2 Polynomials

LEARNING OBJECTIVES

Upon completion of Chapter 2, you should be able to:

1) Add, subtract, multiply and divide polynomials.
2) Recognise and apply special product formulas.
3) Use remainder theorem.
4) Factorise polynomials and algebraic expressions.
5) Use factor theorem.
6) Find the roots of cubic equations.

Relevant sections in e-book:

- R.4 Polynomials and Multiplication of Radicals
- R.5 Factoring
- 3.3 Division of Polynomials and the Remainder and Factor Theorems

Go down deep into anything and you will find mathematics.
-Dean Schlicter-

2.1 What are polynomials?

A **polynomial** in the variable x is given by

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x^1 + a_0 x^0$$

The coefficients a_n, a_{n-1}, a_{n-2}, … , a_1, a_0 are real numbers where $a_n \neq 0$, and the exponents n, n-1, n-2, … , 0 are whole numbers.

The term $a_n x^n$ is called the **leading term**, the coefficient a_n is the **leading coefficient**, and the exponent n is the **degree** of the polynomial.

Monomial: A polynomial with ______ term.

Binomial: A polynomial with ______ terms.

Trinomial: A polynomial with ______ terms.

Here are some examples of polynomials:

$$8x^3, \quad x+2, \quad x^2-5x-3,$$

2.2 Operations on polynomials

2.2.1 Addition and subtraction of polynomials

We add and subtract polynomials by combining **like terms**. Like terms are terms with exactly the same powers.

In subtracting polynomials, note that if a minus sign precedes an expression in brackets, then the sign of every term within the brackets is changed when we remove the brackets.

VIDEO EXAMPLE 2.1

Perform the indicated operation : $\left(-7d^5-4d^4-6\right)+\left(9d^5+2d^4-2\right)$.

VIDEO EXAMPLE 2.2

Perform the indicated operation : $\left(-5t^3+4t^2-3t\right)-\left(2t^3-6t^2+5t\right)$.

2.2.2 Multiplication of polynomials

We multiply two polynomials by using the distributive property, $a\,(b + c) = ab + ac$.

Every term in the first polynomial must be multiplied by each term in the second, then simplify and combine like terms. It is helpful to align terms such that each polynomial is in descending order.

When multiplying terms of the same base, apply the product rule for exponents.

$$a^m \bullet a^n = a^m \times a^n = a^{m+n}$$

VIDEO EXAMPLE 2.3

Perform the indicated operation : $(3y+6)\left(\frac{1}{3}y^2 - 5y - 4\right)$.

2.2.3 Special case products

If a and b are any real numbers or algebraic expressions, then

1. Multiplying conjugates: $(a+b)(a-b) = a^2 - b^2$

The expressions $(a+b)$ and $(a-b)$ are said to be **conjugates** of each other.
The product of two conjugates always results in a **difference of squares** with the square of the first term minus the square of the second term.

2. Squaring a binomial: $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

$$(a-b)^2 = (a-b)(a-b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

The square of a sum of two terms is the first term squared plus 2 times the product of the terms plus the last term squared. These products occur so frequently that it will be helpful to memorize them.

VIDEO EXAMPLE 2.4

Find $(3w^2 - 7z)(3w^2 + 7z)$.

VIDEO EXAMPLE 2.5

Find $(4t^2 + 3p^3)^2$.

2.3 Factoring

Factoring is the opposite of multiplying (expansion). To factor an expression is to write it as a product of other expressions.

FACTORING →

$x^2 - x - 6$ $\qquad$ $= (x-2)(x+3)$

← MULTIPLYING (expansion)

$(x-2)$ and $(x+3)$ are **factors** of $x^2 - x - 6$.

2.3.1 Factor out the greatest common factor

First, determine the greatest common factor, which is an expression of the highest degree that divides each term of the polynomial. Then factor out this common factor.

VIDEO EXAMPLE 2.6

Factorize $10k^2\left(3k^2+7\right)-5k\left(3k^2+7\right)$.

2.3.2 Factoring formulas

By writing the special product formulas backwards, we can obtain the following factoring formulas:

1. Difference of squares: $x^2 - y^2 = (x+y)(x-y)$
2. Perfect square: $x^2 + 2xy + y^2 = (x+y)^2$
3. Perfect square: $x^2 - 2xy + y^2 = (x-y)^2$

2.3.3 Factoring a Difference of squares

The difference of the squares of two terms can be factored as the product of the sum and difference of those terms.

VIDEO EXAMPLE 2.7

Factorize $x^2 - 25$.

2.3.4 Recognizing perfect squares

To recognize a perfect square trinomial,

[1] the first and last terms are squares of expressions.
[2] The middle term is twice the product of the expressions being squared in the first and last terms.

VIDEO EXAMPLE 2.8

Factorize $x^2 + 6x + 9$.

2.3.5 Factoring quadratic polynomials

A quadratic polynomial is a polynomial of degree 2, and is of the form $ax^2 + bx + c$, where $a \neq 0$.

EXERCISE 2.1

Factor $3x^2 + x - 2$.

The following method will help you to obtain the factors more easily.

1. Write down the factors of the first term and the last term in two columns.
2. Cross multiply the factors and write the product in the last column.
3. Add the product of the last column and check that it is the middle term of the expression.

Cross-multiply		*Add*
x	1	$3x$
$3x$	-2	$-2x$
$3x^2$	-2	x

Hence, $3x^2 + x - 2 = (x+1)(3x-2)$.

Study Tip: Check your factorization by multiplying the factors you obtained. If your factorization is correct, you will obtain the polynomial you started with.

EXERCISE 2.2

Factorize the following:

a) $8x^2y^4 + 4x^3y^3 - 2x^4y$

b) $\dfrac{x^2}{9} - \dfrac{1}{64}$

c) $3x^4 - 75x^2$

d) $x^2 + 5x + 6$

2.3.7 Factoring trinomials using substitution

Sometimes, a more complicated trinomial can be factored by substituting one variable for another.

EXERCISE 2.3

a) Factor $x^4 + x^2 - 6$.

b) Factor $2(x+3)^2 - 5(x+3) - 12$.

General strategy for factoring polynomials

[1] If there is a common factor, factor out the greatest common factor.
[2] Determine the number of terms in the polynomial, and try to factorize as follows:

- If there are two terms, is the binomial a difference of two squares? If so, factorise accordingly.
- If there are three terms, is the trinomial a perfect square? If so, factorise accordingly. Otherwise, factor the trinomial using trial and error or substitution.

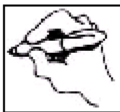
[3] Check to see if any factors with more than one term can be factored further. If so, factor completely.

2.4 Division of polynomials

In division of polynomials, we assume that the denominator is non-zero, as division by 0 is undefined.

2.4.1 Divide a polynomial by a monomial

To divide a polynomial by a monomial, we use the fact that $\dfrac{A+B}{C}=\dfrac{A}{C}+\dfrac{B}{C}$.



EXERCISE 2.4

Calculate $\dfrac{6x^2-8x-7}{2x}$.

2.4.2 Divide a polynomial by a binomial

Before dividing a polynomial by a binomial, we briefly review division of natural numbers.

$$\begin{array}{rrl}
\text{quotient} \longrightarrow & 58 & \\
\text{divisor} \longrightarrow \quad 3 & \overline{)\,176} & \longleftarrow \text{dividend} \\
 & -\underline{(15)} & \\
 & 26 & \\
 & -\underline{(24)} & \\
 & \underline{2} & \longleftarrow \text{remainder}
\end{array}$$

The quotient and remainder can be expressed as $58\frac{2}{3}$.

The result can be checked by multiplying the divisor with the quotient, and adding the remainder, i.e. dividend = divisor x quotient + remainder.

EXERCISE 2.5

$\left(8+30x-27x^2-12x^3+4x^4\right)\div\left(x+2\right)$

EXERCISE 2.6

Divide $2x^3-3x^2-11x+8$ by $x-3$, and check your answer.

Answer:

$$
\begin{array}{rrl}
\text{quotient} \longrightarrow & & 2x^2+3x\ -\ 2 \\
\text{divisor} \longrightarrow & x-3 \,\big) & \overline{2x^3-3x^2-11x+8} \longleftarrow \text{dividend} \\
& & -\underline{(2x^3-6x^2)} \\
& & \qquad 3x^2-11x \\
& & \qquad -\underline{(3x^2-\ 9x)} \\
& & \qquad\qquad -2x+8 \\
& & \qquad\qquad -\underline{(-2x+6)} \\
& & \qquad\qquad\qquad \underline{2} \longleftarrow \text{remainder}
\end{array}
$$

The quotient is $2x^2+3x-2$ and the remainder is 2.

This result can be written as $2x^2+3x-2+\frac{2}{x-3}$, just like how $176\div 3$ can be expressed as $58\frac{2}{3}$.

The result can also be checked, shown as follows:

$$\begin{aligned}
&\frac{(x-3)(2x^2+3x-2)}{x-3}+\frac{2}{x-3} \\
&=\frac{x(2x^2+3x-2)-3(2x^2+3x-2)+2}{x-3} \\
&=\frac{2x^3+3x^2-2x-6x^2-9x+6+2}{x-3} \\
&=\frac{2x^3-3x^2-11x+8}{x-3}
\end{aligned}$$

In general, the process of dividing a polynomial by another is as follows:

[1] Arrange the terms of both the dividend and divisor in descending powers of the variable.

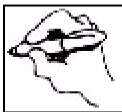
[2] Divide the first term in the dividend by the first term in the divisor. The result is the first term of the quotient.

[3] Multiply every term in the divisor by the first term in the quotient. Write the resulting product beneath the dividend with like terms lined up.

[4] Subtract the product from the dividend.

[5] Bring down the next term in the original dividend and write it next to the remainder to form a new dividend.

[6] Use the new expression as the dividend and repeat the above steps until the remainder can no longer be divided. This will occur when the degree of the remainder is less than the degree of the divisor.



EXERCISE 2.7

Given that $Q(x)$ is a function in x and R is a constant, find $Q(x)$ and R such that

$$\frac{6x^2-7x+3}{2x+1}=Q(x)+\frac{R}{2x+1} .$$

2.4.3 Introduction to function notation

For $y = 2x^3 - 3x^2 - 11x + 8$, the value of y depends on what the value of x is.

The expression can be written as $f(x) = 2x^3 - 3x^2 - 11x + 8$ where $f(x)$ indicates that the function f depends on x.

If $f(x) = 2x^3 - 3x^2 - 11x + 8$,

a) $f(0) = 2(0)^3 - 3(0)^2 - 11(0) + 8 = 8$

b) $f(1)$

c) $f(3)$

EXERCISE 2.8

Find the remainder when $2x^3 - 3x^2 - 11x + 8$ is divided by $x - 3$.

From Exercise 2.5,

$$
\begin{array}{rrl}
\text{quotient} \longrightarrow & & 2x^2 + 3x \; - \; 2 \\
\text{divisor} \longrightarrow & x-3 \big) & \overline{2x^3 - 3x^2 - 11x + 8} \longleftarrow \text{dividend} \\
 & & -\underline{(2x^3 - 6x^2)} \\
 & & \qquad\quad 3x^2 - 11x \\
 & & \qquad -\underline{(3x^2 - \;9x)} \\
 & & \qquad\qquad\quad -2x + 8 \\
 & & \qquad\qquad -\underline{(-2x + 6)} \\
 & & \qquad\qquad\qquad\quad \underline{2} \longleftarrow \text{remainder}
\end{array}
$$

This result can also be written as

$$2x^3 - 3x^2 - 11x + 8 = (x-3)(2x^2 + 3x - 2) + 2$$

i.e. dividend = divisor x quotient + remainder

Let $f(x) = 2x^3 - 3x^2 - 11x + 8$ and $q(x) = 2x^2 + 3x - 2$,

then $f(x) = (x-3)q(x) + 2$.

When $x = 3$,

$$
\begin{aligned}
f(3) &= (3-3)q(3) + 2 \\
&= 0 + 2 \\
&= 2, \text{ which is the remainder!}
\end{aligned}
$$

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials, with $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x) \bullet q(x) + r(x)$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. The polynomial $r(x)$ is called the remainder.

If the divisor is $x-k$, then the division algorithm for polynomials simplifies to

$$f(x) = (x-k)q(x) + r$$

where r is a constant.

If we let $x = k$, then

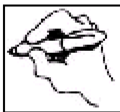
$$\begin{aligned} f(k) &= (k-k)q(k) + r \\ &= 0 \bullet q(k) + r \\ &= r \end{aligned}$$

Therefore, if a polynomial is divided by $x-k$, the remainder is the value of the polynomial at $x = k$.

2.5 Remainder Theorem

Remainder Theorem

If the polynomial $f(x)$ is divided by $x-k$, then the remainder is $f(k)$.



EXERCISE 2.9

Find the remainder when $3x^4 + 6x^3 - 2x + 4$ is divided by $x+2$.

EXERCISE 2.10

The expression $x^3 + 3x^2 - kx + 4$ leaves a remainder of k when it is divided by $x - 2$. Find the value of k.

Self-Directed Learning @SP

Manage Learning

Use appropriate learning strategies

Study tip #2: Teaching others
Is this topic familiar to you? If so, try teaching your peers who are learning it for the first time.
You'll learn more this way!

https://tinyurl.com/ms960-study-tip-2

2.6 Factor Theorem

Using the Division Algorithm when the divisor is $x - k$,

$$f(x) = (x-k)q(x) + r$$

where r is the remainder.

If $r = 0$, then

$$f(x) = (x-k)q(x).$$

Hence for a polynomial function, if $f(k) = 0$, then $(x-k)$ is a factor of $f(x)$.
Conversely, if $(x-k)$ is a factor of $f(x)$ and $r = 0$, then $f(x) = (x-k)q(x)$.
When $x = k$, then

$$\begin{aligned} f(k) &= (k-k)q(k) \\ &= 0 \bullet q(k) \\ &= 0 \end{aligned}$$

Therefore, if $(x-k)$ is a factor of $f(x)$, then $f(k) = 0$.

The Factor Theorem

Let $f(x)$ be a polynomial.

If $f(k) = 0$, then $(x-k)$ is a factor of $f(x)$.

If $(x-k)$ is a factor of $f(x)$, then $f(k) = 0$.

EXERCISE 2.11

Determine whether $x-5$ is a factor of $6x^2-25x-25$.

EXERCISE 2.12

If $x + 1$ and $x - 5$ are factors of $ax^3 - bx^2 + 3x + 10$, determine the values of a and b.

2.7 Find the roots of cubic equations

EXERCISE 2.13

Solve $x^3+2x^2-5x-6=0$

Answer: To solve the equation, we will let $g(x)=x^3+2x^2-5x-6$

$g(1)=(1)^3+2(1)^2-5(1)-6=-8\neq 0$

$\therefore (x-1)$ is not a factor of $g(x)$

$g(2)=(2)^3+2(2)^2-5(2)-6=0$

$\therefore (x-2)$ is a factor of $g(x)$

$$
\begin{array}{r}
x^2+4x+3 \\
x-2\overline{\smash{)}x^3+2x^2-5x-6} \\
\underline{x^3-2x^2} \\
4x^2-5x-6 \\
\underline{4x^2-8x} \\
3x-6 \\
\underline{3x-6} \\
0
\end{array}
$$

Thus the equation can be written as

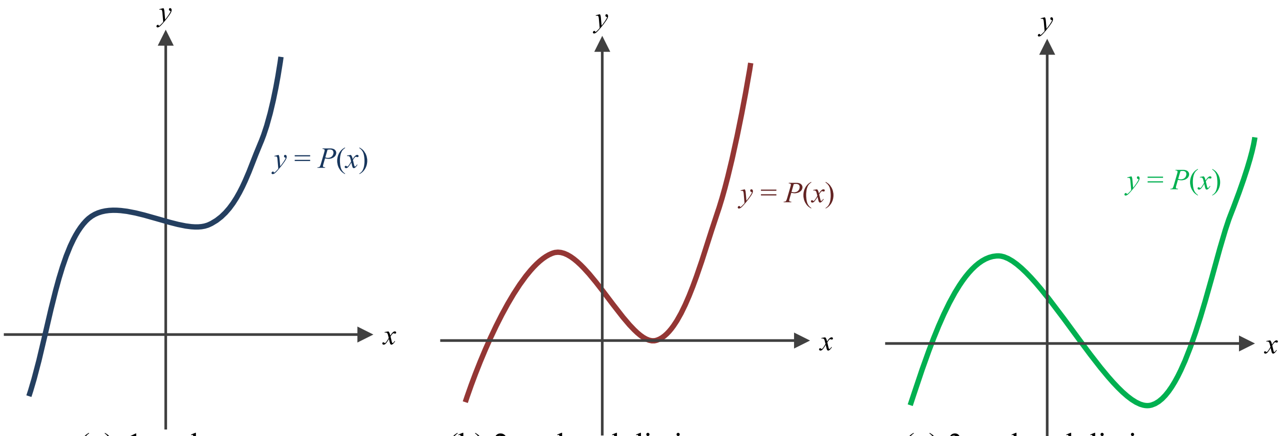
$(x-2)(x^2+4x+3)=0$

$(x-2)(x+1)(x+3)=0$

$x=-3$, $x=-1$ or $x=2$

They are the roots of $g(x)$, ie $g(-3)=0$, $g(-1)=0$ and $g(2)=0$.

The general shapes of the graph $y=ax^3+bx^2+cx+d$ for $a>0$ are shown below.



(a) 1 real root (b) 2 real and distinct roots (c) 3 real and distinct roots

The **roots** of $P(x)=0$ are the x-coordinates of the points of **intersection** of the graph of $y=P(x)$ with the ***x*-axis**.

EXERCISE 2.14

Given that $P(x) = x^3 + 4x^2 - 7x - 10$ and $x + 1$ is a factor of $P(x)$, solve the polynomial function $P(x) = 0$.

Check: Verify your solution by plotting the graph of $y = x^3 + 4x^2 - 7x - 10$ in www.desmos.com.

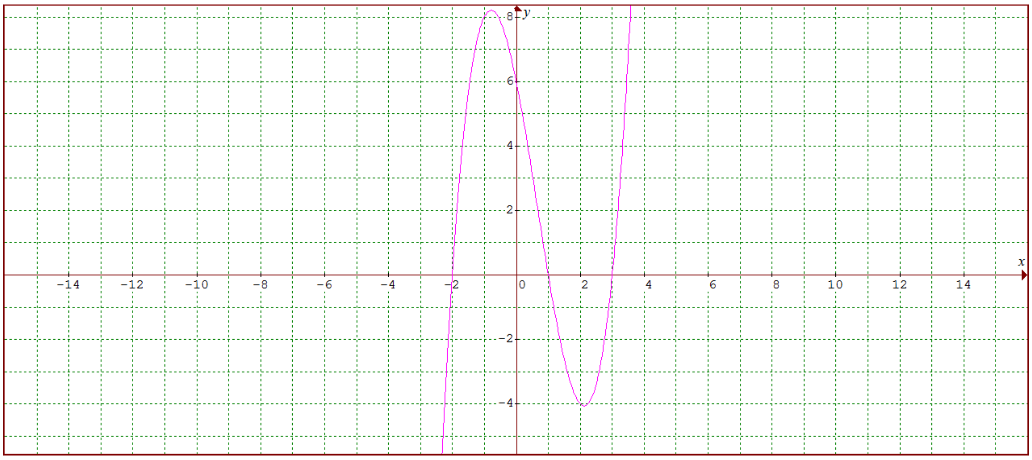
EXERCISE 2.15

Find a polynomial of degree 3 that has roots 1, –2 and 3.

The polynomial from Exercise 2.15 is graphed below.

Note that the roots of P correspond to the x-intercepts of the graph. The roots of the polynomial equation $P(x) = 0$ are also called **zeros** of the polynomial $P(x)$.

For a polynomial function defined by $y = f(x)$, the real zeros of $f(x)$ are the x-intercepts of the graph of the function.



*** * * END OF CHAPTER 2 * * ***

<u>TUTORIAL CHAPTER 2</u>

Multiple Choice Questions

1. If the polynomial g(y) is divided by ($y + h$), the remainder is

   (a) 0 (b) g(0) (c) g(h) (d) g($-h$)

2. The expansion of $(p - q)^2$ is

   (a) $p^2 - q^2$ (b) $p^2 + q^2$ (c) $(p - q)(p + q)$ (d) $p^2 + q^2 - 2pq$

Written solutions

1. Perform the indicated operations:

   (a) $17x^3 - 5x^2 + 4x - 2 - (5x^3 - 9x^2 - 8x + 10)$

   (b) $(3x^2 + 2x)(3x^2 - 2x)$

   (c) $(2x + 5)^2$

   (d) $(9 - 4x)^2$

2. Perform the indicated operations.

   (a) $(3x + 2y)^2 - (3x - 2y)^2$ (b) $[(4x+5)+3y][(4x+5)-3y]$

   (c) $(2x + 3)(2x - 3)(4x^2 + 9)$ (d) $[(5x + y) + 1]^2$

3. Suppose that x represents the larger of two consecutive integers,

   (a) write a polynomial that represents the product of these two integers.

   (b) Write a polynomial that represents the sum of the squares of the two integers and simplify your answer.

4. Find the quotient and remainder of the following:

   (a) $\dfrac{x^2-6x-7}{x-4}$ (b) $\dfrac{x^3-x^2-2x+5}{x-2}$

   (c) $\dfrac{x^6+x^4+x^2+1}{x^2+1}$ (d) $\dfrac{6x^3+2x^2+21x}{2x^2+5}$

5. Use the Remainder Theorem to find the remainder when $f(x)=5x^2-3x+2$ is divided by $x-1$.

6. Find the remainder when $f(x)=-4x^2-6x+7$ is divided by $x+4$.

7. What is the remainder when $6x^{1000}-7x^{562}+12x+25$ is divided by $x+1$?

8. The expression $6x^2-2x+3$ leaves a remainder of 3 when divided by $x-k$. Determine the values of k.

9. The polynomial $8x^3+px^2+qx-9$ has a remainder of -95 when divided by $x+2$ and a remainder of 3 when divided by $2x-3$. Find the values of p and q.

10. Factorize the following polynomials completely.

    (a) $6x^4-12x^3+18x^2$ (b) $x^2(2x-3)+4(2x-3)$

    (c) x^4-81 (d) $16x-x^5$

    (e) $x^3-2x^2+6x-12$ (f) $3x^3-2x^2-9x+6$

11. Factorize the following polynomials completely.

    (a) $10x^2(x+2)-7x(x+2)-6(x+2)$ (b) $x^4-3x^2y^2-4y^4$

    (c) $(x-y)^4-9(x-y)^2$ *(d) $x^2-10x+25-49y^2$

    *(e) $2x^2-12x+18-2y^2$

12. Use the Factor Theorem to decide if $x-2$ is a factor of $f(x)=x^3+3x^2-4x-12$.

13. Determine if $2x-1$ is a factor of $f(x)=2x^4-11x^3+9x^2+7x$.

14. Solve the equation $x^3-2x^2-5x+6=0$ given that 3 is a zero of $f(x)=x^3-2x^2-5x+6$.

15. Solve the equation $4x^3+4x^2-5x-3=0$ given that $-\frac{1}{2}$ is a zero of $f(x)=4x^3+4x^2-5x-3$.

    Verify your solution by plotting the graph of $y=4x^3+4x^2-5x-3$ in www.desmos.com.

16. Find a polynomial of degree 3 which has integer coefficients and roots $-\frac{2}{3}, \frac{1}{2}$ and 4.

17. If x - 1 and x – 2 are factors of $x^3 + px^2 + qx$ -6, determine the values of p and q.

18. The polynomial x^3+px^2+qx+3 is exactly divisible by $x+3$ but it has a remainder of 91 when divided by $x-4$. What is the remainder when it is divided by $x+2$?

19. Let $f(x)=x^3+mx^2+5x-n$ and $g(x)=x^3-x^2-(m+2)x+n$ where m and n are constants. It is given that $x+3$ is a common factor of $f(x)$ and $g(x)$.
    (a) Find the values of m and n,
    (b) Factorise $g(x)$ and hence solve the equation $g(x)=0$.

*20. The width of a rectangular box is three times its height. Its length is 11cm more than the height. Determine the dimensions of the box if its volume is 720cm^3.

ANSWERS

Multiple Choice Questions

1.D   2.D

Written solutions

1. (a) $12x^3+4x^2+12x-12$ (b) $9x^4-4x^2$
   (c) $4x^2+20x+25$ (d) $81-72x+16x^2$

2. (a) $24xy$ (b) $16x^2-9y^2+40x+25$
   (c) $16x^4-81$ (d) $25x^2+10xy+y^2+10x+2y+1$

3. (a) x^2-x (b) $2x^2-2x+1$

4. (a) $Q: x-2$, $R: -15$ (b) $Q: x^2+x$, $R: 5$
   (c) $Q: x^4+1$, $R: 0$ (d) $Q: 3x+1$, $R: 6x-5$

5. 4   6. -33

7. 12

8. 0, 1/3

9. $p = -6, q = -1$

10. (a) $6x^2(x^2 - 2x + 3)$

(b) $(2x - 3)(x^2 + 4)$

(c) $(x - 3)(x + 3)(x^2 + 9)$

(d) $x(4 + x^2)(2 + x)(2 - x)$

(e) $(x - 2)(x^2 + 6)$

(f) $(3x - 2)(x^2 - 3)$

11. (a) $(x + 2)(2x + 1)(5x - 6)$

(b) $(x^2 + y^2)(x + 2y)(x - 2y)$

(c) $(x - y)^2(x - y - 3)(x - y + 3)$

(d) $(x + 7y - 5)(x - 7y - 5)$

(e) $2(x - 3 + y)(x - 3 - y)$

12. $f(2) = 0$, $x - 2$ is a factor.

13. $f(1/2) \neq 0$, $2x - 1$ is not a factor.

14. x = -2, 1, 3

15. x = -1/2, -3/2, 1

16. $6x^3 - 23x^2 - 6x + 8$

17. p = -6, q = 11

18. 7

19. m = 6, n= 12, x= −3 or 2

20. Height = 4cm, Width = 12cm, Length = 15cm