5 Quadratic Equations, Functions & Inequalities

LEARNING OBJECTIVES

Upon completion of Chapter 5, you should be able to:

1) Define a quadratic equation.
2) Solve quadratic equations by factorisation, completing the square and the quadratic formula.
3) Sketch graphs of quadratic functions.
4) Identify key characteristics of graphs of quadratic functions.
5) Solve application problems involving quadratic functions
6) Solve linear inequalities in one unknown, and represent the solution set on the number line.
7) Solve quadratic inequalities
8) State conditions for a quadratic equation to have:
    (i) two real roots
    (ii) two equal roots
    (iii) no real roots
9) State conditions for a given line to:
    (i) intersect a given curve
    (ii) be a tangent to a given curve
    (iii) not intersect a given curve
10) State the conditions for $y = ax^2 + bx + c$ to be always positive (or always negative).
11) Solve a nonlinear system of equations by the substitution method.

Relevant sections in e-book:

- 1.4 Quadratic equations
- 3.1 Quadratic functions and applications
- 9.4 System of nonlinear equations in two variables



Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspect of the world.
- Alfred North Whitehead-

5.1 Definition of a Quadratic Equation

A linear equation in one variable is an equation of the form $ax + b = 0$, where $a \neq 0$. A linear equation is also called a first degree equation. A quadratic equation is identified as a second-degree equation.

Definition of a Quadratic Equation

Let a, b and c represent real numbers where $a \neq 0$. A **quadratic equation** in the variable x is an equation of the form

$$ax^2 + bx + c = 0$$

5.2 Solve quadratic equations

5.2.1 Solve Quadratic Equations by using the Zero Product Property

To solve a quadratic equation, we use the zero product property

Zero Product Property

If $mn = 0$, then $m = 0$ or $n = 0$.

To solve a quadratic equation using the zero product property, set one side of the equation to zero and factor the other side.

VIDEO EXAMPLE 5.1

Use the zero product property to solve the equation $x^2 - x - 12 = 0$.

Solutions:

Factorize $(\quad)(\quad) = 0$

Set each factor to zero: $(\quad) = 0$ or $(\quad) = 0$

This is called the **zero product property:**

Solve for x: $x =$ or $x =$ => these are the solutions of the equation

Check $x =$, LHS = $(\quad)^2 - (\quad) - 12 = 0$ = RHS

Check $x =$, LHS = $(\quad)^2 - (\quad) - 12 = 0$ = RHS

Hence, $x =$ and $x =$ are the solutions of the quadratic equation.

5.2.2 Solve Quadratic Equations by using the Square Root Property

Square Root Property

If $x^2 = k$, then $x = \pm\sqrt{k}$.

The solution set is $\{-\sqrt{k}, \sqrt{k}\}$ or more concisely $\{\pm\sqrt{k}\}$.

To apply the square root property to solve a quadratic equation, first isolate the square term on one side and the constant term on the other side.

VIDEO EXAMPLE 5.2

Solve the following by applying the square root property.

(a) $x^2 = 81$

Solutions: $x = \pm\sqrt{\ }$

$x = \pm$, The solution set is { , }

(b) $5y^2 - 35 = 0$

Solutions: $5y^2 = 35$

$y^2 =$

$y = \pm\sqrt{\ }$ The solution set is { , }

5.2.3 Solve a Quadratic Equation $ax^2 + bx + c = 0$ by Completing the Square and Applying the Square Root Property.

We can solve a quadratic equation as $ax^2 + bx + c = 0$ by completing the square and applying the square root property.

Solve a Quadratic Equation $ax^2 + bx + c = 0$ by Completing the Square and Applying the Square Root Property. Follow the steps below.

Step 1 Divide both sides by a to make the leading coefficient of x^2 as one.
Step 2 Isolate the variable terms on one side of the equation.
Step 3 Complete the square.
- Add the square of one-half the linear term coefficient to both sides.
- Factor the resulting perfect square trinomial.

Step 4 Apply the square root property and solve for x.

VIDEO EXAMPLE 5.3

Solve by completing the square and applying the square root property.

$y^2 + 22y - 4 = 0$ (Note: Leading coefficient of x^2 is one)

VIDEO EXAMPLE 5.4

Solve by completing the square and applying the square root property.

$2x(x-3) = 4 + x$ (Note: Leading coefficient of x^2 is NOT EQUAL to one)

5.2.4 Solving quadratic equations using the quadratic formula

It is possible to write a general formula for the quadratic equation $ax^2 + bx + c = 0$, by employing the "completing the square" method:

$ax^2 + bx + c = 0$	Where a, b and c are real numbers, and $a \neq 0$
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	*Divide throughout by a.*
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	*Adding the term* $\left(\frac{b}{2a}\right)^2$ *to both sides.*
$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$	*Completing the square on the LHS.*
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$	*Expanding the square on the RHS.*
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	*Making a common denominator on the RHS.*
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	*Apply the Square root property* *Note the ± sign!*
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	*The denominator is a perfect square!*
$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$	*Bring the constant over to solve for x.* *The quadratic formula.*

The Quadratic Formula

For a quadratic equation of the form $ax^2 + bx + c = 0 \quad (a \neq 0)$, the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

VIDEO EXAMPLE 5.5

We have shown up till now the following methods to solve a quadratic equation:

1. Factorisation and apply the Zero product rule.
2. Complete the square and apply the square root property.
3. Apply the quadratic formula.

Find the **exact** values of y that satisfy the equation $y^2 + 22y - 4 = 0$ using quadratic formula.

5.3 Graphs of quadratic functions

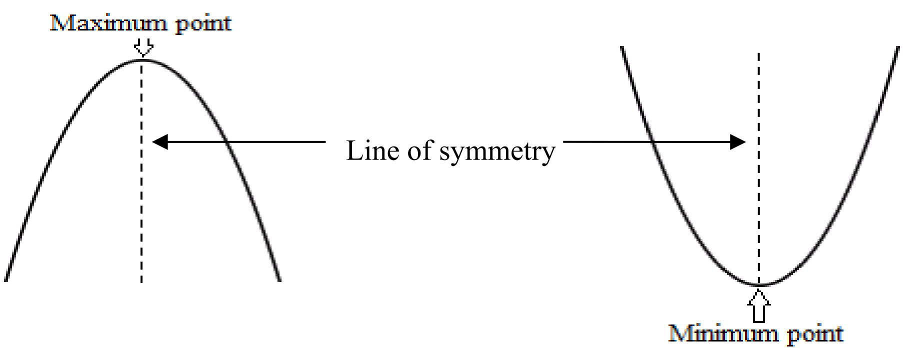
A **quadratic function** is a polynomial function of degree 2,

$$f(x) = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$.

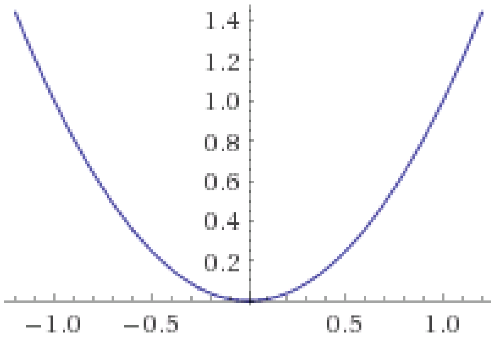
The graph of a quadratic function, $f(x) = ax^2 + bx + c$ is always a smooth, cup-shaped curve. We call it a **parabola**.

The **shape of the graph for quadratic functions** is as follows:



Coefficient of x^2, $a < 0$ Coefficient of x^2, $a > 0$

The simplest case is $y = x^2$. The graph is shown as follows, with the following features.



1. The curve is symmetric about the y-axis. This follows from the fact that the equation $y = x^2$ is not changed if we replace (x,y) by $(-x,y)$.

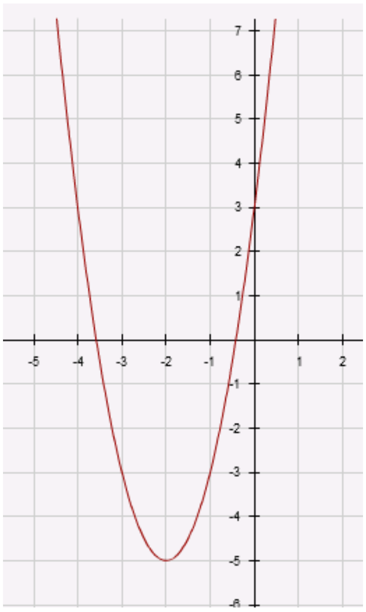
2. The curve reaches its lowest point at (0,0), the point where the curve intersects the line of symmetry. We call this point the ***vertex*** of the parabola. The vertex of a parabola is the point where the parabola crosses its axis of symmetry.

3. The domain is $\{x \mid -\infty < x < \infty\}$. The range is $\{y \mid y \geq 0\}$

For a quadratic function in the standard form $f(x) = ax^2 + bx + c$ where *a, b* and *c* are constants and $a \neq 0$, the x-coordinate of the vertex is $-\dfrac{b}{2a}$ and the corresponding y-coordinate is $f(-\dfrac{b}{2a})$

EXERCISE 5.1

Answer the following questions based on the quadratic graph on the right.



(a) What is the sign of the coefficient of x^2?

(b) Determine the value of y when x is -3.

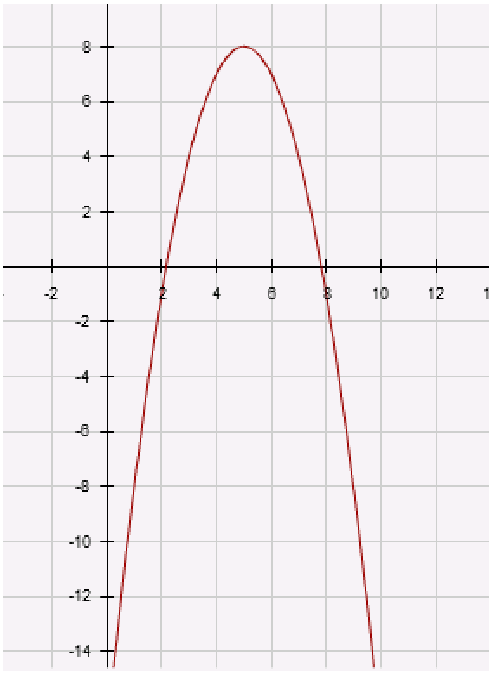
(c) What is the minimum point? State the vertex.

(d) State the symmetry of the graph.

(e) Determine the maximum or minimum y value?

(f) What is the domain and range of the graph?

EXERCISE 5.2



Use the quadratic graph on the left to answer the following questions:

(a) What is the sign of the coefficient of x^2?

(b) Does this graph have a maximum or minimum point? What are the coordinates of this point? State the vertex.

(c) Beth state that the symmetry of this graph is $y = 5$. Do you agree with her?

(d) Determine the maximum or minimum y value?

(e) State the domain and range for this graph.

5.3.1 Sketching the graph of a quadratic function in factorized form, $f(x)=\pm(x-a)(x-b)$

What does $f(x)=0$ mean?

Given that $f(x)=(x-2)(x-5)$.

When $f(x)=0$,

$$0=(x-2)(x-5)$$

$$\Rightarrow x=2 \text{ and } x=5$$

What do $x=2$ and $x=5$ represent?

The x co-ordinate of the vertex is $\dfrac{2+5}{2}=3.5$. The y co-ordinate of the vertex is $f(3.5)=(3.5-2)(3.5-5)=-2.25$. Hence the vertex is at $(3.5,-2.25)$

Hence, Vertex is at $\left(\dfrac{a+b}{2}, f(\dfrac{a+b}{2})\right)$

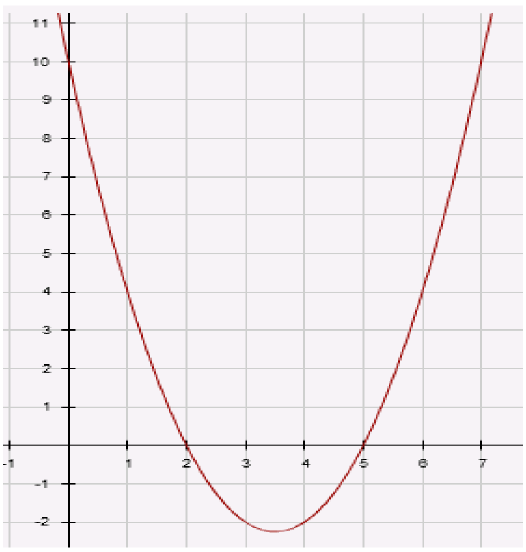
What does $x=0$ mean?

When $x=0$,

$$f(0)=(0-2)(0-5)=10$$

What does '10' represent?

Hence, the curve of $f(x)=(x-2)(x-5)$ is as follows:



EXERCISE 5.3

Sketch the graph of $f(x) = (x+4)(x+8)$.

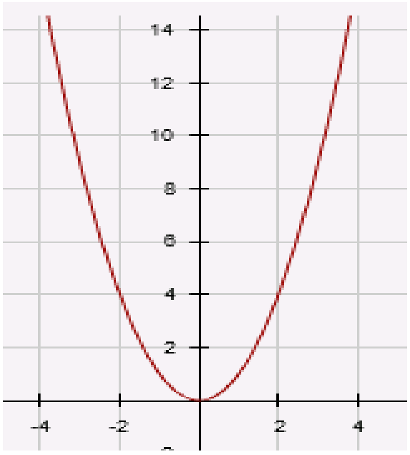
Label the maximum/minimum point, the line of symmetry, the x and y-intercepts (if any) on the graph. State the domain and range of the graph.

EXERCISE 5.4

Sketch the graph of $f(x) = -x^2 + 2x + 3$. Label the maximum/minimum point, the line of symmetry, the x and y-intercepts (if any) on the graph. State the domain and range of the graph.

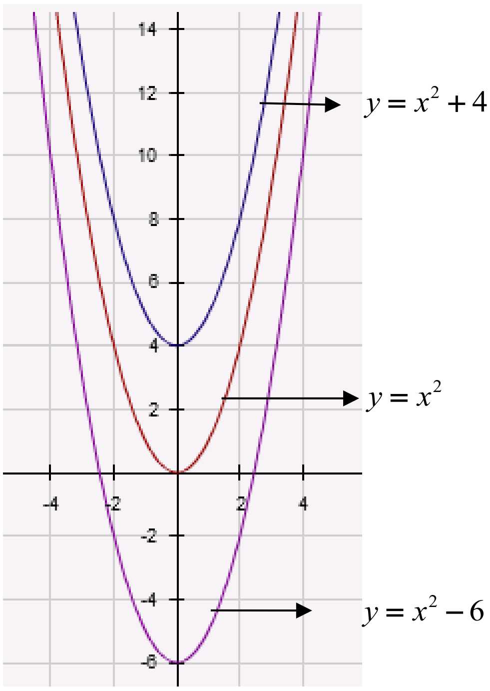
5.3.2 Vertical and horizontal translations of $f(x) = x^2$

The graph below shows the curve of $f(x) = x^2$.

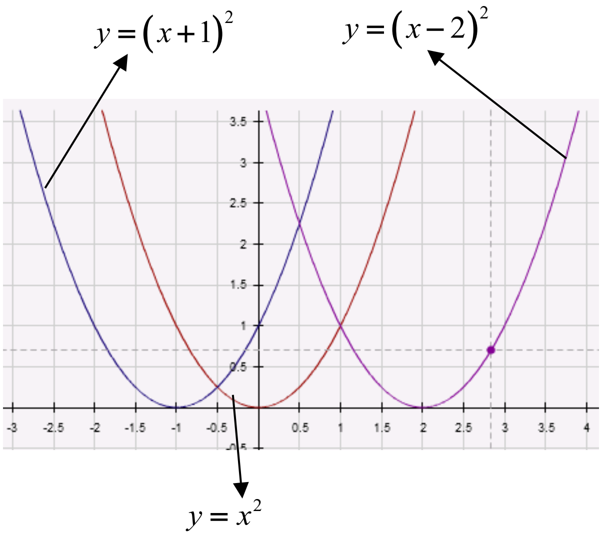


Recall from the previous chapter that we can obtain the graph of $f(x) = \pm(x-h)^2 + k$ by shifting (translating) the graph of $f(x) = x^2$. Use Desmos at https://www.desmos.com/calculator to visualize the shifting (translating) of the graphs below!

$\boxed{f(x) = x^2 + k}$



$\boxed{f(x) = (x-h)^2}$



For $k > 0$,
the graph of $y = x^2 + k$ is the graph of $y = x^2$ shifted **up** by k units.
the graph of $y = x^2 - k$ is the graph of $y = x^2$ shifted **down** by k units.

For $h > 0$,
the graph of $y = (x + h)^2$ is the graph of $y = x^2$ shifted **left** by h units.
the graph of $y = (x - h)^2$ is the graph of $y = x^2$ shifted **right** by h units.

EXERCISE 5.5

Sketch the graph of $f(x) = (x + 2)^2 - 4$.
Label the maximum/minimum point, the line of symmetry, the *x* and *y*-intercepts (if any) on the graph. State the domain and range of the graph.

EXERCISE 5.6

The graph of $f(x) = x^2$ is shifted ten units to the right and four units upward. Write the equation for the final transformed graph.

5.3.3 Sketching the graph of a quadratic function in vertex form, $f(x)=a(x-h)^2+k$

We can write $f(x)=ax^2+bx+c$ in the form $f(x)=a(x-h)^2+k$ by completing the square.

From the previous chapter, we know that the graph of $f(x)=a(x-h)^2+k$ is related to the graph of $y=x^2$ by a vertical shrink or stretch determined by *a*, a horizontal shift determined by *h*, and a vertical shift determined by *k*. Therefore, the graph will be a parabola with vertex (*h*, *k*).

A function defined by $f(x) = ax^2 + bx + c$ ($a \neq 0$) is called a quadratic function.

By completing the square, $f(x)$ can be expressed in vertex form as

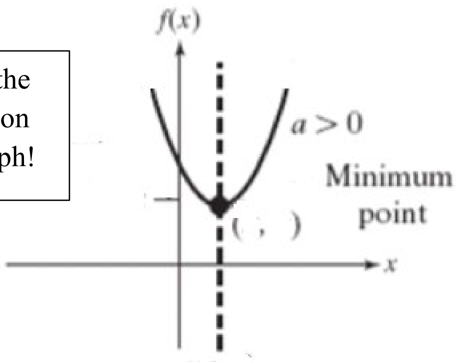
$$f(x) = a(x-h)^2 + k$$

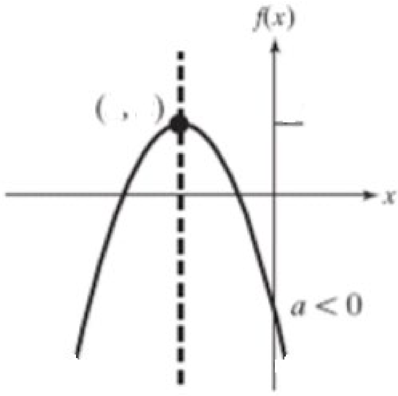
where (h, k) is the vertex of the parabola.

VIDEO EXAMPLE 5.6

If $a > 0$, the parabola opens upwards and the vertex is the _________ point.

Fill in the blanks on the graph!





If $a < 0$, the parabola opens downwards and the vertex is the _________ point.

The axis of symmetry is ___________. This is the vertical line that passes through the vertex.

VIDEO EXAMPLE 5.7

Given $h(x) = 2\left(x+1\right)^2 - 8$,

a. Determine whether the graph of the parabola opens upward or downward.

b. Identify the vertex.

c. Determine the x-intercept.

d. Determine the y-intercept.

e. Sketch the function.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. State the domain and range.

EXERCISE 5.7

Express $y = x^2 - 2x - 6$ in vertex form. Hence, deduce the coordinates of its minimum point.

$$y = x^2 - 2x - 6$$
$$y = x^2 - 2x + \left(\frac{2}{2}\right)^2 - 6 - \left(\frac{2}{2}\right)^2$$
$$y = \left(x - \frac{2}{2}\right)^2 - 6 - \left(\frac{2}{2}\right)^2$$
$$y = (x-1)^2 - 7$$

The coordinates of the minimum point are (1, -7).

Recall: Completing the square

In general, the expression $x^2 \pm bx$ can be written as a perfect square when $\left(\frac{b}{2}\right)^2$ is **<u>added</u>**:

$$x^2 \pm bx = x^2 \pm bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$
$$= \left(x \pm \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

EXERCISE 5.8

Express $f(x) = 2x - x^2 + 7$ in the form $f(x) = a(x-b)^2 + c$ where a, b and c are constants. Sketch the graph, indicating its vertex, line of symmetry as well as x and y-intercepts (if any). State the domain and range of the graph.

Self-Directed Learning @SP

Manage Learning

Use appropriate learning strategies

Study tip #5: Summarizing
By summarizing, you will have to review the information and identify the key ideas.

https://tinyurl.com/ms960-study-tip-5

In summary, there are a few possible ways you can sketch the graphs of quadratic functions:

1. Use the x-intercepts to sketch the graph of $f(x) = \pm(x-a)(x-b)$
2. Use the vertex form $f(x) = a(x-h)^2 + k$ to sketch the graph where (h, k) is the vertex of the parabola.

Summary of different forms of quadratic functions

Standard form	Vertex form	Factorized form
$f(x) =$	$f(x) =$	$f(x) =$
Axis of symmetry $x =$	Axis of symmetry $x =$	Axis of symmetry $x =$
Vertex (,)	Vertex (,)	Vertex (,)

5.4 Applications of quadratic functions

EXERCISE 5.9

The parabolic cable for a suspension bridge is attached to the two towers at points 400 meters apart and 90 meters above the horizontal bridge deck. The cables drop to a point 10 meters above the deck. Find the *xy*-equation of the cable, assuming it is symmetric about the *y*-axis with vertex at (0,10).

EXERCISE 5.10

In a game of tennis, the ball is initially struck 3.6 metres above the ground and hits the ground 6 seconds later. It reaches its greatest height 2 seconds after being hit. The motion of the ball can be modelled by the parabola $y = a(t-h)^2 + k$, where t is the number of seconds after the ball has been hit, and y is the height of the ball above the ground.

Find the greatest height reached by the ball.

5.5 Solving Linear and Quadratic Inequalities

5.5.1 Solving Linear Inequalities in One unknown

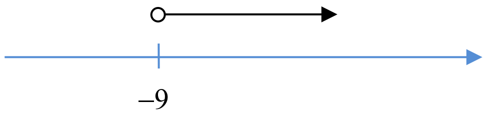
VIDEO EXAMPLE 5.8

Solve $-2x-5\le 3$.

EXERCISE 5.11

Find $4x-3<5x+6$.

$$\begin{aligned} 4x-3&<5x+6\\ 4x-5x&<6+3\\ -x&<9\\ x&>-9 \end{aligned}$$



5.5.2 Simultaneous Linear Inequalities in One variable

VIDEO EXAMPLE 5.9

Solve the compound inequality $-\frac{2}{3}y>-12$ and $2.08\ge 0.65y$.

EXERCISE 5.12

Solve the following pairs of simultaneous inequalities

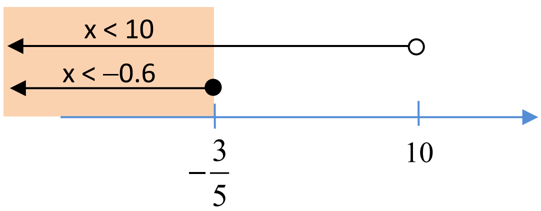
i. $2x-7<x+3$ and $9x-5\le 4x-8$

ii. $6x-15<5x-5$ and $7(x+1)-2(3x+2)>0$

i.

$$
\begin{array}{ccc}
2x-7<x+3 & \text{and} & 9x-5\le 4x-8 \\
2x-7<x+3 & \text{and} & 9x-4x\le 5-8 \\
x<10 & \text{and} & 5x\le -3 \\
x<10 & \text{and} & x\le -\dfrac{3}{5}
\end{array}
$$

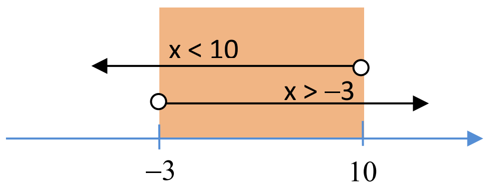
$\therefore$ Solution is $x\le -\dfrac{3}{5}$.



ii.

$$
\begin{array}{ll}
6x-15<5x-5 \text{ and} & 7(x+1)-2(3x+2)>0 \\
6x-5x<15-5 & 7x+7-6x-4>0 \\
x<10 & x>-3
\end{array}
$$

$\therefore$ Solution is $-3<x<10$.



EXERCISE 5.13

Solve $9x+2\le 6x-3$ and $2+\dfrac{x}{3}>4$

5.5.3 Modeling with linear inequalities

Constraints in real-world scenarios can be modelled using linear inequalities.

EXERCISE 5.14

Timothy's scores on three Physics tests were 74, 83, and 70. An *A* grade requires an average score of 80 or better. If the final exam will count twice as much as each test, what does Timothy need to score on the final exam in order to score an *A*?

5.5.4 Solving Quadratic Inequalities

We're going to add one more tool to our arsenal to help us to deal with more problems involving quadratic equations and their roots i.e. solving quadratic inequalities.

An inequality with a quadratic expression in one variable on one side, and a zero on the other side of the inequality sign is called a **quadratic inequality in one variable**.

Let's look at the following example as a demonstration of how to tackle such problems:

EXERCISE 5.15

Find the range of values of x that satisfy the inequality $x^2 - 5x < -4$.

Step 1: Rearrange the terms so that there is a quadratic expression $ax^2 + bx + c$ on one side, and a zero on the other side.

$$x^2 - 5x < -4$$

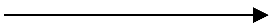
$$x^2 - 5x + 4 < 0$$

Step 2: Factorise the quadratic expression $ax^2 + bx + c$ if possible.

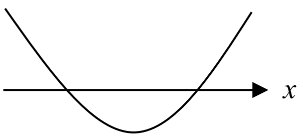
$$(x - 4)(x - 1) < 0$$

Step 3: Sketch the graph of $y = ax^2 + bx + c$.

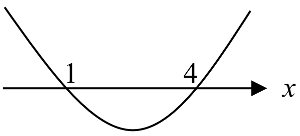
3.1) Draw the x-axis.



3.2) Depending on the sign of a, draw the graph. (i.e. For $a > 0$, draw a "smiley face"  . For $a < 0$, draw a sad face  .)



3.3) Label the roots of the equation on the graph.



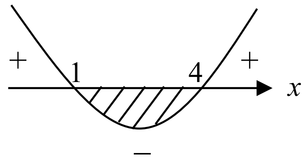
Step 4: Notice that, most of the time, part of the graph lies **above** the x-axis, and part of the graph lies **below** the x-axis.

The part of the graph that is **above** the x-axis is when $y = ax^2 + bx + c > 0$.

The part of the graph **below** the *x*-axis means that $y = ax^2 + bx + c < 0$.

Shade the part of the graph that you are interested in.

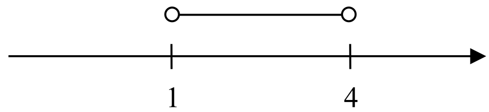
We want $x^2 - 5x + 4 < 0$, so we shade the part of the graph that is **below** the *x*-axis.



Step 5: From the graph, write down the range of values of *x* that describes the shaded region.

The shaded region is between 1 and 4, so the range we're looking for is $1 < x < 4$.

Step 6 (Optional): We can represent the range of values found on the **number line** as such:



The symbol "○" means that the numbers 1 and 4 are **not included** in the solution set.

EXERCISE 5.16

Find the range of values of *x* for which $x^2 \geq 6 - x$.

5.6 Nature of roots and the discriminant of a quadratic equation

VIDEO EXAMPLE 5.10

Recall that for a quadratic equation of the form $ax^2+bx+c=0$ $(a \neq 0)$, the solutions are

$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Case 1: b^2-4ac _____ 0 (2 non-real solutions)

Case 2: b^2-4ac _____ 0 (2 real solutions)

Case 3: b^2-4ac _____ 0 (1 real solution)

Fill in the blanks with the correct mathematical symbols!

Example:

a) $3x^2-4x+6=0$

b) $x^2-5x-9=0$

c) $-1.4m+0.1=-4.9m^2$

Since the value of b^2-4ac can tell us so much about the kind of roots that the quadratic equation has, it is given the name <u>**discriminant**</u>, as it allows us to discriminate (i.e. tell the difference) among the possible types of roots.

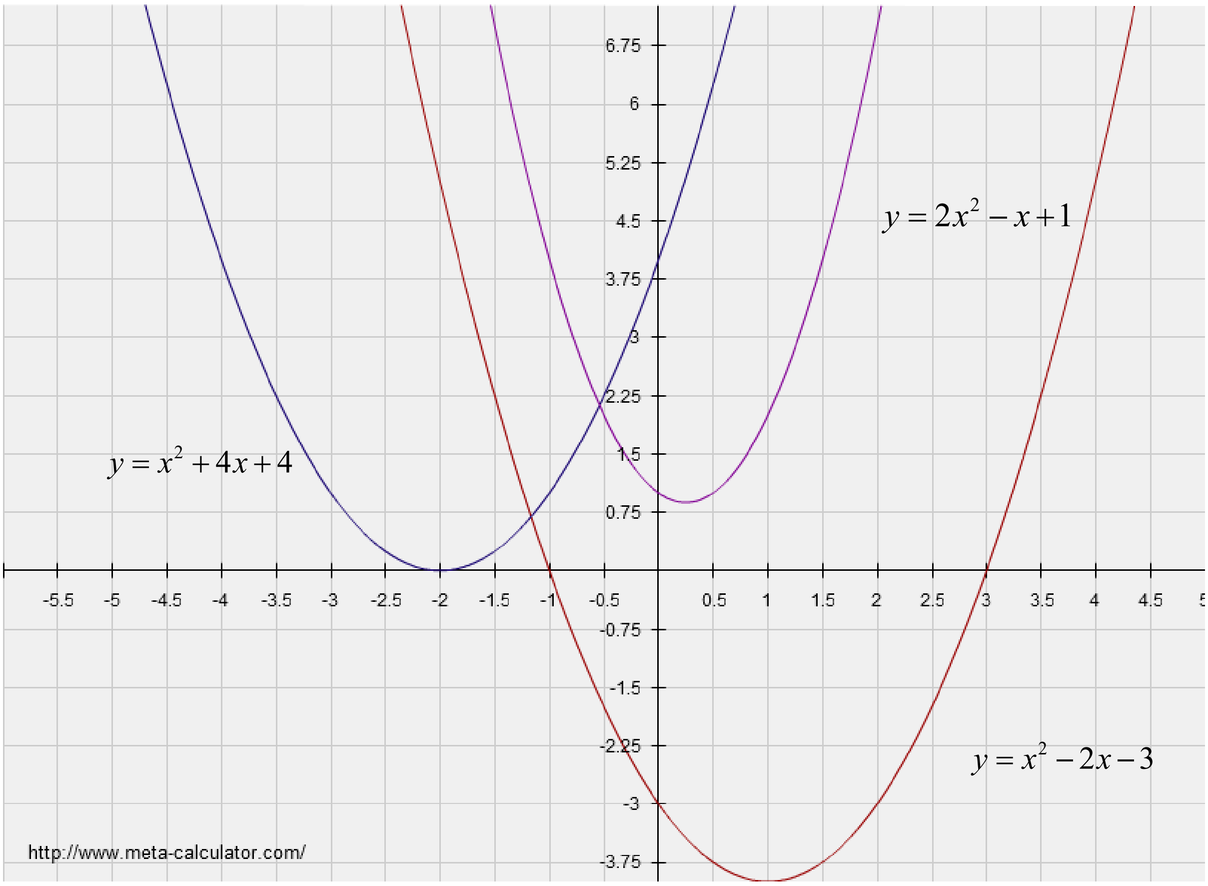
In summary,

Sign of b^2-4ac	Nature of roots	Summary
$b^2-4ac=0$	Real and equal	$b^2-4ac \geq 0$ => Real roots
$b^2-4ac>0$	Real and distinct	
$b^2-4ac<0$	Not real/Imaginary	No real roots

Graphical interpretation of nature of roots

		$b^2 - 4ac$	$\sqrt{b^2 - 4ac}$	$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Number of real roots
A)	$x^2 - 2x - 3 = 0$				
B)	$x^2 + 4x + 4 = 0$				
C)	$2x^2 - x + 1 = 0$				

Let's look at what we obtain if we plot the graphs of the above quadratic functions.



When a quadratic graph like $x^2 - 2x - 3 = 0$ has 2 solutions, it **intercepts** the x-axis twice, and we call the roots **real and distinct** (i.e. not equal) i.e. $x = 3$ or $x = -1$.

When a quadratic graph like $x^2 + 4x + 4 = 0$ has only 1 solution, it **touches** the x-axis only once, and we call the roots **real and equal** i.e. $x = -2$.

When a quadratic graph like $2x^2 - x + 1 = 0$ has no solution, it does **not touch/not intercept** the x-axis, and we say that the roots are **not real**.

EXERCISE 5.17

Find the values of q for which $qx^2 = 2x - q$, where $q \neq 0$, has equal and real roots.

$$qx^2 - 2x + q = 0$$

For equal and real roots the discriminant, $b^2 - 4ac$ is 0:

$$(-2)^2 - 4q^2 = 0$$

$$4q^2 = 4$$

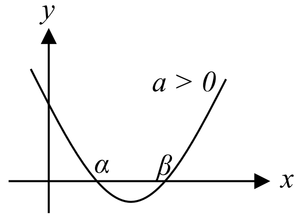
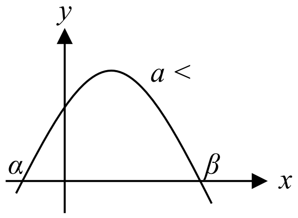
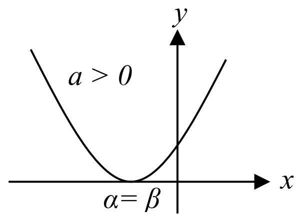
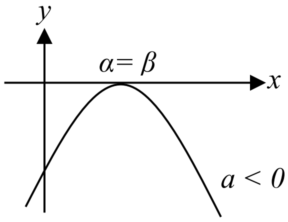
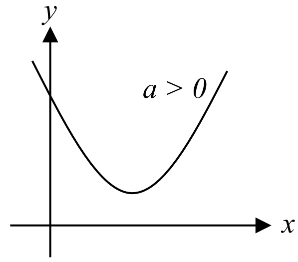
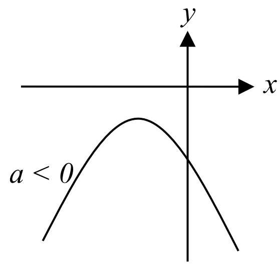
$$q = \pm 1$$

EXERCISE 5.18

Find the range of values of m for which the equation $x^2 - mx + 1 = 0$ has no real roots.

The Discriminant and the Position of the Quadratic Graph

Graphically, the discriminant also tells us the position of the curve $y = ax^2 + bx + c$ relative to the x-axis.

$b^2 - 4ac > 0$ Two distinct real roots => two x-intercepts		
$b^2 - 4ac = 0$ Equal real roots => only one x-intercept (the graph touches the x-axis) => The x-axis is tangent to the curve		
$b^2 - 4ac < 0$ No real roots => no x-intercept => the graph is either 1) entirely **<u>above</u>** the x-axis (i.e. y values are always positive) 2) entirely **<u>below</u>** the x-axis (i.e. y values are always negative)		

EXERCISE 5.19

The curve $y = x^2 + 2kx + (k-2)(k+3)$ lies entirely above the x-axis. Find the range of values of k.

To lie entirely above the x-axis => the curve does not cut the x-axis
=> no real roots
=> discriminant < 0

$$(2k)^2 - 4(k-2)(k+3) < 0$$
$$4k^2 - 4(k^2 + k - 6) < 0$$
$$-4k + 24 < 0$$
$$4k > 24$$
$$k > 6$$

EXERCISE 5.20

a) Find the smallest value of the integer a for which $y = ax^2 + 5x + 3$ is positive for all values of x.

b) Find the biggest value of the integer q for which $y = qx^2 - 5x - 4$ is negative for all values of x.

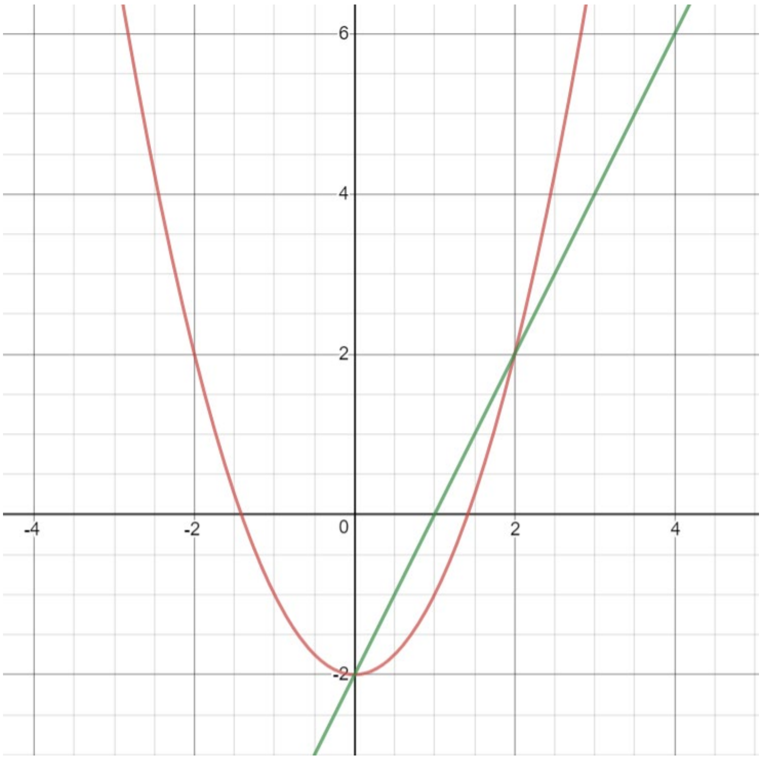
5.7 Solve a nonlinear system of equations by the substitution method

VIDEO EXAMPLE 5.11

Solve the following system of equations by using substitution:

$$y = x^2 - 2 \qquad \ldots (1)$$
$$2x - y = 2 \qquad \ldots (2)$$

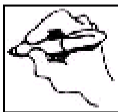
If we plot the graphs of these two equations, we will get something like this:



From the graph, how many solutions to this system of simultaneous equations can you see?

Write down the coordinates of the solutions here: __________, ______________ .

The solution should match what you had solved algebraically.



EXERCISE 5.21

Solve the following simultaneous equations.

$$y^2 + (2x+5)^2 = 26 \quad \ldots (1)$$
$$y + 2x = 1 \quad \ldots (2)$$

How would you plot the graphs on https://www.desmos.com/calculator?

Try it and see how the graphs look like and verify your answers!

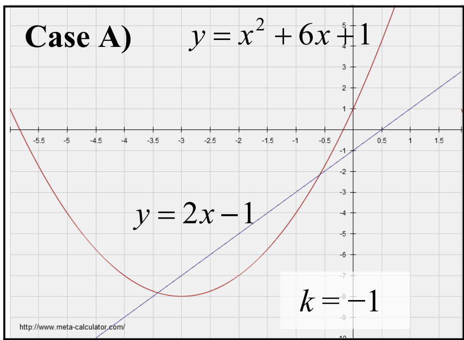
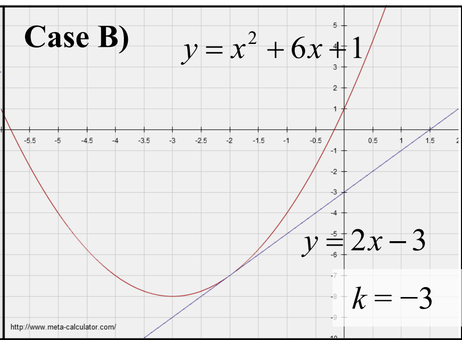
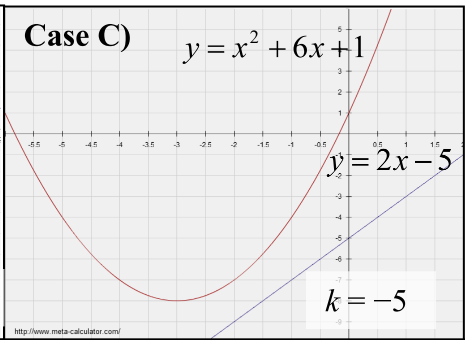
5.8 Intersection Problems leading to Quadratic Equations

Consider the following simultaneous equations:

$$y = x^2 + 6x + 1 \quad \dots (1)$$
$$y = 2x + k \quad \dots (2)$$

where k is a real number.

Three possible values of k, -1, -3 and -5 are chosen, and the graphs of the functions are plotted as shown:

		
Number of intersections = 2	Number of intersections = 1	Number of intersections = 0

Now, let's see what we get if we try to solve the simultaneous equations algebraically:

Case A) $y = x^2 + 6x + 1$ $y = 2x - 1$ => $2x - 1 = x^2 + 6x + 1$ => $x^2 + 4x + 2 = 0$	**Case B)** $y = x^2 + 6x + 1$ $y = 2x - 3$ => $2x - 3 = x^2 + 6x + 1$ => $x^2 + 4x + 4 = 0$	**Case C)** $y = x^2 + 6x + 1$ $y = 2x - 5$ => $2x - 5 = x^2 + 6x + 1$ => $x^2 + 4x + 6 = 0$

Amazingly, we find ourselves back to solving quadratic equations!

Let's consider the nature of the roots of the 3 quadratic equations that we now have:

	Case A) $k = -1$ $x^2 + 4x + 2 = 0$	**Case B)** $k = -3$ $x^2 + 4x + 4 = 0$	**Case C)** $k = -5$ $x^2 + 4x + 6 = 0$
$b^2 - 4ac < 0, = 0$ or > 0			
Number of roots			
Number of intersections between $y = x^2 + 6x + 1$ and $y = 2x + k$			

What relationship do you observe between the number of roots of the simplified quadratic equations and the number of intersections between the original two functions?

Let us now consider the general case:

$$y = x^2 + 6x + 1$$
$$y = 2x + k$$

Equating the two equations, we get:

$$x^2 + 6x + 1 = 2x + k$$
$$x^2 + 4x + (1 - k) = 0$$

Calculating the discriminant, we get:

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4(1)(1 - k) \\ &= 16 - 4 + 4k \\ &= 12 + 4k \end{aligned}$$

Let's consider the following cases:

For **<u>equal roots</u>**: $b^2 - 4ac = 0$ => $12 + 4k = 0$ => $k = -3$ We saw that when $k = -3$ i.e. $y = 2x - 3$, the line is **<u>tangent</u>** to the curve.
For **<u>two distinct roots</u>**: $b^2 - 4ac > 0$ => $12 + 4k > 0$ => $k > -3$ We saw that when $k = -1 > -3$ i.e. $y = 2x - 1$, the two graphs **<u>intersect each other at two points</u>**.
For **<u>zero real roots</u>**: $b^2 - 4ac < 0$ => $12 + 4k < 0$ => $k < -3$ We saw that when $k = -5 < -3$ i.e. $y = 2x - 5$, the two graphs **<u>do not intersect</u>**.

In conclusion:

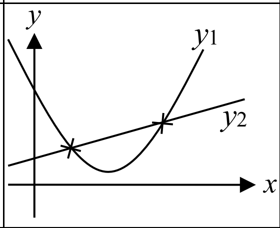
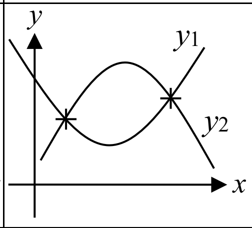
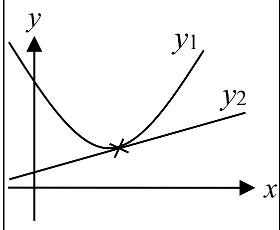
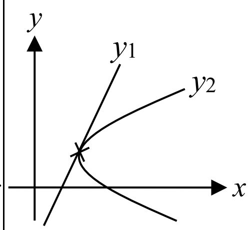
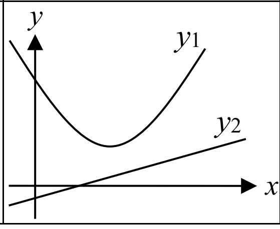
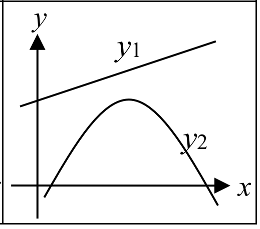
Step 1 When solving the **<u>simultaneous equations</u>** y_1 and y_2, and …

Step 2 we obtain a **<u>quadratic equation</u>** $ax^2 + bx + c = 0$ after rearranging the terms, …

Step 3 we can deduce the **<u>number of intersections</u>** between the graphs of y_1 and y_2 …

Step 4 by considering the **<u>nature of the roots</u>** of $ax^2 + bx + c = 0$

Summary Table

Nature of roots of $ax^2 + bx + c = 0$	Intersections between y_1 and y_2	Example 1	Example 2
$b^2 - 4ac > 0$	2 intersection points		
$b^2 - 4ac = 0$	1 intersection point / the graphs are **<u>tangent</u>** to each other (just touching)		
$b^2 - 4ac < 0$	No intersection		

EXERCISE 5.22

Find the values of m for which the line $y = mx - 9$ is a tangent to the curve $4y = x^2$.

EXERCISE 5.23

Find the range of values of q for which the line $y = x + 2$ intersects the curve $y^2 + (x + q)^2 = 2$ at two distinct points.

Apply learning across different contexts

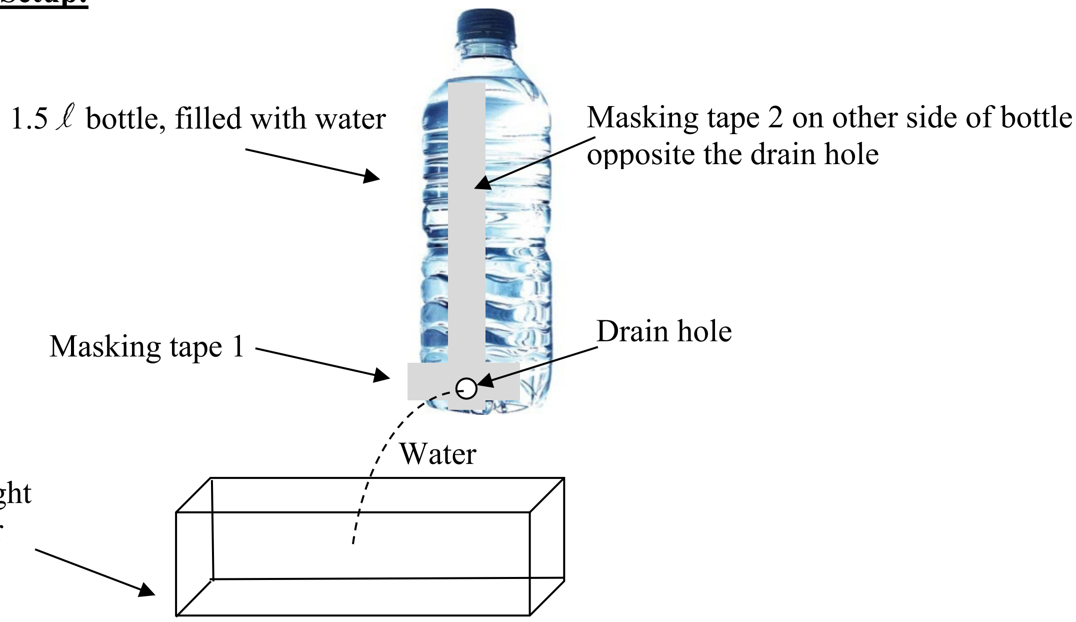
LEARNING ACTIVITY 1

Before conducting this activity, your group (of 4 or 5 students) should have the following materials at hand:

Apparatus list:

- Transparent 1.5 litres bottle
- Masking Tape
- Stopwatch (use hp)
- Calculator
- Container to collect water
- Marker
- Ruler
- Laptop (Microsoft Excel)

Proposed Setup:



Experiment Procedure:

a) Put a piece of masking tape horizontally over the hole.

b) Fill the bottle with water, while keeping a finger pressed on the hole to prevent leakage.

c) Place the bottle on the table, with the container some distance away to collect the water.

d) Paste another masking tape on the other side of the bottle opposite the drain hole. Mark the water height, h mm, on the masking tape with a marker.

e) Once ready, the team member holding the stopwatch will announce the start of the experiment, while simultaneously starting the stopwatch. The time from the start of the experiment is t seconds. At this moment, another team member is to remove the masking tape covering the drain hole.

f) Every 10 seconds (or any other suitable interval, depending on the size of the drain hole), the team member with the stopwatch will call for the other member(s) to mark the current water height on the masking tape.

g) Repeat step (f) until the water level reaches the level of the drain hole i.e. $h = 0$.

h) Remove the masking tape and paste it on a piece of paper. Label the value of h on the tape at each interval.

i) Repeat the experiment to collect another set of readings for h.

j) Tabulate your data with the following columns.

t/s	h_1/mm	h_2/mm	h_{avg}/mm

<u>Analysis of results:</u>

1. Use Microsoft Excel to plot the graph of h_{avg}/mm against t/s.

2. Plot an appropriate trendline that fits your data.

3. Display the equation of the trendline and the R^2 value.

4. Let's try to calculate a best-fit quadratic equation ourselves, and see if we can get an equation that is similar to that calculated by the software.

   4.1) Choose any three points from your table. Let x be the time taken in seconds, and y be the height of the water in mm. Try to choose "easy" points from your table!

   4.2) Assuming that the equation that can describe your curve is $y = ax^2 + bx + c$, substitute the three pairs of values you chose to obtain 3 equations with 3 unknowns, namely a, b and c. Write your equations down.

   4.3) Solve for a, b and c. You may solve the system using any method you wish.

   Equation of best-fit calculated:

5. Compare the equation you calculated with the equation given by the software. Could you think of any reason for the discrepancies between the equations?

6. Research and explain the significance of the R^2 value. Hence, explain whether the quadratic model is a good representation of the rate of water flow from the bottle.

7. Using the quadratic model that you have found, determine how long it will take to empty the water bottle.

8. What are some sources of error in the experiment?

9. How can you get a better approximation for the quadratic model?

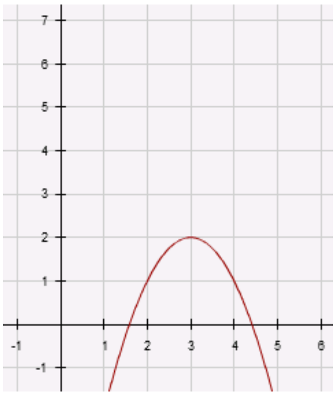
*** * * END OF CHAPTER 5 * * ***

TUTORIAL TOPIC 5

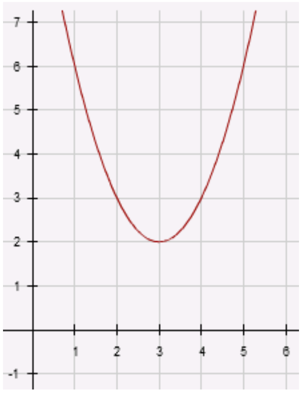
Multiple Choice Questions: Choose the best option.

1. Which of the following is a possible graph of $(x - 3)^2 + 2$?

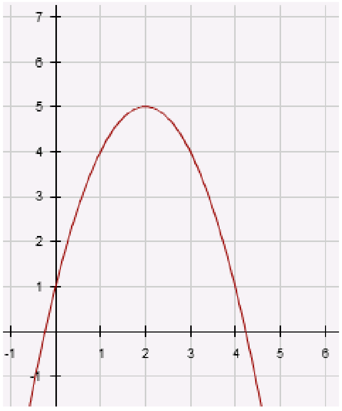
(a)



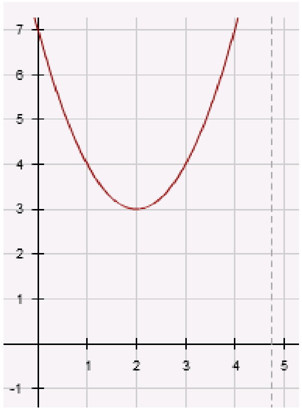
(b)



(c)



(d)



2. Given that $x = 3$ is a solution to the quadratic equation $2x^2 + x = p$, find the value of p.

(a) 3 (b) -3

(c) 21 (d) 7

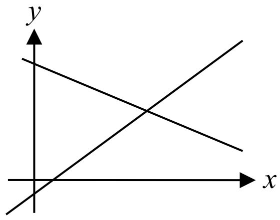
3. Which of the following equations has no real roots?

(a) $3x^2 - 4x + 7 = 0$ (b) $-2x^2 + 5x + 6 = 0$

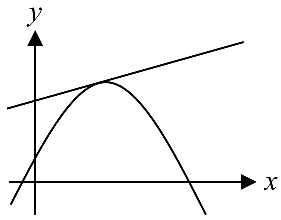
(c) $4x^2 + 4x + 1 = 0$ (d) $-x^2 + 5 = 0$

4. Which of the following pairs of graphs, on solving, will give a quadratic equation with a discriminant that is strictly positive?

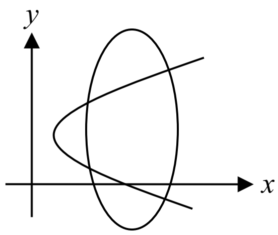
(a)



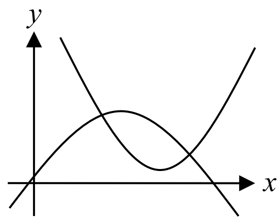
(b)



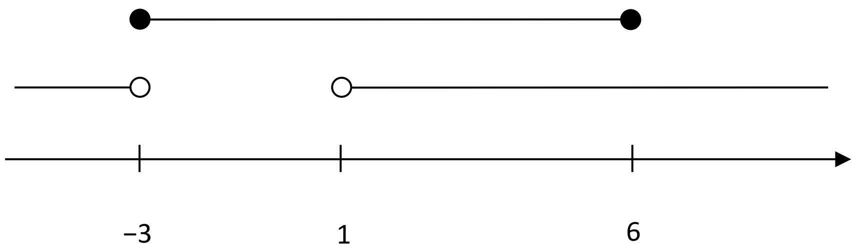
(c)



(d)



5. How can the following solution expressed on a number line be expressed in inequalities?



(a) $x < -3,\ -3 \le x \le 6,\ x > 1$

(b) $1 < x \le 6$

(c) $-3 < x \le 6$

(d) $1 \le x \le 6$

Written solutions

I) *Solving Quadratic Equations*

1. Solve for the possible value(s) of the unknowns:
   (a) $2x^2 + x - 6 = 0$ (b) $3y^2 - 4y - 7 = 0$
   (c) $3x^2 + 5x - 1 = 0$ (d) $4y^2 + 7y - 3 = 0$

2. **Complete the square** for the following expressions:
   (a) $x^2 - 4x + 5$ (b) $2x^2 - 4x + 13$
   (c) $2x^2 + 5x - 4$ (d) $-5x^2 + 4x + 12$

3. **Complete the square** for the following equations. Hence, solve for the exact value(s) of x by using **completing the square**:
   (a) $x^2 - 6x - 4 = 0$ (b) $2x^2 + 8x - 3 = 0$
   (c) $5 - 2x - x^2 = 0$ (d) $4x^2 - 2x - 5 = 0$

4. Solve for the possible value(s) of the unknowns:
   (a) $x + \frac{1}{x} = 2$ (b) $\frac{(x-2)}{6} = \frac{x}{(x+5)}$
   (c) $\frac{y-6}{y+2} = \frac{y-1}{2y-5}$ (d) $\frac{1}{y+4} + \frac{1}{y-4} = \frac{3}{5}$

II) *Sketching of Quadratic Functions/Applications of Quadratic Functions*

5. Sketch the graph of each parabola, indicating the coordinates of the vertex.
   (a) $y = 3x^2$ (b) $y = 3 + x^2$
   (c) $y = (3 + x)^2$ (d) $y = -2(x - 3)^2 + 4$

6. State the domain, range, line of symmetry and the coordinates of vertex for the following functions.
   (a) $y = 2x^2 - 8x + 4$
   (b) $y = (x + 2)(x + 4)$
   (c) $y = -(x + 9)^2 - 2$

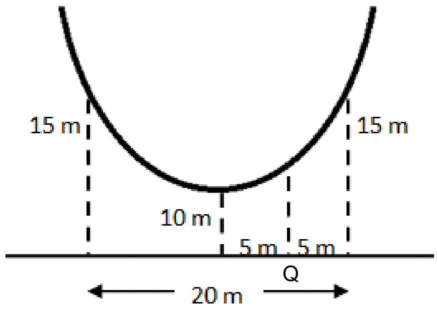
7. Express $f(x) = 2x^2 + 6x + 1$ in the form $f(x) = a(x - b)^2 + c$ where a, b and c are constants. Hence find the maximum/minimum point and the line of symmetry of $f(x)$.

8. Find the quadratic function of the parabola with
   (a) vertex (2,-3) and passing through the point (9, -10).
   (b) vertex (-2, 0) and passing through (6, -8).

9. Find the quadratic function of the parabola passing through (1,2) and (-2, -7) and having y-axis as the line of symmetry.

10. Find the quadratic function of the parabola that passes through (-1, -2), (0, 3) and (2,7).

11. If the curve shown below is part of a cable that follows the quadratic function, find the possible height of a tree that can be planted at point Q without the tree touching the cable.



[Hint: Begin by finding the equation of the curve, assuming its vertex is at (0,10).]

12. Starting at (0, 0), a ball travels along the path $y = ax^2 + bx$. Find a and b if the ball reaches a height of 75 at $x = 50$ and its maximum height is at $x = 100$.

13. A rectangle is 25 cm long and 16 cm wide. When its length decreases by x cm and its width increases by x cm, its area increased by 20 cm^2. Find a quadratic equation in x and solve it to find the possible values of x.

14. Dolphin-lovers would take the boat out to sea just to see these magnificent mammals leaping in and out of the sea. It was observed that the height, h metres, of the dolphin's leap above the sea level at time t seconds is given by the equation $h = 5t - t^2 - 1$. Find the times when the dolphin exits/enters the sea, and hence calculate the length of time when the dolphin is above sea level.

15. Gaius decided to cycle from Town A to Town C. He departed from Town A and travelled for 50 km at a constant speed of x km/h, and he reached Town B.
    (a) Write down the time, in terms of x, taken for his cycled journey from A to B.
    (b) At B, his bicycle unexpectedly broke down. He walked the remaining 6 km from B to C at a constant speed of $(x - 16)$ km/h. Write down the time, in terms of x, taken for him to walk from B to C.
    (c) Given that the total time for the whole journey from A to C is 4 hours, write down an equation in x and show that it reduces to $x^2 - 30x + 200 = 0$.
    (d) Solve the equation $x^2 - 30x + 200 = 0$.
    (e) Find the time, in hours and minutes, that Gaius would have taken if he had completed the whole journey by bicycle at the original constant speed.

16. A company makes fancy golf carts has fixed overhead costs of \$12,000 per year and direct cost (labour and materials) of \$90 per cart. It sells its carts to a certain retailer at a nominal price of \$120 each. However, the company offers a discount of 1% for 100 carts, 2% for 200 carts and in general, (x/100) % for x carts. Assume the retailer will buy as many carts as the company can produce but that its production facilities limit this to a maximum of 1800 units.
    (a) Write a formula for C, the cost of producing x carts.
    (b) Show that its total revenue R in dollars is $R = 120x - 0.012x^2$
    (c) What is the smallest number of carts it can produce and still break-even? (Break-even is a term when there is no profit).
    (d) What number of carts will produce the maximum profit and what is this profit?

III) *Inequalities/Discriminants/Nature of roots*

17. Sarah scored 70 in test 1, 65 in mid-year examination, 80 in test 3, 90 in assignments and 85 in projects. The weightage for test 1, test 3, assignments and projects was10% each. The weightage for the mid-year examination is 20%. An "***A***" grade requires an average score of 80 or better. How much does she have to score in the final examination in order to get an "***A***"?

18. Company A rents out photocopiers at \$800 a month, but a non-refundable lump sum f \$1000 is payable. Company B charges \$850 a month, and requires a non-refundable lump sum payment of \$850. After how many months will Company A's offer be more attractive?

19. After queuing for an hour, Jordan is finally seated in a restaurant. However, he realised that he only has \$20 cash with him. As he has to pay 10% service tax and 7% GST, what is the maximum price of the food item(s) that he can order ?

20. Find the range of values of x for which
    (a) $x(2x-3) < 20$ (b) $2 + 3x > 5x^2$
    (c) $x^2 + 3x - 4 > 4x + 2$ (d) $x(2x+5) > 18$

21. Find the set of values of x for which $(x-12)^2 > 2x$.

22. Find the value(s) or range of values of k for which the equation
    (a) $x^2 + 2kx + (k-2)(k+1) = 0$ has no real roots,
    (b) $3x^2 = 4x + k - 2$ has distinct real roots,
    (c) $(x+2)(2x-3) = k+1$ has real roots,
    (d) $x^2 + k^2 = 3kx - 4$ has equal roots,
    (e) $k(x+2)(x-3) = x - 4k + 1$ has two unequal real roots,
    (f) $2x^2 - kx = -2$ has no real roots.

23. Show that the solutions of the equation $x^2 + mx = 2 - m$ are real and distinct for all real values of m.

24. Find the range of values of k for which
    (a) $2x^2 + 3x + k$ is always positive,
    (b) $kx^2 + 3x + 4k$ is always negative.

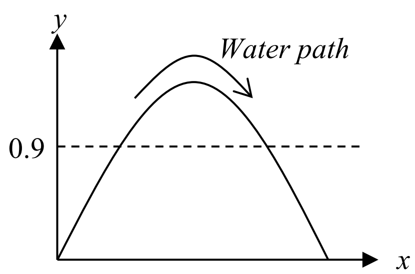
25. The curve $y = (k-6)x^2 - 8x + k$ cuts the x-axis at two points and has a maximum point. Find the range of values of k.

26. Solve for the possible value(s) of the unknowns:
    (a) $x + 2y = 10$
    $2y^2 - 7y + x = 1$
    (b) $2x + 6 - y = 0$
    $x^2 + x - y = 0$
    (c) $3y - 2x = 1$
    $(2x+3)^2 + (y-2)^2 = 26$
    (d) $\frac{2}{3x} = 2 + \frac{1}{y}$
    $3x - y - 3 = 0$

27. Find the values of k for which
    (a) the line $y = kx$ is a tangent to the curve $x^2 = 2y - 9$,
    (b) the line $y = k(x-1)$ intersects the curve $y = x^2 + 6x + k$ at two distinct points,
    (c) the line $y - x = k - 1$ does not meet the curve $(y-1)^2 = 4x$,
    (d) the curve $y^2 = 2x - 1$ meets the line $x + 2y = k - 3$.

28. The curve $y = 3kx^2 + (k-5)x - 2$ lies entirely below the curve $y = 5x^2$. Find the range of values that k can take.

29. An engineer was tasked to design a water fountain. Knowing that the trajectory of the water follows a parabola, the engineer worked out his equations and modelled the water path with the equation $y = x - \frac{5x^2}{24}$, taking the outlet of the water as the origin, and measuring x and y in metres from the origin. The x-axis is hence the ground level.



His superior posed him the following questions:

(a) When will the water path attain a height of above 0.9 m above the ground?

(b) If the height of the water trajectory is always below h, what will be the smallest integer value of h?

Problem-solving Assignment 2

The goal of this series of problem-solving assignments is to develop problem-solving skills, not just to test your ability to get the answer. It's ok to try hard and not succeed at first (only your effort is evaluated), but <u>you must try</u>.

Question 1

A retailer has learnt from experience that if she charges x dollars for a toy truck, she can sell $300 - 100x$ of them. The trucks cost $2 each.

Determine what she should charge for each truck to maximize her profit.

1. Understand the problem • State the given conditions and quantities. • Identify the unknown that you are asked to find. • If applicable, draw a diagram to describe the scenario.	
2. Devise a plan • Break down the problem into smaller parts. • Identify which are the relevant concepts that can be applied.	
3. Implement the plan • Carry out the plan, showing each step clearly. • Any graph or diagram should be clearly labelled.	
4. Look back • Ask yourself - "Does it answer the question that was asked?" - "Does the answer make sense?" • Determine if there is any other easier way of finding the solution.	

ANSWERS:

Multiple Choice Questions

1.B 2.C 3.A 4.D 5.B

Written solutions:

1. (a) $x = 1\frac{1}{2}\ \ or\ \ -2$ (b) $y = 2\frac{1}{3}\ \ or\ \ -1$
   (c) $x = -1.85,\ 0.180$ (d) $y = 0.356, -2.11$

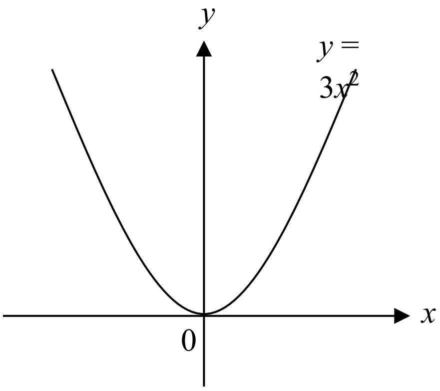
2. (a) $y = (x-2)^2 + 1$ (b) $y = 2(x-1)^2 + 11$
   (c) $y = 2\left(x + \frac{5}{4}\right)^2 - \frac{57}{8}$ (d) $y = -5\left(x - \frac{2}{5}\right)^2 + 12\frac{4}{5}$

3. (a) $x = 3 \pm \sqrt{13}$ (b) $x = -2 \pm \sqrt{\frac{11}{2}}$
   (c) $x = -1 \pm \sqrt{6}$ (d) $x = \frac{1}{4} \pm \frac{\sqrt{21}}{4}$

4. (a) $x = 1$ (b) $x = -2\ \ or\ \ 5$
   (c) $y = 2\ \ or\ \ 16$ (d) $y = -2\frac{2}{3}\ \ or\ \ 6$

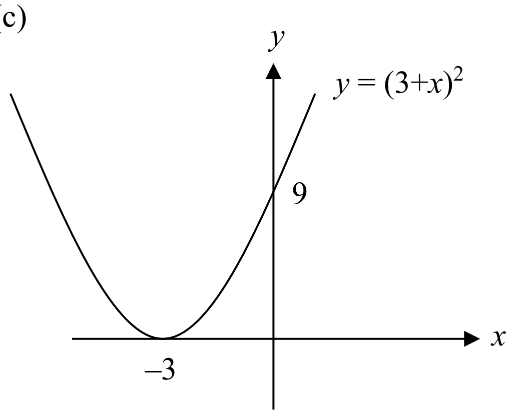
5.

(a)



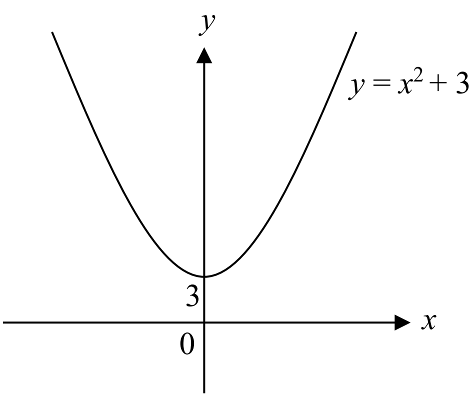
Vertex = (0, 0)

(c)



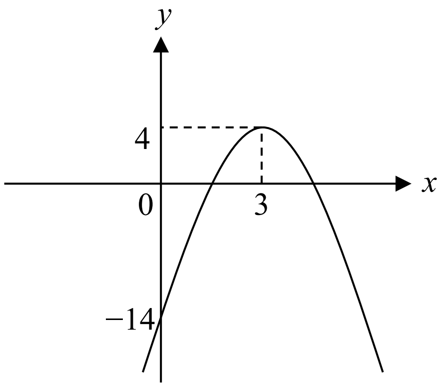
Vertex = (–3, 0)

(b)



Vertex = (0, 3)

(d)



Vertex = (3, 4)

6. (a) Domain = $(-\infty, \infty)$ Range = $[-4, \infty)$
Line of symmetry: $x = 2$ Vertex = $(2, -4)$
(b) Domain = $(-\infty, \infty)$ Range = $[-1, \infty)$
Line of symmetry: $x = -3$ Vertex = $(-3, -1)$
(c) Domain = $(-\infty, \infty)$ Range = $(-\infty, -2]$
Line of symmetry: $x = -9$ Vertex = $(-9, -2)$

7. $f(x) = 2\left(x + \frac{3}{2}\right)^2 - \frac{7}{2}$

The coordinates of the minimum point are $\left(-\frac{3}{2}, -\frac{7}{2}\right)$. Line of symmetry is $x = -\frac{3}{2}$.

8. (a) $y = -\frac{1}{7}(x-2)^2 - 3$ (b) $y = -\frac{1}{8}(x+2)^2$

9. $y = -3x^2 + 5$ 10. $y = -x^2 + 4x + 3$

11. $y < 11.25$ m 12. $a = -0.01$, $b = 2$

13. $x = 4$ or 5 14. $t = 0.21, 4.79$ s , Duration = 4.58s

15. (a) $\frac{50}{x}\,h$ (b) $\frac{6}{x-16}\,h$ (d) $x = 20$ or 10 (e) $2\,h$ 48 min

16. (a) $C = 12000 + 90x$
(c) 500 carts
(d) 1250 carts. Maximum profit = $6750.

17. At least 86.25 marks 18. After 3 months

19. $16.99

20. (a) $-2.5 < x < 4$ (b) $-\frac{2}{5} < x < 1$ (c) $x < -2$ or $x > 3$ (d) $x > 2$ or $x < -4.5$

21. $x < 8$ *or* $x > 18$

22. (a) $k < -2$ (b) $k > \frac{2}{3}$ (c) $k \geq -7\frac{1}{8}$
(d) $k = \pm\frac{4}{\sqrt{5}}$ (e) $k \neq -\frac{1}{3}$ (f) $-4 < k < 4$

24. (a) $k > \frac{9}{8}$ (b) $k < -\frac{3}{4}$ (Note that k must be negative.)

25. $-2 < k < 6$

26. (a) $(x = 4,\ y = 3), (x = 7,\ y = \frac{3}{2})$ (b) $(x = 3,\ y = 12), (x = -2,\ y = 2)$
(c) $(x = 1,\ y = 1), (x = -\frac{16}{5},\ y = -\frac{9}{5})$ (d) $\left(x = \frac{1}{2},\ y = -\frac{3}{2}\right), (x = \frac{2}{3}, y = -1)$

27. (a) $k = \pm 3$ (b) $k < 2$ *or* $k > 18$ (c) $k > 3$ (d) $k \geq \frac{3}{2}$

28. $-15 < k < 1$

29. (a) $1.2 < x < 3.6$
 (b) $h = 2$

Problem-solving Assignment 2:

1. She should charge $2.50 for each truck to maximize her profit.