MAT20306 - Advanced Statistics

Lecture 9: Kruskal-Wallis and Factorial ANOVA

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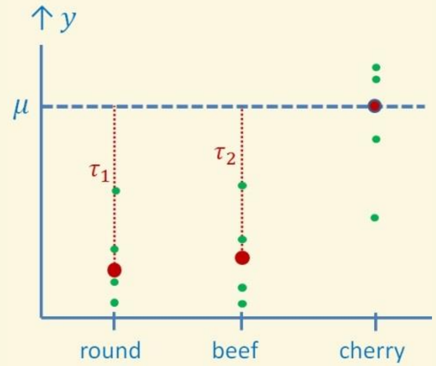
DETOUR

Kruskal-Wallis test (O&L section 8.6)

- Suppose that we want to compare t treatments, but the data do not seem to satisfy the assumption of normality.
- Similar to the case of two treatments (t = 2), we can replace observations by rank numbers.



- The null hypothesis and alternative hypothesis are:

  H_0: the t population distributions are the same,
  H_a: not all the t population distributions are the same.

- For t = 2 there is Wilcoxon's rank-sum test from week 1.

- Again, we can think of a shift alternative, i.e. all distributions are of the same form, but under H_a some are shifted relative to each other.

Kruskal-Wallis test, continued

- The response has to be continuous or at least ordinal.
- Observations are ranked over all samples from low (lowest rank) to high (highest rank), possibly including mid ranks, just like we did for Wilcoxon's rank-sum test.

- The classical test statistic can be written as in the book, but it is very complex. You only have to know the following form:

$$H = \frac{SSBranks}{TSSranks/(N-1)} \sim H_0 \sim \chi^2_{t-1}$$

- The R.R. is right sided. Table 7, 2nd page, has critical values.

- Kruskall-Wallis test is similar to the F-test applied to the rank numbers

- Indeed, we could first obtain the ranks in the data set, and then apply the F-test to these ranks.

Example, Clerics & mental illness (O&L section 8.3 / 8.6)

Three groups of clerics are tested on their knowledge about causes of mental illness. Response *y* *(non-Normal)* is score on written tests.

Ranks

	clerictype	N	Mean Rank
score	Methodist	10	18.25
	Catholic	10	16.75
	Pentecostal	10	11.50
	Total	30	

Test Statistics[a,b]

	score
Chi-Square	3.254
df	2
Asymp. Sig.	.197

a. Kruskal Wallis Test

b. Grouping Variable: clerictype

H_0: 3 identical "knowledge" distributions
T.S.: $H = SSB_{ranks}/(TSS_{ranks}/29)$.
Under H_0: $H \sim \chi_2^2$, approximately; use RPV
$H = 3.254$
RPV = $P_{H0}(H \geq 3.254) = 0.197$.
Conclusion: H_0 not rejected, H_a not proven.
It is not shown ($\alpha=0.05$) that there are systematic differences in knowledge about causes of mental illness between the clerics of the three religions.


END
DETOUR

Map of two-way ANOVA

tomato example

Two-way ANOVA model

Sums of squares

F-tests for interaction & main effects

Illustrative examples

Taste of tomatoes revisited

	taste	type	ripe
1	46.46	r	lo
2	36.96	r	lo
3	25.44	r	sh
4	28.10	r	sh
5	48.15	b	lo
6	31.78	b	lo
7	24.83	b	sh
8	28.47	b	sh
9	57.07	c	lo
10	70.87	c	lo
11	53.42	c	sh
12	38.08	c	sh

Response y = sweet taste

Two factors now:

Type of tomato:
beef, cherry or round tomatoes.

Ripening duration: long or short.

How to handle two factors together?

Taste of tomatoes revisited

	taste	type	ripe
1	46.46	r	lo
2	36.96	r	lo
3	25.44	r	sh
4	28.10	r	sh
5	48.15	b	lo
6	31.78	b	lo
7	24.83	b	sh
8	28.47	b	sh
9	57.07	c	lo
10	70.87	c	lo
11	53.42	c	sh
12	38.08	c	sh

six treatments:
combinations of three types and two durations

two tomatoes for each treatment

we could perform a one-way ANOVA, with six treatments

Exploring structure in treatments

We want to explore the structure in the 6 treatments.

For instance, is the effect of ripening the same for the three types of tomato?

If so, we can say something about ripening without having to bother about the type of tomato.

If not, we have to describe the effect of ripening for each type of tomato separately.

Interaction

When the effect of ripening depends upon the type of tomato, we say that there is interaction between the experimental factors for type and ripening.

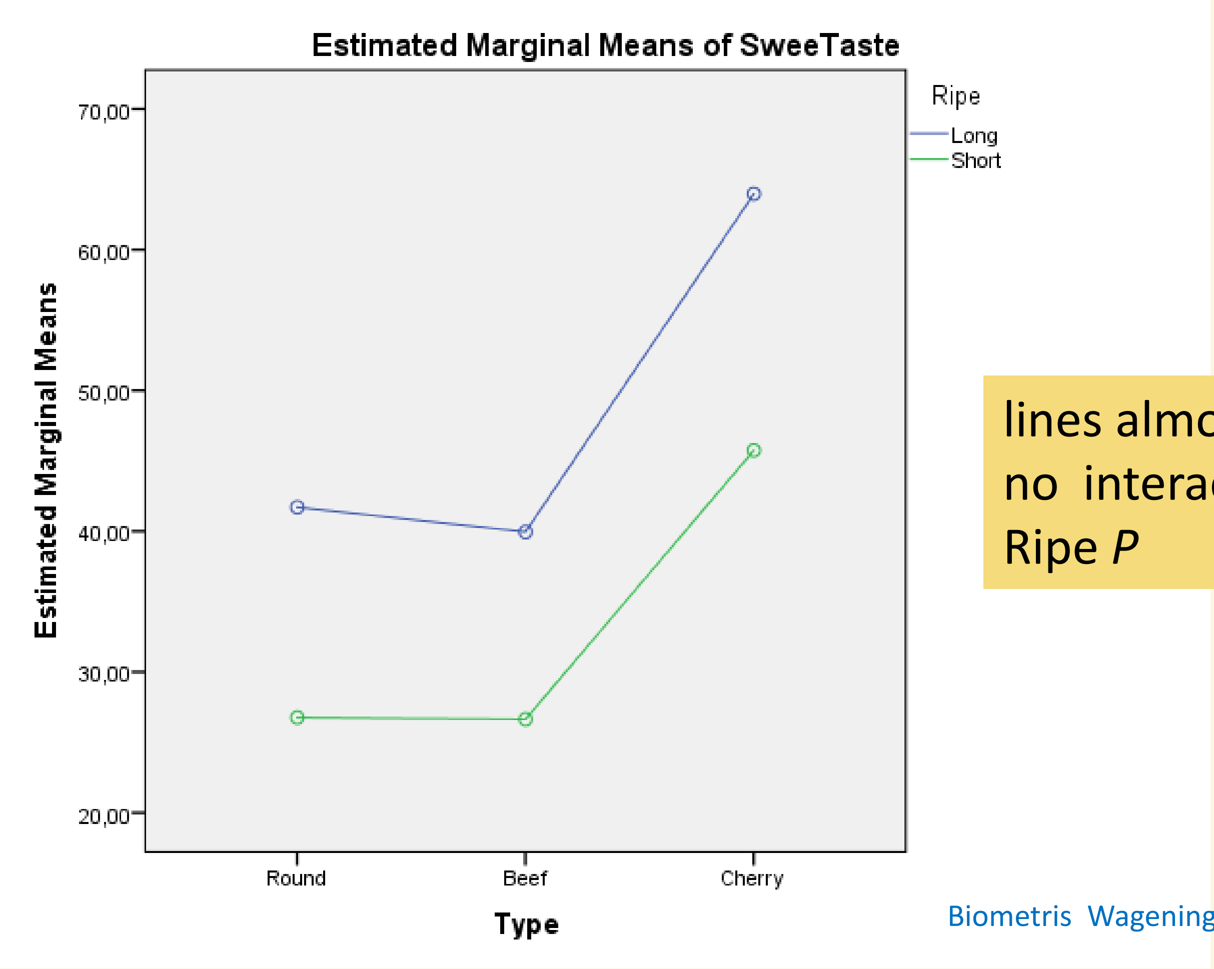
We want to know more about interaction.

The one-way ANOVA is of little use in this respect, because it does not explore the structure of the treatments.

Profile plot - Interaction

Profile plot is plot of means versus levels of factor A (or B), with means at common levels of factor B (or A)connected by lines.

Visual aid in explaining interaction.



lines almost parallel $\rightarrow$ suggests no interaction between Type *and* Ripe *P*

Map of two-way ANOVA

Illustrative example

Two-way ANOVA model

Sums of squares

F-tests for interaction & main effects

Illustrative examples

Two-way ANOVA model without interaction

Model without interaction is the Additive Model: $y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$

Easy and concise conclusions , e.g. for tomatoes same effect of duration can be reported for all types of tomato.

More accurate inference, e.g. for tomatoes long vs. short averaged over types, and not per type.

But the model has to be appropriate, otherwise conclusions can be misleading.

	taste	type	ripe
1	46.46	r	lo
2	36.96	r	lo
3	25.44	r	sh
4	28.10	r	sh
5	48.15	b	lo
6	31.78	b	lo
7	24.83	b	sh
8	28.47	b	sh
9	57.07	c	lo
10	70.87	c	lo
11	53.42	c	sh
12	38.08	c	sh

Cornerstone representation of two-way ANOVA

type	*ripe* lo	sh
r	41.71	26.77
b	39.96	26.65
c	63.97	45.75

Sample means $\bar{y}_{ij.}$ are NOT ALWAYS estimates for $\mu_{ij} = \mu + \tau_i + \beta_j$

What are estimates for separate parameters μ, τ_i, β_j ?

type	*Ripe* lo	sh
r	$\mu+\tau_1+\beta_1$	$\mu+\tau_1+\beta_2$
b	$\mu+\tau_2+\beta_1$	$\mu+\tau_2+\beta_2$
c	$\mu+\tau_3+\beta_1$	$\mu+\tau_3+\beta_2$

$\Rightarrow$

type	*Ripe* lo	Sh
r	$\mu+\tau_1+\beta_1$	$\mu+\tau_1$
b	$\mu+\tau_2+\beta_1$	$\mu+\tau_2$
c	$\mu+\beta_1$	μ

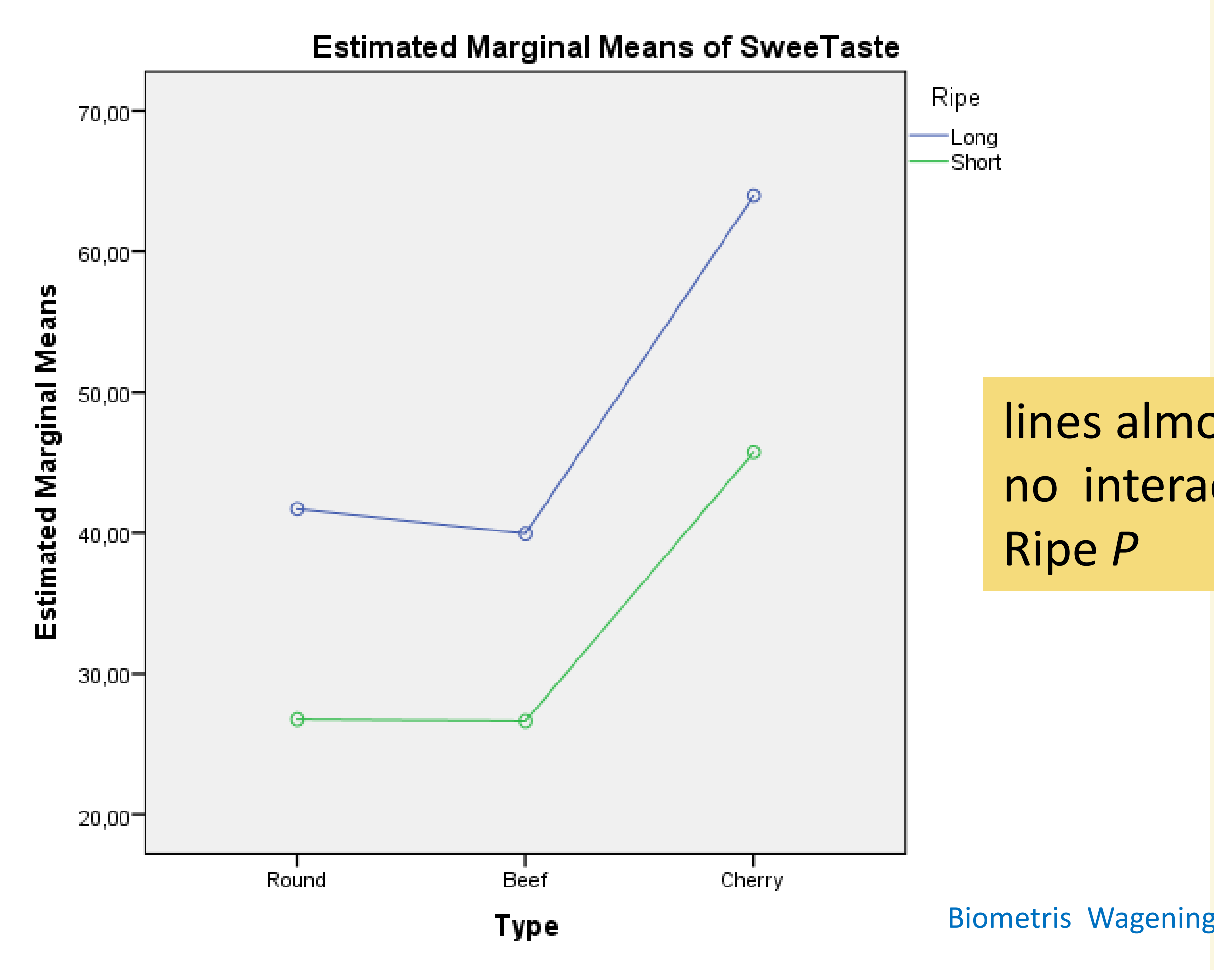
SPSS takes CHERRY and SHORT as a reference

- μ is the expected sweetness for type CHERRY and duration SHORT
- no need for τ_3, β_2, i.e. $\tau_3 = 0, \beta_2 = 0$.

Profile plot - Interaction

Profile plot is plot of means versus levels of factor A (or B), with means at common levels of factor B (or A)connected by lines.

Visual aid in explaining interaction.



lines almost parallel $\rightarrow$ suggests no interaction between Type *and* Ripe *P*

Two-way ANOVA model with interaction

Two ways of writing model with interaction:

Means Model: $y_{ijk} = \mu_{ij} + \epsilon_{ijk}$ (no factorial structure, one-way ANOVA)

Effects Model: $y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ijk}$ (factorial structure)

$\tau\beta_{ij}$ is an interaction term

correction on top of "additive" part $\mu + \tau_i + \beta_j$ with main effects

Effect of factor A may differ at different levels of factor B and vice versa

interaction is explicit in the effects model

Effects model with interaction, tomato example

Model: $y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ijk}$

	ripe	
type	lo	sh
r	41.71	26.77
b	39.96	26.65
c	63.97	45.75

	ripe	
type	lo	sh
r	$\mu+\tau_1+\beta_1+\tau\beta_{11}$	$\mu+\tau_1+\beta_2+\tau\beta_{12}$
b	$\mu+\tau_2+\beta_1+\tau\beta_{21}$	$\mu+\tau_2+\beta_2+\tau\beta_{22}$
c	$\mu+\tau_3+\beta_1+\tau\beta_{31}$	$\mu+\tau_3+\beta_2+\tau\beta_{32}$

Sample means $\bar{y}_{ij.}$ are estimates for $\mu_{ij} = \mu + \tau_i + \beta_j + \tau\beta_{ij}$

What are estimates for separate parameters μ, τ_i, β_j, $\tau\beta_{ij}$?

6 sample means, but 1 + 3 + 2 + 6 = 12 parameters!

Cornerstone representation of two-way ANOVA

type \ *ripe*	lo	sh
r	$\mu+\tau_1+\beta_1+\tau\beta_{11}$	$\mu+\tau_1+\beta_2+\tau\beta_{12}$
b	$\mu+\tau_2+\beta_1+\tau\beta_{21}$	$\mu+\tau_2+\beta_2+\tau\beta_{22}$
c	$\mu+\tau_3+\beta_1+\tau\beta_{31}$	$\mu+\tau_3+\beta_2+\tau\beta_{32}$

$\Rightarrow$

type \ *ripe*	lo	Sh
r	$\mu+\tau_1+\beta_1+\tau\beta_{11}$	$\mu+\tau_1$
b	$\mu+\tau_2+\beta_1+\tau\beta_{21}$	$\mu+\tau_2$
c	$\mu+\beta_1$	μ

SPSS takes CHERRY and SHORT as a reference

- μ is the population mean for type CHERRY and duration SHORT
- no need for τ_3, β_2, i.e. $\tau_3 = 0, \beta_2 = 0$.

- use interactions on top of main effects to get other two means
- no need for interactions in 3rd row & 2nd column, i.e.
  $\tau\beta_{31} = \tau\beta_{32} = \tau\beta_{22} = \tau\beta_{12} = 0$
- 6 parameters left: $\mu, \tau_1, \tau_2, \beta_1, \tau\beta_{11}, \tau\beta_{21}$

Example cornerstone representation in two-way ANOVA

type \ *ripe*	lo	sh
r	41.71	26.77
b	39.96	26.65
c	63.97	45.75

type \ *ripe*	lo	Sh
r	$\mu+\tau_1+\beta_1+\tau\beta_{11}$	$\mu+\tau_1$
b	$\mu+\tau_2+\beta_1+\tau\beta_{21}$	$\mu+\tau_2$
c	$\mu+\beta_1$	μ

μ is estimated by mean taste cherry, short: 45.75

τ_1 is estimated by the taste difference of round vs cherry after short ripening: 26.77–45.75 = -18.98

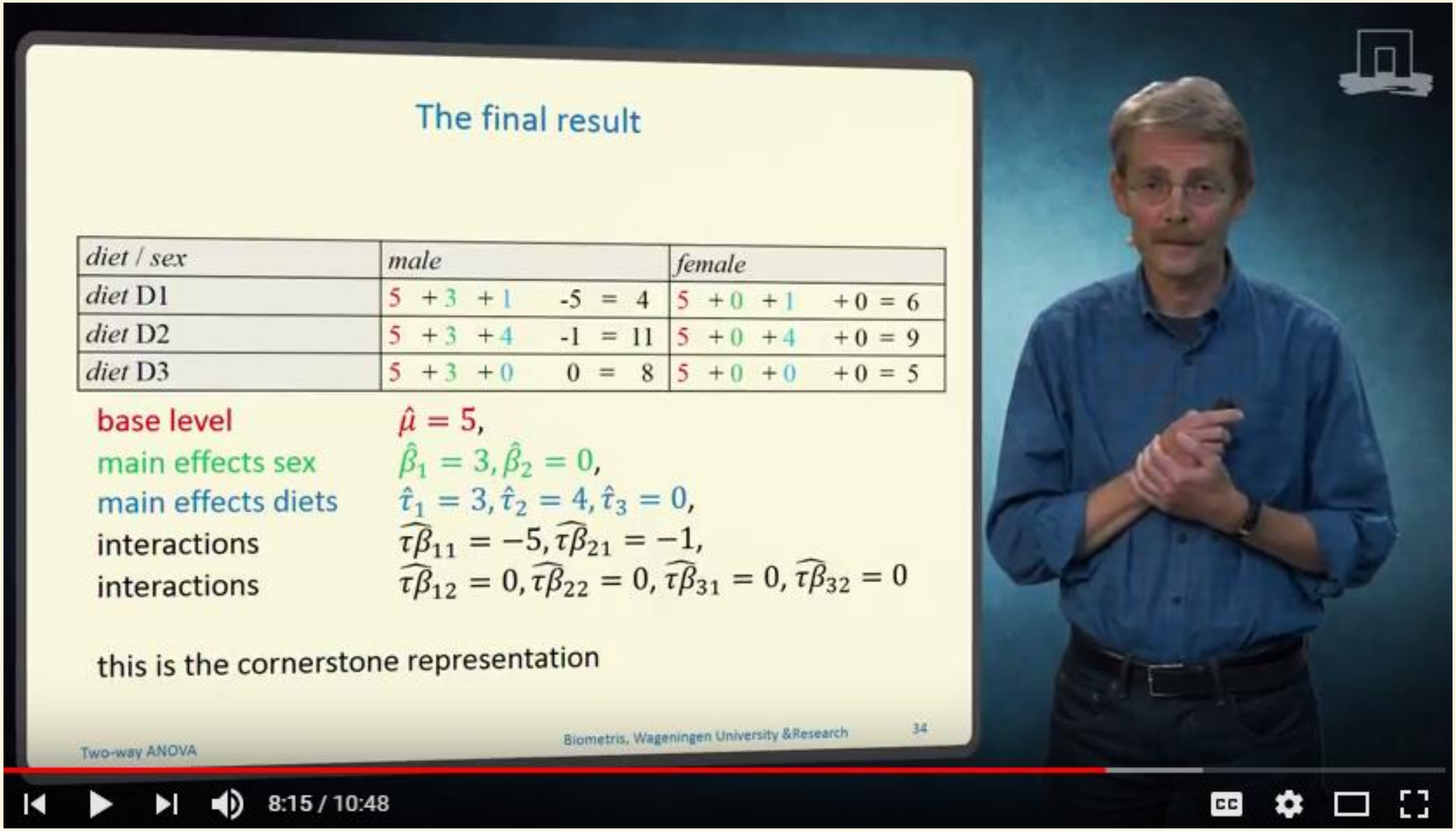
τ_2 is estimated by the taste difference of beef vs cherry after short ripening: 26.65–45.75 = -19.10

β_1 is estimated by the taste diff of long vs short ripening for cherry tomatoes: 63.97-45.75 = 18.22

Estimation of $\tau\beta_{21}$ added to the estimation of $\mu+\tau_2+\beta_1$ to get mean of beef-long (= 39.96),
so: 39.96 – (45.75 - 19.10 + 18.22) = -4.91

Estimation of $\tau\beta_{11}$ added to the estimation of $\mu+\tau_1+\beta_1$ to get mean of round-long (=41.71) ...
Try yourself !!!

Estimations - Cornerstone representation



Cornerstone representation in SPSS

Parameter Estimates

Dependent Variable: SweeTaste

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Intercept	45,750	5,789	7,903	,000	31,585	59,915
[Type=1.00]	-18,980	8,187	-2,318	,060	-39,013	1,053
[Type=2.00]	-19,100	8,187	-2,333	,058	-39,133	,933
[Type=3.00]	0[a]	.	.	.	.	.
[Ripe=1.00]	18,220	8,187	2,226	,068	-1,813	38,253
[Ripe=2.00]	0[a]	.	.	.	.	.
[Type=1.00] * [Ripe=1.00]	-3,280	11,578	-,283	,786	-31,610	25,050
[Type=1.00] * [Ripe=2.00]	0[a]	.	.	.	.	.
[Type=2.00] * [Ripe=1.00]	-4,905	11,578	-,424	,687	-33,235	23,425
[Type=2.00] * [Ripe=2.00]	0[a]	.	.	.	.	.
[Type=3.00] * [Ripe=1.00]	0[a]	.	.	.	.	.
[Type=3.00] * [Ripe=2.00]	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

$\hat{\mu} = 45.75$

$\hat{\tau}_1 = -18.98$
$\hat{\tau}_2 = -19.10$
$\tau_3 = 0$

$\hat{\beta}_1 = 18.22$
$\beta_2 = 0$

$\widehat{\tau\beta}_{11} = -3.28$
$\widehat{\tau\beta}_{21} = -4.91$
…
$\tau\beta_{32} = 0$

By default, SPSS uses last level of each factor as reference.

Different statistical software may choose a different level as a reference, usually by default the first (R package) or the last level.


KEEP
CALM
THERE'S
MORE TO
COME

Estimate treatment means (of interaction model)

$$\hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + \hat{\tau\beta}_{ij} = \bar{y}_{ij.}$$

1a. In a two-factor interaction model, μ_{ij} has estimator and s.e. :

$$\bar{y}_{ij\cdot} \qquad se(\bar{y}_{ij\cdot}) = \hat{\sigma} / \sqrt{n_{ij}}$$

1b. The difference of 2 treatment means: μ_{ij} - μ_{kl} have estimator and s.e.

$$\hat{\mu}_{ij} - \hat{\mu}_{kl} = \bar{y}_{ij\cdot} - \bar{y}_{kl\cdot}$$
$$(= \hat{\tau}_i - \hat{\tau}_k + \hat{\beta}_j - \hat{\beta}_l + \tau\hat{\beta}_{ij} - \tau\hat{\beta}_{kl})$$

$$se(\bar{y}_{ij.} - \bar{y}_{kl\cdot}) = \hat{\sigma}\sqrt{\tfrac{1}{n_{ij}} + \tfrac{1}{n_{kl}}}$$

2. $\mu_{p\cdot} = (\mu_{p1} + \mu_{p2} + \ldots + \mu_{pb}) / b$ is estimated using the corresponding observed treatment means $\bar{y}_{ij\cdot}$. The s.e. tougher to compute (use SPSS)

2a. If $n_{p1} = n_{p2} =..=n_{pb}$, then

$$\hat{\mu}_{p.} = \bar{y}_{p..} \quad with\ s.e.(\bar{y}_{p..}) = \hat{\sigma} / \sqrt{n_{p.}}$$

2b. $n_{p1}=n_{p2}=..=n_{pb}$ and $n_{q1}=n_{q2}=..=n_{qb}$: $\mu_{p\cdot}$ - $\mu_{q\cdot}$ has estimator and s.e.

$$\hat{\mu}_p - \hat{\mu}_q = \bar{y}_{p..} - \bar{y}_{q..} \quad with \quad se(\bar{y}_{p..} - \bar{y}_{q..}) = \hat{\sigma}\sqrt{\tfrac{1}{n_p} + \tfrac{1}{n_q}}$$

Summary of two-way model

Model with interaction: $y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ij}$

- Focus on interaction parameters $\tau\beta_{ij}$
- Effects of factor *A* depend on level of factor *B* and vice versa
- Non-parallel lines in profile plot

Additive model without interaction: $y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ij}$

- Simpler conclusions: effects of factor *A* can be reported separately from factor *B* and vice versa
- Parallel lines in profile plots

Map of two-way ANOVA

Illustrative examples

Two-way ANOVA model

Sums of squares

F-tests for interaction & main effects

Illustrative examples

Splitting *SSTreat* / SSModel - 1

Effects model: $\underline{y}_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \underline{\epsilon}_{ijk}$,

$\underline{\epsilon}_{ijk} \sim N(0, \sigma_\epsilon^2)$, independent.

We want a test for interaction, this will be an F-test.

We will split *SSTreat* into three *SS*:

SSA	*SS* for main effects *A*
SSB	*SS* for main effects *B*
SSAB	*SS* for interaction between *A* and *B*

SSA + SSB + SSAB = SSTreat

Splitting *SSTreat* / SSModel - 2

Use idea of extra *SS* again.

SS are in general order dependent:

SS for main effects *A*, *A* first,

SS for main effects *B*, *B* after A

not the same as

SS for main effects *B*, *B* first,

SS for main effects *A*, *A* after B

Except when design is balanced, i.e. equal no. of observations per treatment (= combinations of levels of A and B)

ANOVA table, two factors, interaction model

a=nr. of levels in factor A b=nr. of levels in factor B N=nr. of experimental units

Source	SS	df	MS	F
treat / model	SSTreat	df = ab -1	MSTreat	MSTreat/MSE
main effects *A*	*SSA*	df = $a - 1$	MSA	MSA/MSE
main effects *B*	*SSB*	df = $b - 1$	MSB	MSB/MSE
interaction effects *AB*	*SSAB*	df = $(a-1)(b-1)$	MSAB	MSAB/MSE
residual error	*SSE*	df = $N - ab$	MSE	
Total	*TSS*	df = $N - 1$		

Mean squares: *MSA* = *SSA* / (*a* -1), *MSB* = *SSB* / (*b* -1) … etc.
F-values are calculated for each factor or interaction: F = MS.../ MSE

From the ANOVA table we estimate as usual: $\hat{\sigma}^2 = \text{MSE}$

The system can be extended to 3 factors *A*, *B* and *C* (or more), see O&L.

Splitting *SSTreat*, tomato example

We start with the *SS* for interaction.

Compare *SSE* from models with and without interaction (extra SS from complete versus reduced model):

CM : $y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ijk}$ (interaction model)

RM: $y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$ (additive model)

Tests of Between-Subjects Effects

Dependent Variable: SweeTaste

Source	Type I Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1919,880[a]	5	383,976	5,729	,028
Intercept	19978,128	1	19978,128	298,067	,000
Type	1187,419	2	593,709	8,858	,016
Ripe	719,975	1	719,975	10,742	,017
Type * Ripe	12,486	2	6,243	,093	,912
Error	402,154	6	67,026		
Total	22300,162	12			
Corrected Total	2322,034	11			

a. R Squared = ,827 (Adjusted R Squared = ,682)

So: *SSTypexRipe* = 12.49

Splitting *SSTreat*, tomato example

We start with the *SS* for interaction.

Compare *SSE*s from models with and without interaction:

CM : $y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ijk}$ (interaction model)

RM: $y_{ijk} = \mu + \tau_i + \beta_j \qquad\qquad + \epsilon_{ijk}$ (additive model)

```
> CM.ia <- lm(taste ~ type + ripe + type:ripe)
> deviance(CM.ia)
[1] 402.1538
> RM.ia <- lm(taste ~ type + ripe)
> deviance(RM.ia)
[1] 414.6399
> SS.ia <- deviance(RM.ia)-deviance(CM.ia)
[1] 12.48602
```

So: $SS_{type\ x\ ripe}$ = 414.6399 – 402.1538 = 12.49

Splitting *SSTreat*, tomato example

SS for main effect of *type*

Restrict attention to additive model

CM : $y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$
RM: $y_{ijk} = \mu + \beta_j + \epsilon_{ijk}$

SS for main effect of *ripe*

Restrict attention to additive model

CM : $y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$
RM: $y_{ijk} = \mu + \tau_i + \epsilon_{ijk}$

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a. R Squared = ,827 (Adjusted R Squared = ,682)

So: *SSType* = 1187.42 and *SSRipe* = 719.98

SSType+*SSRipe*+*SSTypexRipe* =1187.42+719.98+12.49 = 1919.88 = *SSTreat*

Design is balanced (two obs. per treatment), so split is unique.

Splitting *SSTreat*, tomato example

SS for main effect of *type*

Restrict attention to additive model

CM : $y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$

RM: $y_{ijk} = \mu + \qquad \beta_j + \epsilon_{ijk}$

```
> CM.m1 <- lm(taste ~ type + ripe)
> deviance(CM.m1)
[1] 414.6399
> RM.m1 <- lm(taste ~ ripe)
> deviance(RM.m1)
[1] 1602.059
> SS.type <- deviance(RM.m1)-deviance(CM.m1)
[1] 1187.419
```

SS for main effect of *ripe*

Restrict attention to additive model

CM : $y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$

RM: $y_{ijk} = \mu + \tau_i \qquad + \epsilon_{ijk}$

```
> CM.m2 <- lm(taste ~ type + ripe)
> deviance(CM.m2)
[1] 414.6399
> RM.m2 <- lm(taste ~ type)
> deviance(RM.m2)
[1] 1134.615
> SS.ripe <- deviance(RM.m2)-deviance(CM.m2)
[1] 719.975
```

So: *SStype* = 1187.42 and *SSripe* = 719.98

SStype+*SSripe*+*SStype x ripe* =1187.42+719.98+12.49 = 1919.88 = *SSTreat*

Design is balanced (two obs. per treatment), so split is unique.

Summary of steps to obtain *SS*

Tests of Between-Subjects Effects

Dependent Variable: SweeTaste

Source	Type I Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1919,880[a]	5	383,976	5,729	,028
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Type * Ripe	12,486	2	6,243	,093	,912
Error	402,154	6	67,026		
Total	22300,162	12			
Corrected Total	2322,034	11			

a. R Squared = ,827 (Adjusted R Squared = ,682)

SS are:
type I *SS* =
accumulated *SS* =
sequential *SS*

Sums of squares constructed sequentially:

$$y_{ijk} = \mu + \epsilon_{ijk}$$

+type → 1187.42

$$y_{ijk} = \mu + \tau_i + \epsilon_{ijk}$$

+ripe → 719.98

$$y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$$

+ type x ripe → 12.49

$$y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ijk}$$

Summary of steps to make *SS*

```
> anova(lm(taste ~ type + ripe + type:ripe))

Analysis of Variance Table
Response: taste
           Df  Sum Sq Mean Sq F value Pr(>F)
type        2 1187.42  593.71  8.8579 0.0162
ripe        1  719.98  719.98 10.7418 0.0169
type:ripe   2   12.49    6.24  0.0931 0.9124
Residuals   6  402.15   67.03
```

SS are: type I SS = accumulated SS = sequential SS

Same as previous because design is balanced.

In the R table above sums of squares are constructed like this:

$$y_{ijk} = \mu + \epsilon_{ijk}$$

+ type → 1187.42

$$y_{ijk} = \mu + \tau_i + \epsilon_{ijk}$$

+ ripe → 719.98

$$y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk}$$

+ type x ripe → 12.49

$$y_{ijk} = \mu + \tau_i + \beta_j + \tau\beta_{ij} + \epsilon_{ijk}$$

Map of two-way ANOVA

tomatoe example

Two-way ANOVA model

Sums of squares

F-tests for interaction & main effects

Illustrative examples

F-test for interaction

```
> anova(lm(taste ~ type + ripe + type:ripe))
Analysis of Variance Table
Response: taste
          Df  Sum Sq Mean Sq F value Pr(>F)
type       2 1187.42  593.71  8.8579 0.0162
ripe       1  719.98  719.98 10.7418 0.0169
type:ripe  2   12.49    6.24  0.0931 0.9124
Residuals  6  402.15   67.03
```

Tests of Between-Subjects Effects

Dependent Variable: SweeTaste

Source	Type I Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1919,880[a]	5	383,976	5,729	,028
Intercept	19978,128	1	19978,128	298,067	,000
Type	1187,419	2	593,709	8,858	,016
Ripe	719,975	1	719,975	10,742	,017
Type * Ripe	12,486	2	6,243	,093	,912
Error	402,154	6	67,026		
Total	22300,162	12			
Corrected Total	2322,034	11			

a. R Squared = ,827 (Adjusted R Squared = ,682)

$H_0: \tau\beta_{11} = \tau\beta_{12} = 0$ versus H_a: not all $\tau\beta_{ij} = 0$

test statistic: $F = \frac{MSType \times Ripe}{MSE}$ *with outcome* $= \frac{6.24}{67.030} = 0.093$

P-value = 0.91 > 0.05, from F-distribution, df1 = 2, df2 = 6

do not reject H_0,

effect of ripening cannot be shown to be different across tomato types

additive model is good enough

continue with tests for main effects for type and ripe

F-tests for main effects, type

Tests of Between-Subjects Effects

Dependent Variable: SweeTaste

Source	Type I Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1919,880[a]	5	383,976	5,729	,028
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Error	402,154	6	67,026		
Total	22300,162	12			
Corrected Total	2322,034	11			

a. R Squared = ,827 (Adjusted R Squared = ,682)

```
> anova(lm(taste ~ type + ripe + type:ripe))
Analysis of Variance Table
Response: taste
          Df  Sum Sq Mean Sq F value Pr(>F)
type       2 1187.42  593.71  8.8579 0.0162
ripe       1  719.98  719.98 10.7418 0.0169
type:ripe  2   12.49    6.24  0.0931 0.9124
Residuals  6  402.15   67.03
```

$H_0\colon \tau_1 = \tau_2 = 0$ vs. H_a: not all $\tau_i = 0$

Test statistic / outcome: $F = \frac{MSType}{MSE} = 8.86$

P-value $= 0.016 < 0.05$, from F-distr. df1 = 2, df2 = 6, reject H_0,

expected sweetness differs among types (irrespective of ripe)

F-tests for main effects, ripe

Tests of Between-Subjects Effects

Dependent Variable: SweeTaste

Source	Type I Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1919,880[a]	5	383,976	5,729	,028
Intercept	19978,128	1	19978,128	298,067	,000
Type	1187,419	2	593,709	8,858	,016
Ripe	719,975	1	719,975	10,742	,017
Type * Ripe	12,486	2	6,243	,093	,912
Error	402,154	6	67,026		
Total	22300,162	12			
Corrected Total	2322,034	11			

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```
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type:ripe  2   12.49    6.24  0.0931 0.9124
Residuals  6  402.15   67.03
```

$H_0\colon \beta_1 = 0$ vs. $H_a\colon \beta_1 \neq 0$

Test statistic / outcome: $F = \frac{MSRipe}{MSE} = 10.74$

P-value $= 0.017 < 0.05$, from F-distr. df1 = 1, df2 = 6, reject H_0, expected sweetness differs between long and short (irrespective of type)

Map of two-way ANOVA

tomato example

Two-way ANOVA model

Sums of squares

F-tests for interaction & main effects

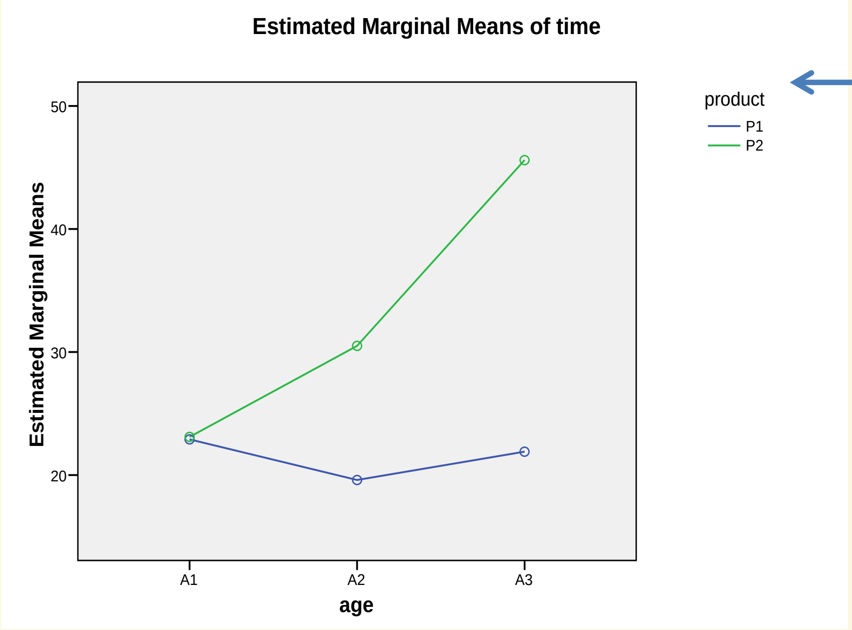
Illustrative examples

Further examples, attention span commercial

y = attention span of a child for a television commercial

3 age classes (A_1: 5 - 6, A_2: 7 - 8, A_3: 9 - 10 years)
2 types of product (P_1: cereal, P_2: video game)

10 representative children of each age class, randomly assigned to products



clear interaction

Two factors:

A (age at 3 levels)

P (product at 2 levels)

6 treatments

Attention span, SPSS

Tests of Between-Subjects Effects

Dependent Variable: time

Source	Type II Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	4705.733[a]	5	941.147	6.398	.000
Intercept	44608.267	1	44608.267	303.228	.000
age	1303.033	2	651.517	4.429	.017
product	2018.400	1	2018.400	13.720	.001
age * product	1384.300	2	692.150	4.705	.013
Error	7944.000	54	147.111		
Total	57258.000	60			
Corrected Total	12649.733	59			

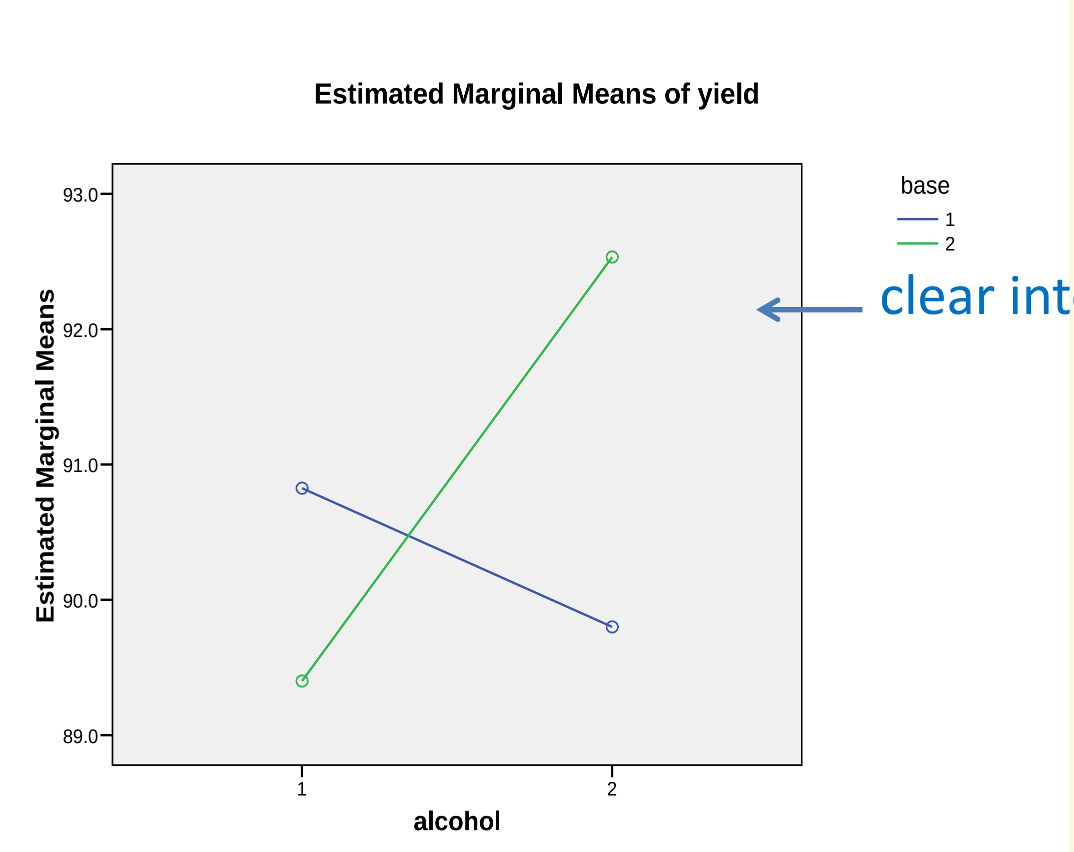
a. R Squared = .372 (Adjusted R Squared = .314)

interaction Age and Product significant (P-value = 0.013)

do not look at main effects!

Further examples, Alcohol and base

y = % yield of a chemical process
2 types of alcohol (A_1, A_2)
2 types of base (B_1, B_2)



clear interaction

Two experimental factors:

A (alcohol at two levels)

B (base at two levels)

4 treatments

Alcohol and base, SPSS

Tests of Between-Subjects Effects

Dependent Variable: yield

Source	Type II Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	18.295[a]	3	6.098	2.460	.123
Intercept	114898.921	1	114898.921	46341.110	.000
alcohol	2.006	1	2.006	.809	.389
base	1.467	1	1.467	.592	.460
alcohol * base	14.821	1	14.821	5.978	.035
Error	24.794	10	2.479		
Total	114942.010	14			
Corrected Total	43.089	13			

a. R Squared = .425 (Adjusted R Squared = .252)

interaction Alcohol and Base significant (P-value = 0.035)

do not look at main effects (in this case very misleading)!