the form $y = ax^n$ where n = -1, 0, 1, 2 and 3.
7) Describe characteristics of graphs such as symmetry, intersection with the axes, turning points and asymptotes.
8) Deduce the effect of transformations on the graph of $y = f(x)$ as represented by $y = kf(x), y = f(x) + k$ and $y = f(x+k)$, where k is a constant.
9) Find instantaneous rates of change with tangent lines.
10) Solve application problems involving linear functions.

Relevant sections in e-book:

- 2.3 Functions and Relations

"He's right! When you look at it that way, it's not so bad!"

The essence of mathematics is not to make simple things complicated,
but to make complicated things simple.
-S. Gudder

4.1 Relations

In the physical world, many quantities that are subject to change are related to other variables. For example:

- The cost of mailing a package is related to the weight of a package.
- The minimum braking distance of a car depends on the speed of the car.
- The perimeter of a rectangle is a function of its length and width.
- The test score that a student earns is related to the number of hours of study.

We often see situations where one variable is somehow linked to the value of another variable. For example, a person's level of education is linked to annual income, engine size is linked to gas mileage, the amount of hours put into revising a topic is linked to the marks obtained in an exam, etc. Can you give a few more examples?

In mathematics we can express the relationship between two values as a set of ordered pairs.

Definition of a **Relation**.

A set of ordered pairs (x, y) is called a **relation** in x and y.

•The set of x values in the ordered pairs is called the **domain** of the relation.
•The set of y values in the ordered pairs is called the **range** of the relation.

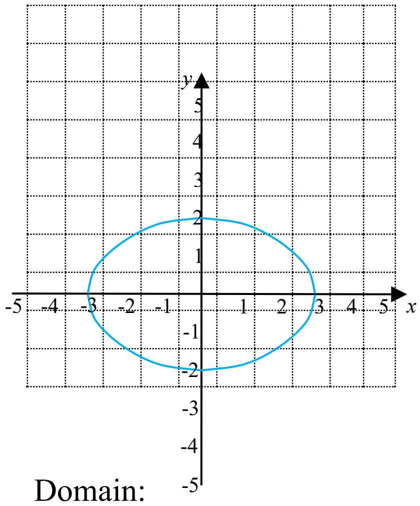
VIDEO EXAMPLE 4.1

a. $\{(3, 1), (4, 2), (-1, 2)\}$

Domain:

Range:

b.



Domain:

Range:

c. $y = x + 1$

Domain:

Range:

4.2 Functions

4.2.1 Definition of function

In many applications, we prefer to work with relations that assign one and only one y value for a given value of x. Such a relation is called a function.

Definition of a **Function**.

Given a relation in x and y, we say that y is a **function** of x if for each value of x in the domain, there is exactly one value of y in the range.

VIDEO EXAMPLE 4.2

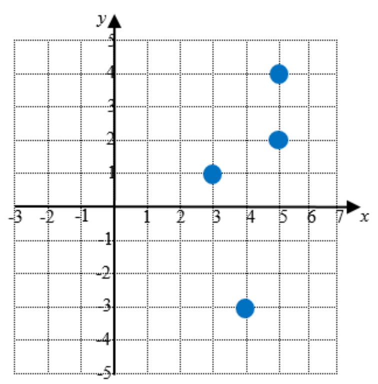
Determine whether the relation defines y as a function of x.

a. $\{(5, 2), (4, -3), (3, 1), (5, 4)\}$

b. $\{(3, 1), (4, 2), (-1, 2)\}$

VIDEO EXAMPLE 4.3

$\{(5, 2), (4, -3), (3, 1), (5, 4)\}$

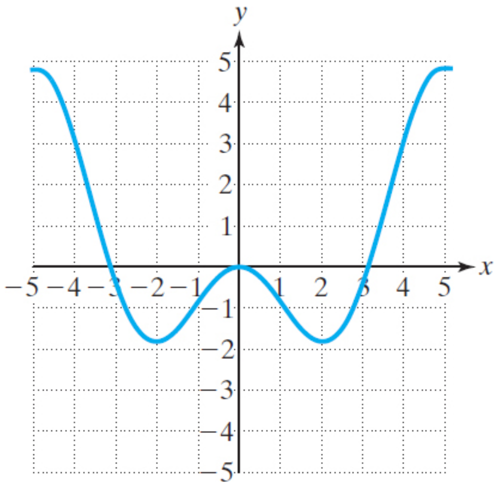


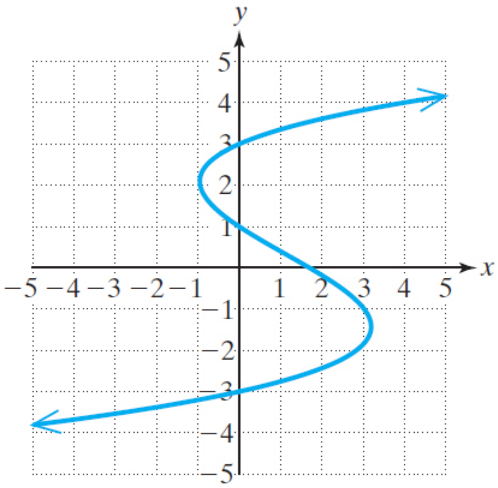
4.2.2 Determine Whether a Relation Is a Function

Using the **Vertical Line Test**.

Consider a relation defined by a set of points (x, y) graphed on a rectangular coordinate system. The graph defines y as a function of x if no **vertical line** intersects the graph in more than one point.

VIDEO EXAMPLE 4.4





Note for the exercise below: $(x-a)^2 + (y-b)^2 = r^2$ is an equation of a circle of radius r and with the centre of the circle at (a, b).

EXERCISE 4.1

Determine if the relation is a function.

{(-3, 4), (-1, -2), (2, 3), (2, 1), (2, -4)}

4.3 Function notation

A function may be defined by an equation with two variables. For example, the equation $y = x - 2$ defines y as a function of x. This is because for any real number x, the value of y is the unique number that is 2 less than x.

When a function is defined by an equation, we often use function notation. For example, the equation $y = x - 2$ may be written in function notation as

$f(x) = x - 2$ read as "f of x equals $x - 2$".

With function notation,

- f is the name of the function,
- x is an input variable from the domain,
- $f(x)$ is the function value (or y value) corresponding to x.

Avoiding Mistakes: The notation $f(x)$ does not imply multiplication of f and x.

VIDEO EXAMPLE 4.5

Given $y = x^2 + 3x$, evaluate the function for the given values of x.

a. $f(-2)$

b. $f(-1)$

c. $f(0)$

d. $f(1)$

e. $f(2)$

VIDEO EXAMPLE 4.6

Given $f(x) = x^2 + 3x$, evaluate $f(a+4)$.

EXERCISE 4.2

Given that $f(x) = x^2 - \frac{2}{x}$, evaluate and simplify each expression.

(a) $f(2)$

(b) $f(-1)$

(c) $f\left(\frac{1}{2}\right)$

(d) $f\left(\frac{1}{x}\right)$

EXERCISE 4.3

Given that $f(x) = 5x^2 + 3x$, find

(a) $f(x+h)$

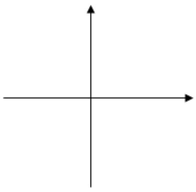
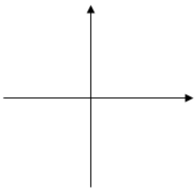
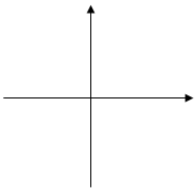
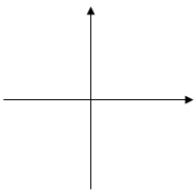
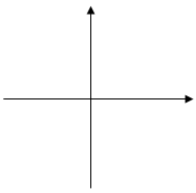
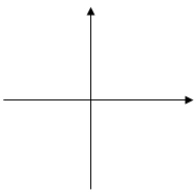
(b) $\dfrac{f(x+h) - f(x)}{h}$

4.4 Graphical representation of functions

Let's look at graphs of $f(x) = x^n$ where $n = -1, 0, 1, 2, 3$. We can use an online graphing tool to help us. One recommended website is Desmos at https://www.desmos.com/calculator.

VIDEO EXAMPLE 4.7

Complete the following table.

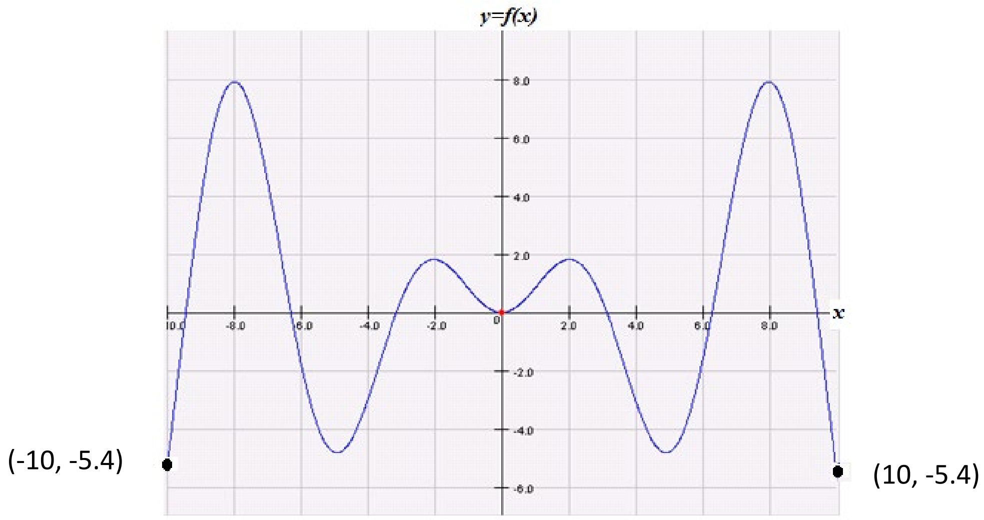
n	Type of function	Algebraic representation	Numerical representation	Visual representation	Key features (intercepts, symmetries, asymptotes, vertex etc)	Domain & Range
-1	Reciprocal	$f(x)=\frac{1}{x}$	x | $f(x)$ -10 | -0.1 | 0 | 0.1 | 10 |			
0						
1	Linear	$f(x)=x$				
1	Absolute	$f(x)=\lvert x\rvert$				
2						
3	Cubic	$f(x)=x^3$	x | $f(x)$ -2 | -1 | 0 | 1 | 2 |			

Visualizing domain and range

Domain is the set of inputs, found on the horizontal axis (x-axis).
Range is the set of outputs, found on the vertical axis (y-axis).

The most common method for visualizing a function is its graph. The graph of f consists of all the points (x, y) in the coordinate plane such that $y = f(x)$. The graph f allows us to picture the domain of f on the x-axis and its range on the y-axis, see Exercise 4.4.

EXERCISE 4.4

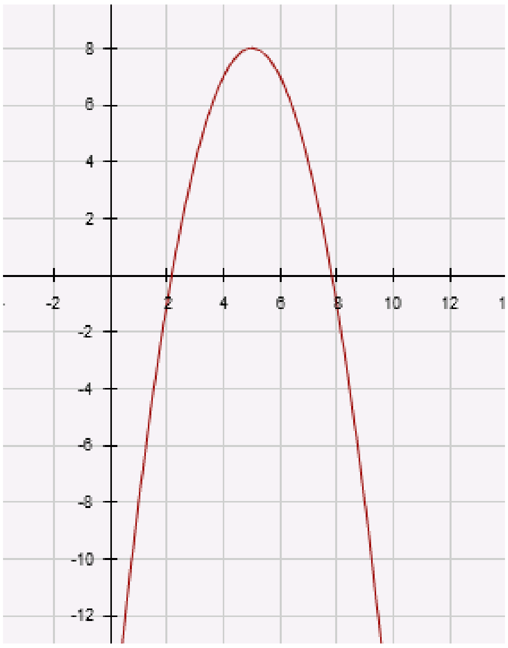


(a) Determine $f(0)$.

(b) Given that $f(x)$ is 8, what is/are the corresponding value(s) of x?

(c) State the domain and range for the above graph.

EXERCISE 4.5



(a) State the domain for this graph.

To find the domain, we look for the inputs on the x-axis that correspond to a point on the graph. We see that they include all values between $-\infty$ and ∞.

The domain is $\{x \mid -\infty < x < \infty\}$ or $(-\infty, \infty)$

In general, the **set-builder notation**:

$\{x \mid condition\}$ means "the set of all elements x such that x satisfies the *condition*".

In **interval notation**, a parenthesis) or (indicates that an endpoint is <u>not</u> included in an interval.

(b) State the range for this graph.

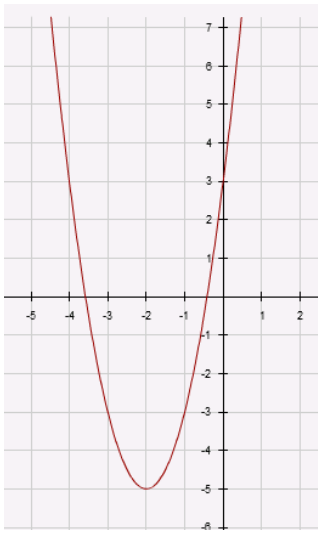
To find the range, we look for the outputs on the y-axis that correspond to a point on the graph. We see that they include all values less than 8.

The range is $\{y \mid y \le 8\}$ or $(-\infty, 8]$.

In **interval notation**, a bracket] or [indicates that an endpoint <u>is</u> included in the interval.

EXERCISE 4.6

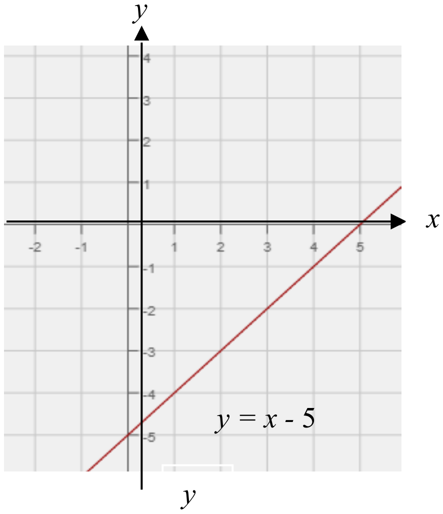
(a) What is the domain of this graph?



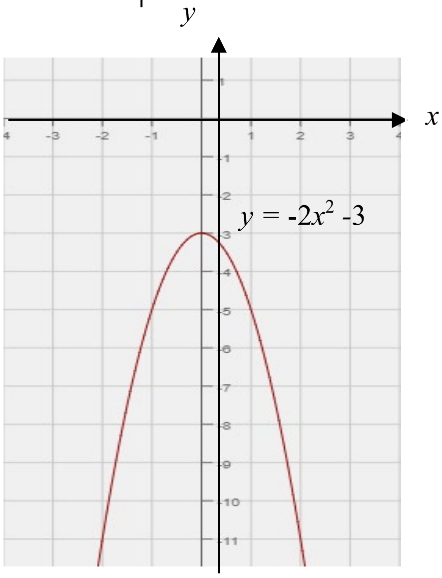
(b) What is the range of this graph?

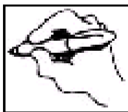
EXERCISE 4.7

1. State the domain and range of the given graph.



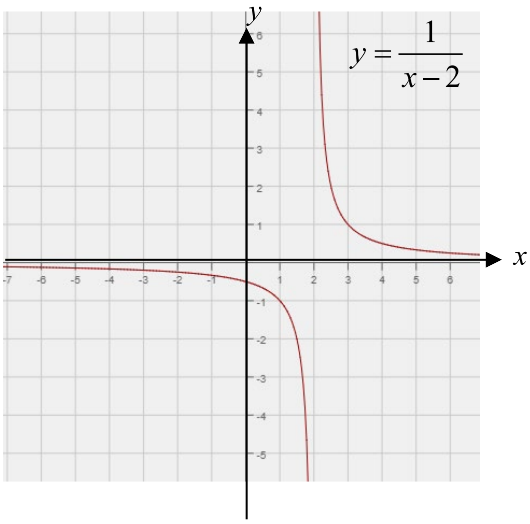
2. State the domain and range of the given graph.





EXERCISE 4.8

State the asymptotes, domain and range of the given graph.



In the given graph,
$x = 2$ is the vertical asymptote,
$y = 0$ is the horizontal asymptote.

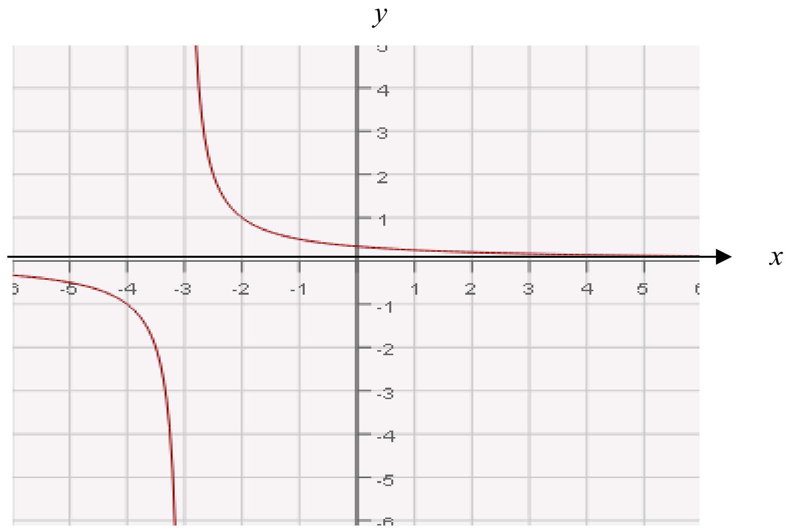
An **asymptote** is a line that a curve approaches, as it heads towards infinity.

The domain is $\{x \mid x \neq 2\}$.

The range is $\{y \mid y \neq 0\}$.

EXERCISE 4.9

State the asymptotes, domain and range of the given graph.



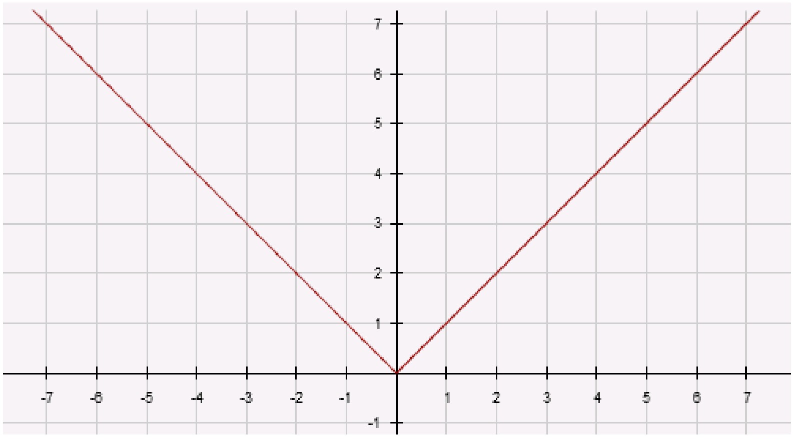
$$y = \frac{1}{x+3}$$

The absolute value function

If $f(x) = x$, then $y = |x|$ has a two-part rule

$$y = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

For $x < 0$, the graph of this function coincides with the line $y = -x$.
For $x \geq 0$, the graph of this function coincides with the line $y = x$.



EXERCISE 4.10

Sketch the graphs of the following.

(a) $y = |2x - 7|$

(b) $y = |-2x - 7|$

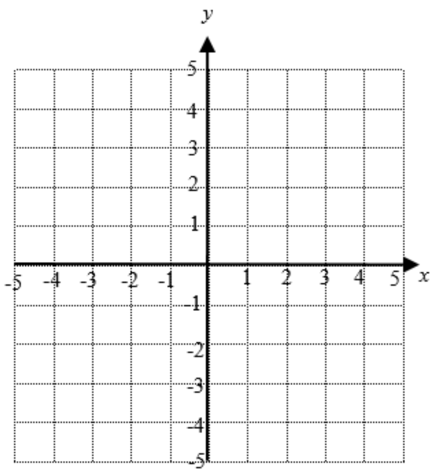
4.5 Transformation of functions

4.5.1 Vertical Translation

EXERCISE 4.11

Sketch the graph of $f(x) = |x| + 1$.

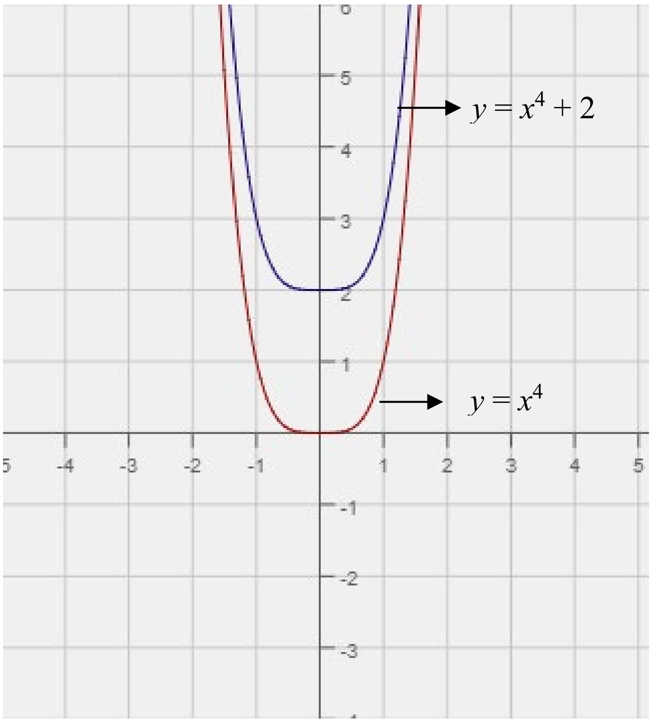
x	$y = \|x\|$	$f(x) = \|x\| + 1$
-3		
-2		
-1		
0		
1		
2		
3		



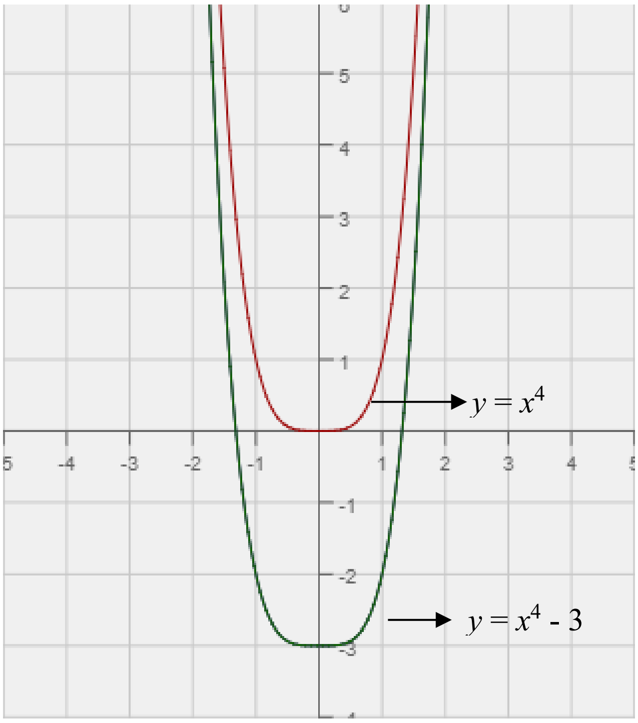
Consider a function defined by $y = f(x)$ and k is a positive real number,

- The graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted k units ______________
- The graph of ____________ is the graph of $y = f(x)$ ___________________________

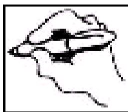
Below is another example that illustrates the graphs of $y = f(x)$, $y = f(x) + k$ and $y = f(x) - k$ for $k > 0$.



The graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted **up** by k units.



The graph of $y = f(x) - k$ is the graph of $y = f(x)$ shifted **down** by k units.

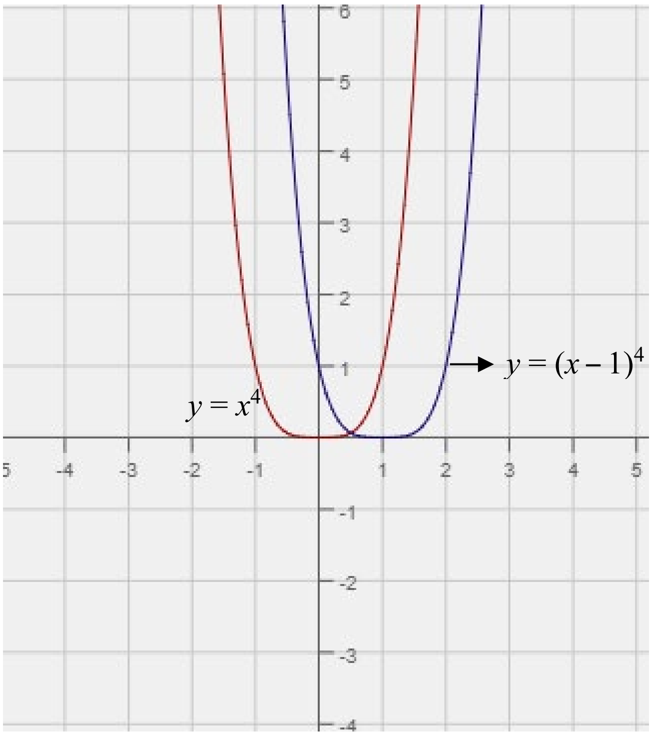


EXERCISE 4.12

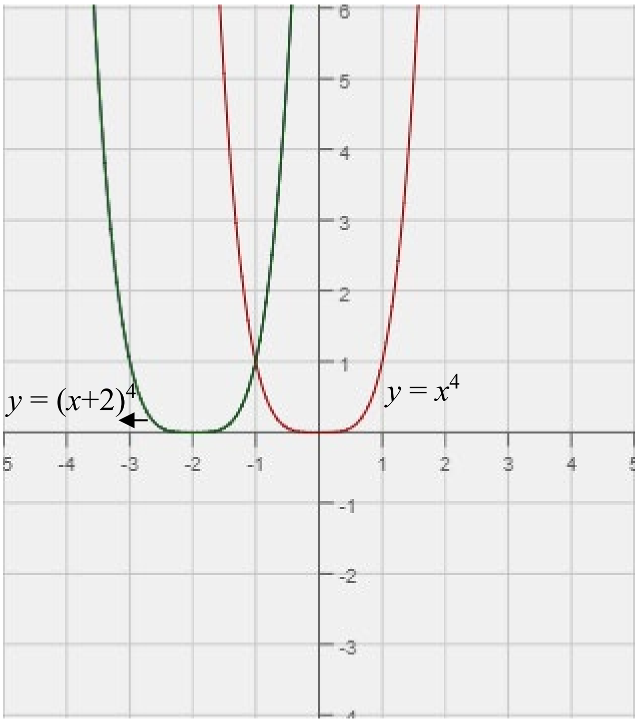
On the same axes, sketch the graphs of $y = x^2$, $y = x^2 + 4$ and $y = x^2 - 1$

4.5.2 Horizontal Translation

Suppose that we have a function given by $y = f(x)$, let's explore the graphs of the new functions $y = f(x - h)$ and $y = f(x + h)$ for $h > 0$.



The graph of $y = f(x - h)$ is the graph of $y = f(x)$ shifted **right** by h units.



The graph of $y = f(x + h)$ is the graph of $y = f(x)$ shifted **left** by h units.

Consider a function defined by $y = f(x)$ and h is a positive real number,

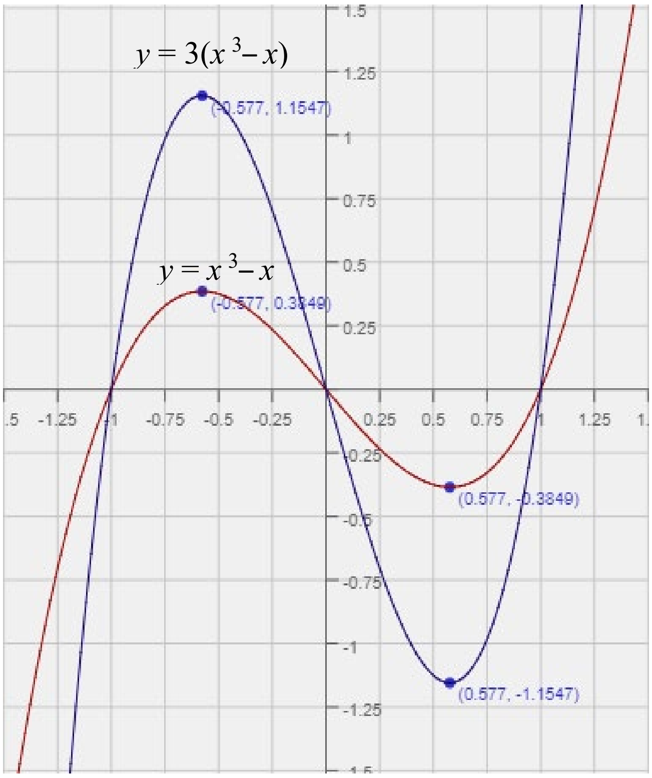
- The graph of $y = f(x - h)$ is the graph of $y = f(x)$ shifted h units _____________
- The graph of ____________ is the graph of $y = f(x)$ ____________________________

EXERCISE 4.13

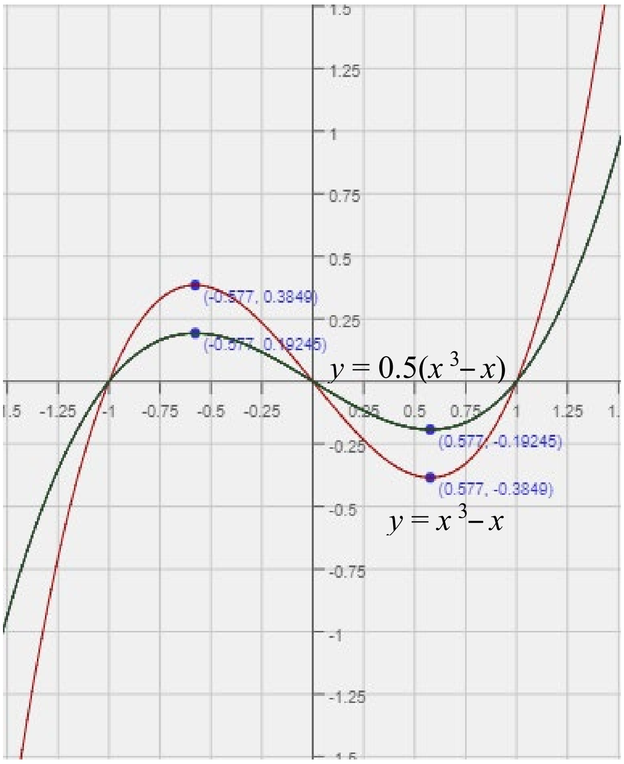
On the same axes, sketch the graphs of $y = x^2$, $y = (x + 3)^2$ and $y = (x - 6)^2$.

4.5.3 Vertical Stretching and Shrinking

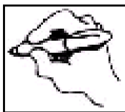
Suppose that we have a function given by $y = f(x)$, let's explore the graphs of the new functions $y = a\,f(x)$.



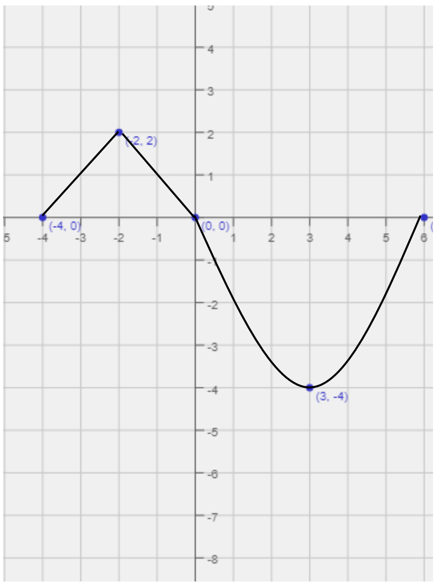
For $a > 1$,the graph of $y = k\,f(x)$ can be obtained from the graph of $y = f(x)$ by **stretching** vertically.



For $0 < a < 1$, the graph of $y = k\,f(x)$ can be obtained from the graph of $y = f(x)$ by **shrinking** vertically.



EXERCISE 4.14



Given the graph of $y = f(x)$ on the left, sketch the graphs of

a) $g(x) = 2f(x)$ and

b) $h(x) = 0.5f(x)$.

4.6 Application and Problem Solving Using Functions

"Solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable." – Polya 1981, p. ix

One of the main reasons for studying mathematics is to use it to solve problems.

To solve most real-life application problems, we need to express the problem in mathematical symbols using expressions or equations, that is, to create a **mathematical model** of the situation.

Based on George Polya's *How To Solve It*, the following are some guidelines for problem solving:

1. Understand the problem

- State the given conditions and quantities.
- Identify the unknown that you are asked to find. Introduce suitable notation: choose a variable to represent the unknown, and represent any other quantity to be found in terms of this variable.
- If applicable, draw a diagram to describe the scenario.

2. Devise a plan

- Break down the problem into smaller parts.
- Identify which are the relevant concepts that can be applied.
- The following are some strategies that may be useful:
    - Write an equation that describes the relationship between the unknown and given quantities.
    - Make a table.
    - Work backward.
    - Use guess and check.

3. Implement the plan

- Carry out the plan, showing each step clearly.
- Any graph or diagram should be clearly labelled.

4. Look back (reflect)

- Check if your solution answer the problem's question.
- Ask yourself "Does it answer the question that was asked?" "Does the answer make sense?" "Is the answer reasonable?"
- Determine if there is any other easier way of finding the solution.

EXERCISE 4.15

An airplane leaves the airport and flies due east at a speed of 180 km/h. Two hours later, a private jet leaves the airport and flies due east at a speed of 900 km/h. How far from the airport will the jet overtake the airplane?

1. Understand the problem • State the given conditions and quantities. • Identify the unknown that you are asked to find. • If applicable, draw a diagram to describe the scenario.	
2. Devise a plan • Break down the problem into smaller parts. • Identify which are the relevant concepts that can be applied.	
3. Implement the plan • Carry out the plan, showing each step clearly. • Any graph or diagram should be clearly labelled.	
4. Look back • Substitute your answer back into the problem and check if it satisfies the given conditions.	

EXERCISE 4.16

Ryan borrowed money at 5% simple interest. At the end of 1 year, he owed a total of $1365 in principal and interest. How much did he borrow?

EXERCISE 4.17

Colin has invested $4000 in a 1-year fixed deposit account at a bank, at an interest rate of 4.2%. He recently discovered that another bank is offering a 1-year fixed deposit account at an interest rate of 6%. How much must he invest at the other bank if he wants to receive an overall interest of $500 from both his fixed deposit accounts ?

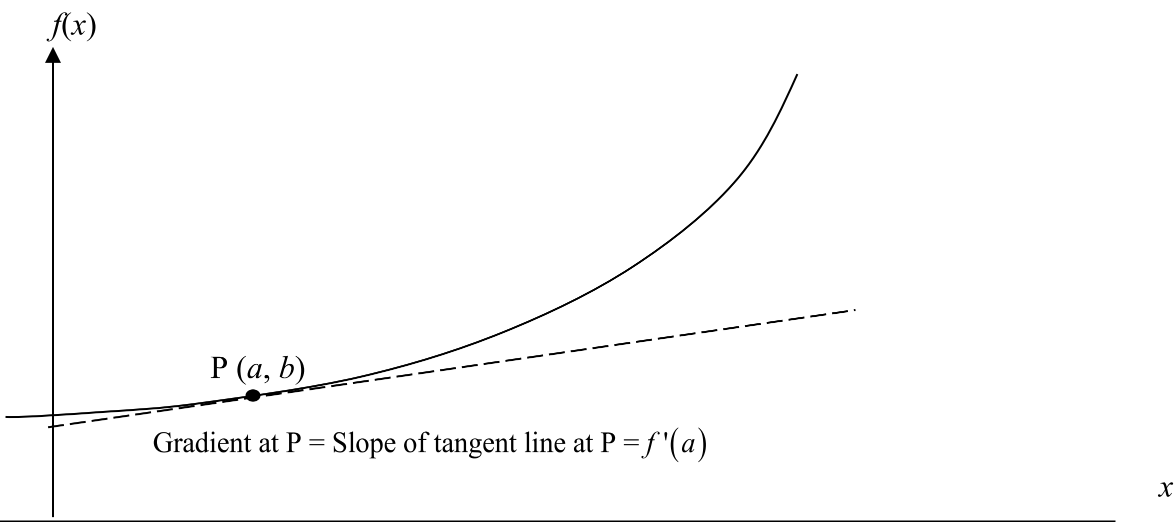
4.7 Rates of change

Many real world problems involve changing quantities – the speed of a rocket, the inflation of a currency, the number of bacteria in a culture, the shock intensity of an earthquake, the voltage of an electrical signal and so forth.

The rate of change is the amount of change that occurred over time. Examples of application on the rate of change are:

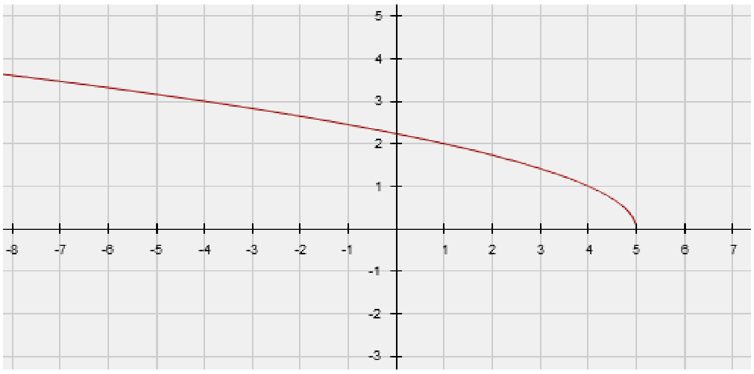
- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- An engineer might be interested in the rate at which the length of a metal rod changes with temperature.

In this section, you will learn how to estimate the gradient of a curve by sketching a straight line that touches the curve at exactly one point (so-to-speak), say point P where $x = a$. This straight line is called the tangent line. So the (estimated) **<u>instantaneous rate of change</u>** of the curve at point P is the same as the slope of the tangent line.

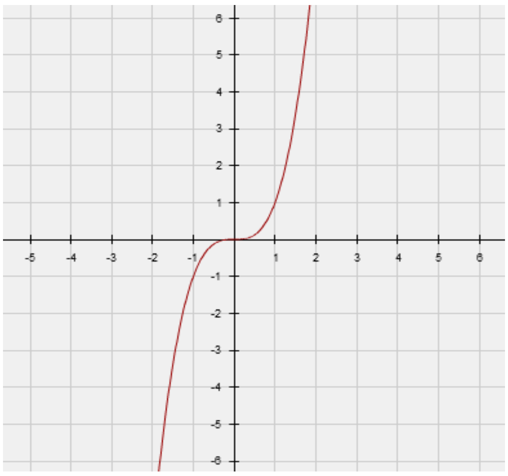


EXERCISE 4.18

(i) Find $f(1)$ and $f'(1)$ for the following curve.



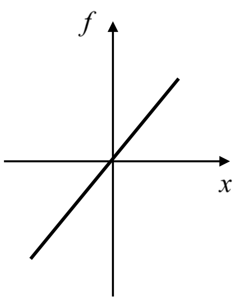
(ii) Find $f(-1)$ and $f'(-1)$ for the following curve.



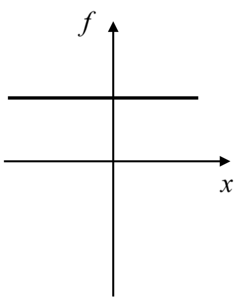
In general, $f'(a)$ is the slope of the tangent line to the graph at $x = a$.

Recap on properties of the slope of a tangent line:

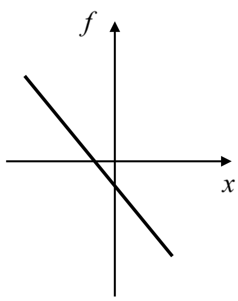
1. The value tells us the steepness of the slope.
2. The sign of f' tells us whether f is (a) increasing, or (b) constant, or (c) decreasing as x increases.



(a) $f' > 0$

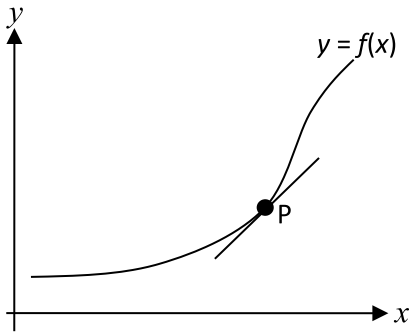


(b) $f' = 0$

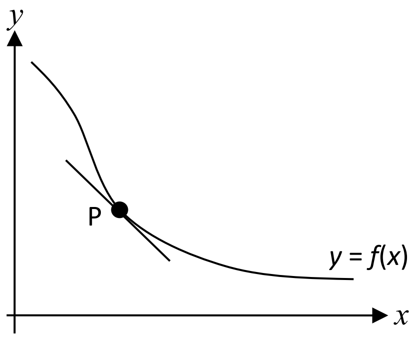


(c) $f' < 0$

If $f'(x)$ has a **positive** value, then we will have a **rate of increase** of y w.r.t x.



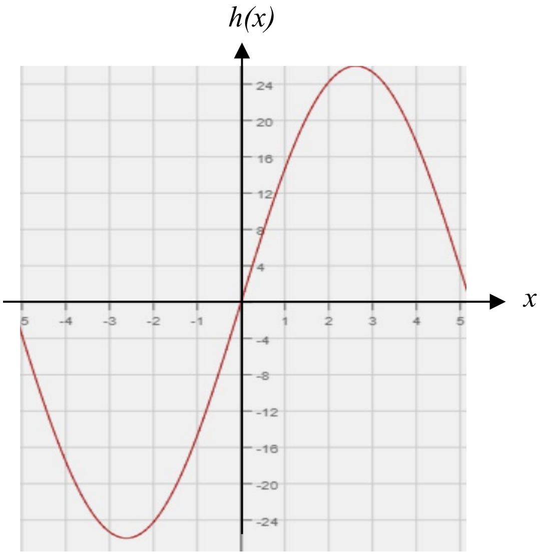
If $f'(x)$ has a **negative** value, then we will have a **rate of decrease** of y w.r.t. x.



EXERCISE 4.19

In which of the following intervals is $h(x) > 0$ and $h'(x) > 0$?

(a) $-4 < x < -3$
(b) $-2 < x < -1$
(c) $1 < x < 2$
(d) $3 < x < 4$



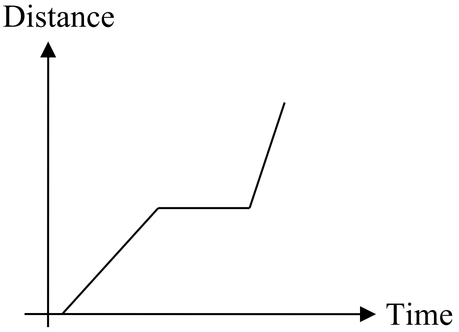
*** * * END OF CHAPTER 4 * * ***

TUTORIAL TOPIC 4

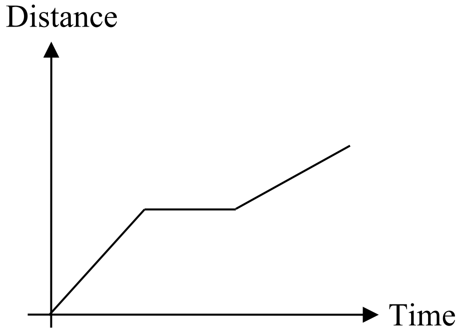
Multiple Choice Questions: Choose the best option.

1. After hiking all morning, a hiker stops to rest and take a snack. He then resumes his hike at a slower pace. Which of the following graphs best describes his hike?

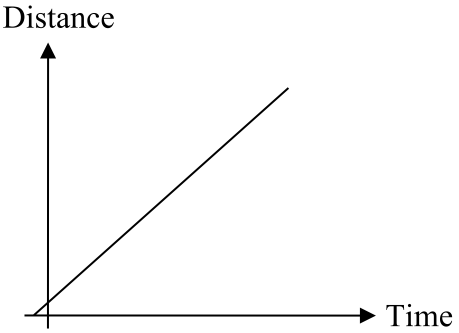
(a)



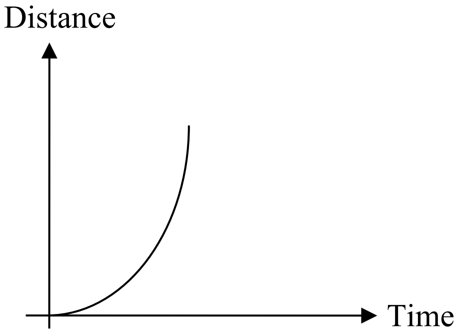
(b)



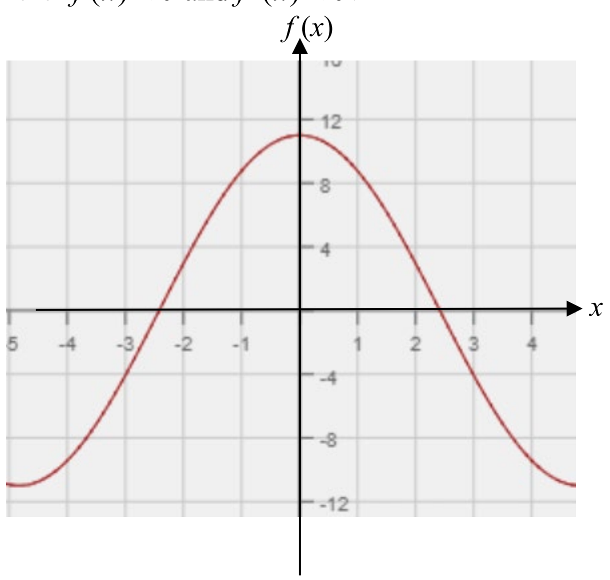
(c)



(d)



2. Select the interval where $f(x) < 0$ and $f'(x) < 0$.



(a) $3 < x < 4$

(b) $1 < x < 2$

(c) $-2 < x < -1$

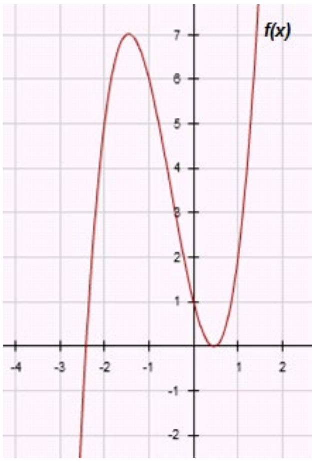
(d) $-4 < x < -3$

Written solutions

1. If $g(x)=\dfrac{1}{x-4}$, evaluate the following.
   (a) $g(4.01)$
   (b) $g\left(\dfrac{1}{x}\right)$
   (c) $g\left(x^2\right)$

2. If $f(x)=2x^2-2$, find the following:
   (a) $f\left(3x\right)$
   (b) $f\left(-x\right)$
   (c) $\dfrac{f(x+h)-f(x)}{h}$

3. Answer the following questions based on the given graph.



   (a) Given that $x=-1$, what is the corresponding value of $f(x)$?
   (b) Determine the value(s) of x when $f(x)=6$.

4. (a) Describe how the graph of $g(x)=\left(x-3\right)^2$ can be obtained from the graph of $f(x)=x^2$. Hence, sketch the graph of $g(x)$.

   (b) Describe how the graph of $h(x)=x^2+\dfrac{1}{2}$ can be obtained from the graph of $f(x)=x^2$. Hence, sketch the graph of $h(x)$.

5. (a) Describe how the graph of $g(x)=\dfrac{1}{x}-4$ can be obtained from the graph of $f(x)=\dfrac{1}{x}$. Hence, sketch the graph of $g(x)$.
   (b) Describe how the graph of $h(x)=\dfrac{1}{x+2}$ can be obtained from the graph of $f(x)=\dfrac{1}{x}$ Hence, sketch the graph of $h(x)$.

6. Given that $f(x)=x^2$, find the following functions, sketch the graphs and state its domain and range:
   (a) $f(x)+5$ (b) $f(x)-2$ (c) $f(x+1)$

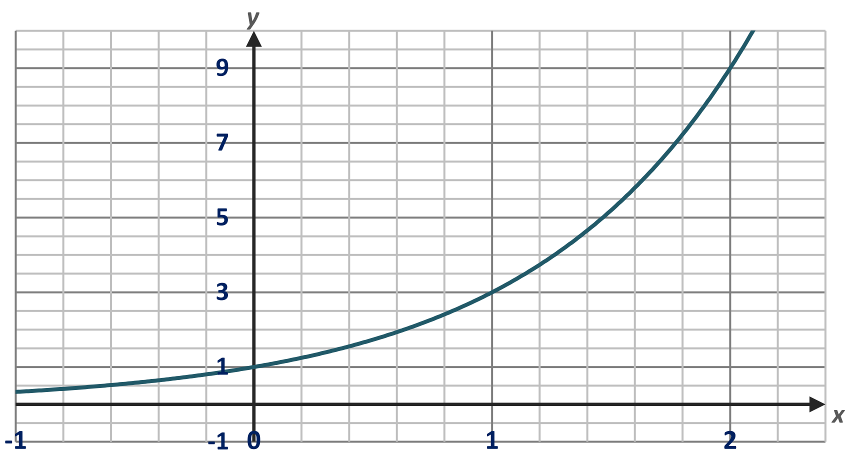
7. Sketch the graphs of the following functions, then visually estimate the domain and range.
   (a) $f(x)=5-3x$
   (b) $g(x)=(x+1)^2$
   (c) $h(x)=\dfrac{1}{x-1}$

8. Given $g(x)=\dfrac{1}{x}$, find $g(x-3)$ and $g(x+3)$, hence state its domain and range.

9. Find the domain and range of the following functions:
   (a) $f(x)=x^2-1$
   (b) $g(x)=\dfrac{1}{x+6}$
   (c) $h(x)=\sqrt{x-1}$

10. Write the equation of each graph after going through the indicated transformation(s).
    (a) The graph of $f(x)=x^2$ is shifted 3 units to the left and 5 units downward.
    (b) The graph of $f(x)=x^3$ is shifted 2 units to the right and 8 units upward.
    (c) The graph of $f(x)=x^2$ is shifted 1 unit to the right, stretched vertically by a factor of 5 and shifted 9 units upward.

11. The diagram shows the graph of $y=3^x$.
    (a) Find the gradient of the curve at $x=1$.
    (b) By adding a suitable line, find the value of x such that $x>0$, and $x+4=3^x$.



12. Jim earns \$1800 per month. If he works more than 44 hours in a week, he is paid 1.5 times his regular salary for the overtime hours he worked. If he earned \$2475 in March, how many overtime hours did he work per week ?

13. A coach travels from Singapore to Kuala Lumpur. It sets off at 9.30am, and travels at a constant speed of 65 km/h. A car makes the same journey, travelling 10 km/h faster but leaving 10 minutes later. When does the car overtake the coach?

14. A car salesman earns commission for selling a car. The commission given on the first \$50,000 is 1% and on the remainder is 3%. If the commission earned is \$800, calculate the price of the car.

Problem-solving Assignment 1

The goal of this series of problem-solving assignments is to develop problem-solving skills, not just to test your ability to get the answer. It's ok to try hard and not succeed at first (only your effort is evaluated), but you must try.

Question 1

If Alan invests \$5000 at 3% interest per year, how much additional money must he invest at 5.5% annual interest to ensure that the interest he receives each year is 4.5% of the total amount invested?

1. Understand the problem • State the given conditions and quantities. • Identify the unknown that you are asked to find. • If applicable, draw a diagram to describe the scenario.	
2. Devise a plan • Break down the problem into smaller parts. • Identify which are the relevant concepts that can be applied.	
3. Implement the plan • Carry out the plan, showing each step clearly. • Any graph or diagram should be clearly labelled.	
4. Look back • Substitute your answer back into the problem and check if it satisfies the given conditions.	

Question 2

A MRT train starts from Toa Payoh and increases speed at a steady rate until it reaches 25 m/s. It keeps at this speed for 30 seconds, and slowly decrease its speed at a steady state until it stops at Novena. The whole journey takes 2 mins. Calculate the distance between Toa Payoh and Novena.

1. Understand the problem • State the given conditions and quantities. • Identify the unknown that you are asked to find. • If applicable, draw a diagram to describe the scenario.	
2. Devise a plan • Break down the problem into smaller parts. • Identify which are the relevant concepts that can be applied.	
3. Implement the plan • Carry out the plan, showing each step clearly. • Any graph or diagram should be clearly labelled.	
4. Look back • Ask yourself - "Does it answer the question that was asked?" - "Does the answer make sense?" • Determine if there is any other easier way of finding the solution.	

ANSWERS:

Multiple Choice

1. B
2. C

Written solutions:

1. (a) 100 (b) $\frac{x}{1-4x}$ (c) $\frac{1}{(x-2)(x+2)}$

2. (a) $18x^2-2$ (b) $2x^2-2$ (c) $4x+2h$

3. (a) $f(x)=6$ (b) $x=-1$ or approximately -1.8 or 1.4

4. (a) Shift the graph of $f(x)$ to the right by 3 units.
   (b) Shift the graph of $f(x)$ up by $\frac{1}{2}$ units.

5. (a) Shift the graph of $f(x)$ down by 4 units.
   (b) Shift the graph of $f(x)$ to the left by 2 units.

6. (a) x^2+5; $\{x|-\infty<x<\infty\}$, $\{y:y\geq 5\}$ (b) x^2-2; $\{x|-\infty<x<\infty\}$, $\{y:y\geq -2\}$
   (c) $(x+1)^2$; $\{x|-\infty<x<\infty\}$, $\{y:y\geq 0\}$

7. (a) $\{x|-\infty<x<\infty\}$, $\{f(x)|-\infty<f(x)<\infty\}$
   (b) $\{x|-\infty<x<\infty\}$, $\{g(x)\,|\,g(x)\geq 0\}$ (c) $\{x|x\neq 1\}$, $\{h(x)\,|\,h(x)\neq 0\}$.

8. (a) $g(x-3)=\dfrac{1}{x-3}$; $\{x:x\neq 3\}$, $\{y:y\neq 0\}$ (b) $g(x+3)=\dfrac{1}{x+3}$; $\{x:x\neq -3\}$, $\{y:y\neq 0\}$

9. (a) $\{x|-\infty<x<\infty\}$, $\{f(x):f(x)\geq -1\}$ (b) $\{x:x\neq -6\}$, $\{g(x):g(x)\neq 0\}$
   (c) $\{x:x\geq 1\}$, $\{h(x):h(x)\geq 0\}$

10. (a) $f(x)=(x+3)^2-5$, (b) $f(x)=(x-2)^3+8$ (c) $f(x)=5(x-1)^2+9$

11. 3.4, x~1.55

12. 11 hours of overtime per week

13. 10.45am

14. \$60,000

15. $y=f(x_0)+f'(x_0)(x-x_0)$; $x_1=x_0-\dfrac{f(x_0)}{f'(x_0)}$

Problem-solving Assignment 1:

1. \$7500

2. 1875m.

Use appropriate learning strategies

Study tip #4: Math study skills
Studying for MST? Check out some
Math study skills here!

https://tinyurl.com/ms960-study-tip-4