MAT20306 - Advanced Statistics

Lecture 8: One way ANOVA

Biometris

Quantitative Methods brought to Life

A qualitative explanatory variable

?

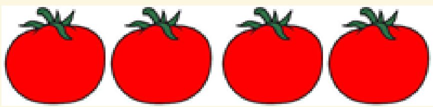
response variable y $\longleftarrow$ explanatory variable x

In regression y and x are both quantitative variables.

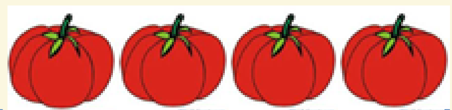
Now, suppose that y is quantitative, but x is qualitative.

Example: sweet taste of tomatoes

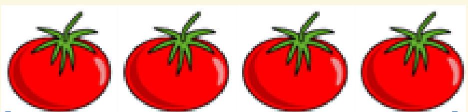
Does sweet taste differ between three types of tomato?



$n_1 = 4$



$n_2 = 4$



$n_3 = 4$

Three random samples from three populations: round, beef or cherry tomatoes.

Response y = sweet taste, as scored by judges (scale 0 to100).

Explanatory variable x = type of tomato (round, beef, or cherry).

Sweet taste of tomatoes, aim of experiment

	taste	type
1	25.44	r
2	28.10	r
3	46.46	r
4	36.96	r
5	24.83	b
6	28.47	b
7	48.15	b
8	31.78	b
9	53.42	c
10	70.87	c
11	57.07	c
12	38.08	c

Response y = taste

Explanatory variable x = type of tomato

Aim:
Inference about systematic differences in taste between the types of tomato.

Provide estimates, standard errors, tests, and confidence intervals.

In more general terms

t populations of units, with t random samples of sizes $n_1 \ldots n_t$.

- y_{ij} = response for j-th experimental unit receiving treatment *i*,
- $\bar{y}_{i\cdot}$ = mean response for units receiving treatment *i*
- n_i = number of experimental units receiving treatment *i*,
- $N = n_T = n_1 + n_2 + \ldots + n_t$ = total number of experimental units.

For the tomatoes: $t = 3, n_1 = 4, n_2 = 4$ and $n_3 = 4$.

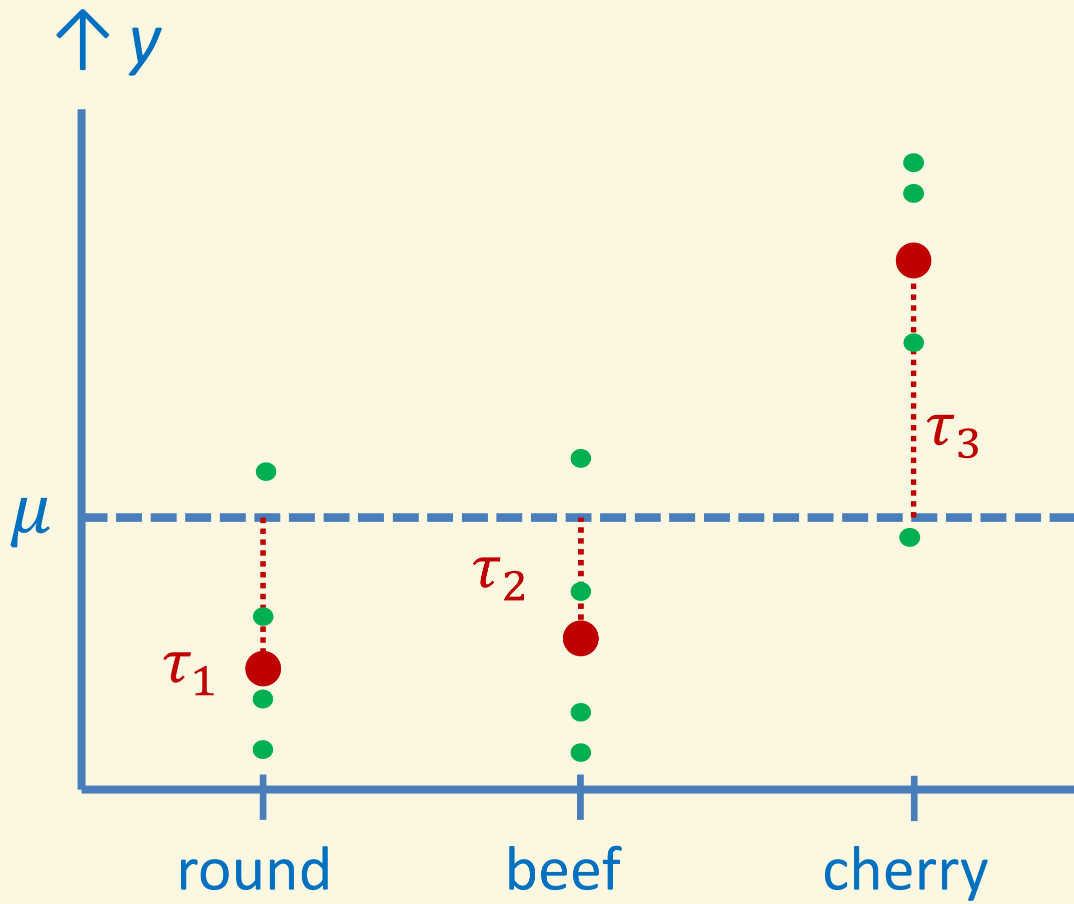
The (unknown) population means for y are $\mu_1 \ldots \mu_t$.
Are there differences between $\mu_1 \ldots \mu_t$?

If so, how large are these differences?
Provide tests, estimates, standard errors, confidence intervals for differences.

Building the model

$$y_{ij} = \mu_i + \varepsilon_{ij}$$

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}$$

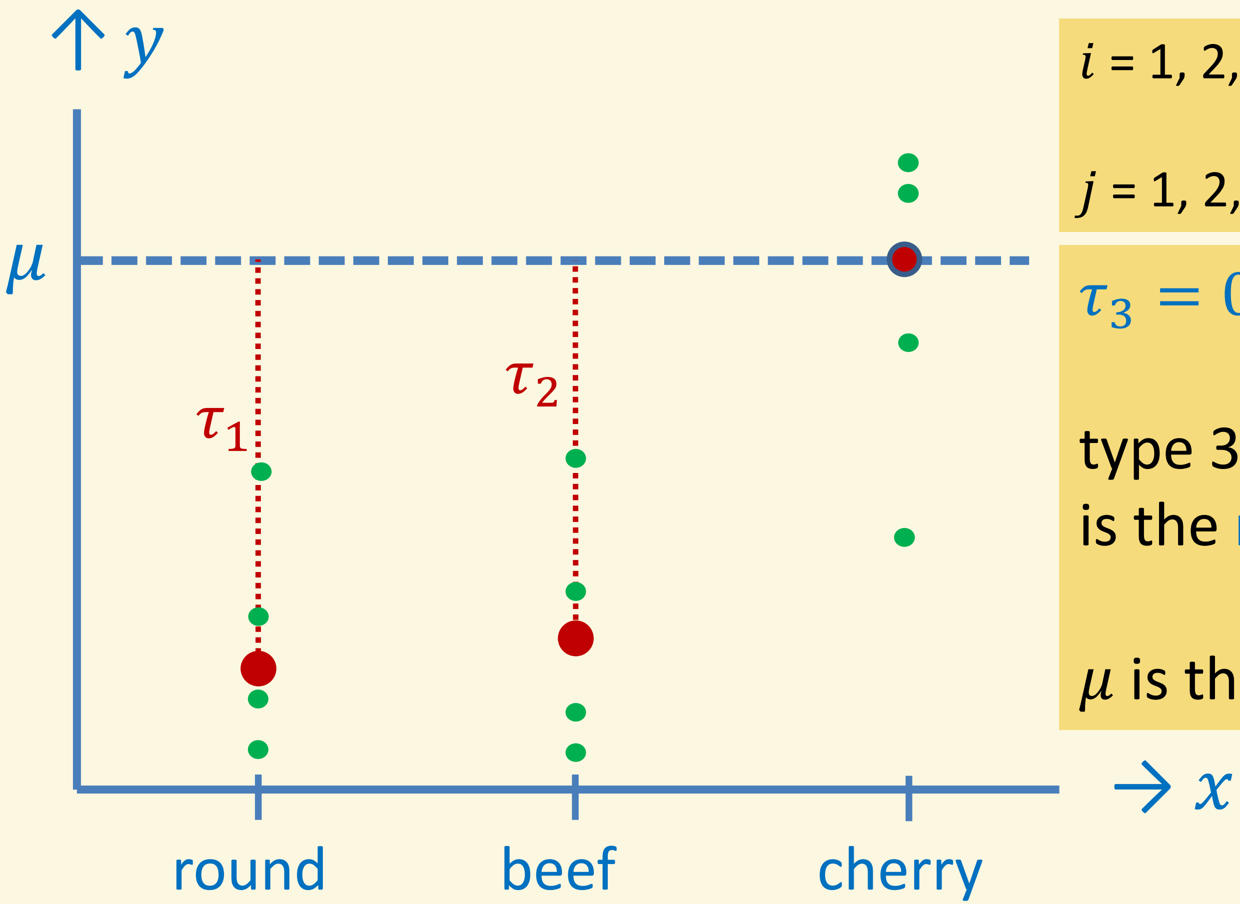


i = 1, 2, 3 for type

j = 1, 2, 3, 4 for tomato

Too many parameters!

Cornerstone representation



$$y_{ij} = \mu + \tau_i + \epsilon_{ij}$$

i = 1, 2, 3 for type

j = 1, 2, 3, 4 for tomato

$\tau_3 = 0$

type 3 (cherry tomato) is the reference

μ is the mean of type 3

Cornerstone representation: interpretation of model parameters

$y_{ij} = \mu + \tau_i + \epsilon_{ij}$

i = 1, 2, 3 for type
j = 1, 2, 3, 4 for tomato

$\tau_3 = 0$

type 3 (cherry tomato)
is the reference

type 1 and 2 relative to 3

$\mu_1 = \mu + \tau_1$	mean type 1
$\mu_2 = \mu + \tau_2$	mean type 2
$\mu_3 = \mu$	mean type 3
$\mu = \mu_3$	reference level
$\tau_1 = \mu_1 - \mu_3$	type 1 versus 3
$\tau_2 = \mu_2 - \mu_3$	type 2 versus 3

Model assumptions about error terms ϵ

Same assumptions as in regression:

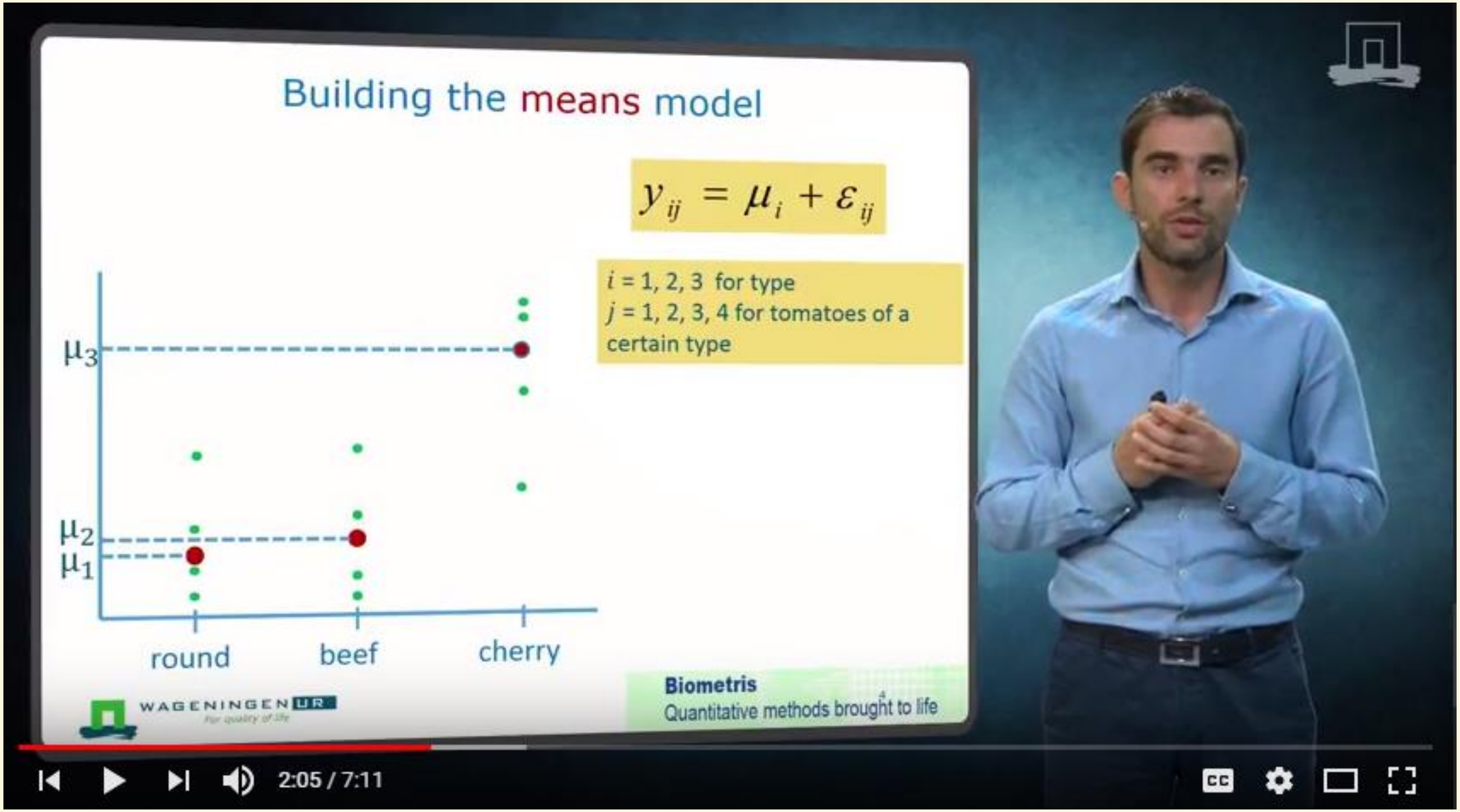
- ϵ's are mutually independent,
  i.e. we need random samples from the populations

- ϵ's are normally distributed around 0 with constant variance σ_ϵ^2

For the tomato example there are four parameters in the model:

μ, τ_1, τ_2 and σ_ϵ^2 (assuming that cherry is the reference, i.e. $\tau_3 = 0$).

Building the model



Least squares estimation

Formally, again we can minimize the sum of squares

$$SS = \sum_{i=1}^{t} \sum_{j=1}^{n_i} \left(y_{ij} - (\mu + \tau_i)\right)^2$$

We simply get sample means as estimators for μ's,

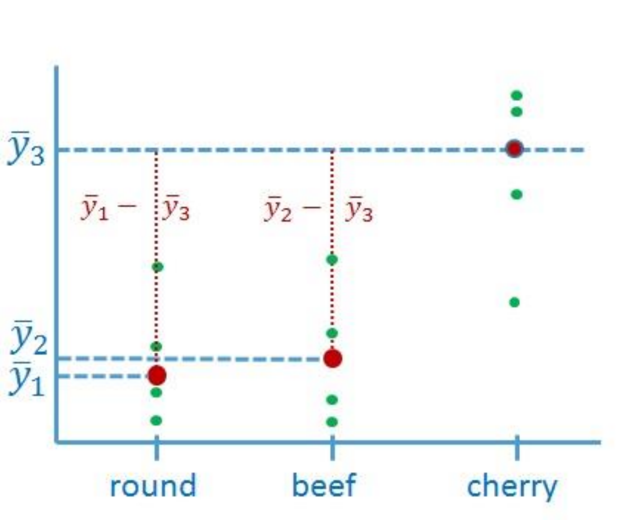
and differences between sample means as estimators for τ's.

E.g. $\hat{\mu}_1 = \bar{y}_{1.}$ $\hat{\tau}_1 = \hat{\mu}_1 - \hat{\mu}_3 = \bar{y}_{1.} - \bar{y}_{3.}$

Parameter estimates with the least squares method

SweeTaste

	N	Mean	Std. Deviation	Std. Error
Round	4	34,2400	9,51957	4,75978
Beef	4	33,3075	10,29405	5,14703
Cherry	4	54,8600	13,47648	6,73824
Total	12	40,8025	14,52907	4,19418



$$\hat{\mu}_1 = \bar{y}_{1.}$$

$$\hat{\tau}_1 = \hat{\mu}_1 - \hat{\mu}_3 = \bar{y}_{1.} - \bar{y}_{3.}$$

Parameter Estimates

Dependent Variable: taste

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	54,860	5,614	9,772	,000	42,160	67,560
[type=1]	-20,620	7,939	-2,597	,029	-38,580	-2,660
[type=2]	-21,553	7,939	-2,715	,024	-39,513	-3,592
[type=3]	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

Standard error of a mean / difference of means

To refresh your memory:

$$se(\hat{\mu}_1) = \sqrt{Var(\hat{\mu}_1)} = \sqrt{Var(\bar{y}_{1.})} = \sqrt{\widehat{\sigma}_\epsilon^2 / n_1}$$

- it shows how accurate the estimator is (the smaller, the better)

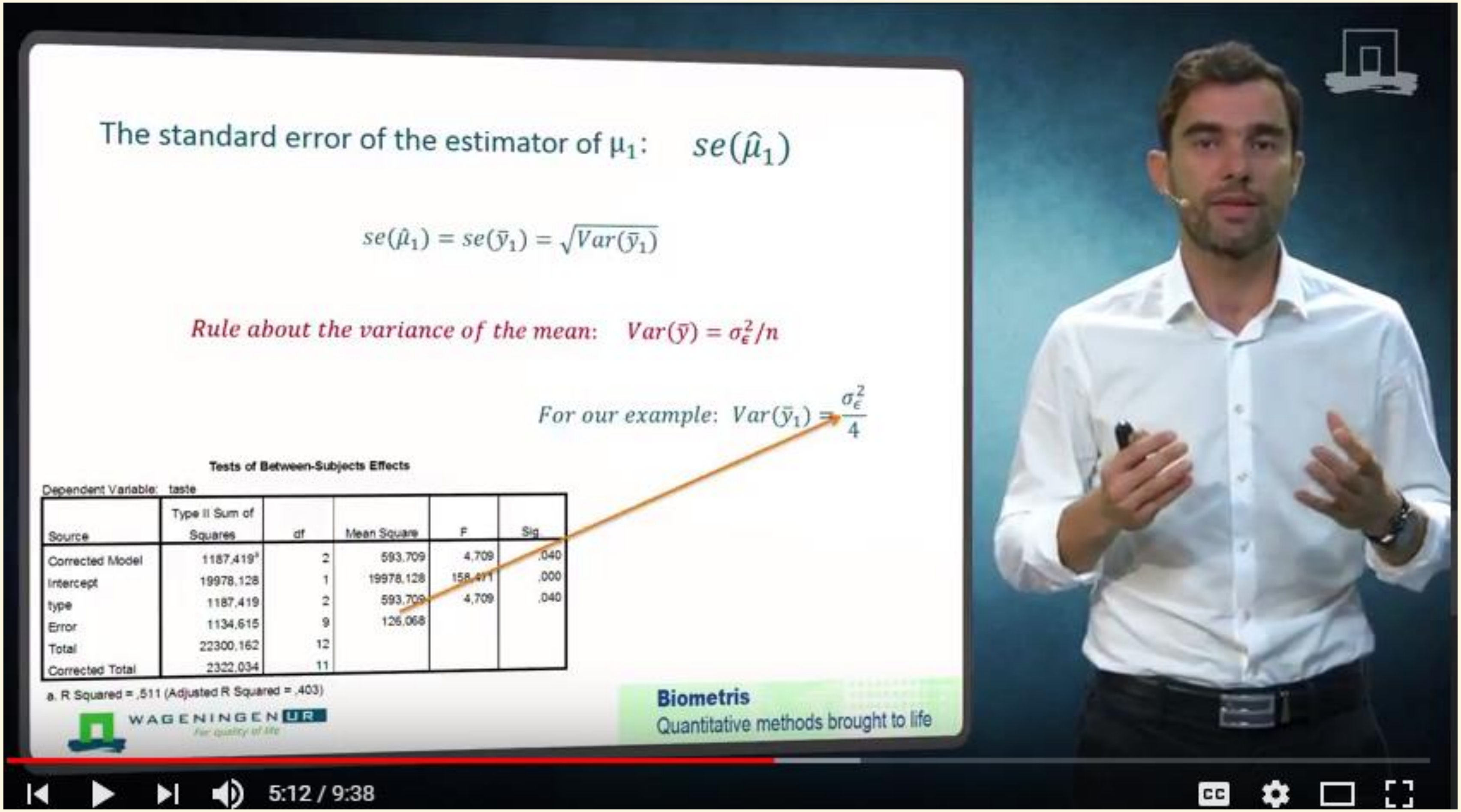
Consider for example $\tau_1 = \mu_1 - \mu_3$

$$se(\hat{\tau}_1) = \sqrt{Var(\hat{\tau}_1)} = \sqrt{Var(\bar{y}_{1.} - \bar{y}_{3.})} =$$

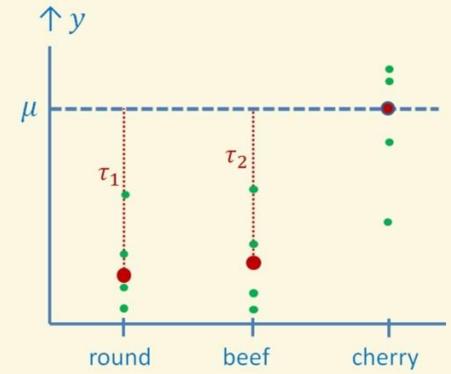
$$\sqrt{Var(\bar{y}_{1.}) + Var(\bar{y}_{3.})} = \sqrt{\frac{\widehat{\sigma}_\epsilon^2}{n_1} + \frac{\widehat{\sigma}_\epsilon^2}{n_3}}$$

For the tomato example $n_1 = 4, n_3 = 4$: $se(\hat{\tau}_1) = \sqrt{2\widehat{\sigma}_\epsilon^2 / 4}$.

Standard error of a mean / difference of means



Estimation of σ_ϵ^2



We estimate the variance from each sample:

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \left(y_{ij} - \bar{y}_{i.}\right)^2, i = 1 \ldots t.$$

σ_ϵ^2 is the same for all treatments (populations), so variances are pooled:

$$\hat{\sigma}_\epsilon^2 = \frac{(n_1 - 1)s_1^2 + \cdots + (n_t - 1)s_t^2}{(n_1 - 1) + \cdots + (n_t - 1)} =$$

$$\frac{1}{n_1 + \cdots + n_t - t} \sum_{i=1}^{t} \sum_{j=1}^{n_i} \left(y_{ij} - \bar{y}_{i.}\right)^2 = \frac{1}{N - t} \sum_{i=1}^{t} \sum_{j=1}^{n_i} e_{ij}^2 = MSE$$

where $N = n_1 + \cdots + n_t$ is the total sample size.
For the tomatoes $N = n_1 + n_2 + n_3 = 4 + 4 + 4 = 12$ and $t = 3$.

Map of one-way ANOVA

- an illustrative example
- the one-way ANOVA model
- **Sums of squares and ANOVA table**
- F-test in ANOVA table
- t-test and Fisher's LSD
- Experiment wise error control, multiple comparisons procedures

One-way ANOVA = regression with dummies

	taste	type	x_1	x_2
1	25.44	r	1	0
2	28.10	r	1	0
3	46.46	r	1	0
4	36.96	r	1	0
5	24.83	b	0	1
6	28.47	b	0	1
7	48.15	b	0	1
8	31.78	b	0	1
9	53.42	c	0	0
10	70.87	c	0	0
11	57.07	c	0	0
12	38.08	c	0	0

x_1 and x_2 are dummy variables for round and beef tomatoes.

No dummy variable for cherry tomatoes.

Multiple regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

is the same as one-way ANOVA.

Compare parameters in ANOVA and regression

		one-way ANOVA: mean taste	Regression: mean taste		
model	i	$\mu_i = \mu + \tau_i$	x_1	x_2	$\mu_y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$
round	1	$\mu_1 = \mu + \tau_1$	1	0	$\mu_1 = \beta_0 + \beta_1 * 1 + \beta_2 * 0 = \beta_0 + \beta_1$
beef	2	$\mu_2 = \mu + \tau_2$	0	1	$\mu_2 = \beta_0 + \beta_1 * 0 + \beta_2 * 1 = \beta_0 + \beta_2$
cherry	3	$\mu_3 = \mu + 0 = \mu$	0	0	$\mu_3 = \beta_0 + \beta_1 * 0 + \beta_2 * 0 = \beta_0$

- β_0 in regression same as μ in ANOVA = μ_3= mean taste for cherry
- β_1 in regression same as τ_1 in ANOVA = $\mu_1 - \mu_3$
- β_2 in regression same as τ_2 in ANOVA = $\mu_2 - \mu_3$

Sums of squares in ANOVA table

One-way ANOVA can be fitted by regression, so again we have sums of squares *SSR* / *SSB*, *SSE* / *SSW*, *SST*.

But *SSR* will now be called *SSTreat*.

Again, *SS* relate to response y, systematic and random parts:

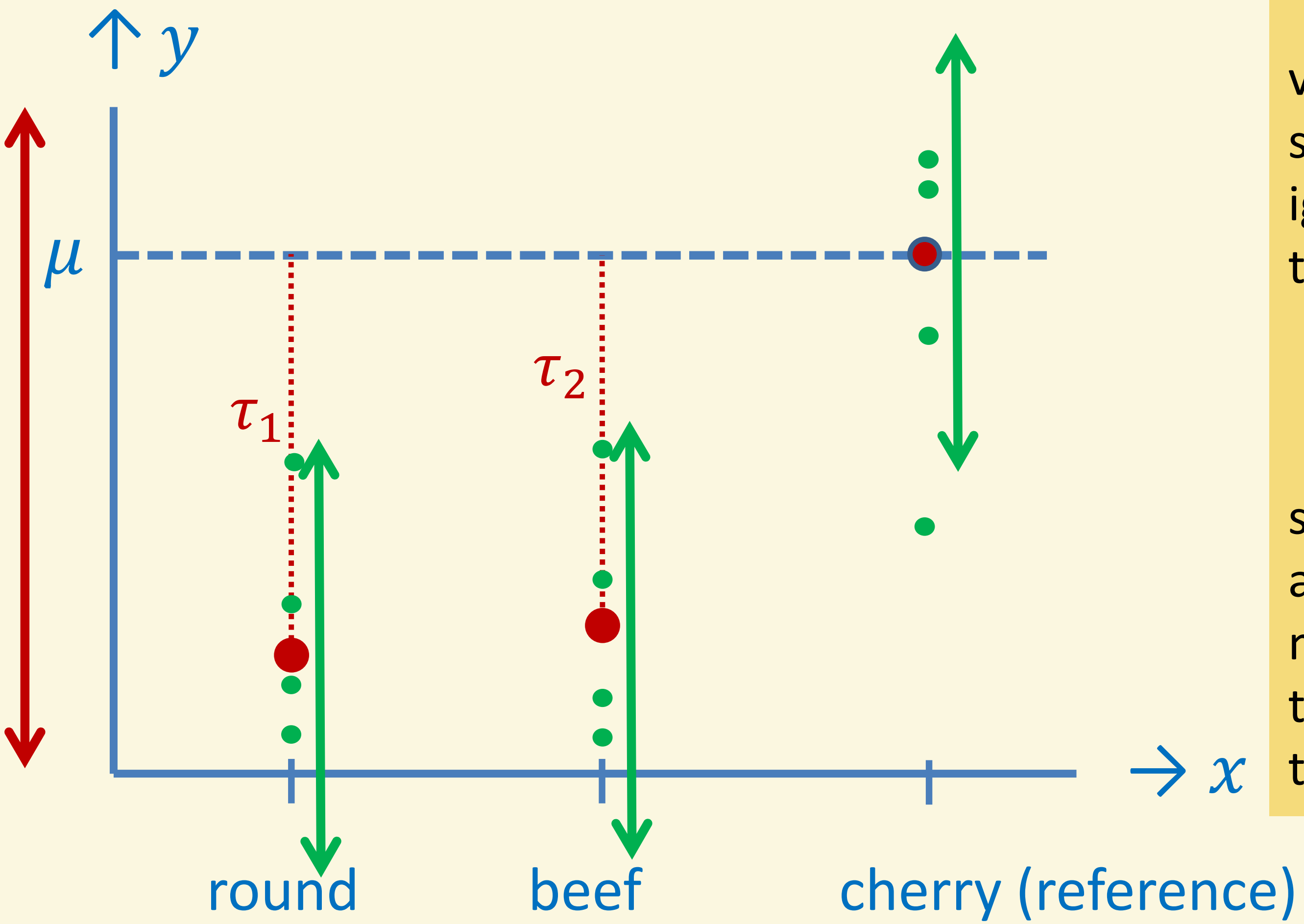
$$y = \mu + \tau + \epsilon$$

total sum of squares *SST*

treatment sum of squares *SSTreat*

error (or residual) sum of squares *SSE*

Variation around overall mean & separate means



variation around overall sample mean of y ignoring types of tomato, relates to *SST*

smaller variation around separate means, i.e. within each type of tomato, relates to *SSE / SSW*

ANOVA table in SPSS / R, tomato example

Tests of Between-Subjects Effects

Dependent Variable: taste

Source	Type II Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1187,419[a]	2	593,709	4,709	,040
Intercept	19978,128	1	19978,128	158,471	,000
type	1187,419	2	593,709	4,709	,040
Error	1134,615	9	126,068		
Total	22300,162	12			
Corrected Total	2322,034	11			

a. R Squared = ,511 (Adjusted R Squared = ,403)

R output

```
Response:taste
          Df  Sum Sq  Mean Sq  F value Pr(>F)
tomtype    2  1187.50  593.75  4.7114  0.03981
Residuals 9  1134.21  126.02
```

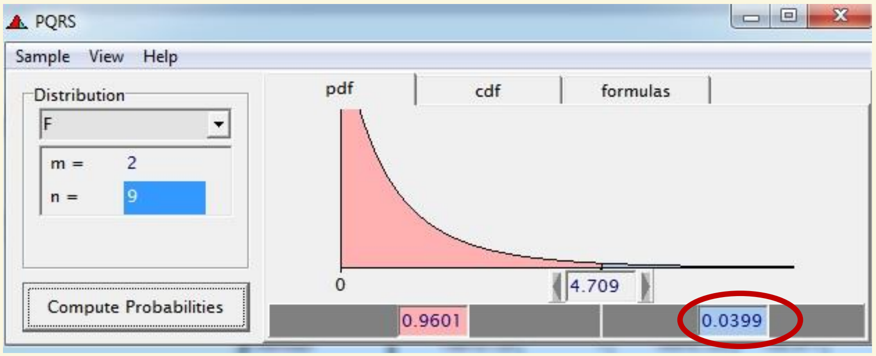
F-test for tomato example

- F-test is for predictive value of the model, $H_0: \beta_1 = \beta_2 = 0$.
- β_1, β_2 in regression are the same as τ_1, τ_2 in one-way ANOVA
- So, $H_0: \beta_1 = \beta_2 = 0$ is the same as $H_0: \tau_1 = \tau_2 = 0$.
- $H_0: \tau_1 = \tau_2 = 0$ is the same as $H_0: \mu_1 = \mu_2 = \mu_3$.

Tests whether the three types of tomato have the same expected sweet taste or not.

TS: $F = MSR/MSE$ becomes $F = MSTreat/MSE$.

F-test for tomato example, P-value



Tests of Between-Subjects Effects

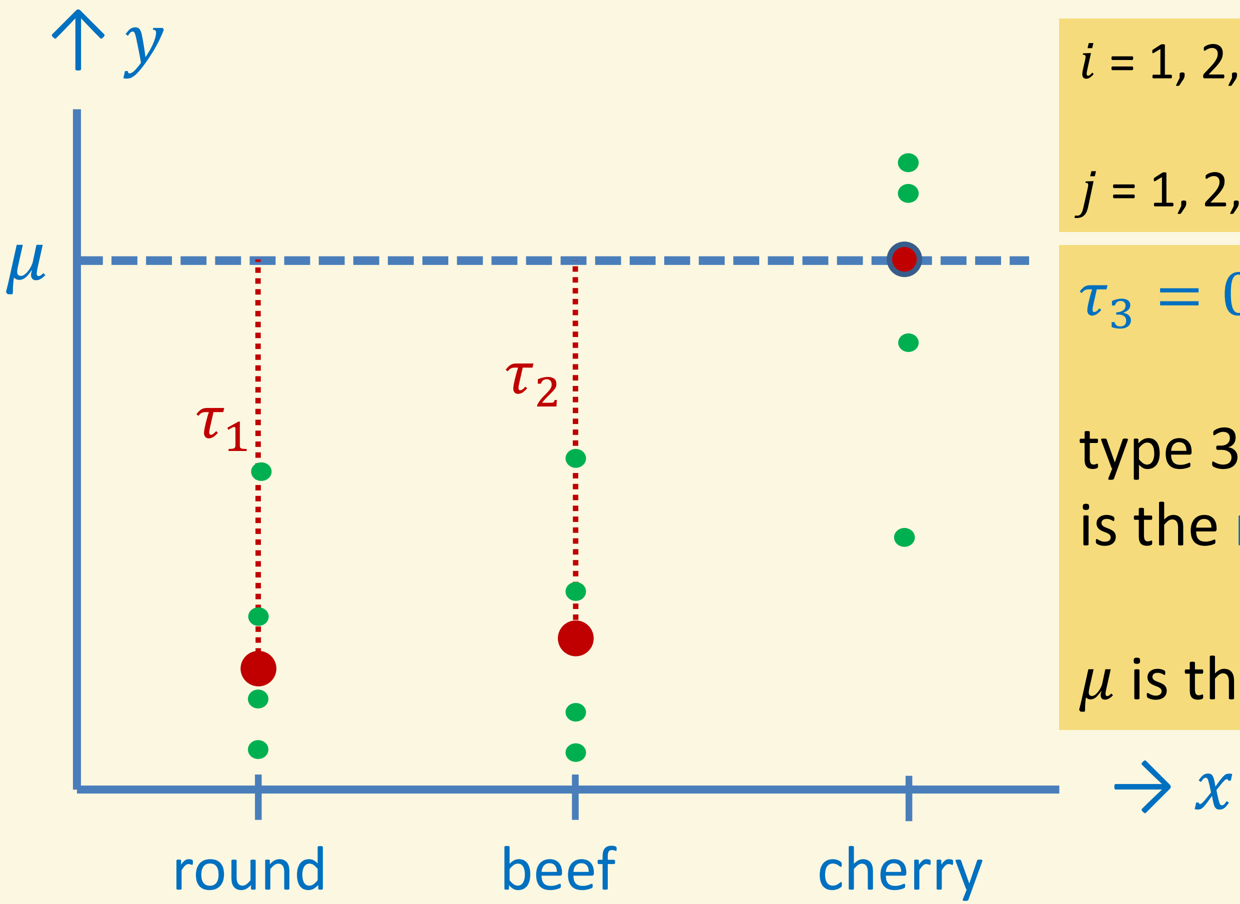
Dependent Variable: taste

Source	Type II Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1187,419[a]	2	593,709	4,709	,040
Intercept	19978,128	1	19978,128	158,471	,000
type	1187,419	2	593,709	4,709	,040
Error	1134,615	9	126,068		
Total	22300,162	12			
Corrected Total	2322,034	11			

a. R Squared = ,511 (Adjusted R Squared = ,403)

Under H_0 F follows F-distribution, df1 = 2, df2 = 12 - 1 - 2 = 9.
Right rejection region.
Outcome F = 4.709.
P-value = area to the right of 4.709, is 0.0399, is below 0.05.

H_0 is rejected: we have enough evidence that at least two of the types of tomato have different expected taste.



$$y_{ij} = \mu + \tau_i + \epsilon_{ij}$$

i = 1, 2, 3 for type

j = 1, 2, 3, 4 for tomato

$\tau_3 = 0$

type 3 (cherry tomato) is the reference

μ is the mean of type 3

What next?

F-test shows (P-value = 0.04 < 0.05) differences among the three population means for taste of the types of tomato.

Are all means different, or are two means the same and is the third different from the other two?

To find out more, we perform pairwise comparisons between the three sample means.

Map of one-way ANOVA

- an illustrative example
- the one-way ANOVA model
- Sums of squares and ANOVA table
- F-test in ANOVA table
- t-test and Fisher's LSD
- Experiment wise error control, multiple comparisons procedures

Pairwise comparison by t-test, tomato example

For example, round vs. cherry : $H_0\!: \tau_1 = 0$ vs. $H_a\!: \tau_1 \neq 0$

$$t = \hat{\tau}_1 / se(\hat{\tau}_1) = (\bar{y}_{1.} - \bar{y}_{3.}) / se(\bar{y}_{1.} - \bar{y}_{3.}) =$$

$$(\bar{y}_{1.} - \bar{y}_{3.}) / \sqrt{\frac{\hat{\sigma}_\epsilon^2}{n_1} + \frac{\hat{\sigma}_\epsilon^2}{n_2}} = (\bar{y}_{1.} - \bar{y}_{3.}) / \sqrt{\frac{2\hat{\sigma}_\epsilon^2}{4}} =$$

$$(\bar{y}_{1.} - \bar{y}_{3.}) / \sqrt{\frac{2 * MSE}{4}}$$

Refer to t-distribution, 9 degrees of freedom (from *SSE*).

t-test, tomato example, continued

Comparisons round versus cherry ($\mu_1 - \mu_3$) and beef versus cherry ($\mu_2 - \mu_3$) are in the output from SPSS / R below.

Parameter Estimates

Dependent Variable: taste

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	54,860	5,614	9,772	,000	42,160	67,560
[type=1]	-20,620	7,939	-2,597	,029	-38,580	-2,660
[type=2]	-21,553	7,939	-2,715	,024	-39,513	-3,592
[type=3]	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

```
Coefficients:
              Estimate   Std. Error   t value     Pr(>|t|)
(Intercept)     54.861        5.613     9.774     4.33e-06
tomtype2r      -20.619        7.938    -2.598     0.0289
tomtype2b      -21.555        7.938    -2.715     0.0238
```

Both pairwise comparisons are significant ($\alpha = 0.05$):
P-values are 0.0289 and 0.0238.

We conclude that expected taste for cherry tomatoes is higher than for round or beef tomatoes.

Missing comparison: $\mu_1 - \mu_2 = \tau_1 - \tau_2$ round vs. beef tomatoes !!!

Pairwise comparisons in SPSS, tomato example

Parameter Estimates

Dependent Variable: taste

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Intercept	54,860	5,614	9,772	,000	42,160	67,560
[type=1]	-20,620	7,939	-2,597	,029	-38,580	-2,660
[type=2]	-21,553	7,939	-2,715	,024	-39,513	-3,592
[type=3]	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

Multiple Comparisons

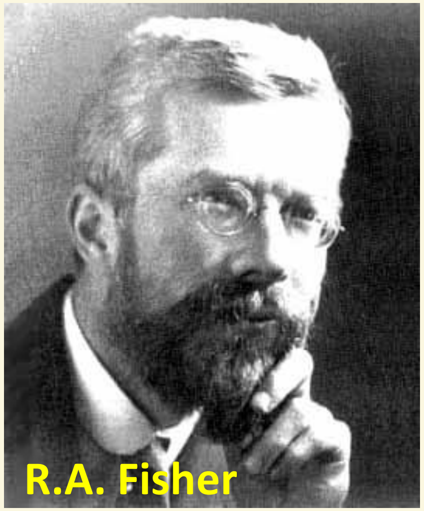
Dependent Variable: taste

LSD

(I) type	(J) type	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
round	beef	,93250	7,93941	,909	-17,0277	18,8927
	cherry	-20,62000*	7,93941	,029	-38,5802	-2,6598
beef	round	-,93250	7,93941	,909	-18,8927	17,0277
	cherry	-21,55250*	7,93941	,024	-39,5127	-3,5923
cherry	round	20,62000*	7,93941	,029	2,6598	38,5802
	beef	21,55250*	7,93941	,024	3,5923	39,5127

*. The mean difference is significant at the 0.05 level.

Fisher's LSD method - 1



A difference, e.g. round vs. cherry, is significant when

$$| \ (\hat{\tau}_1 / se(\hat{\tau}_1)) \ | > t_{dfE}$$

t_{dfE} from t-distribution (e.g. $\alpha/2$ = 0.025 to the right, df = 9 from *SSE*)

Same as:

$$| \ (\bar{y}_{1.} - \bar{y}_{3.}) \ | > t_{dfE} \ se(\hat{\tau}_1)$$

$t_{dfE} \ se(\hat{\tau}_1)$ is called the least significant difference (*LSD*).

Two means that differ more than the *LSD* are significantly different.

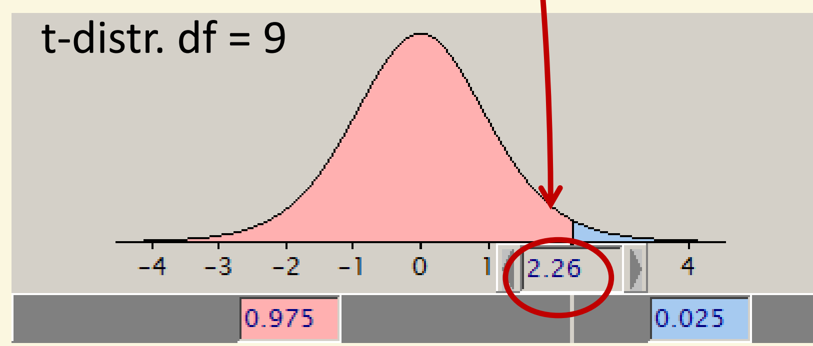
Fisher’s LSD method - 2

For equal sample sizes, say *n*, same *LSD* for all pairs:

$$LSD = t_{dfE}\, se(\hat{\tau}_1) = t_{dfE}\sqrt{2MSE/n}.$$

For tomato example: *n* = 4, *MSE* = 126.02 and t_{dfE}= 2.26

LSD =2.26 $\sqrt{2 * 126.02/4}$ = 17.94



Two means more than 17.94 apart

are significantly ($\alpha = 0.05$) different.

Fisher's F-protected LSD

When pairwise comparisons by the LSD method are only performed after a significant result with the F-test, the LSD method is F-protected.

Although F-protected LSD is used quite often, its theoretical basis is somewhat weak.

Notation of significant differences, tomato example

type	means	
ro	34.24	a
be	33.31	a
ch	54.86	b

LSD = 17.94

Common letter implies means are not significantly different.

type

ro	be	ch
34.24^a	33.31^a	54.86^b

Common superscript implies means are not significantly different.

type

be	ro	ch
33.31	34.24	54.86

Common underline implies means are not significantly different (means in increasing order).

One way ANOVA –SPSS output

Parameter Estimates

Dependent Variable: taste

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Intercept	54,860	5,614	9,772	,000	42,160	67,560
[type=1]	-20,620	7,939	-2,597	,029	-38,580	-2,660
[type=2]	-21,553	7,939	-2,715	,024	-39,513	-3,592
[type=3]	0[a]	.	.	.	.	.

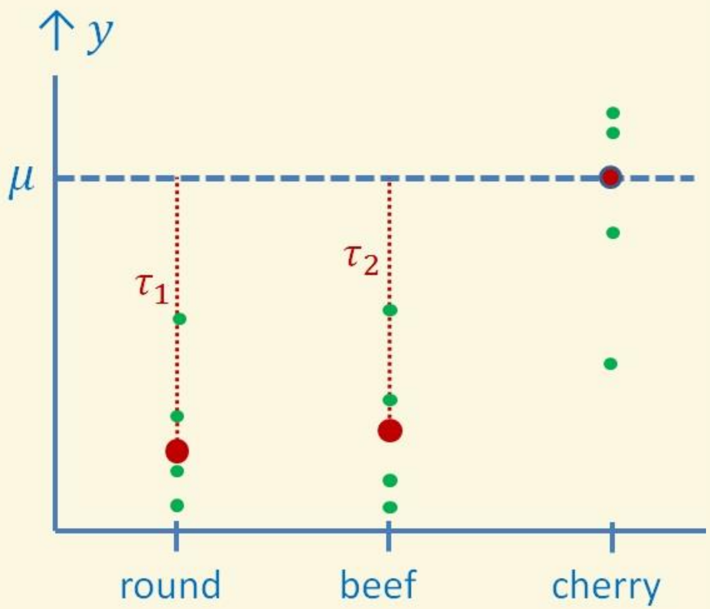
a. This parameter is set to zero because it is redundant.

Tests of Between-Subjects Effects

Dependent Variable: taste

Source	Type II Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1187,419[a]	2	593,709	4,709	,040
Intercept	19978,128	1	19978,128	158,471	,000
type	1187,419	2	593,709	4,709	,040
Error	1134,615	9	126,068		
Total	22300,162	12			
Corrected Total	2322,034	11			

a. R Squared = ,511 (Adjusted R Squared = ,403)



Multiple Comparisons

Dependent Variable: taste

LSD

(I) type	(J) type	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
round	beef	,93250	7,93941	,909	-17,0277	18,8927
	cherry	-20,62000*	7,93941	,029	-38,5802	-2,6598
beef	round	-,93250	7,93941	,909	-18,8927	17,0277
	cherry	-21,55250*	7,93941	,024	-39,5127	-3,5923
cherry	round	20,62000*	7,93941	,029	2,6598	38,5802
	beef	21,55250*	7,93941	,024	3,5923	39,5127

*. The mean difference is significant at the 0.05 level.

Map of one-way ANOVA

an illustrative example

the one-way ANOVA model

Sums of squares and ANOVA table

F-test in ANOVA table

t-test and Fisher's LSD

Experiment wise error control, multiple comparisons procedures

Experimentwise error rate - 1

- Suppose that $\mu_1 = \mu_2 = \mu_3$.
- With each pairwise comparison we have a probability α that we wrongly decide that two means are different.
- The total probability that at least once we wrongly decide that two means are different is called the experimentwise error rate.
- This experimentwise error rate α_{exp} will be larger than α.
- The more pairwise comparisons we make, e.g. with $t = 10$ treatments we have 10 * 9 / 2 = 45 pairs, the larger α_{exp} will tend to be.
- This has worried people in the past.

Experimentwise error - 2

Many methods have been proposed to keep the experimentwise error rate small, e.g. $\alpha_{exp} \leq 0.05$.

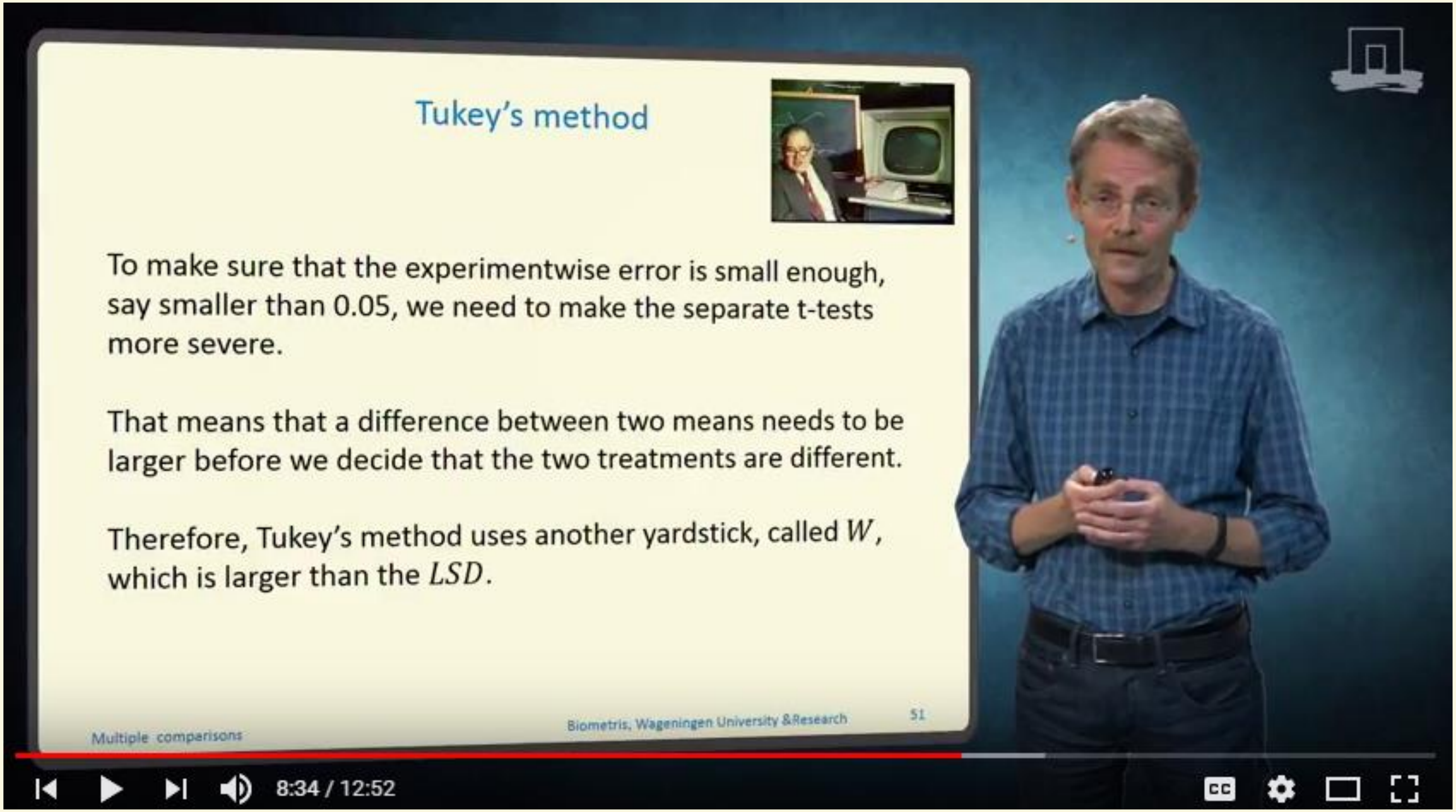
LSD method, even with F-protection, offers no guarantee that $\alpha_{exp} \leq 0.05$.

We discuss one method that do guarantee that $\alpha_{exp} \leq 0.05$:

- Tukey's method method of choice for all pairwise comparisons

This method uses a "yardstick" larger than the *LSD*.

Standard error of a mean / difference of means



Example: control of weeds

Response y = yield of hay (tons per acre)

Five treatments ($t = 5$): a control and four agents to control weeds:

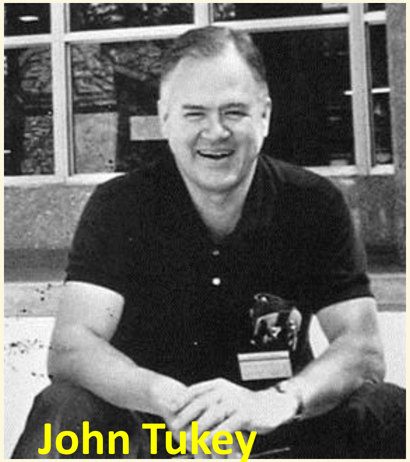
- 1 = control
- 2, 3 = biological agents
- 4, 5 = chemical agents

N = 30 plots (total number of observations)
n = 6 plots (sample sizes) randomly assigned to each treatment

ANOVA table

	SS	df	MS	F	P-value
Treatment	0.3648	4	0.0912	5.96	0.0026
Error	0.3825	25	0.0153		
Total	0.7472	29			

Tukey's procedure



Two treatments differ significantly, when the difference between their sample means exceeds yardstick W.

For treatments 1 and 2 one needs:

$$|(\bar{y}_{1.} - \bar{y}_{2.})| > W,$$

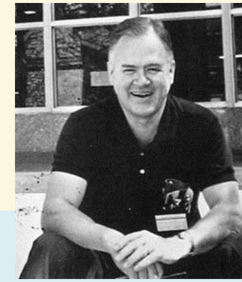
$$where \quad W = q\sqrt{\frac{s_\varepsilon^2}{n}}.$$

q depends on number of treatments (t), df from *SSE* and desired α_{exp}.

q from the so-called studentized range distribution (table 10 in O&L).

For unequal sample sizes: replace $1/n$ by harmonic mean $(\frac{1}{n_1} + \frac{1}{n_2})$ / 2 (approximate Tukey-Kramer method).

Tukey versus Fisher



Rank sample means from low to high:

Treatment	1	2	3	4	5
Sample mean	1.175	1.293	1.328	1.415	1.500

Tukey:
table 10, O&L, t = 5, df = 25 (from SSE), α = 0.05: $q \approx 4.17$
W = 4.17 * √(0.0153 / 6) = 0.21.

Tukey: 1 2 3 4 5

Fisher:
LSD = 2.060 * √(2*0.0153) / 6) = 0.15.

Fisher: 1 2 3 4 5

Note that $W > LSD$; LSD method (even with F-protection) does not offer full experimentwise error protection.

Tukey: table 10, O&L , $t = 5$, df = 25 (from *SSE*), $\alpha = 0.05$: $q \approx 4.17$

$W = 4.17 * \sqrt{(0.0153 / 6)} = 0.21$.

Fisher: table 2 $LSD = 2.060 * \sqrt{(2*0.0153) / 6)} = 0.15$.

Error df	α	t = Number of Treatment Means 2	3	4	5	6	7	8
5	.05	3.64	4.60	5.22	5.67	6.03	6.33	6.58
	.01	5.70	6.98	7.80	8.42	8.91	9.32	9.67
6	.05	3.46	4.34	4.90	5.30	5.63	5.90	6.12
	.01	5.24	6.33	7.03	7.56	7.97	8.32	8.61
7	.05	3.34	4.16	4.68	5.06	5.36	5.61	5.82
	.01	4.95	5.92	6.54	7.00	7.37	7.68	7.94
8	.05	3.26	4.04	4.53	4.89	5.17	5.40	5.60
	.01	4.75	5.64	6.20	6.62	6.96	7.24	7.47
9	.05	3.20	3.95	4.41	4.76	5.02	5.24	5.43
	.01	4.60	5.43	5.96	6.35	6.66	6.91	7.13
10	.05	3.15	3.88	4.33	4.65	4.91	5.12	5.30
	.01	4.48	5.27	5.77	6.14	6.43	6.67	6.87
11	.05	3.11	3.82	4.26	4.57	4.82	5.03	5.20
	.01	4.39	5.15	5.62	5.97	6.25	6.48	6.67
12	.05	3.08	3.77	4.20	4.51	4.75	4.95	5.12
	.01	4.32	5.05	5.50	5.84	6.10	6.32	6.51
13	.05	3.06	3.73	4.15	4.45	4.69	4.88	5.05
	.01	4.26	4.96	5.40	5.73	5.98	6.19	6.37
14	.05	3.03	3.70	4.11	4.41	4.64	4.83	4.99
	.01	4.21	4.89	5.32	5.63	5.88	6.08	6.26
15	.05	3.01	3.67	4.08	4.37	4.59	4.78	4.94
	.01	4.17	4.84	5.25	5.56	5.80	5.99	6.16
16	.05	3.00	3.65	4.05	4.33	4.56	4.74	4.90
	.01	4.13	4.79	5.19	5.49	5.72	5.92	6.08
17	.05	2.98	3.63	4.02	4.30	4.52	4.70	4.86
	.01	4.10	4.74	5.14	5.43	5.66	5.85	6.01
18	.05	2.97	3.61	4.00	4.28	4.49	4.67	4.82
	.01	4.07	4.70	5.09	5.38	5.60	5.79	5.94
19	.05	2.96	3.59	3.98	4.25	4.47	4.65	4.79
	.01	4.05	4.67	5.05	5.33	5.55	5.73	5.89
20	.05	2.95	3.58	3.96	4.23	4.45	4.62	4.77
	.01	4.02	4.64	5.02	5.29	5.51	5.69	5.84
24	.05	2.92	3.53	3.90	4.17	4.37	4.54	4.68
	.01	3.96	4.55	4.91	5.17	5.37	5.54	5.69

df	Right-Tail Probability (α) .40	.25	.10	.05	.025
1	.325	1.000	3.078	6.314	12.706
2	.289	.816	1.886	2.920	4.303
3	.277	.765	1.638	2.353	3.182
4	.271	.741	1.533	2.132	2.776
5	.267	.727	1.476	2.015	2.571
6	.265	.718	1.440	1.943	2.447
7	.263	.711	1.415	1.895	2.365
8	.262	.706	1.397	1.860	2.306
9	.261	.703	1.383	1.833	2.262
10	.260	.700	1.372	1.812	2.228
11	.260	.697	1.363	1.796	2.201
12	.259	.695	1.356	1.782	2.179
13	.259	.694	1.350	1.771	2.160
14	.258	.692	1.345	1.761	2.145
15	.258	.691	1.341	1.753	2.131
16	.258	.690	1.337	1.746	2.120
17	.257	.689	1.333	1.740	2.110
18	.257	.688	1.330	1.734	2.101
19	.257	.688	1.328	1.729	2.093
20	.257	.687	1.325	1.725	2.086
21	.257	.686	1.323	1.721	2.080
22	.256	.686	1.321	1.717	2.074
23	.256	.685	1.319	1.714	2.069
24	.256	.685	1.318	1.711	2.064
25	.256	.684	1.316	1.708	2.060