# 6 Logarithmic & Exponential Functions

## LEARNING OBJECTIVES

Upon completion of Chapter 6, you should be able to:

1) Define a logarithmic function.
2) Convert from logarithmic to exponential form and vice versa.
3) Apply laws of logarithm to simplify expressions.
4) Solve logarithmic equations.
5) Sketch graphs of exponential functions of the form $y = ka^x$, where $a$ is a positive integer.
6) Solve exponential equations.
7) Solve application problems involving logarithmic and exponential functions.

Relevant sections in e-book:

- 4.3 Logarithmic functions
- 4.4 Properties of Logarithms
- 4.5 Exponential and Logarithmic Equations and Applications


***The method of logarithms, by reducing to a few days the labour of many months, doubles as it were, the life of astronomer, besides freeing him from the errors and disgust inseparable from long calculations.***

***- P. S. Laplace –***

## 6.1 Logarithmic Functions

> If $x$ and $b$ are positive real numbers such that $b \neq 1$, then $y$ is called the logarithm of $x$ to base $b$ where
>
> $$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

For instance, let us consider $\quad 3^2 = 9$.

The exponent is 3. In order to make the exponent the subject (i.e. to write it in terms of the other two numbers), we have to make use of logarithms.

$$2 = \log_3 9$$ (read as "2 is the logarithm of 9 to base 3")

Exponent is made the subject

**VIDEO EXAMPLE 6.1**

(a) $y = \log_2 8$

(b) $y = \log_5 25$

**VIDEO EXAMPLE 6.2**

Write each equation in exponential form.

(a) $\log_8 64 = 2$

(b) $\log\left(\dfrac{1}{10000}\right) = -4$

(c) $\log_4 1 = 0$

> $y = \log_b x$ is the same as $b^y = x$.

**VIDEO EXAMPLE 6.3**

Write each equation in logarithmic form.

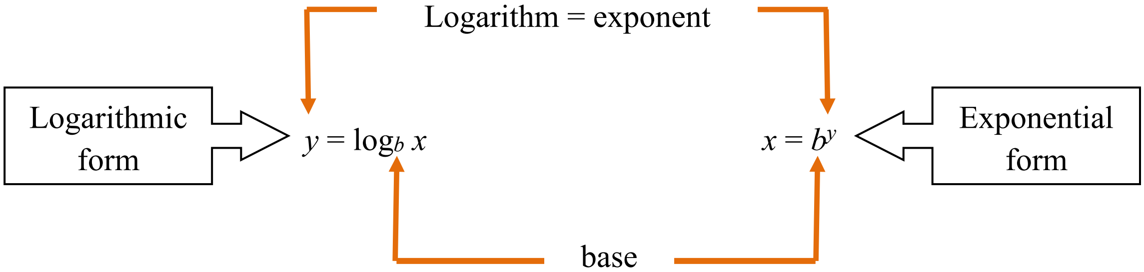
(a) $5^3 = 125$

(b) $\left(\dfrac{1}{5}\right)^{-3} = 125$

(c) $10^9 = 1000{,}000{,}000$

> $y = \log_b x$ is the same as $b^y = x$.

Logarithmic and exponential form can be interchanged as depicted in the following diagram.




## 6.2 Laws of logarithms

Since $y = \log_a x$ is equivalent to $x = a^y$, we can derive the laws of logarithms from the laws of indices. The laws of logarithms are as follows:

Suppose that $n$ is a real number
and $a$, $b$, $x$ and $y$ are positive,
and $a \neq 1$ and $b \neq 1$,

Property 1: $\log_a xy = \log_a x + \log_a y$ (Product law)

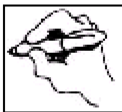
Property 2: $\log_a \frac{x}{y} = \log_a x - \log_a y$ (Quotient law)

Property 3: $\log_a x^n = n \log_a x$ (Power law)

Property 4: $\log_a x = \frac{\log_b x}{\log_b a}$ (Change of base law)

Property 5: $\log_a a = 1$

Property 6: $\log_a 1 = 0$


**EXERCISE 6.1**

Evaluate each of the following:

(a) $\log_6 2 + \log_6 3$

(b) $\log_3 10 + \log_3 0.1$

**EXERCISE 6.2**

Simplify each of the following:

(a) $\log_4 48 - \log_4 12$

(b) $\log\left(x^2 - 9\right) - \log\left(x - 3\right)$

**EXERCISE 6.3**

Evaluate each of the following without using a calculator.

(a) $\log_b \sqrt[3]{b}$

(b) $\dfrac{\log_b 32}{\log_b \dfrac{1}{4}}$

**EXERCISE 6.4**

Evaluate each of the following without using a calculator.

(a) $\log_6 4 + \log_6 9$

(b) $\dfrac{1}{3}\log_5 8 - \log_5 10$

**EXERCISE 6.5**

Simplify each of the following to a single logarithm.

(a) $\log_a 3 + 2\log_a x - 5\log_a y$

(b) $\log x - \log y - \log z$

(c) $2 - \log z$

**EXERCISE 6.6**

Given that $x = \log_b 3$ and $y = \log_b 5$, find the following in terms of $x$ and $y$:

(a) $\log_b 15$

(b) $\log_b \left(3\sqrt{5}\right)$

(c) $\log_b 0.6$

(d) $\dfrac{\log_b 25}{\log_b 3b^2}$

**EXERCISE 6.7**

Evaluate $\log_3 343 \times \log_{49} 16 \times \log_8 27$.

**EXERCISE 6.8**

If $4\log\left(x\sqrt{y}\right)-\log y=1+2\log x$, where $x$ and $y$ are positive, express $y$ in terms of $x$.

## 6.3 Solve logarithmic equations

An equation that contains a variable within a logarithmic expression is called a logarithmic equation. For example, $\log_2\left(3x-4\right)=\log_2\left(x+2\right)$ and $\log_5 x=1$ are logarithmic equations.

If $b$, $x$ and $y$ are positive real numbers with $b\neq 1$, then

$$\log_b x=\log_b y \text{ implies that } x=y.$$

### 6.3.1 Solve a logarithmic equation by using equivalence property

**VIDEO EXAMPLE 6.4**

Solve the equation: $\log\left(x^2+7x\right)=\log 18$.

### 6.3.2 Solve a logarithmic equation by writing in exponential form

**VIDEO EXAMPLE 6.5**

Solve the equation $\log_8\left(3y-5\right)+10=12$

**VIDEO EXAMPLE 6.6**

Solve the equation $\log_2 w - 3 = -\log_2 (w+2)$

**EXERCISE 6.9**

Solve the equation $\log_5 (3x + 8) = 1 + \log_5 x$.

**EXERCISE 6.10**

Solve the equation $\lg (7x - 1) + \lg (x + 2) = 2$.

**EXERCISE 6.11**

Solve the equation $\log_2 (x + 1) - \log_4 (x - 3) = 2$.

**EXERCISE 6.12**

Solve the equation $\log_9 x - \log_3 x = 2$.

## 6.4 Applications of logarithms

**EXERCISE 6.13**

In chemistry, the pH of a solution is defined to be

$$pH = -\log[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per litre.

(a) Complete the following table.

| **pH** | **$[H^+]$** |
|---|---|
| | 1.0 |
| | 0.1 |
| | 0.01 |
| | 0.001 |
| | 0.0001 |
| | 0.00001 |
| | 0.000001 |
| | 0.0000001 |
| | 0.00000001 |
| | 0.000000001 |
| | 0.0000000001 |
| | 0.00000000001 |
| | 0.000000000001 |
| | 0.0000000000001 |
| | 0.00000000000001 |

(b) When $[H^+]$ decreases by a factor of 10, how does the pH change accordingly?

(c) What is the advantage of using the pH scale compared to $[H^+]$?

When a physical quantity varies over a large range, taking its logarithm will give us a more manageable set of numbers to work with.

The hydrogen ion concentration of a sample of human blood was measured to be $[H^+] = 3.2 \times 10^{-8}$ moles per litre. Is the sample acidic or basic?

## 6.5 Graphs of exponential functions

You have been taught exponential expressions in Chapter 1, such as $a^y, x^{-t}, y^z$.

To draw the graph of an exponential function, we can set up the table of values and plot the points on a coordinate plane and then join the points with a smooth curve.

**VIDEO EXAMPLE 6.7**

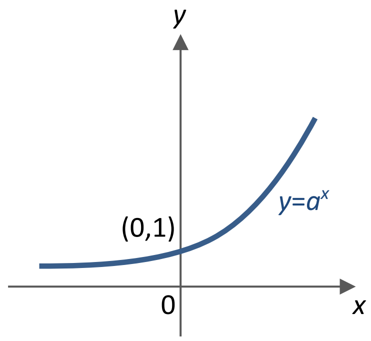
Draw the graph of $f(x) = 4^x$ for $-3 \le x \le 3$.

| *x* | *f*(*x*) |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| -1 | |
| -2 | |
| -3 | |

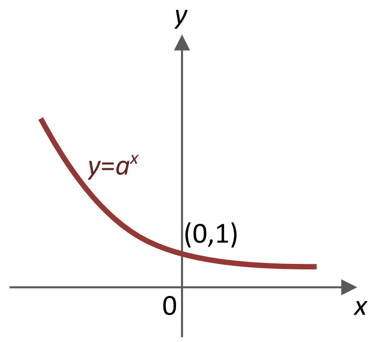
**VIDEO EXAMPLE 6.8**

Draw the graph of $h(x) = \left(\frac{1}{4}\right)^x$ for $-3 \le x \le 3$.

| *x* | *h*(*x*) |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| -1 | |
| -2 | |
| -3 | |



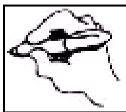

$a > 1$




$0 < a < 1$

For the graph of $f(x) = a^x$, where $a > 0$ and $a \neq 1$:

1. $f(x) > 0$ for all values of $x$.
2. The graph intersects the $y$-axis at the point (0,1).
3. The graph does not touch the $x$-axis at all. It can get very close the $x$-axis.
4. When $a > 1$, $f(x) = a^x$ is an increasing function. When $0 < a < 1$, $f(x) = a^x$ is a decreasing function.
5. The domain is $\{x \mid -\infty < x < \infty\}$ and the range is $\{f(x) \mid f(x) > 0\}$.



**EXERCISE 6.14**

Sketch the following graphs and state their domain and range:

(a) $y = 5^x$

(b) $y = \left(\frac{1}{3}\right)^x$

## 6.6 Exponential equations

A basic exponential equation is of the form

$$a^x = b$$

where $a > 0$ and $a \neq 1$ and $b > 0$.

Taking logarithm (usually of base 10) on both sides, we have

$$\lg a^x = \lg b$$
$$x \lg a = \lg b$$

Thus the solution of the equation is $x = \frac{\lg b}{\lg a}$.

Graphically, the solution of the exponential equation $a^x = b$ is the intersection of the graphs $y = a^x$ and $y = b$.

**EXERCISE 6.15**

Solve the following equations:

(a) $5^x = 25$

(b) $5^x = 11$

(c) $2(3)^x = 7$

**EXERCISE 6.16**

Solve the equation $5^{x+2} = 5^x + 18$.

$$5^{x+2} = 5^x + 18$$
$$5^x \times 5^2 = 5^x + 18$$
$$5^x \times 5^2 - 5^x = 18$$
$$25(5^x) - 5^x = 18$$
$$24(5^x) = 18$$
$$5^x = \frac{3}{4}$$
$$\lg 5^x = \lg \frac{3}{4}$$
$$x \lg 5 = \lg \frac{3}{4}$$
$$x = \frac{\lg \frac{3}{4}}{\lg 5}$$
$$= -0.179 \text{ (from the calculator, 3 s. f.)}$$

**EXERCISE 6.17**

Solve the equation $3^x + 3^{x+3} = 7$.

**EXERCISE 6.18**

Solve the equation $4^x - 2^{x+2} = 5$.

**EXERCISE 6.19**

Solve the equation $10^{2x-5} = 36 - 10^{2x-5}$.

## 6.7 Applications of exponential functions

**EXERCISE 6.20**

A loan of \$6000 was taken with 2% interest compounded annually, the total amount payable, \$$Z$ at the end of $t$ years would be given by $Z = 6000(1.02)^t$ . Find $t$ when the total amount payable first exceeds \$9000.

*** * * END OF CHAPTER 6 * * ***

**TUTORIAL CHAPTER 6**

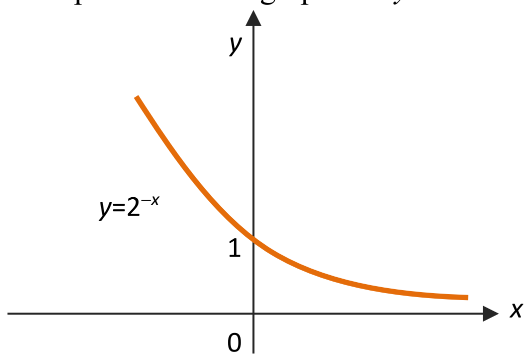
*Multiple Choice Questions*

1. Which of the following is equal to lg $(a + b)^3$?
   (a) 3(lg $a$)(lg $b$)
   (b) 3(lg $a$ + lg $b$)
   (c) lg $a$ + 3 lg $b$
   (d) 3 lg $(a + b)$

2. If $xy = 10^{p+q}$, then $\log_a (xy)$ is equal to
   (a) $p + q$
   (b) $p + q \log_a 10$
   (c) $(p + q) / \log_a 10$
   (d) $(p + q) \log_a 10$

*Written solutions*

1. The diagram below shows the graph of $y = 2^{-x}$, sketch the graph of $y = 2^x$ on the same axes. What is the relationship between the graphs of $y = 2^x$ and $y = 2^{-x}$?




2. Sketch the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 1 - x$ for $-2 \le x \le 2$ on the same axes. State the number of solutions of the equation $\left(\frac{1}{2}\right)^x + x - 1 = 0$.

3. Express each of the following in logarithmic form:
   (a) $7^3 = 343$ (b) $5^{-2} = \frac{1}{25}$ (c) $10^x = \sqrt{2}$

4. Express each of the following in exponential form:
   (a) $\log_6 216 = 3$ (b) $\log 0.01 = -2$ (c) $\log_b a = c$

5. Evaluate the following logarithms:
   (a) $\log_2 4$
   (b) $\log_3 27$
   (c) $\log_2 \frac{1}{8}$
   (d) $\log_3 \sqrt{3}$
   (e) $\log_9 3$
   (f) $\log_{\sqrt{b}} b^2$
   (g) $\log_b \frac{1}{b^2}$
   (h) $\frac{\log_b 9}{2\log_b 27}$

6. Evaluate the following expressions:
   (a) $(\log_8 1 - \log_8 8)^8$
   (b) $\log_8 4 + \log_8 2$
   (c) $\log_3 135 - \log_3 15$
   (d) $\log_7 56 - 3\log_7 2$
   (e) $\log_7 4 + 2\log_7 3 - 2\log_7 6$

7. Simplify and express each of the following as a single logarithm.
   (a) $3\log_2 5 - 2\log_2 7$
   (b) $\frac{1}{2}\log_5 64 + \frac{1}{3}\log_5 27 - \log_5 (x^2 + 4)$
   (c) $\frac{5}{6}\log_3 x + \frac{2}{3}\log_3 y - \frac{1}{2}\log_3 x - \log_3 y$
   (d) $3\lg\left(\frac{y^2}{x}\right) - 2\lg y + \frac{1}{4}\lg(x^4 y^8)$
   (e) $\frac{2}{3}\lg(x+5) + 2\lg(x+1) - \lg(x^2 + 6x + 5)$

8. Given that $\log_x 2 = A$ and $\log_x 3 = B$, find the following in terms of $A$ and $B$.
   (a) $\log_x \frac{3}{2}$
   (b) $\log_x 6$
   (c) $\log_x 16$
   (d) $\log_x 27$
   (e) $\log_x \frac{1}{4}$
   (f) $\log_x \frac{1}{27}$
   (g) $\log_x 24$
   (h) $\log_x 54$
   (i) $\log_x \frac{8}{9}$
   (j) $\log_x \sqrt[3]{3}$

9. Given that $\log x = p$ and $\log y = q$, find the following in terms of $p$ and $q$.
   (a) $\log(x^2 y)$
   (b) $\log\left(\frac{x}{100y}\right)$
   (c) $\log\sqrt{10xy^3}$
   (d) $xy^2$

10. Solve for $y$ in terms of $x$.

(a) $\log y = 2 + 3\log x$

(b) $3 + \log_2(x - y) = \log_2(x + 2y)$

(c) $2\log_9\left(x\sqrt{y}\right) = \dfrac{3}{2} - \log_9 x^2 y + \log_9 \dfrac{x}{y}$

(d) $\log_a(x + y) = \log_a x + \log_a y$

(e) $x = \log_a\left(y + \sqrt{y^2 - 1}\right)$

11. Evaluate the following:

(a) $\log_3 32 \bullet \log_2 27$

(b) $\dfrac{\log_4 25}{\log_8 \dfrac{1}{125}}$

12. Solve the following equations

(a) $\lg(x + 3) + \lg(x - 3) = \lg 16$

(b) $\log_7 x + \log_7 (x - 6) = 1$

(c) $\log_2(x + 3) = 3 - \log_2(x+5)$

(d) $\log_3(x - 5) + \log_3 \frac{1}{4} = 2 - \log_3(2x + 4)$

(e) $\log_9(4x + 1) = \log_3(x + 3) + \log_3 0.6$

(f) $\log_4 x + \log_x 32 = \frac{19}{6}$

(g) $\log_8(\log_4(\log_2 x)) = 0$

13. Solve the following exponential equations.

(a) $9^x = 4$

(b) $6^x = \frac{1}{13}$

(c) $7(3^x) = 26$

(d) $5^{x+1} - 5^x = 28$

(e) $7^{x+2} = 7^x + 16$

(f) $3^{2x} + 3^x - 20 = 0$

(g) $6(7^{2x}) - 17(7^x) + 5 = 0$

14. Solve the following for $x$.

(a) $2^{\log_2 x} = 16$

(b) $2^{\log_x 2} = 16$

(c) $x^{\log_2 x} = 16$

(d) $\log_2 x^2 = 2$

(e) $(\log_2 x)^2 = 1$

(f) $x = (\log_2 x)^{\log_2 x}$

15. Solve the simultaneous equations.

(a) $2^x = 8(2^y)$

$\lg(2x - y) = 0$

(b) $\log_3(x - y) = 1$

$5^x \times 125 = \dfrac{1}{25^y}$

16. Show that $\log_2 x = \log x \log_2 10$.

17. When the intensity of sound is I units, the loudness if the sound D in decibels (dB), is given by the formula $D = 10\log_{10}\dfrac{I}{I_0}$,

where $I_0 = 10^{-16}$ units, which is the minimum intensity that can be heard.

(a) If the intensity of sound of a conversation is $3.2 \times 10^{-10}$ units, find its loudness correct to the nearest decibel.

(b) If the noise level of an aeroplane is 130 dB, find the intensity of the noise.

18. \$10000 was deposited in a CPF Special Account which has 4% interest rate compounded annually. How much money will be in the account 20 years later?

19. It was found that the percentage of carbon C-14, $C$, contained in the bones of an animal $n$ years after it has dies is given by $C = 2^{-kn}$ where $k$ is a positive constant. The percentage of C-14 contained in the bones after the animals has been dead for 5668 years was 50%. How long was the animal dead if the percentage of carbon-14 found in the bones was 76%?

**ANSWERS:**

*Multiple Choice Questions*

1.D 2.D

*Written Solutions*

1. Reflection about $y$–axis

2. 2 solutions

5. (a) 2 (b) 3 (c) -3 (d) $\frac{1}{2}$ (e) $\frac{1}{2}$
   (f) 4 (g) -2 (h) $\frac{1}{3}$

6. (a)1 (b)1 (c) 2 (d) 1 (e) 0

7. (a) $\log_2 \frac{125}{49}$ (b) $\log_5\left(\frac{24}{x^2+4}\right)$ (c) $\log_3\left(\frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}\right)$ (d) $\lg\left(\frac{y^6}{x^2}\right)$ (e) $\lg\left(\frac{x+1}{(x+5)^{1/3}}\right)$

8. (a) $B - A$ (b) $A+B$ (c) $4A$ (d) $3B$ (e) $-2A$
   (f) $-3B$ (g) $3A + B$ (h) $3B + A$ (i) $3A - 2B$ (j) $\frac{1}{3}B$

9. (a) $2p+q$ (b) $p-2-q$ (c) $\frac{1+p+3q}{2}$ (d) $10^{p+2q}$

10. (a) $y=100x^3$ (b) $y=\frac{7x}{10}$ (c) $y=\frac{3}{x}$ (d) $y=\frac{x}{x-1}$ (e) $y=\frac{a^{2x}+1}{2a^x}$

11. (a) 15 (b) $-1$

12. (a) 5 (b) 7 (c) $-1$ (d) 7 (e) 2, $\frac{28}{9}$
    (f) 8, 10.1 (g) 16

13. (a) 0.631 (b) $-1.43$ (c) 1.19 (d) 1.21 (e) -0.565 (f) 1.26
    (g) $-0.565$ or 0.471

14. (a) 16 (b) $2^{\frac{1}{4}}$ (c) 4, $\frac{1}{4}$ (d) 2 (e) 2, $\frac{1}{2}$ (f) 4

15. (a) $x=-2,\ y=-5$ (b) $x = 1,\ y=-2$

17. (a) 65 dB (b) $10^{-3}$ units

18. $21,911.23

19. $k=\frac{1}{5668}$, 2244 years