3 Rational Expressions

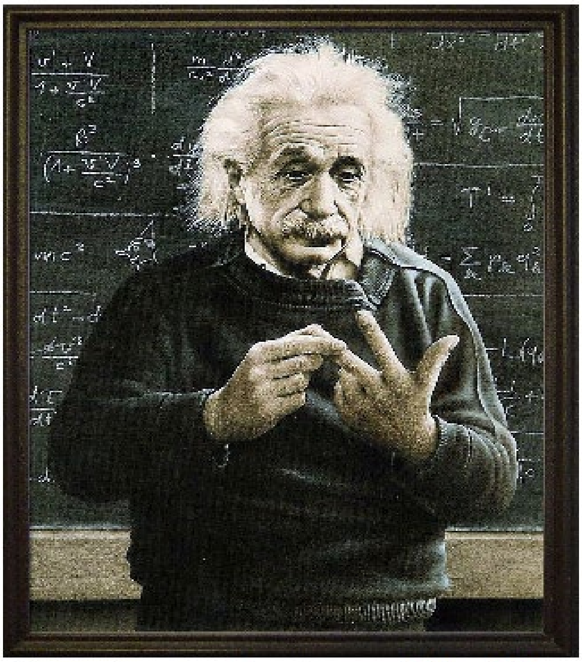
LEARNING OBJECTIVES

Upon completion of Chapter 3, you should be able to:

1) Simplify rational expressions.
2) Multiply rational expressions
3) Divide rational expressions
4) Add and subtract rational expressions
5) Simplify complex fractions
6) Resolve an algebraic fraction where the denominator has linear factors into partial fractions
7) Resolve an algebraic fraction where the denominator has irreducible quadratic factors into partial fractions

Relevant sections in e-book:

- R.6 Rational Expressions and More Operations on Radicals
- 9.3 Partial Fraction Decomposition



Do not worry about your difficulties in mathematics, I assure you that mine are greater.

Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspect of the world.
- Alfred North Whitehead-

3.1 What are rational expressions?

This is yet another chapter to better prepare you for the study of calculus.

Almost every concept in calculus is first studied in the context of polynomials. Why, you may wonder?

It is because polynomials are the simplest of all mathematical expressions!

You have been taught polynomials and the possible operations on polynomials in Chapter 2.

In this chapter, Chapter 3, we will take a look at rational expressions.

A **rational expression** is a quotient of two polynomials, $\frac{F(x)}{G(x)}$.

The domain consists of all values x such that $G(x) \neq 0$.

Here are some examples of rational expressions:

$$\frac{5}{x},\ \frac{x}{x-3},\ \frac{x-2}{x^2-25},\ \text{and}\ \frac{x^2+1}{x^2+2x-3}.$$

The set of real numbers for which an algebraic expression is defined is called the **domain**.

Because rational expressions indicate division and division by zero is undefined, we have to exclude from the domain numbers that make the denominator zero.

For example, in the expression $\frac{5}{x}$, x cannot equal 0 since the denominator will then be 0.

Similarly, in $\frac{x}{x-3}$, we have to exclude 3 from the domain.

A rational expression is simplified if its numerator and denominator have no common factor. For example, $\frac{x}{x+2}$ is simplified, but $\frac{x^2}{x^2+2x}$ is not.

To simplify a rational expression, we factor numerator and denominator and divide out, or cancel out, common factors.

3.2 Simplify rational expressions

A rational expression is simplified (in lowest terms) if the only factors shared by the numerator and denominator are 1 or -1.

To simplify rational expressions,

- Factor the numerator and denominator completely.
- Divide the numerator and denominator by any common factors.

VIDEO EXAMPLE 3.1

Simplify $\dfrac{x^2-9}{x^2-4x-21}$.

3.3 Multiply rational expressions

To multiply rational expressions, we use the following property:

$$\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd} \text{ for } b \neq 0,\ d \neq 0$$

To multiply rational expressions,

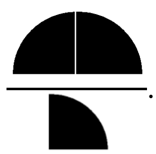
- Factor the numerator and denominator completely.
- Divide the numerator and denominator by any common factors.
- Multiply the remaining factors in the numerator, and multiply the remaining factors in the denominator.

VIDEO EXAMPLE 3.2

Perform the indicated operation and simplify : $\dfrac{3m+3n}{12m-36n} \bullet \dfrac{m^2-mn-6n^2}{m^2+2mn+n^2}$

3.4 Divide rational expressions

Look at the expression $\frac{1}{2} \div \frac{1}{4}$. An alternative representation of that expression is $\dfrac{\frac{1}{2}}{\frac{1}{4}}$.

It can also shown using pictorial representation, such as .

In words, the expression is asking ***how many quarters are there in a half***? Graphically, we know it is 2! There are two quarters in a half.

To divide rational expressions, we use the following property:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \bullet \frac{d}{c} = \frac{ad}{bc} \text{ for } b \neq 0,\ c \neq 0, d \neq 0$$

We divide rational expressions by inverting the divisor (the second or bottom fraction), and multiplying.

And the mathematical operation of $\frac{\frac{1}{2}}{\frac{1}{4}}$ is $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = 2$

The mathematical operations may seem absurd to some but the results of the mathematical operation should be meaningful! “Two quarters in a half”

Hopefully with the pictorial representation, you could better appreciate the results of dividing rational expression now.

VIDEO EXAMPLE 3.3

Perform the indicated operation and simplify : $\frac{x^3-64}{16x-x^3} \div \frac{2x^2+8x+32}{x^2+2x-8}$

3.5 Add and Subtract Rational Expressions

3.5.1 Same numerator

When adding or subtracting two rational expressions with a common denominator, we add or subtract the numerators while keeping the common denominator.

Addition		**Subtraction**
$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$	and	$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$, $b \neq 0$

VIDEO EXAMPLE 3.4

Perform the indicated operation and simplify: $\frac{5}{y+3}+\frac{y^2+y+24}{y^2-9}$

3.5.2 Different denominators

EXERCISE 3.1

Which method is better?

Compare the following 2 methods for adding and subtracting rational expressions with different denominators, and explain which would be the better method.

Method 1	Method 2
Find a common denominator, subtract the numerators then simplify.	Find the Least Common Denominator (LCD). Rewrite each fraction as an equivalent expression with the LCD. Add or subtract the numerators over the LCD. Simplify the answer.
$\frac{3}{x-1}-\frac{6}{x^2-1}$ $=\frac{3(x^2-1)}{(x-1)(x^2-1)}-\frac{6(x-1)}{(x+1)(x^2-1)}$ $=\frac{3x^2-3}{(x-1)(x^2-1)}-\frac{6x-6}{(x+1)(x^2-1)}$ $=\frac{3x^2-3-(6x-6)}{(x-1)(x^2-1)}$ $=\frac{3x^2-3-6x+6}{(x-1)(x^2-1)}$ $=\frac{3x^2-6x+3}{(x-1)(x^2-1)}$ $=\frac{3(x^2-2x+1)}{(x-1)(x^2-1)}$ $=\frac{3(x-1)^2}{(x-1)(x^2-1)}$ $=\frac{3(x-1)(x-1)}{(x-1)(x-1)(x+1)}$ $=\frac{3}{(x+1)}$	$\frac{3}{x-1}-\frac{6}{x^2-1}$ $=\frac{3}{x-1}-\frac{6}{(x+1)(x-1)}$ $=\frac{3(x+1)}{(x+1)(x-1)}-\frac{6}{(x+1)(x-1)}$ $=\frac{3x+3}{(x+1)(x-1)}-\frac{6}{(x+1)(x-1)}$ $=\frac{3x+3-6}{(x+1)(x-1)}$ $=\frac{3x-3}{(x+1)(x-1)}$ $=\frac{3(x-1)}{(x+1)(x-1)}$ $=\frac{3}{(x+1)}$

When adding or subtracting two rational expressions with different denominators,

- find the Least Common Denominator (LCD) .
  - Factor each denominator completely.
  - List all different factors that appear in any of the denominators.
  - The LCD is the product of all the different factors.
- Rewrite each fraction as an equivalent expression with the LCD.
- Add or subtract the numerators over the LCD.
- Simplify the answer.

EXERCISE 3.2

Perform the indicated operation and simplify.

a) $\dfrac{1}{x-2}+\dfrac{x}{x^2-4}$

b) $\dfrac{5}{x-3}+\dfrac{x+2}{3-x}$

c) $\dfrac{2x+9}{x-5}-\dfrac{4}{x^2-3x-10}$

3.6 Simplify Complex Fractions

A complex fraction is a rational expression that contains fractions in its numerator, denominator or both. Here are some examples of complex fractions:

$$\frac{\frac{2}{5}}{x},\ \frac{\frac{x}{y}}{x+1},\ \text{and}\ \frac{1+\frac{1}{x}}{1-\frac{1}{x}}.$$

EXERCISE 3.3

Simplify $\dfrac{\dfrac{1}{27x}+\dfrac{1}{9}}{\dfrac{1}{3}+\dfrac{1}{9x}}$

EXERCISE 3.4

Simplify $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}$.

Method 1

- Simplify the numerator. Combine the terms in the numerator into a single fraction.
- Simplify the denominator. Combine the terms in the denominator into a single fraction.
- Perform division indicated by the main fraction bar: Invert and multiply. If possible, simplify.

Method 2

- Find the Least Common Denominator of all fractions appearing within the complex fraction.
- Multiply both the numerator and denominator of the complex fraction by the LCD.
- If possible, simplify.

EXERCISE 3.5

Simplify $\dfrac{\dfrac{1}{x+y}-\dfrac{1}{x}}{y}$.

3.7 Partial Fractions

When $P(x)$ and $D(x)$ are polynomials, where $D(x) \neq 0$, $\dfrac{P(x)}{D(x)}$ is called an algebraic fraction.

If the degree of $P(x)$ is less than the degree of $D(x)$, $\dfrac{P(x)}{D(x)}$ is said to be a **proper algebraic fraction**.

For example, $\dfrac{3}{x}, \dfrac{2x-3}{x^2+2x+3}, \dfrac{x^3}{x^4+2x^3+3}$ are proper algebraic fractions.

If the degree of $P(x)$ is more than or equal to the degree of $D(x)$, $\dfrac{P(x)}{D(x)}$ is said to be a **improper algebraic fraction**.

For example, $\dfrac{3x^2}{x^2+2x+1}, \dfrac{x^3-3}{x^2+2x+3}, \dfrac{x^3-9x}{x^2+2x-8}$ are improper algebraic fractions.

As shown earlier, the sum or difference of two algebraic fractions is usually a more complicated fraction. For example, $\dfrac{3}{x+1}+\dfrac{4}{2x-5}=\dfrac{10x-11}{(x+1)(2x-5)}$.

By reversing the process, the algebraic fraction $\dfrac{P(x)}{D(x)}$ may be expressed as a sum of simpler fractions with those factors of $D(x)$ as denominators.

These simpler fractions are called <u>**Partial Fractions**</u> of the given algebraic fraction.
Let us look at some methods of decomposing an algebraic fraction into partial fractions.

3.7.1 Proper Fractions with the Denominator $D(x) = (ax + b)(cx + d)$

To each of the distinct linear factors ($ax + b$) and ($cx + d$) of $D(x)$, there corresponds to a partial fraction of the form $\dfrac{A}{ax+b}$ and $\dfrac{B}{cx+d}$, where A and B are constants.

VIDEO EXAMPLE 3.6

Express $\dfrac{6w-7}{w^2+w-6}$ in partial fractions.

EXERCISE 3.6

Express $\dfrac{x-43}{(2x+5)(x-4)}$ in partial fractions.

3.7.2 Proper Fractions with the Denominator $D(x) = (ax + b)(cx + d)^2$

In the denominator $D(x)$, when the linear factor $(cx + d)$ is repeated twice, there correspond two partial fractions of the form $\dfrac{B}{cx+d}$ and $\dfrac{C}{\left(cx+d\right)^2}$, where B and C are constants.

VIDEO EXAMPLE 3.7

Set up the form for the partial fraction decomposition for $\dfrac{x^2+26x+100}{x\left(x+5\right)^2}$.

EXERCISE 3.7

Express $\dfrac{5x^2-8x-5}{\left(x-1\right)\left(x^2-1\right)}$ in partial fractions.

3.7.3 Proper Fractions with the Denominator $D(x) = (ax + b)(x^2 + c^2)$

In the denominator $D(x)$ with quadratic factor $x^2 + c^2$ (which cannot be factorised into 2 linear factors), there corresponds a partial fraction of the form $\dfrac{Ax+B}{x^2+c^2}$, where A and B are constants.

EXERCISE 3.8

Express in $\dfrac{8x^2+x+20}{x^3+4x}$ partial fractions.

Let $\dfrac{8x^2+x+20}{x^3+4x} = \dfrac{8x^2+x+20}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$

Multiply by $x(x^2+4)$

$8x^2+x+20 = A\left(x^2+4\right)+\left(Bx+C\right)x$..............(1)

Let $x=0$

$0+0+20 = A\left(0^2+4\right)+0$

$A=5$

How can we find B and C? Two methods are possible.

Method 1: Substitution of values

Sub $x=1$ and $A=5$ into (1)

$8(1)^2+1+20 = 5\left(1^2+4\right)+\left(B(1)+C\right)1$

$B+C=4$.....................................(2)

Sub $x=2$ and $A=5$ into (1)

$8(2)^2+2+20 = 5\left(2^2+4\right)+\left(B(2)+C\right)2$

$4B+2C=54$.................................(3)

Solve simultaneous equations (2) and (3)

$B=3, C=1$

Method 2: Comparing Coefficients

From (1), we expand the RHS of equation.

$8x^2+x+20 = A\left(x^2+4\right)+\left(Bx+C\right)x$..............(1)

$= Ax^2+4A+Bx^2+Cx$

$\boxed{8x^2}+x+20 = \boxed{(A+B)x^2}+Cx+4A$

By comparing Coefficients:

$\boxed{x^2 \text{ terms}}$: $A+B=8 \rightarrow$ Putting $A=5, \therefore B=8-5=3$

x terms : $C=1$

$\therefore \dfrac{8x^2+x+20}{x^3+4x} = \dfrac{5}{x}+\dfrac{3x+1}{x^2+4}$

EXERCISE 3.9

Express $\dfrac{-x^2+13x-33}{(2x+5)(x^2+4)}$ in partial fractions.

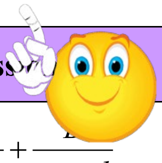
Self-Directed Learning @SP — Manage Learning — *Use appropriate learning strategies*

To express a compound algebraic fraction into partial fractions:

Step 1: Determine if the compound fraction is a proper or improper. If it is improper, perform long division.
Step 2: Ensure that the denominator is completely factorized.
Step 3: Express the proper fraction in partial fractions according to the cases below.
Step 4: Solve for unknown constants by substituting values of x and/or comparing coefficients and/or Cover-Up Rule.

Study tip #3: Use concrete examples
Fill up the following table with your own specific examples!

https://tinyurl.com/ms960-study-tip-3



Case	Denominator of fraction	Algebraic Fraction	Expression
1	Linear Factors	$\dfrac{mx+n}{(ax+b)(cx+d)}$	$\dfrac{A}{ax+b}+\dfrac{B}{cx+d}$
		E.g $\dfrac{3x+4}{(2x+3)(x+2)}$	
2	Repeated Linear Factors	$\dfrac{mx+n}{(ax+b)(cx+d)^2}$	$\dfrac{A}{ax+b}+\dfrac{B}{cx+d}+\dfrac{C}{(cx+d)^2}$
3	Irreducible Quadratic Factor (Quadratic factor which cannot be factorized)	$\dfrac{mx+n}{(ax+b)(x^2+c)}$	$\dfrac{A}{ax+b}+\dfrac{Bx+C}{x^2+c}$

*** * * END OF CHAPTER 3 * * ***

TUTORIAL CHAPTER 3

Multiple Choice Questions: Choose the best option.

1. $\dfrac{x}{(x-1)^3}$ can be expressed as

   (a) $\dfrac{A}{x-1}+\dfrac{Bx+C}{(x-1)^2}+\dfrac{Dx^2+Ex+F}{(x-1)^3}$

   (b) $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}$

   (c) $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}$

   (d) $\dfrac{A}{x-1}+\dfrac{Bx+C}{(x-1)^3}$

2. $\dfrac{x+1}{(x-2)(x^2+9)}$ can be expressed as

   (a) $\dfrac{A}{x-2}+\dfrac{Bx}{x^2+9}$

   (b) $\dfrac{A}{x-2}+\dfrac{B}{x^2+9}$

   (c) $\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+9}$

   (d) $\dfrac{A}{x-2}+\dfrac{B}{x+3}+\dfrac{C}{x-3}$

3. $\dfrac{2}{x^2(x^2+x+3)}$ can be expressed as

   (a) $\dfrac{A}{x}+\dfrac{B}{x^2+x+3}$

   (b) $\dfrac{A}{x^2}+\dfrac{Bx+C}{x^2+x+3}$

   (c) $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^2+x+3}$

   (d) $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+x+3}$

Written solutions

1. Simplify each rational expression.

   (a) $\dfrac{5x-10}{x^2-4x+4}$ (b) $\dfrac{x^2+x-12}{x^2-5x+6}$

2. Perform the multiplication and simplify.

   (a) $\dfrac{4x}{x^2-4}\cdot\dfrac{x+2}{8x}$ (b) $\dfrac{t^2+2t-3}{t^2-2t-3}\cdot\dfrac{3-t}{3+t}$

   (c) $\dfrac{x^2-9}{x}\div\dfrac{x+3}{x-3}$ (d) $\dfrac{\dfrac{t^3}{t+1}}{\dfrac{t}{t^2+2t+1}}$

3. Perform the indicated operation and simplify.

   (a) $1+\dfrac{x}{x-3}$ (b) $\dfrac{3x-2}{x+4}-1$

   (c) $\dfrac{1}{y^2}+\dfrac{1}{y^2+y}$ (d) $\dfrac{7}{2x-3}-\dfrac{5}{(2x-3)^2}$

   (e) $\dfrac{x}{x^2-4}-\dfrac{1}{2-x}$ (f) $\dfrac{1}{x+5}+\dfrac{1}{25-x^2}$

4. Find the sum of (a) $1+\dfrac{1}{y}$ and (b) $1+\dfrac{1}{y}+\dfrac{1}{y^2}$.

   Hence, deduce the sum of $1+\dfrac{1}{y}+\dfrac{1}{y^2}+...+\dfrac{1}{y^n}$.

5. Simplify each complex fraction.

   (a) $\dfrac{\dfrac{1}{x}+\dfrac{1}{2}}{\dfrac{1}{3}+\dfrac{x}{6}}$ (b) $\dfrac{3-\dfrac{12}{y}}{1-\dfrac{16}{y^2}}$

   (c) $\dfrac{1+\dfrac{1}{b-1}}{1-\dfrac{1}{b-1}}$ (d) $\dfrac{2-\dfrac{1}{x+3}}{2+\dfrac{1}{x+3}}$

   (e) $\dfrac{a^{-2}-b^{-2}}{a^{-1}+b^{-1}}$ (f) $\dfrac{a^{-1}+b^{-1}}{(a+b)^{-1}}$

6. When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R_T is given by the equation $R_T = \dfrac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}}$. Simplify the equation.

7. In the study of optics, the focal length of a lens, f, is given by $f=(p^{-1}+q^{-1})^{-1}$, where p is the object's distance from the lens, and q is the image distance from the lens. Simplify the given equation such that there are no negative exponents.

8. Express the following proper algebraic fractions in partial fractions:

(a) $\dfrac{5x+3}{x(x+1)}$ (b) $\dfrac{x+2}{(2x-3)(x-4)}$

(c) $\dfrac{3x-29}{(2x-3)(x-5)^2}$ (d) $\dfrac{30x^2-x+3}{(x-1)(3x+1)^2}$

(e) $\dfrac{x^2-x+4}{x(x^2+4)}$ (f) $\dfrac{2x^2-9x+41}{(x+5)(x^2+9)}$

9. Express the following algebraic fractions in partial fractions:

(a) $\dfrac{5x+11}{x^2+4x+3}$ (b) $\dfrac{x}{x^2-4}$

(c) $\dfrac{7x^2+x-14}{4x^3-7x^2}$ (d) $\dfrac{-3x^2+23x-36}{x^3-6x^2+9x}$

(e) $\dfrac{x-5}{x^3-x^2+x-1}$ (f) $\dfrac{5x^2+7x-9}{4-8x+x^2-2x^3}$

10. Express $\dfrac{1}{x^2-a^2}$ in partial fractions, where a is a positive constant.

*11.(a) Resolve $\dfrac{2}{n(n+2)}$ into partial fractions.

(b) Use your answer in (a) to rewrite the infinite sum $\dfrac{2}{1(3)}+\dfrac{2}{2(4)}+\dfrac{2}{3(5)}+\dfrac{2}{4(6)}\ldots$

Hence, deduce the value of the infinite sum.

ANSWERS

Multiple Choice Questions:

1.B 2.C 3.D

Written Solutions

1. (a) $\frac{5}{x-2}$ (b) $\frac{x+4}{x-2}$

2. (a) $\frac{1}{2(x-2)}$ (b) $\frac{1-t}{t+1}$ (c) $\frac{(x-3)^2}{x}$ (d) $t^2(t+1)$

3. (a) $\frac{2x-3}{x-3}$ (b) $\frac{2(x-3)}{x+4}$ (c) $\frac{2y+1}{y^2(y+1)}$

   (d) $\frac{2(7x-13)}{(2x-3)^2}$ (e) $\frac{2x+2}{(x-2)(x+2)}$ (f) $\frac{6-x}{(5-x)(5+x)}$

4. $\frac{1+y}{y}$; $\frac{1+y+y^2}{y^2}$; $\frac{1+y+y^2+...+y^n}{y^n}$

5. (a) $\frac{3}{x}$ (b) $\frac{3y}{(y+4)}$ (c) $\frac{b}{b-2}$ (d) $\frac{2x+5}{2x+7}$

   (e) $\frac{b-a}{ab}$ (f) $\frac{(a+b)^2}{ab}$

6. $\frac{R_1R_2}{R_1+R_2}$ 7. $\frac{pq}{p+q}$

8. (a) $\frac{3}{x}+\frac{2}{x+1}$ (b) $\frac{-7}{5(2x-3)}+\frac{6}{5(x-4)}$ (c) $\frac{-2}{2x-3}+\frac{1}{x-5}-\frac{2}{(x-5)^2}$ (d) $\frac{2}{x-1}+\frac{4}{3x+1}-\frac{5}{(3x+1)^2}$

8. (e) $\frac{1}{x}-\frac{1}{x^2+4}$ (f) $\frac{4}{x+5}-\frac{2x-1}{x^2+9}$

9. (a) $\frac{3}{x+1}+\frac{2}{x+3}$ (b) $\frac{1}{2(x+2)}+\frac{1}{2(x-2)}$ (c) $\frac{1}{x}+\frac{2}{x^2}+\frac{3}{4x-7}$ (d) $\frac{-4}{x}+\frac{1}{x-3}+\frac{2}{(x-3)^2}$

   (e) $\frac{-2}{x-1}+\frac{2x+3}{x^2+1}$ (f) $\frac{-1}{1-2x}-\frac{3x+5}{x^2+4}$

10. $\frac{-1}{2a(x+a)}+\frac{1}{2a(x-a)}$

11. (a) $\frac{1}{n}-\frac{1}{n+2}$ (b) $\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+...$

    Value of infinite sum = $\frac{3}{2}$