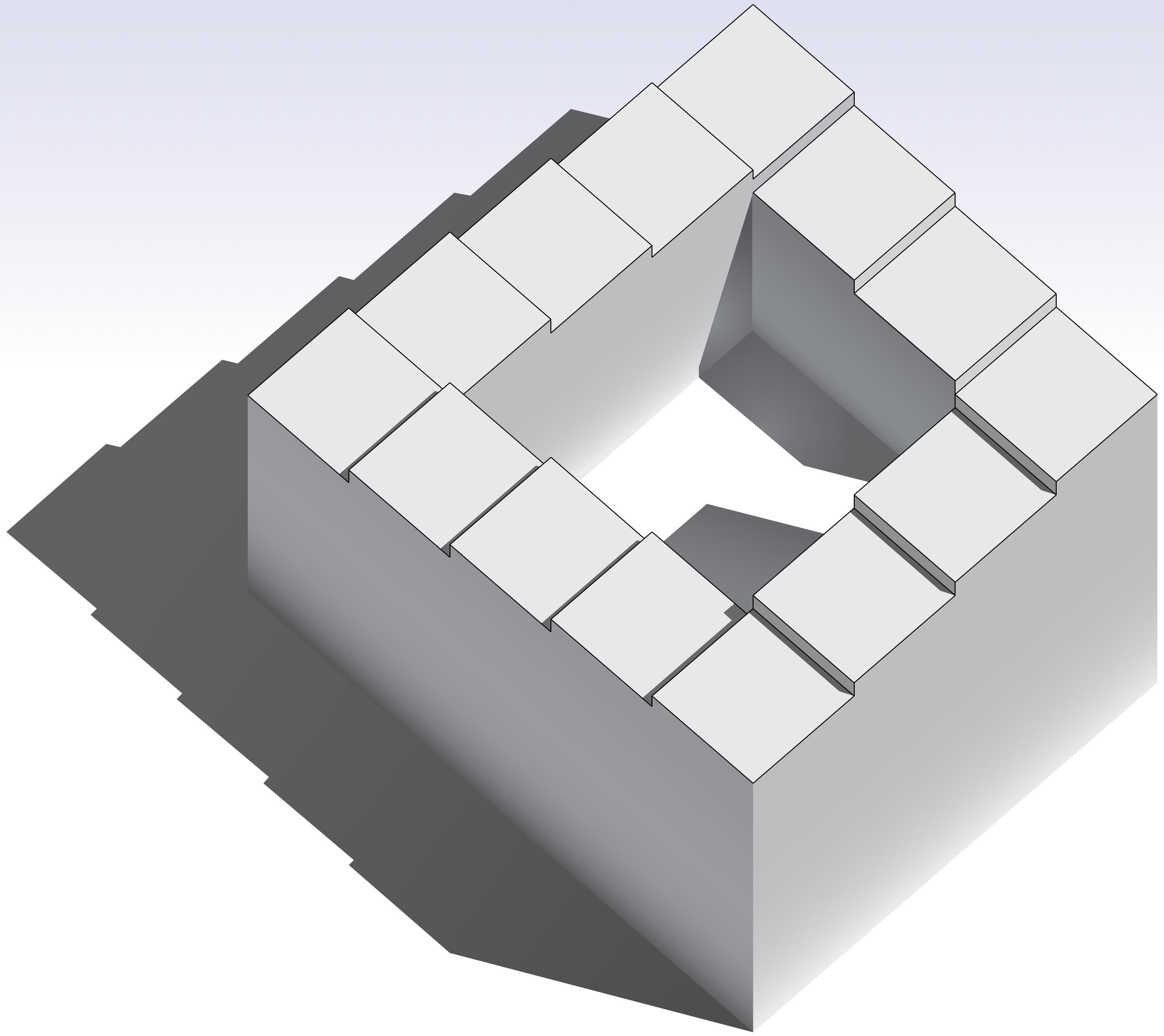
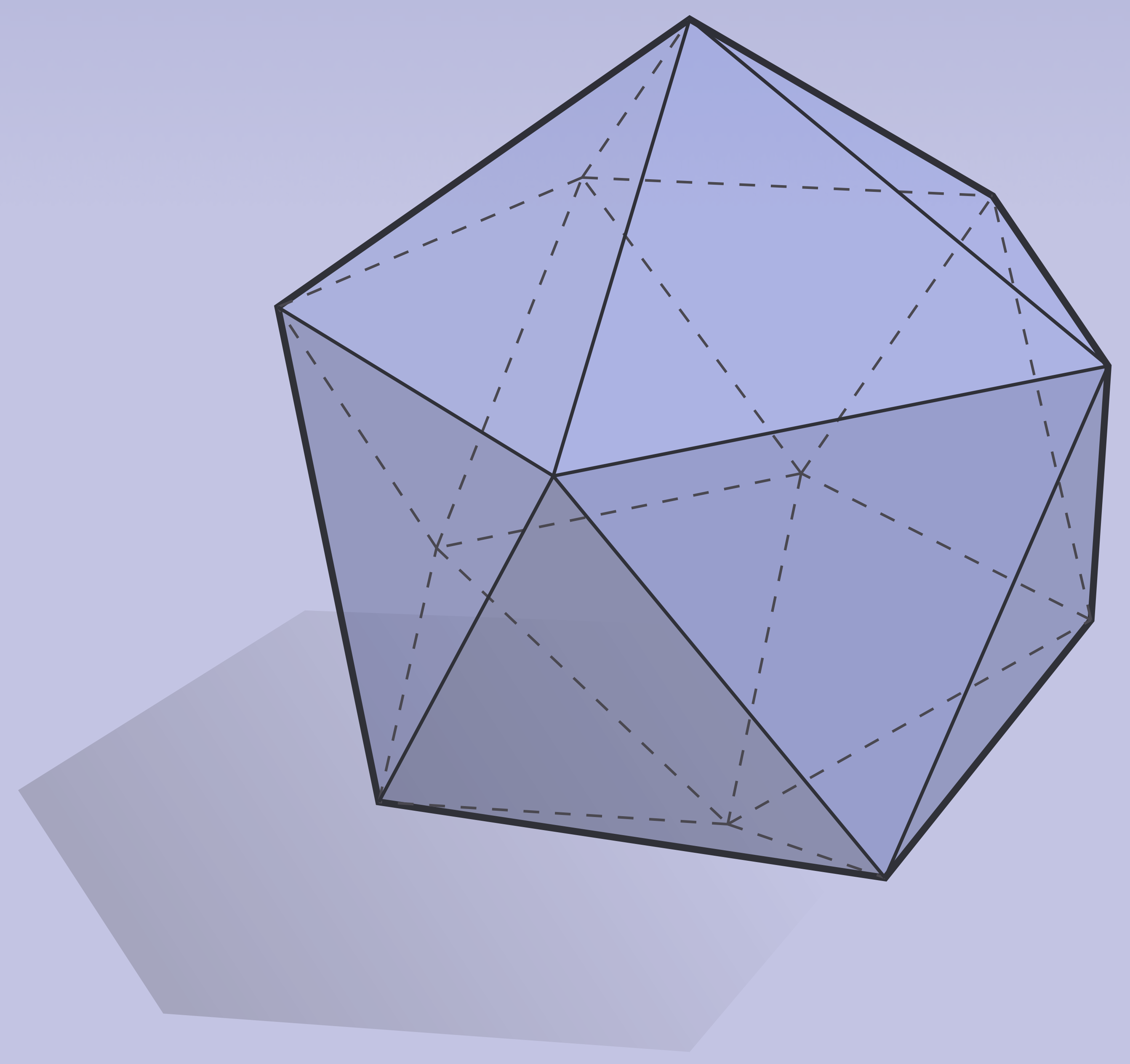


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

Lecture 12:
Smooth Surfaces I



Discrete Differential Geometry: An Applied Introduction

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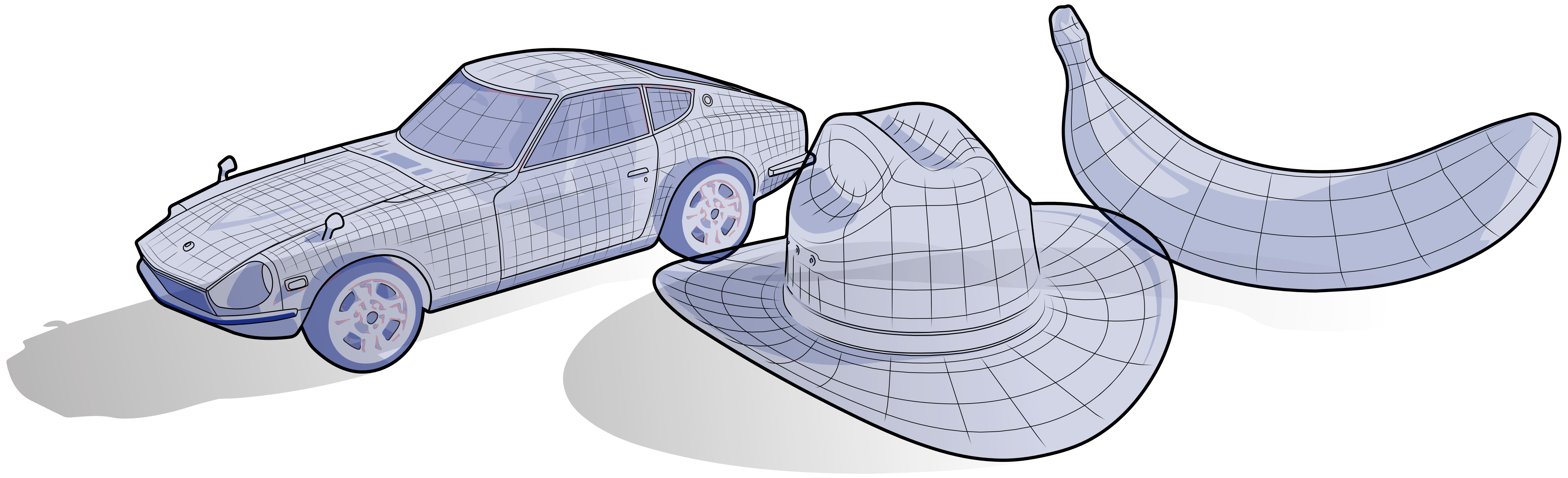
From Curves to Surfaces

- **Previously:** saw how to talk about 1D curves (both smooth and discrete)
- **Today:** will study 2D curved surfaces (both smooth and discrete)
  - Some concepts remain the same (*e.g.*, differential); others need to be generalized (*e.g.*, curvature)
  - Still use exterior calculus as our *lingua franca*

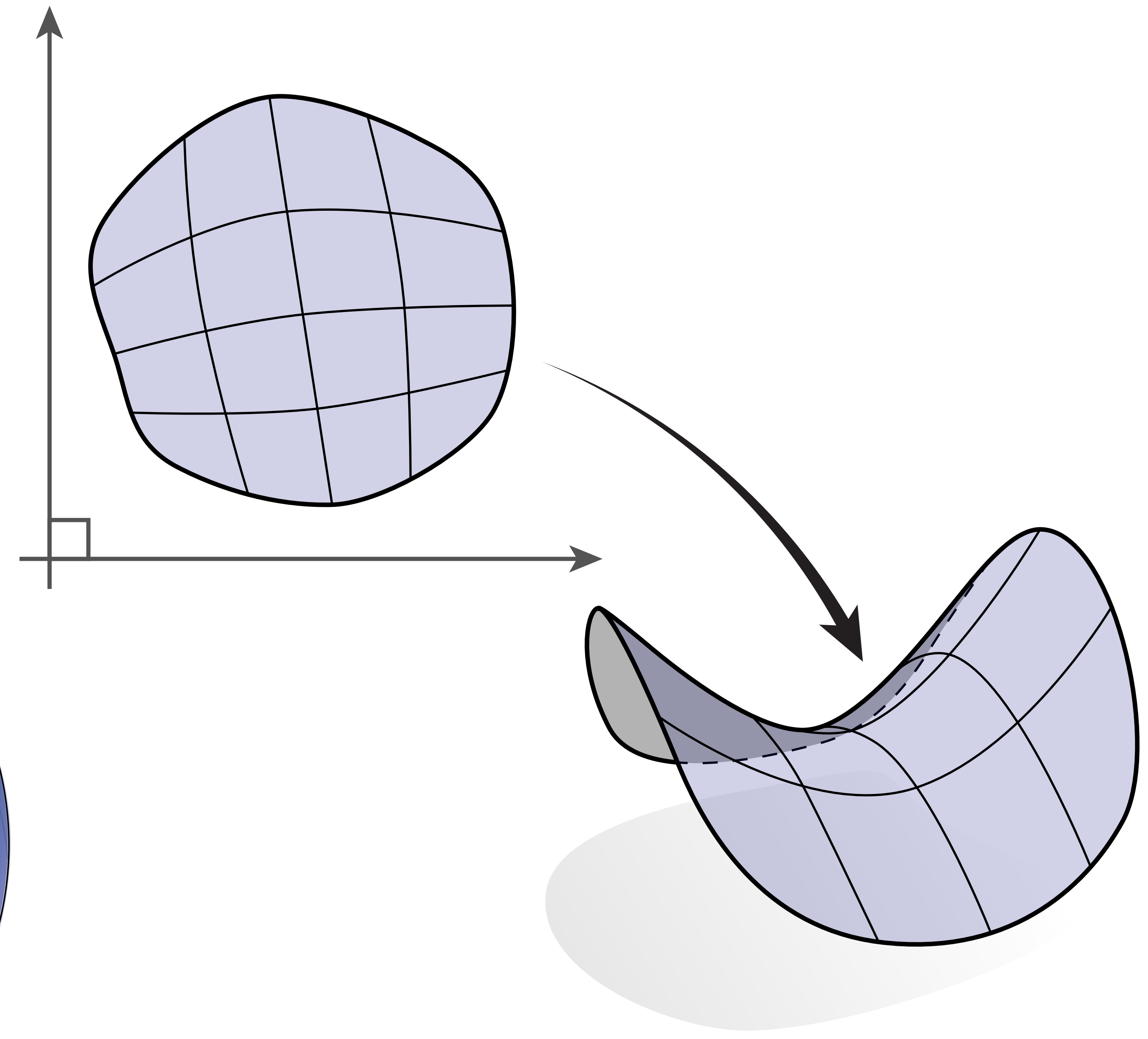
(Curves)

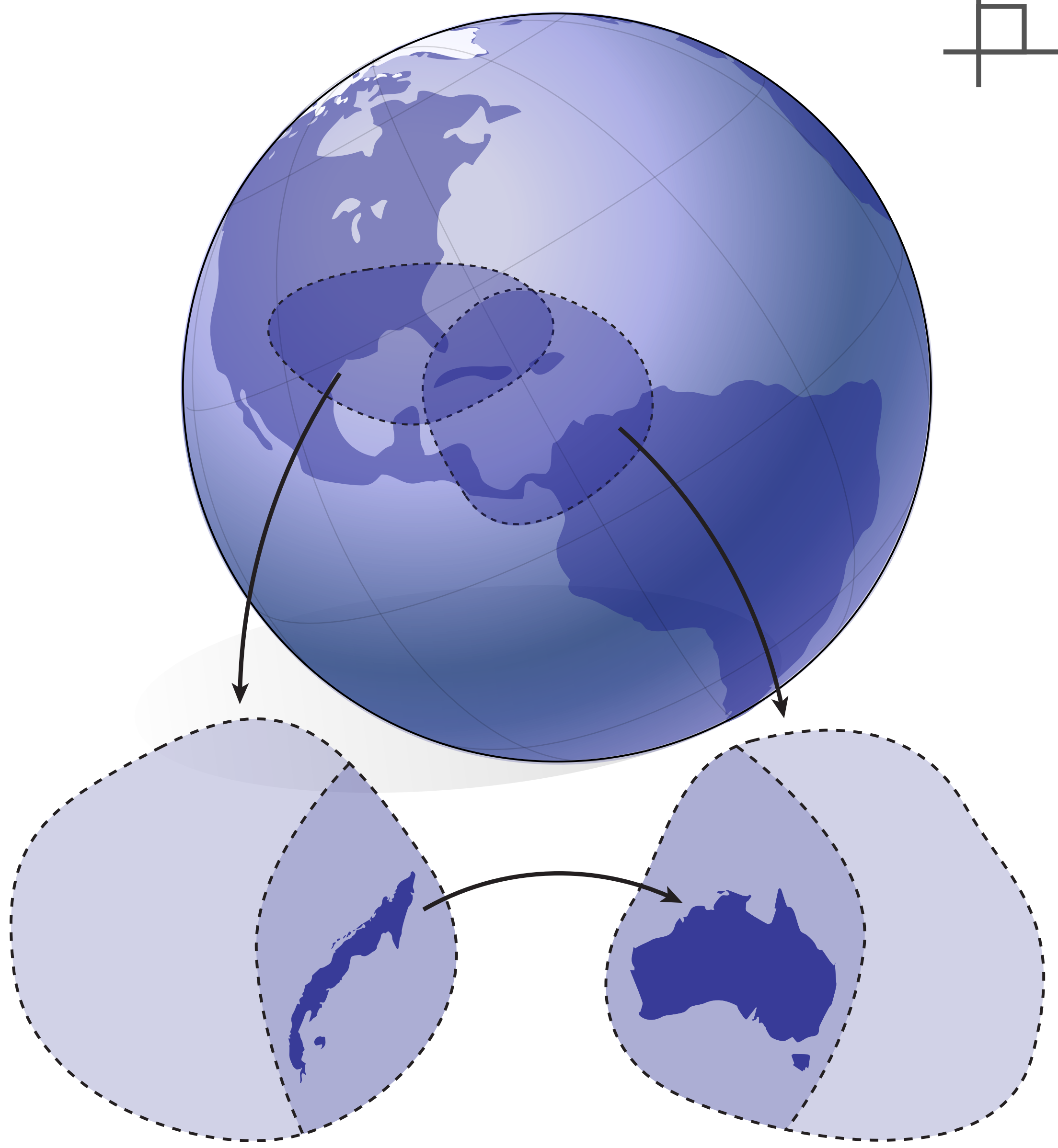


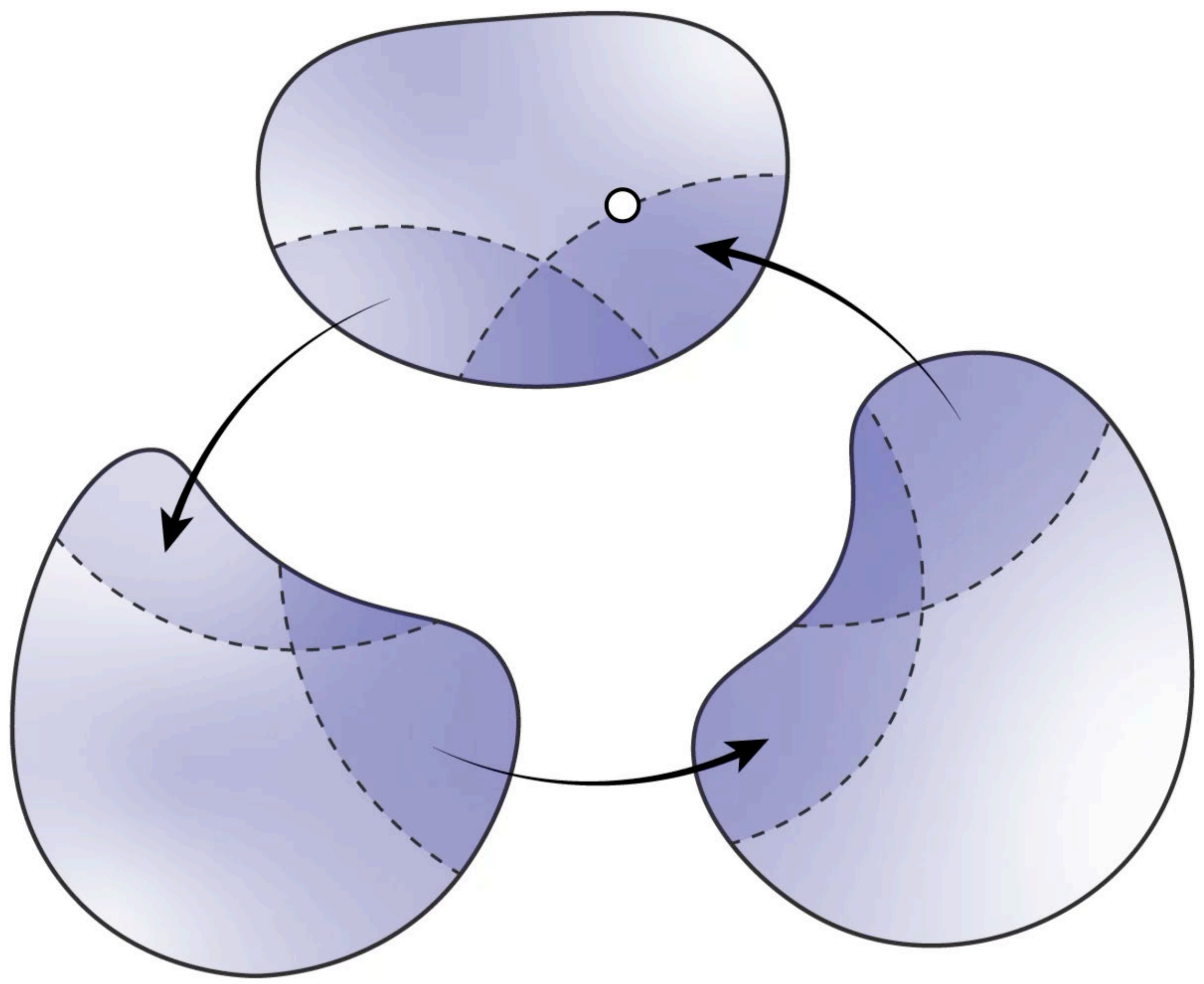
(Surfaces)

Surfaces—From Local to Global to Intrinsic…

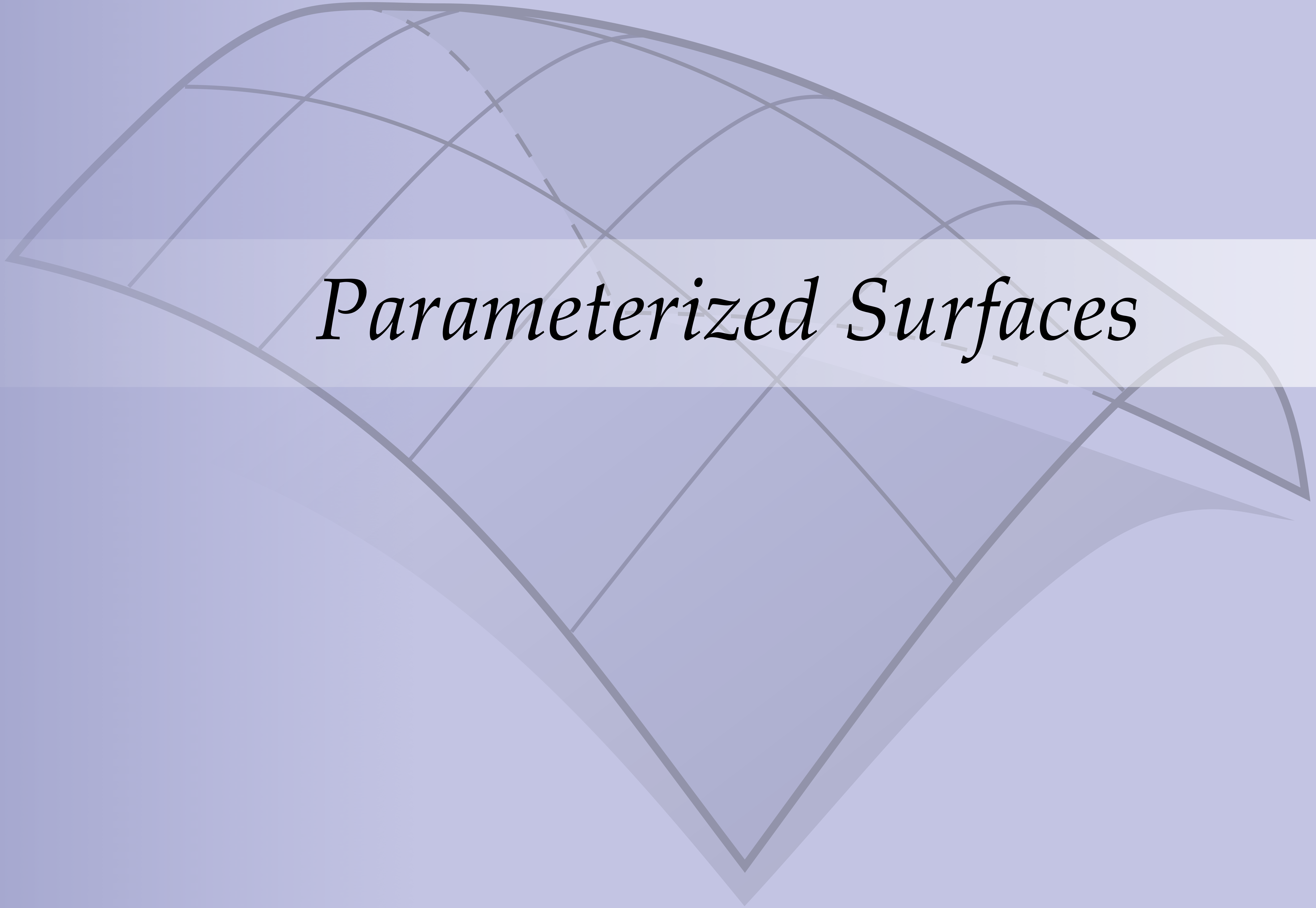
- **Local picture.** Will initially describe surfaces in terms of the geometry of a local patch.
  - As with curves, parameterization gives an *extrinsic* description: how does it sit in space?
- **Global picture.** Can piece several local pieces together to describe a whole surface, rather than one patch. (Still extrinsic.)
- **Intrinsic picture.** From here, can "throw away" embeddings into space—induced *Riemannian metric* retains a "memory" of the shape
  - In fact, we never needed an embedding at all! Can describe manifolds purely *intrinsically*.
- **(Discrete picture.)** mesh geometry via edge lengths (*intrinsic*), rather than vertex positions (*extrinsic*).





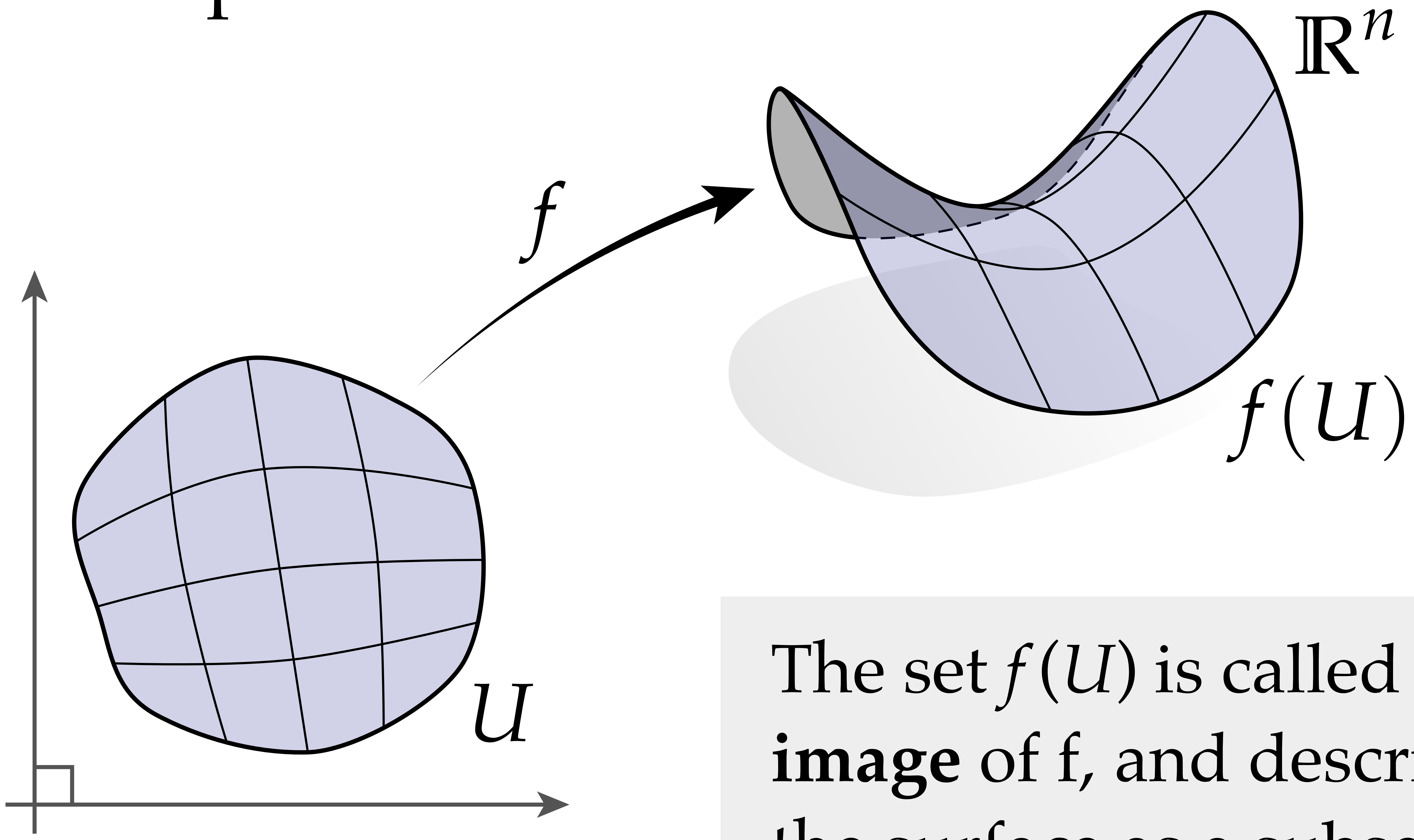


Parameterized Surfaces



Parameterized Surface

A **parameterized surface** is a map $f : U \to \mathbb{R}^n$ from a two-dimensional region $U \subset \mathbb{R}^2$ into space.



The set $f(U)$ is called the **image** of f, and describes the surface as a subset of $\mathbb{R}^n$ rather than as a map.

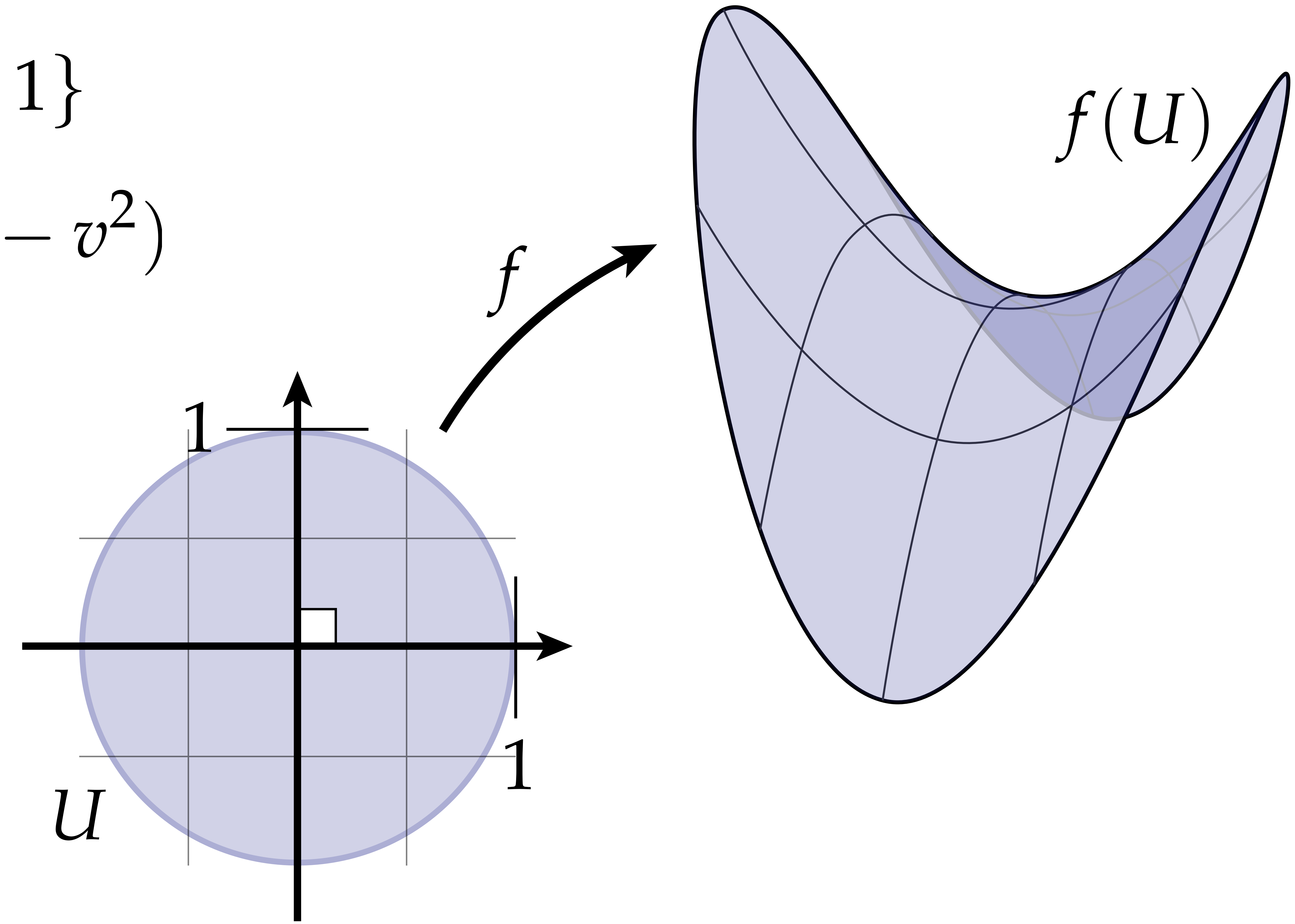
*Continuous, differentiable, smooth, …

Parameterized Surface—Example

For example, can express a *saddle* as a parameterized surface:

$$U := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f : U \to \mathbb{R}^3; (u,v) \mapsto (u, v, u^2 - v^2)$$



Reparameterization

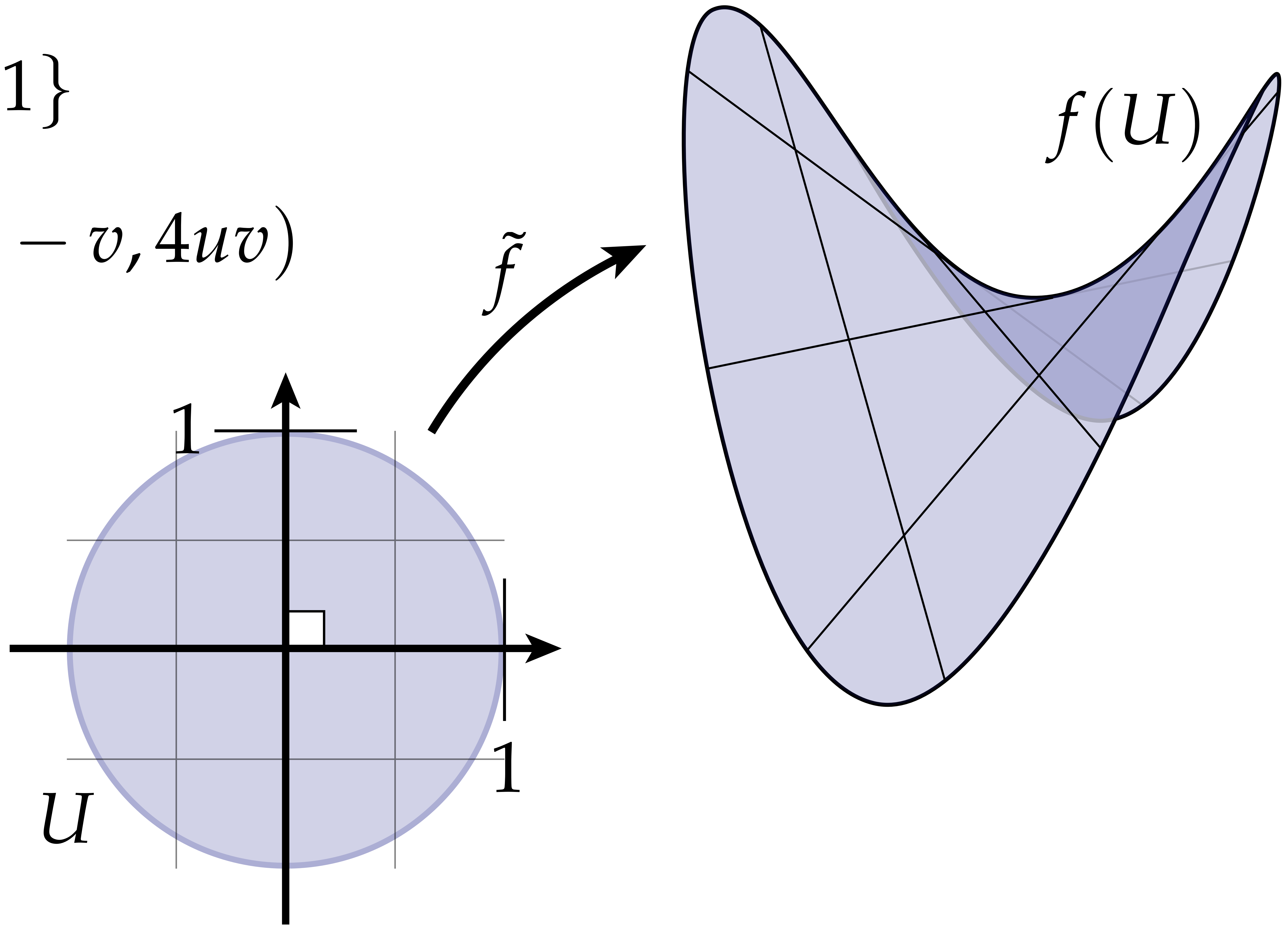
Many different parameterizations describe the same surface:

$$U := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$\tilde{f} : U \to \mathbb{R}^3; (u,v) \mapsto (u+v, u-v, 4uv)$$

This *"reparameterization symmetry"* can be a major challenge in applications—*e.g.*, trying to decide if two parameterized surfaces (or meshes) describe the same shape.

Analogy: graph isomorphism



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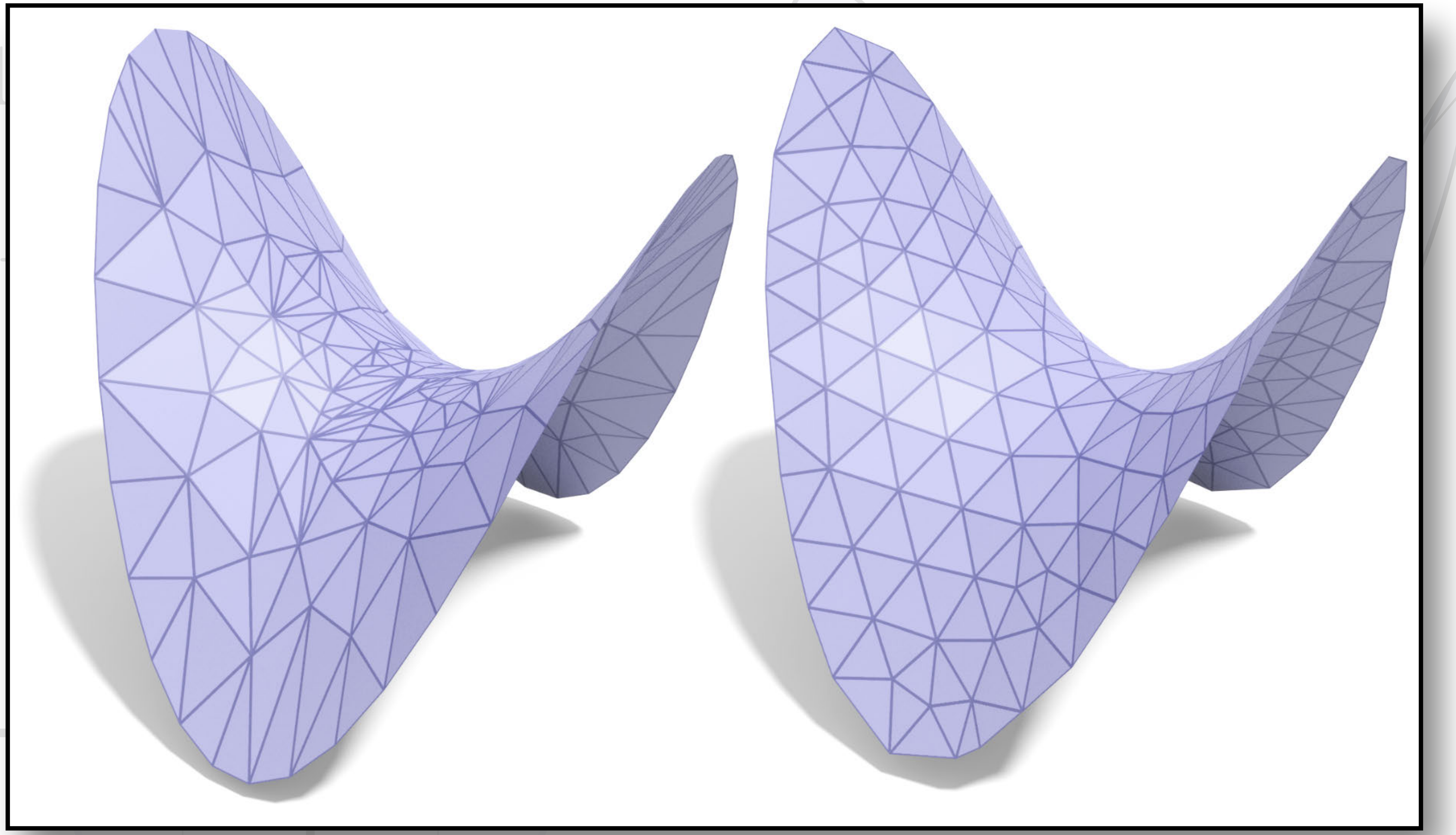
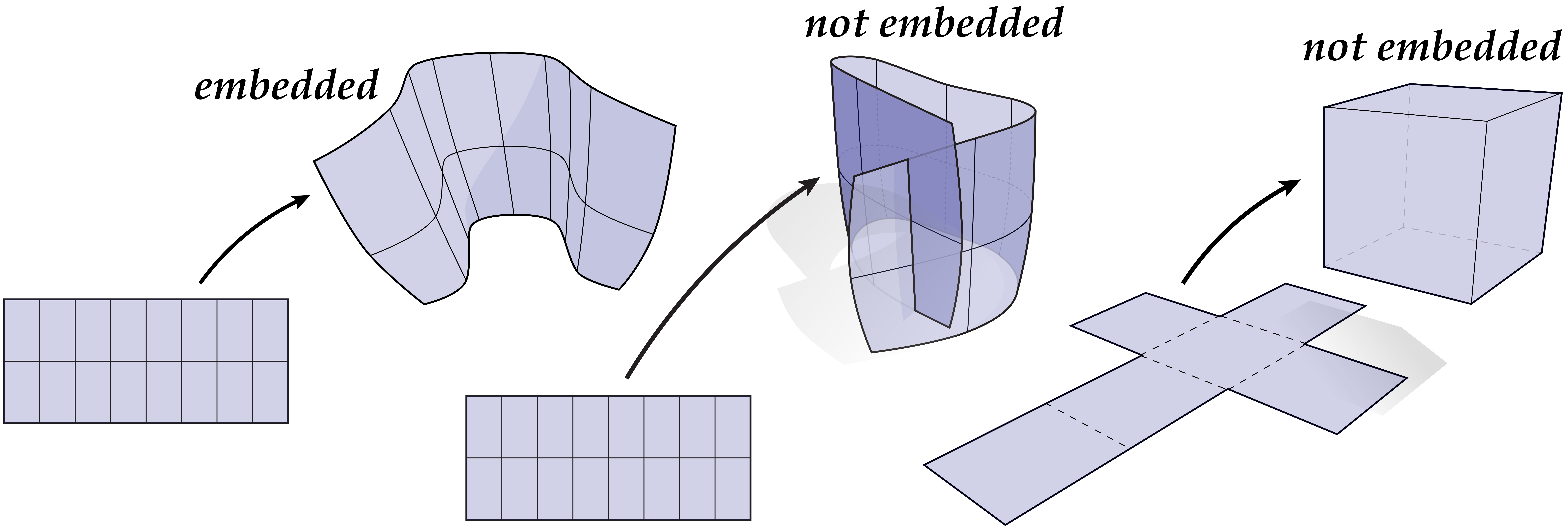


This *"reparameterization symmetry"* can be a major challenge in applications—*e.g.*, trying to decide if two parameterized surfaces (or meshes) describe the same shape.

Analogy: graph isomorphism

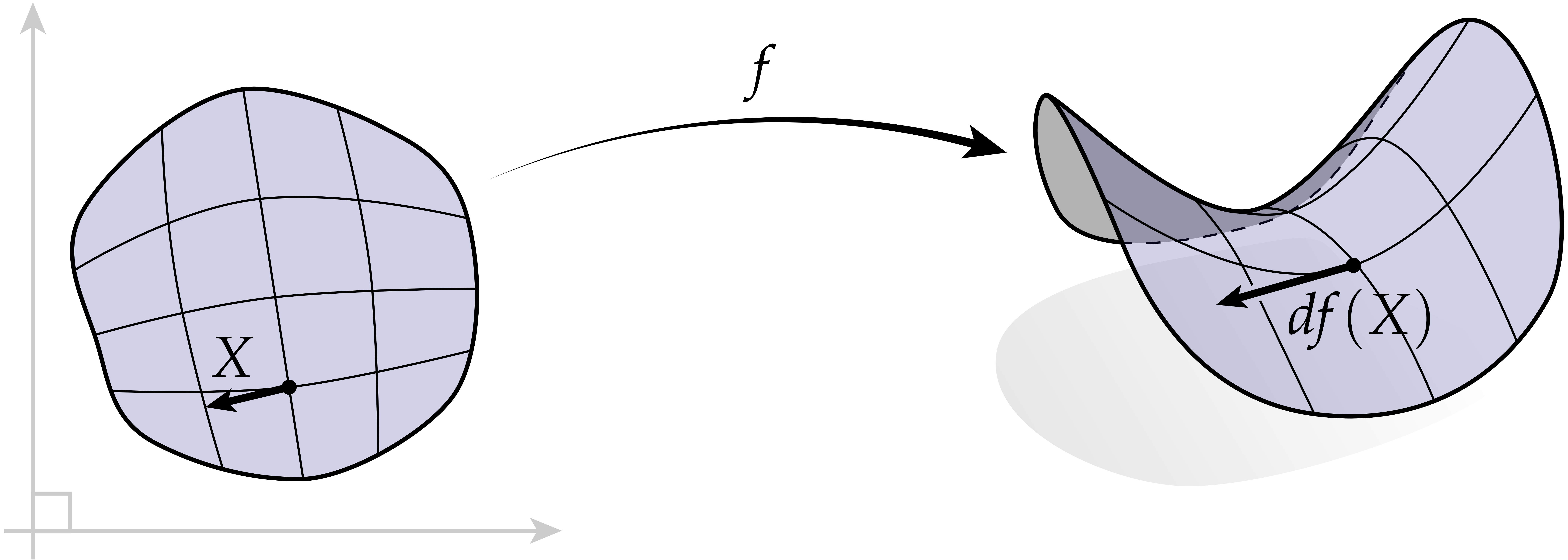
Embedded Surface

- Loosely speaking, an embedding "preserves the topology" of the domain
- More precisely, a parameterized surface f is an **embedding** if it is a continuous bijection onto its image $f(U)$, with continuous inverse



Differential of a Surface

Intuitively, the *differential* of a parameterized surface tells us how tangent vectors on the domain get "stretched out" into space:



We say that df *"pushes forward"* vectors X into $\mathbb{R}^n$, yielding vectors $df(X)$

Differential in Coordinates

More explicitly, the differential is the exterior derivative of the parameterization:

$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u, v, u^2 - v^2)$$

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv =$$

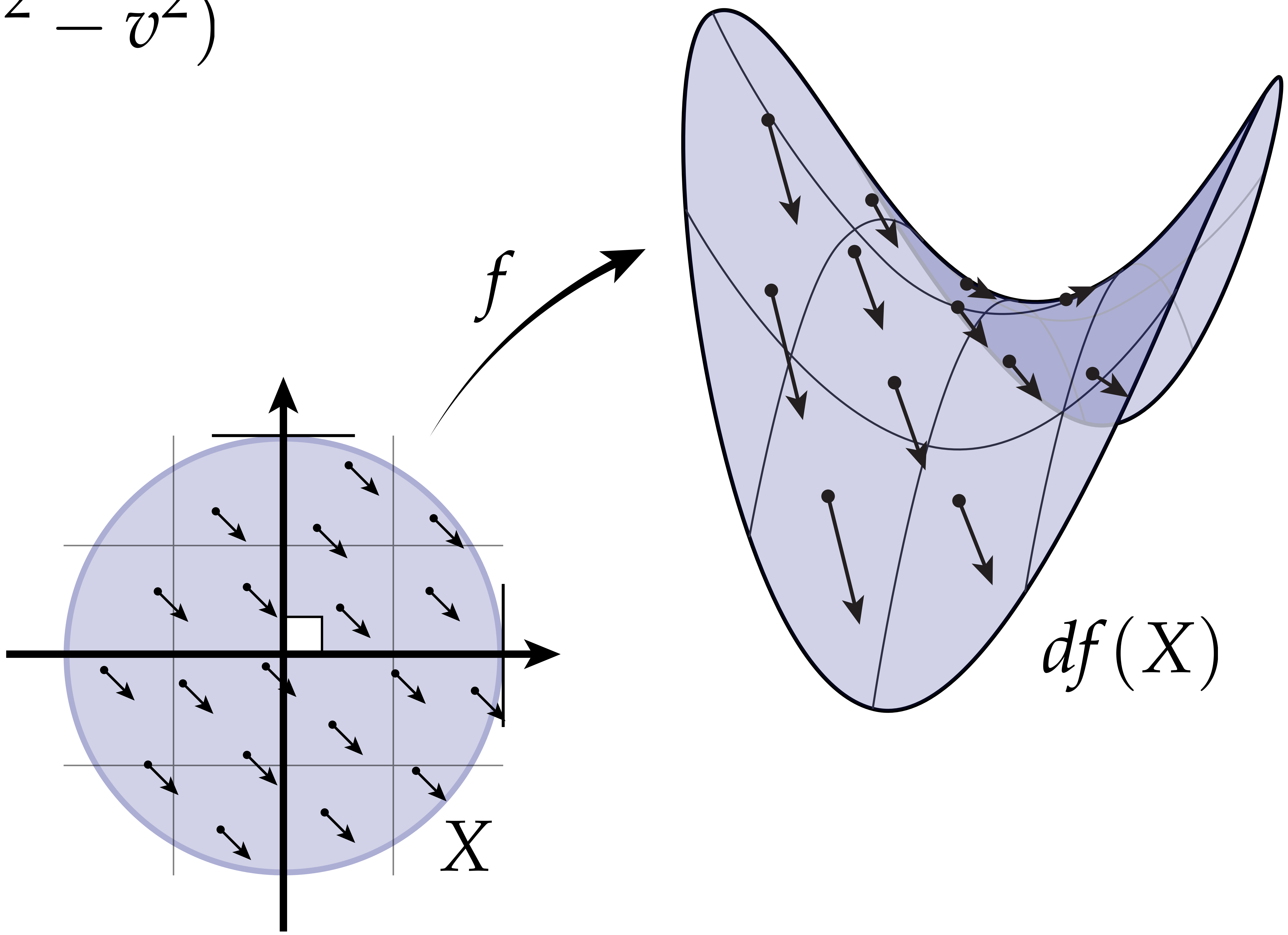
$$(1, 0, 2u)du + (0, 1, -2v)dv$$

To "push forward" a vector field X:

$$X := \tfrac{3}{4}\left(\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y}\right)$$

$$df(X) = \tfrac{3}{4}(1, -1, 2(u+v))$$

E.g., at $u{=}v{=}0$: $(\frac{3}{4}, -\frac{3}{4}, 0)$



Differential—Matrix Representation (Jacobian)

Definition. Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$, and let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. Then the *Jacobian* of f is the matrix

$$J_f := \begin{bmatrix} \partial f^1/\partial x^1 & \cdots & \partial f^1/\partial x^n \\ \vdots & \ddots & \vdots \\ \partial f^m/\partial x^1 & \cdots & \partial f^m/\partial x^n \end{bmatrix},$$

where $f^1, \ldots, f^m$ are the components of f w.r.t. some coordinate system on $\mathbb{R}^m$. This matrix represents the differential in the sense that $df(X) = J_f X$.

(In solid mechanics, also known as the *deformation gradient*.)

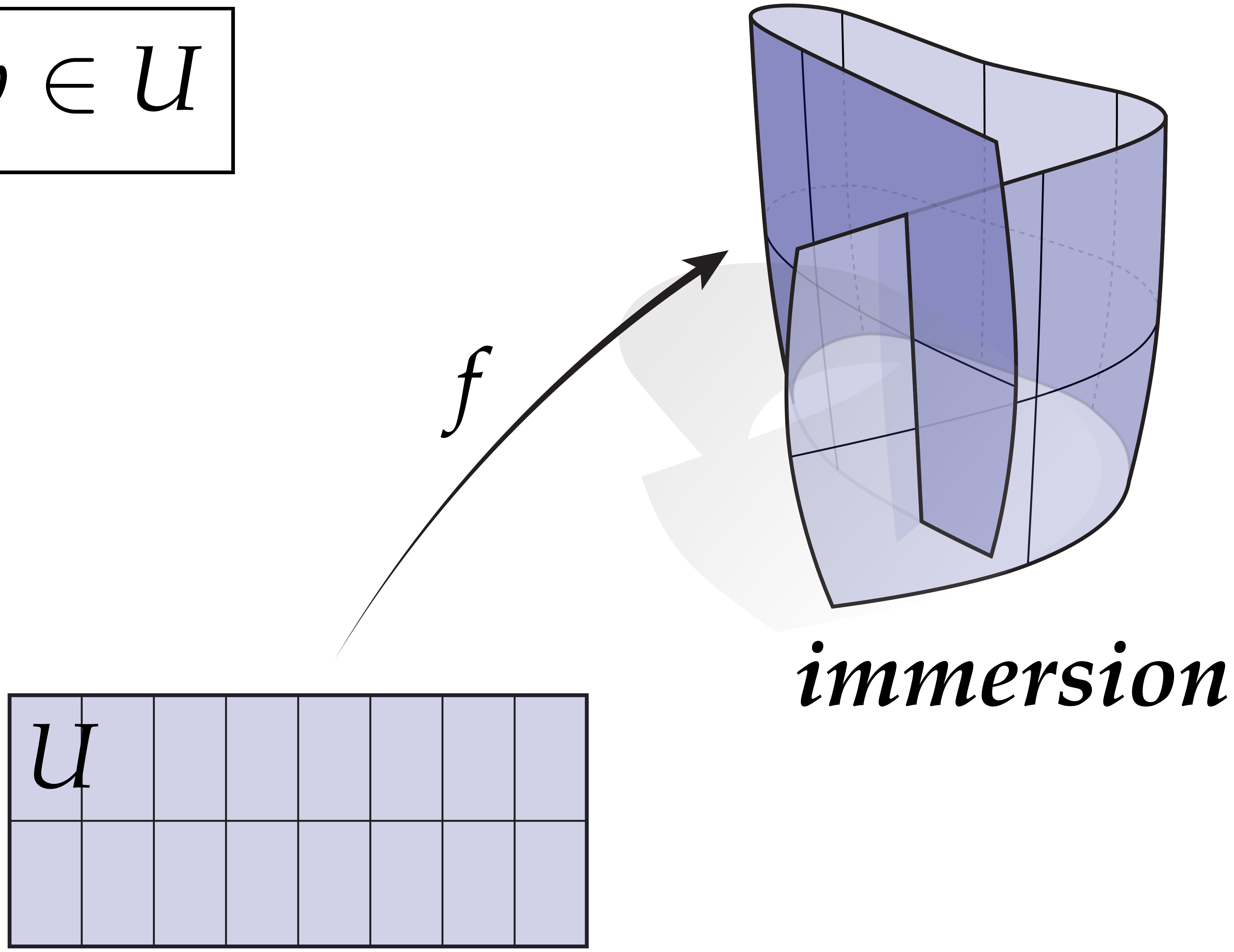
Note: does not generalize to infinite dimensions! (*E.g.*, maps between functions.)

Immersed Surface

A map $f : U \to \mathbb{R}^n$ is an *immersion* if its differential *df* is nondegenerate

$$\boxed{df(X)|_p = 0 \iff X|_p = 0 \qquad \forall p \in U}$$

Key idea: as long as surface is immersed, quantities like tangents, normals, *etc.*, are well-defined—*even if there are self-intersections!*



<u>**Fact.**</u> Any immersion is locally an embedding and vice versa (C^1).

Intuition: no region of the surface gets "pinched" / "squashed"

Immersion — Example

Consider the standard parameterization of the sphere:

$$f(u,v) := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

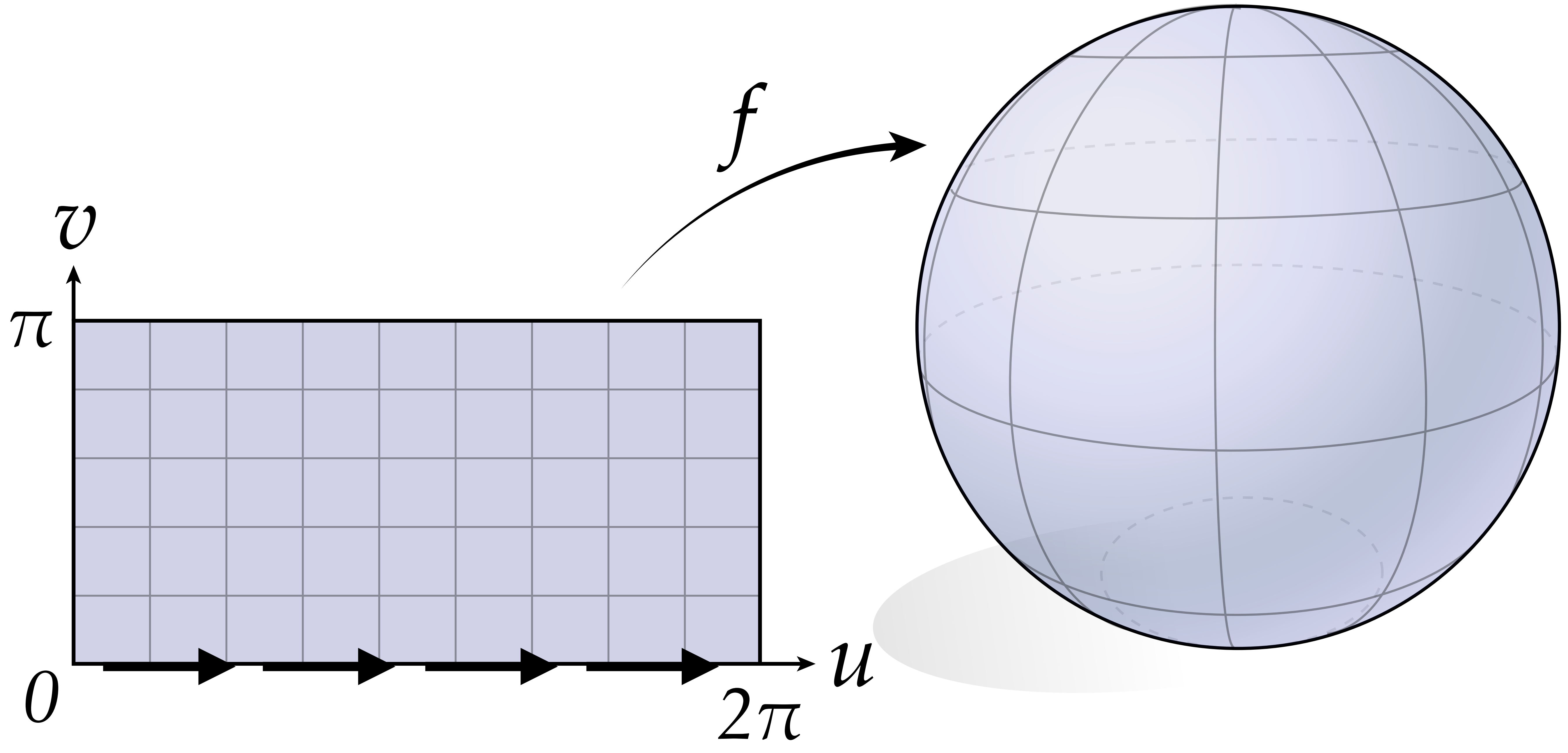
$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = \begin{array}{lcccr} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &)\; du\; + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &)\; dv \end{array}$$

Q: Is f an immersion?

A: No: when $v = 0$ we get

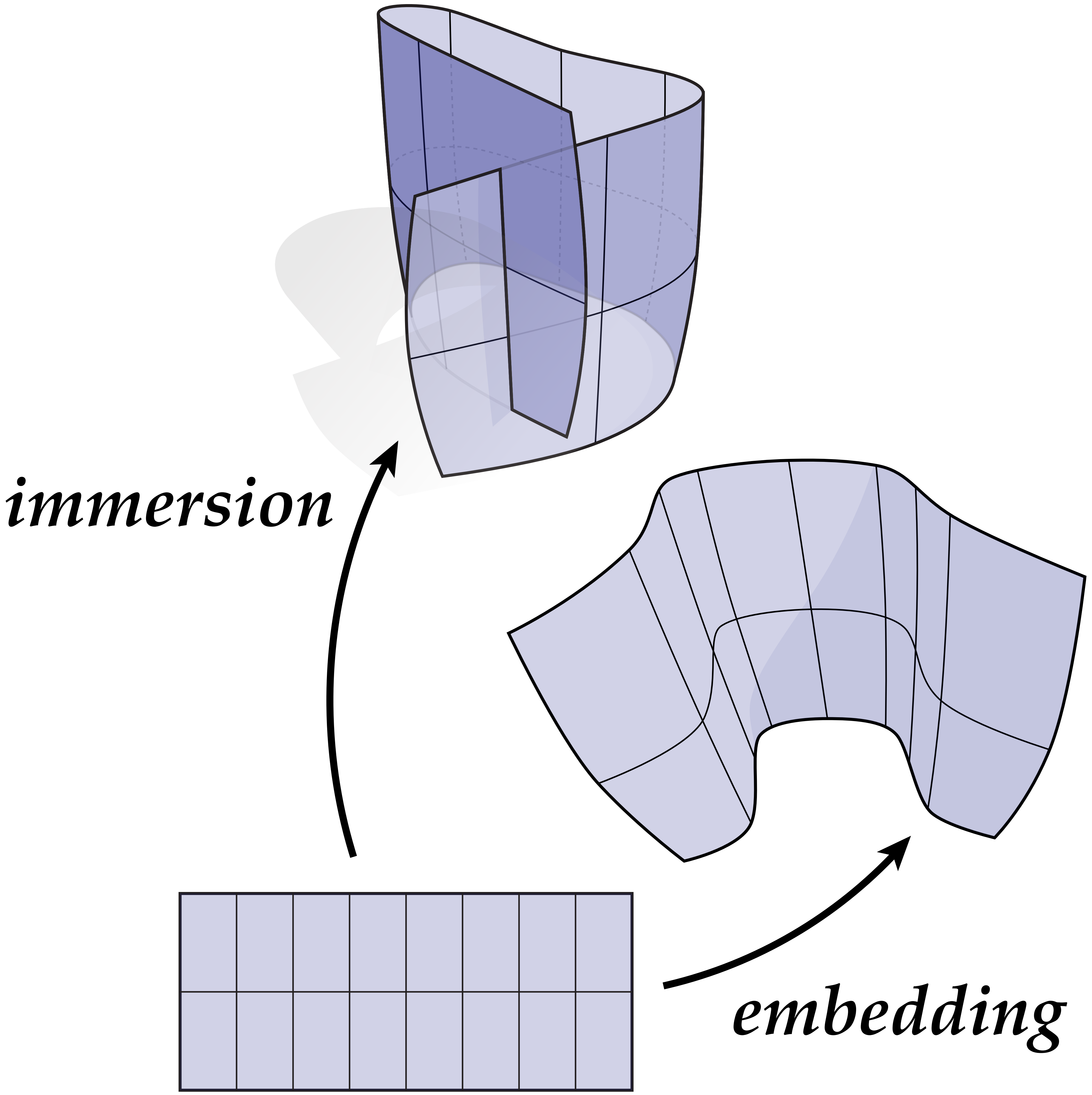
$$\begin{array}{lcccr} (& 0, & 0, & 0 &)\; du\; + \\ (& \cos(u), & \sin(u), & -\sin(v) &)\; dv \end{array}$$

Can't walk "east/west" at poles!



Immersion vs. Embedding

- Immersions are fairly common notion of "nice" / "regular surface: can pass through themselves (non-physical) but still provide <u>local</u> quantities like tangents, normals, metric, etc.
- Immersions also natural model for how we often think about discrete geometry: <u>local</u> quantities (angles, lengths, etc.) are perfectly well-defined, even if there happen to be self-intersections
- Ensuring a surface is immersed **(local)** typically easier than ensuring it is embedded **(global)**
  - roughly speaking: sum of two immersions is "usually" immersion; less likely for embeddings
  - *e.g.*, mesh with random vertex coordinates will likely be immersed but not embedded



Regular Homotopy

- *Regular homotopy* notion of "nice motion"
  - no pinches, sharp creases, or stops
- More formally:

$$f_0, f_1 : U \to \mathbb{R}^3$$

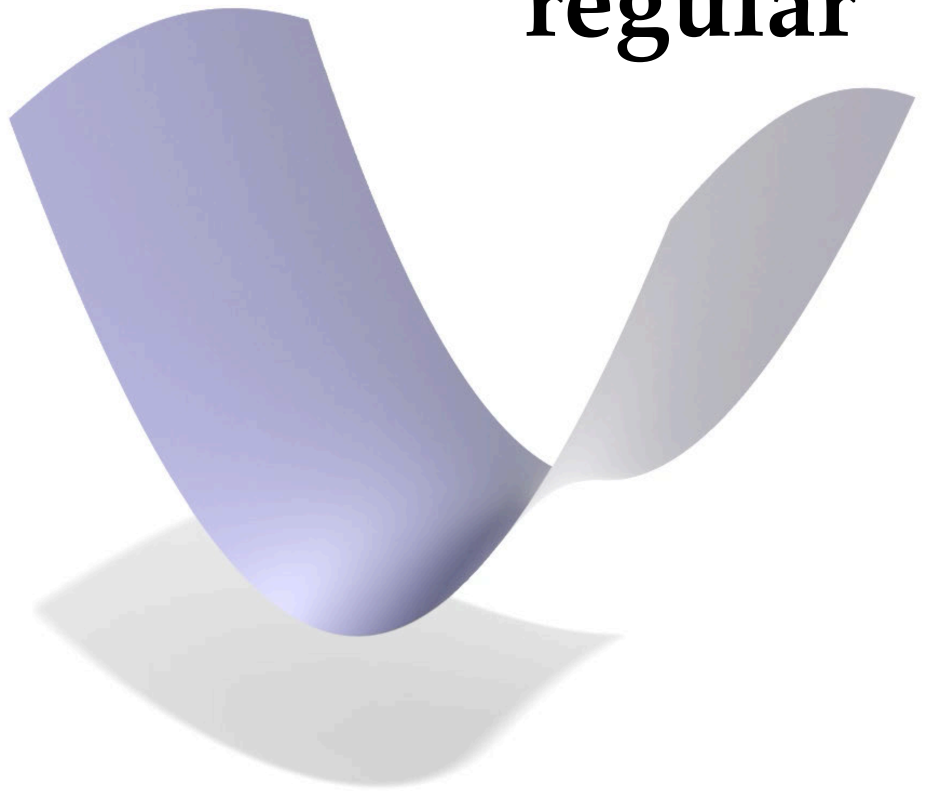
$h : U \times [0,1] \to \mathbb{R}^3$ continuous

$$h(x,0) = f_0(x)$$

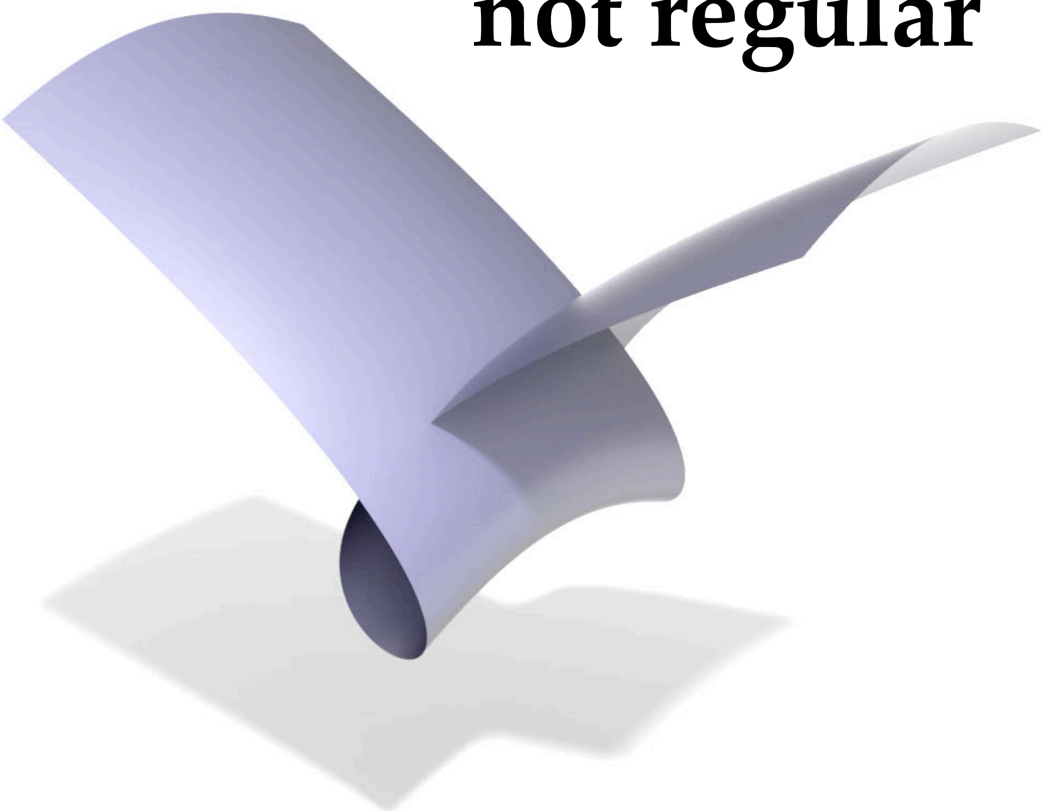
$$h(x,1) = f_1(x)$$

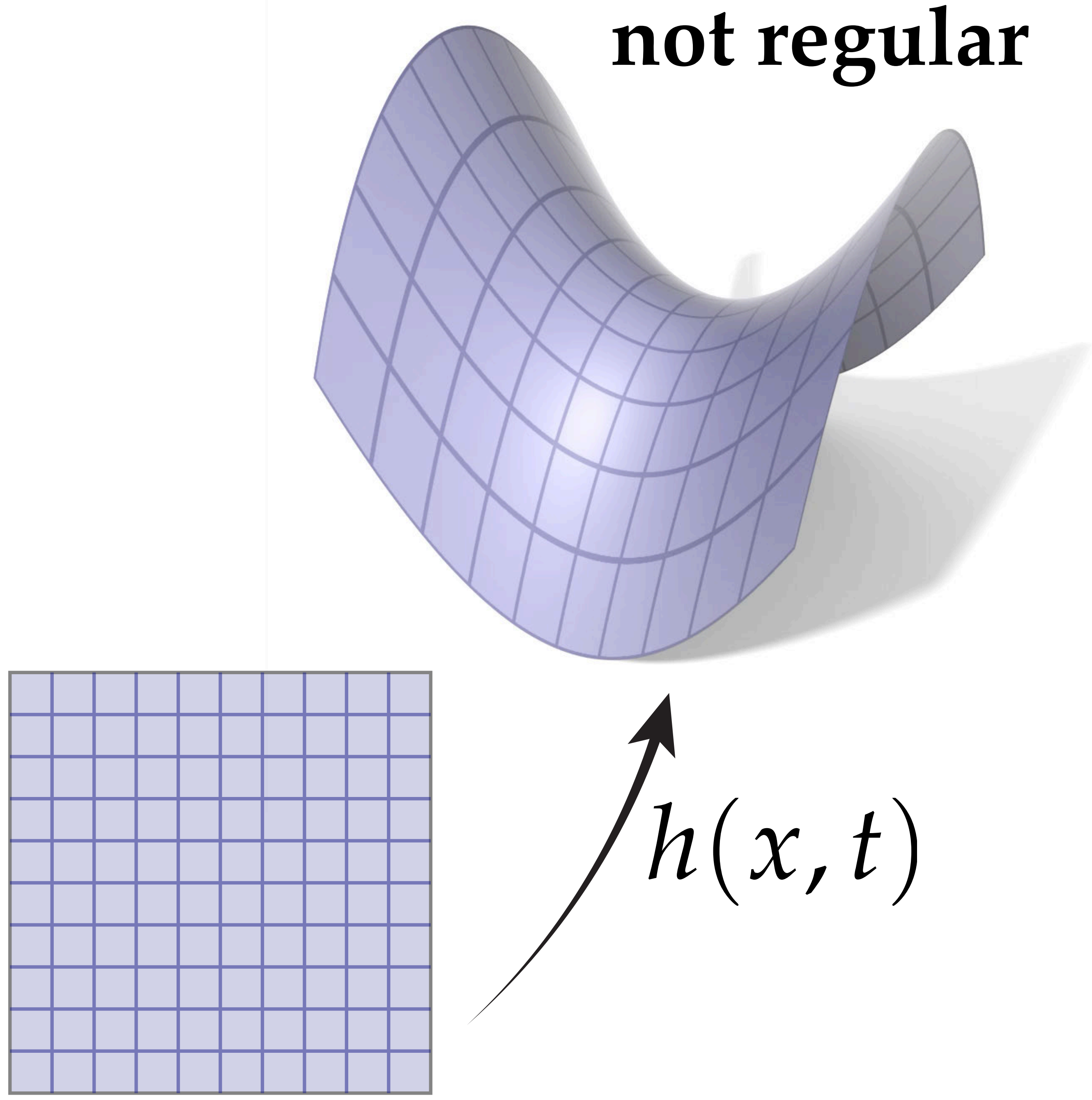
$$\boxed{h(\cdot,t) \text{ is an immersion for all } t}$$

regular



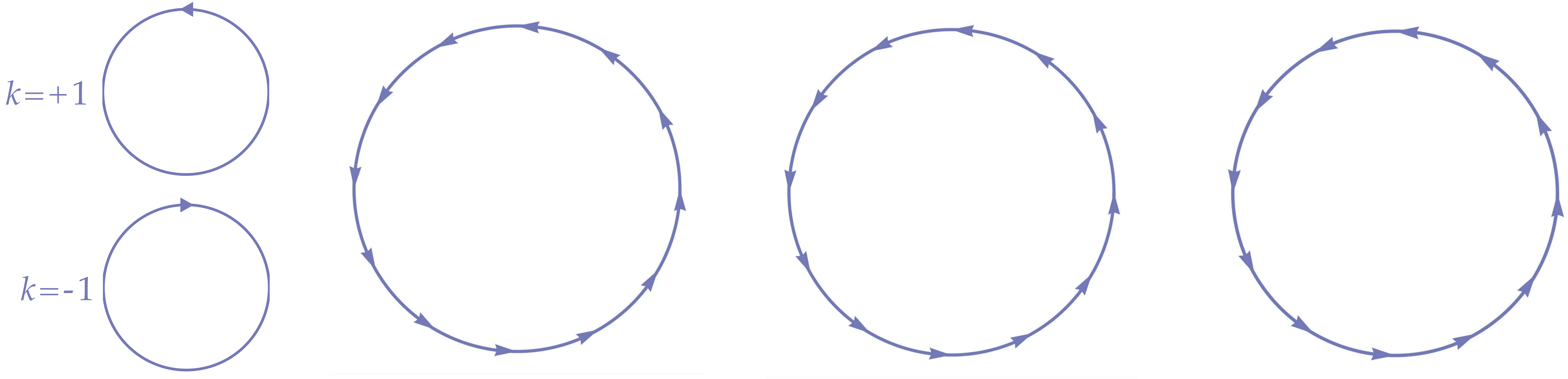
not regular





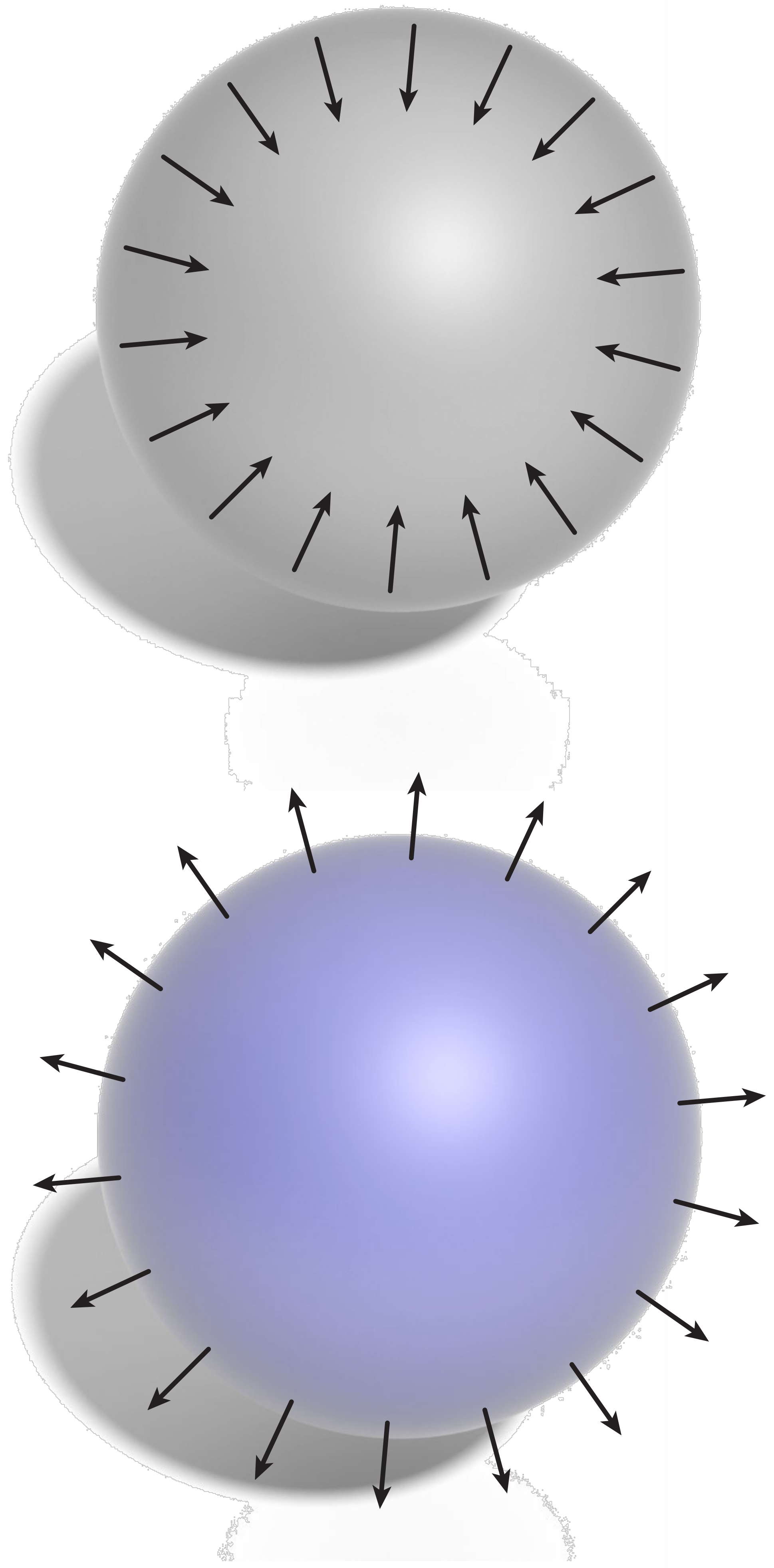
Review: Circle Eversion

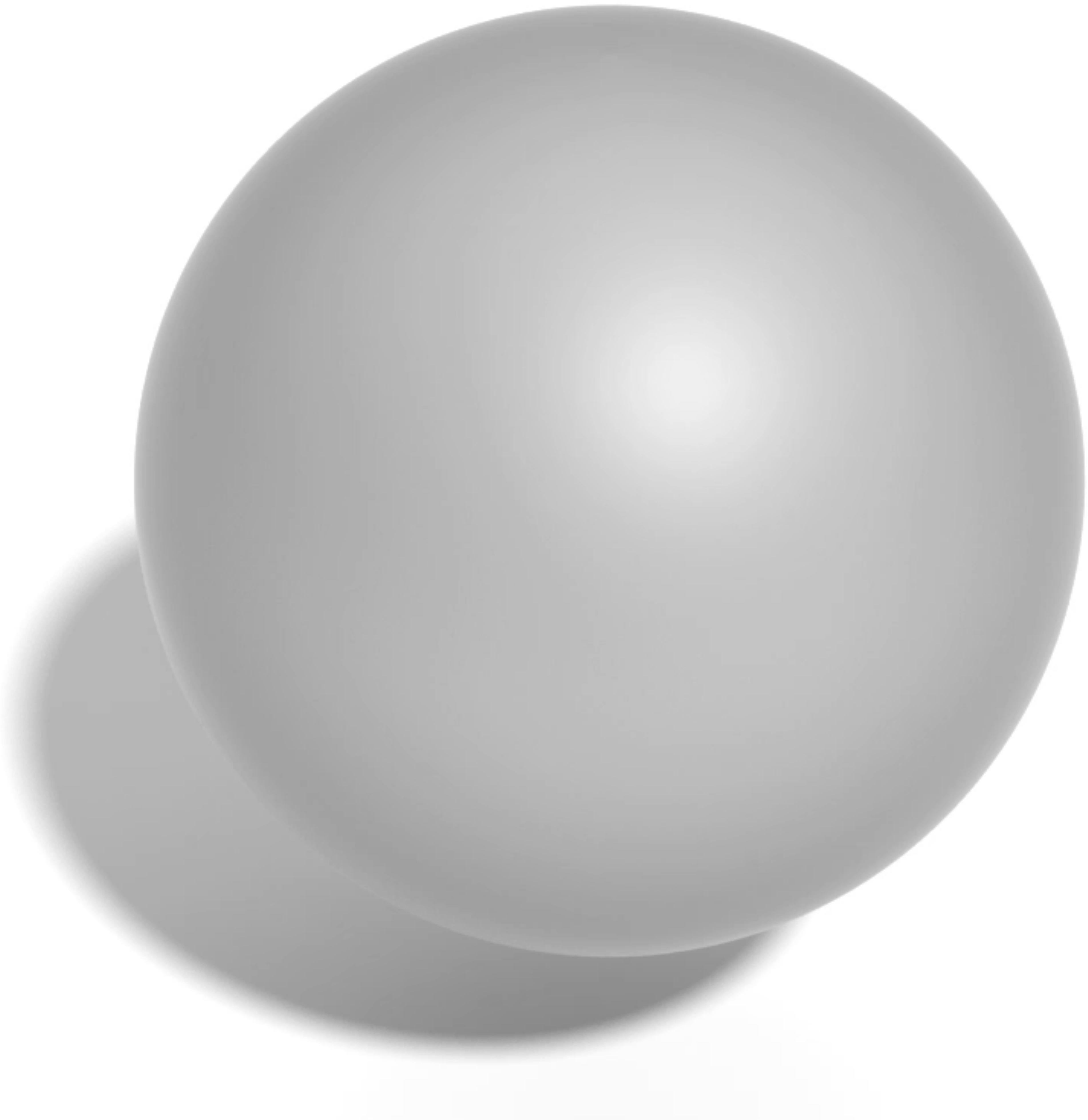
- (Whitney-Graustein) For curves in 2D, turning number determines regular homotopy class
  - e.g., can't turn circle inside-out while remaining immersed



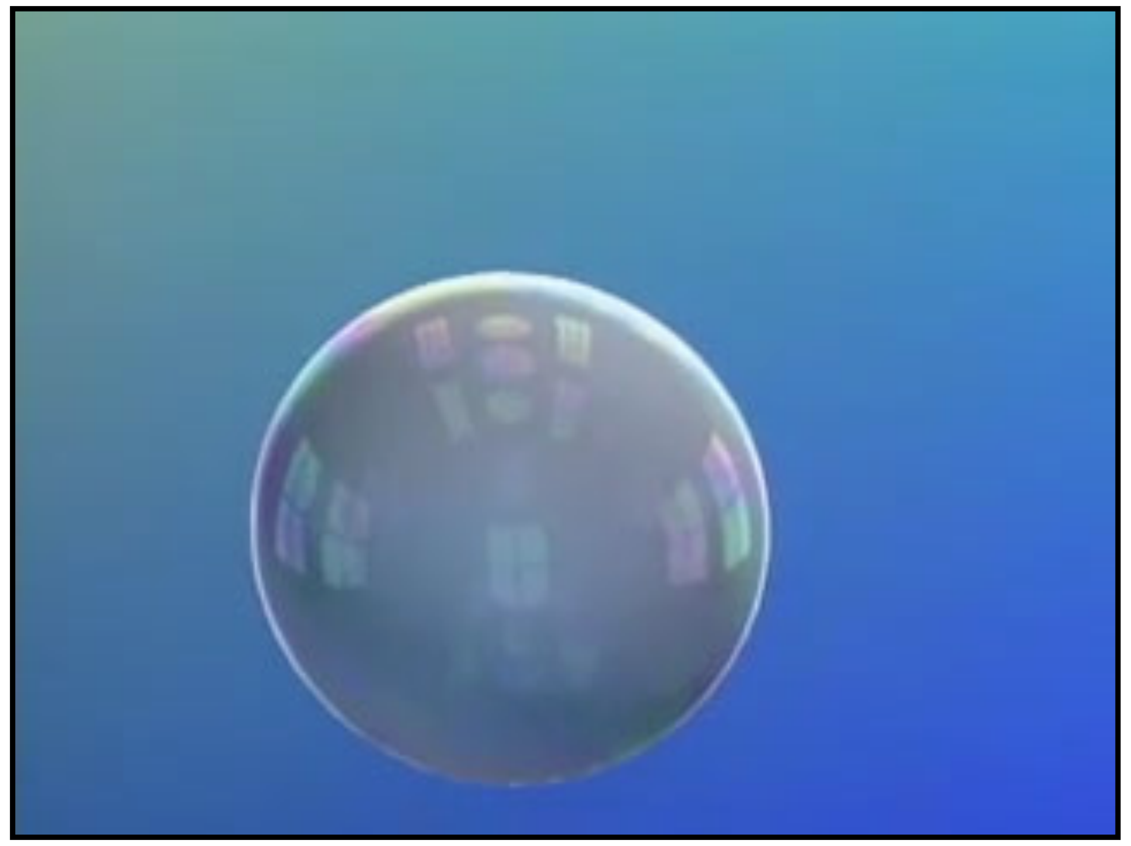
What about surfaces in 3D? (Can you turn the sphere inside out?)

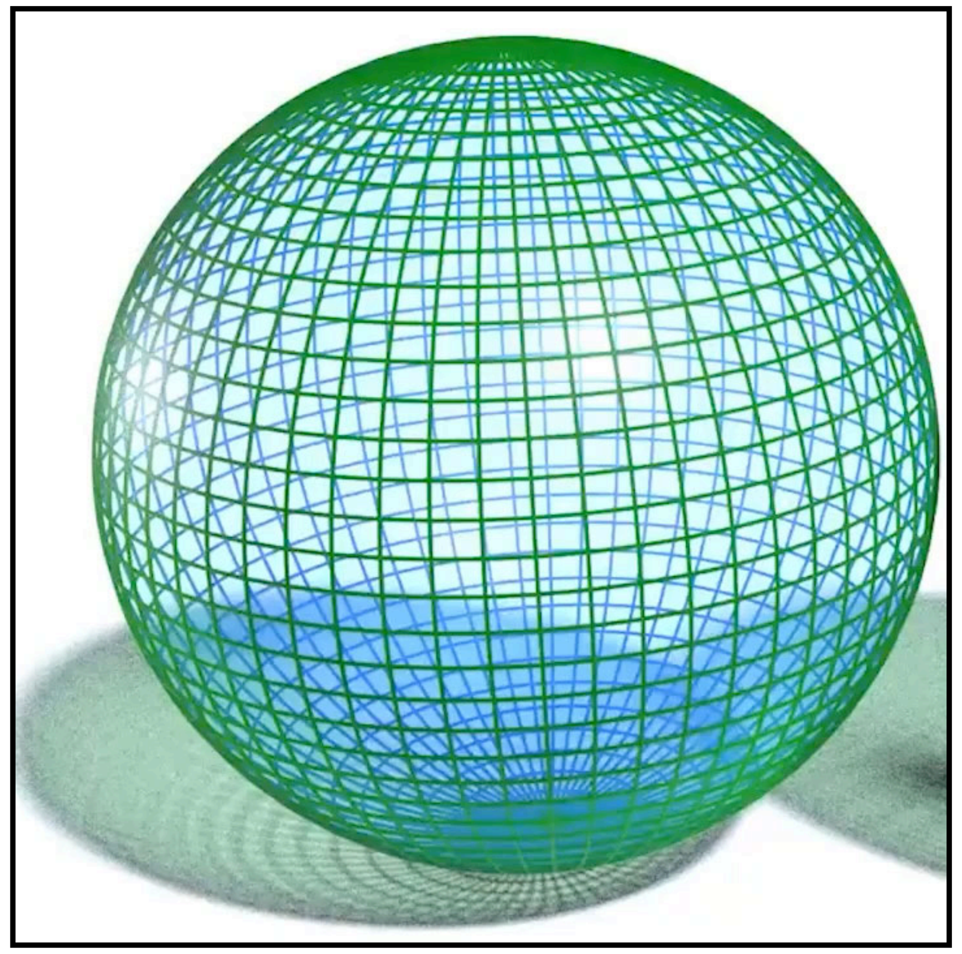
Sphere Eversion





"Optiverse" eversion
Francis/Kusner/Sullivan

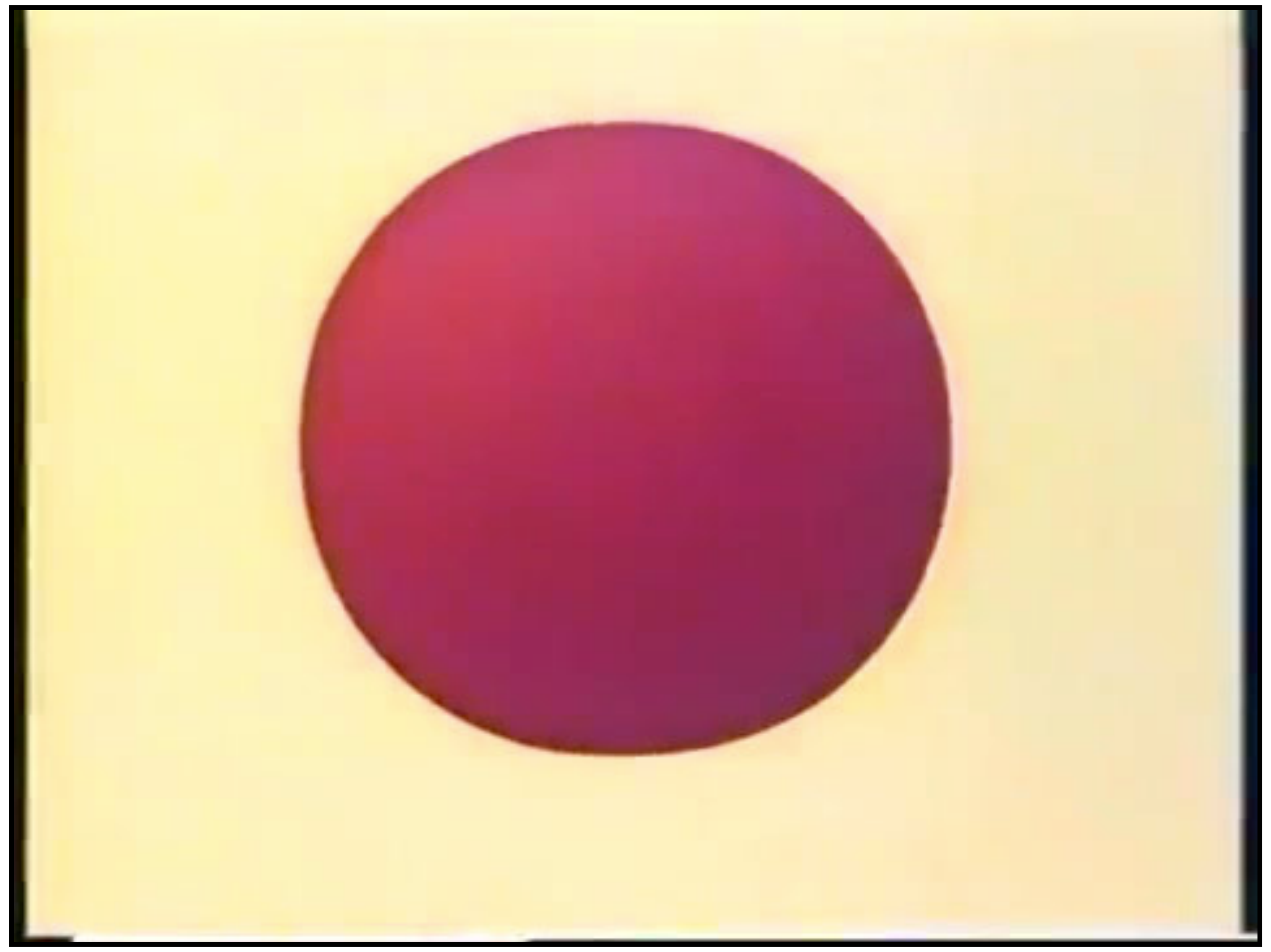




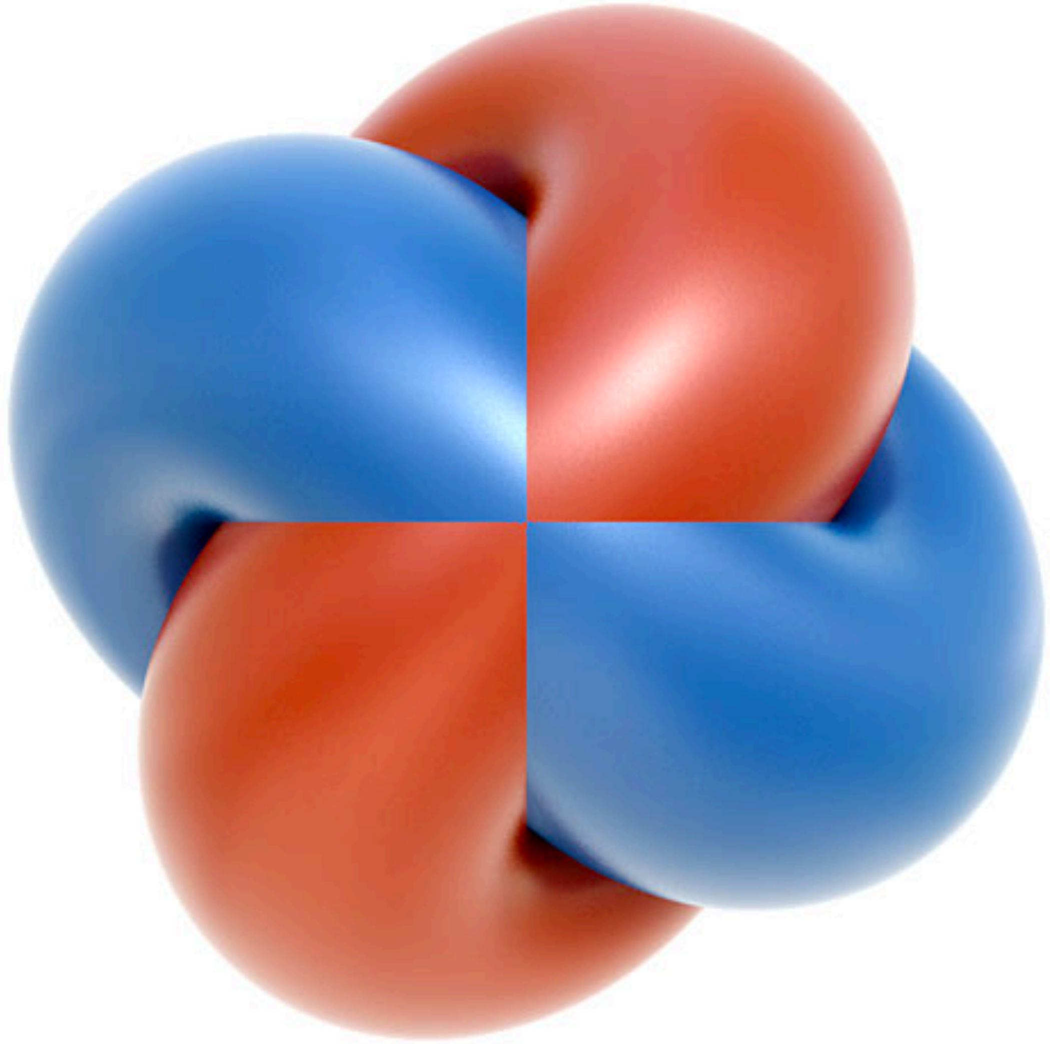
ruled eversion
(Bednorz & Bednorz; Padilla)

See video: *Outside In* (Thurston/Geometry Center, 1994)

Morin Sphere Eversion



"Turning the Sphere Inside Out" (1976)
(director: Nelson Max)

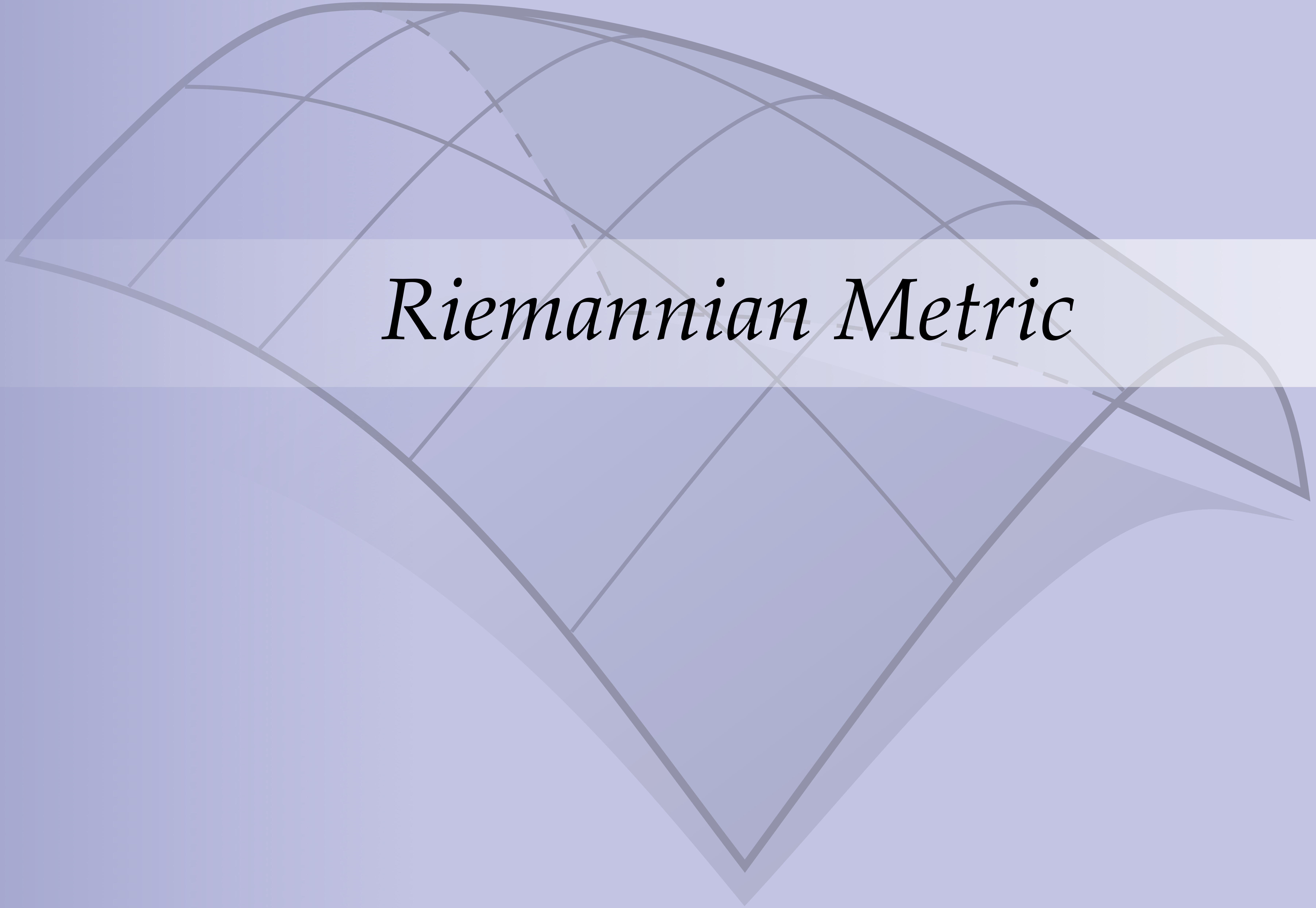


3D prints by Arnaud Chéritat

"Our spatial imagination is framed by manipulating objects … You act on objects with your hands, not with your eyes"

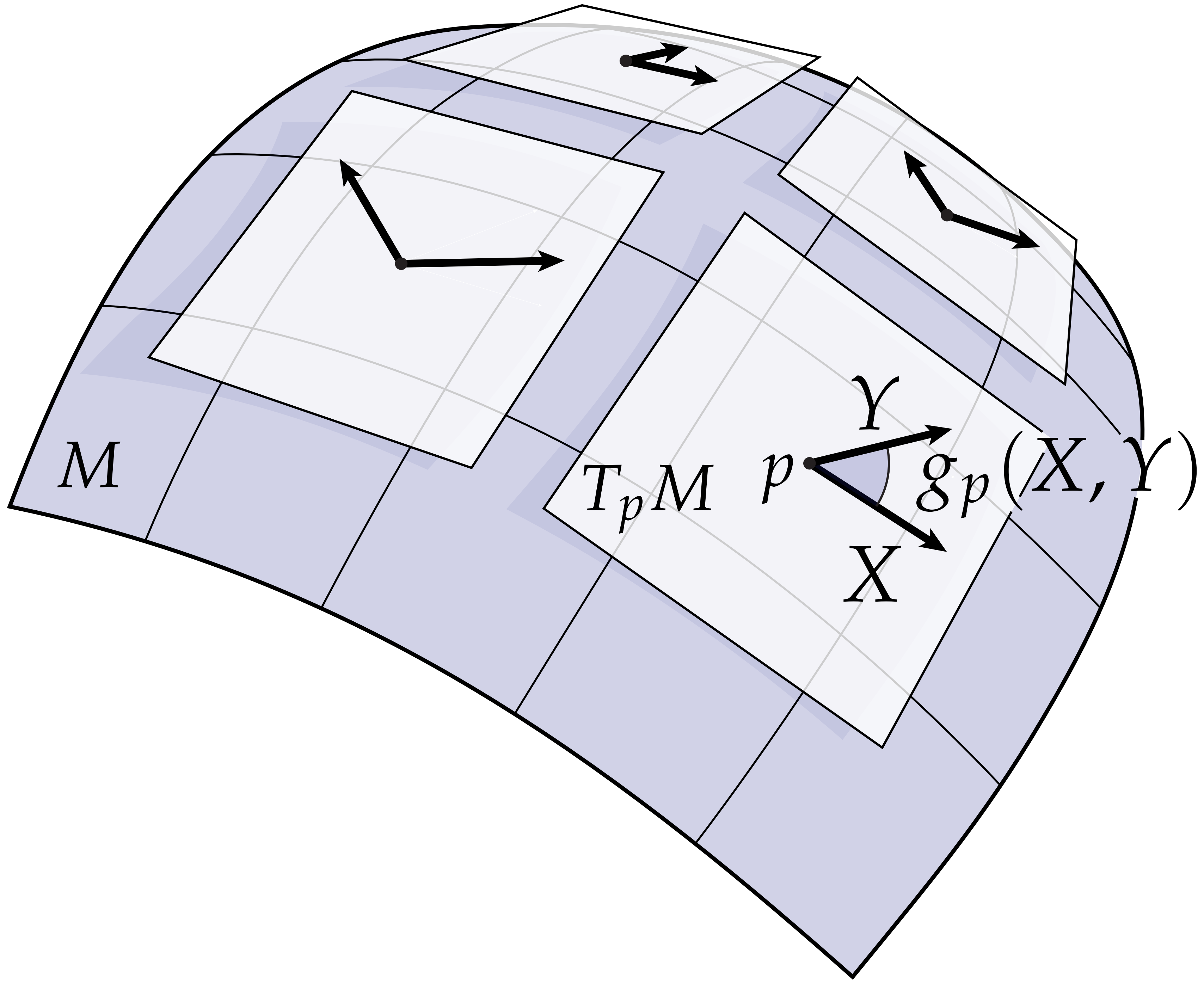
—Bernhard Morin

Riemannian Metric



Riemann Metric

- Many quantities on manifolds (curves, surfaces, *etc.*) ultimately boil down to measurements of *lengths* and *angles* of tangent vectors X, Y
- This information is encoded by the so-called *Riemannian metric** $g(X,Y)$
  - abstractly: smoothly-varying positive-definite bilinear form
- For immersed surface, can (and will!) describe more concretely/geometrically

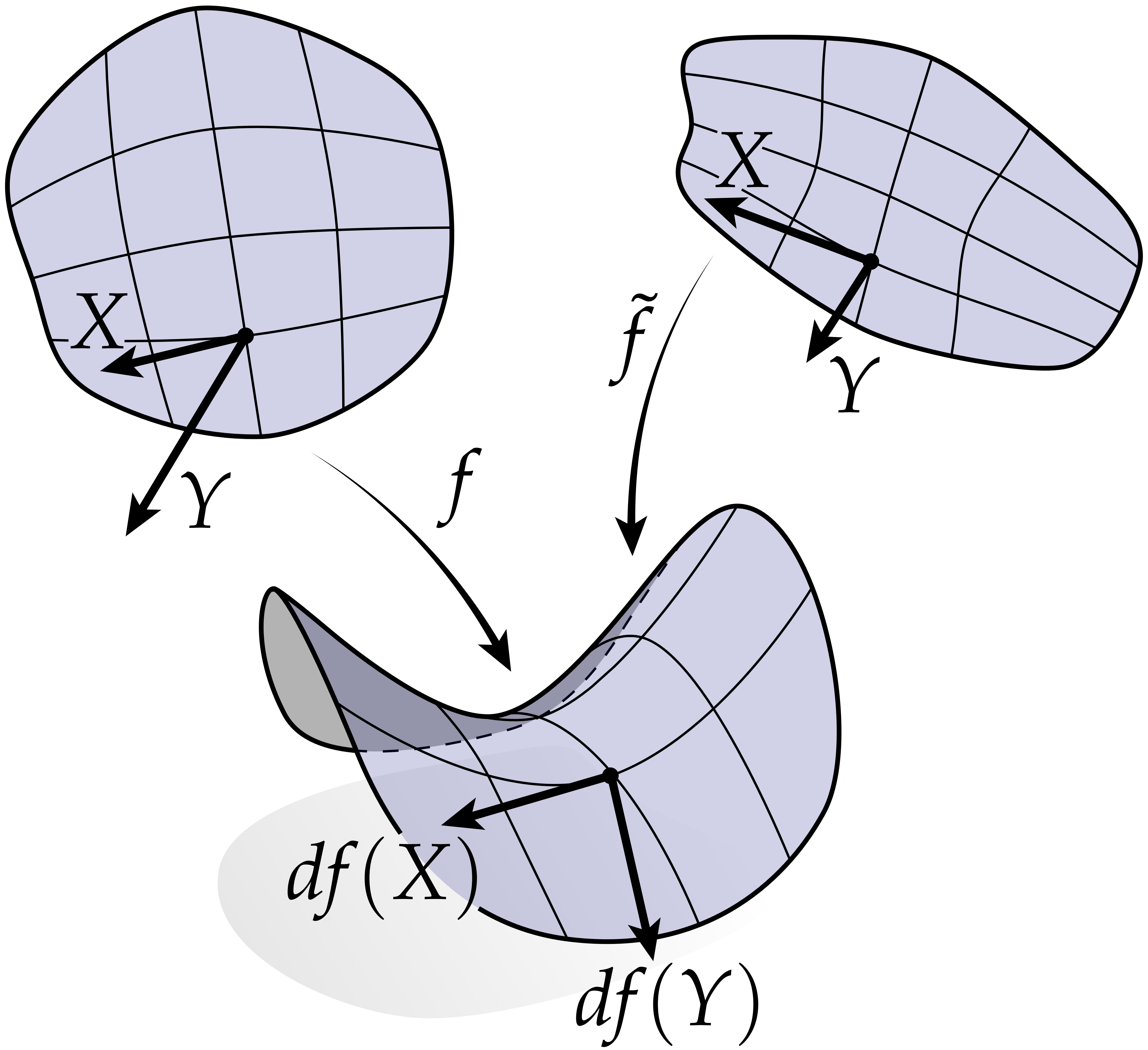


***Note:** *not* the same as a point-to-point distance metric $d(x,y)$

Metric Induced by an Immersion

- For an immersion f, how should we measure inner product of vectors X, Y represented in its domain U?
  - should **not** use the usual inner product on the plane! (Why not?)
- Planar inner product tells us *nothing* about actual length & angle on the surface—gives the same result for any parameterization!
- Instead, use **induced metric**

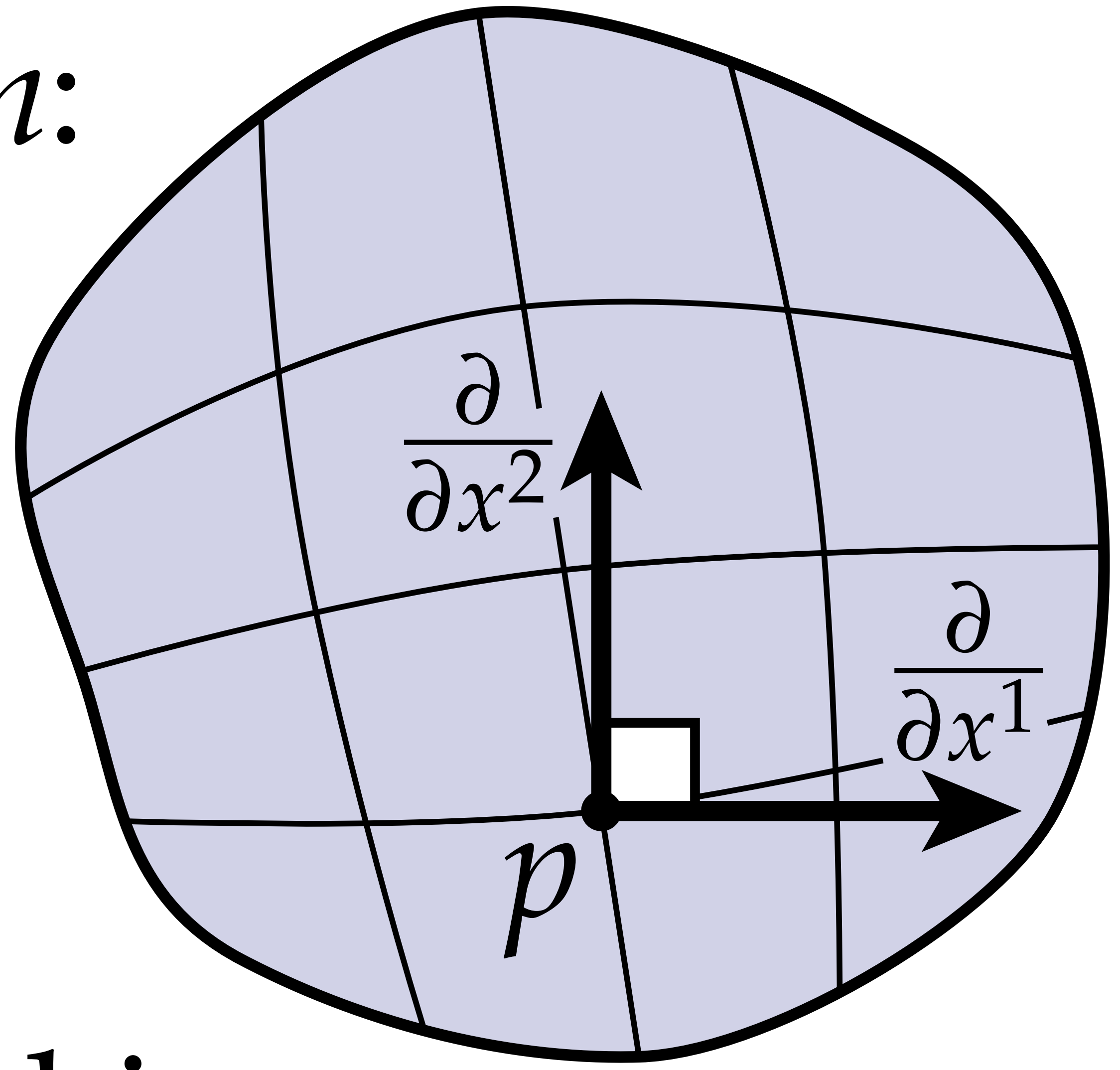
$$g(X,Y) := \langle df(X), df(Y) \rangle$$



Key idea: induced metric accounts for "stretching"

Induced Metric—Matrix Representation

- Metric is a bilinear map from a pair of vectors to a scalar, which we can represent as a 2x2 matrix **I** called the *first fundamental form*:



$$g(X, Y) = X^T \mathbf{I} Y$$

$$\Rightarrow \mathbf{I}_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle df\left(\frac{\partial}{\partial x^i}\right), df\left(\frac{\partial}{\partial x^j}\right)\right\rangle$$

- Alternatively, can express first fundamental form via Jacobian:

$$g(X, Y) = \langle df(X), df(Y)\rangle = (J_f X)^\mathsf{T}(J_f Y) = X^\mathsf{T}(J_f^\mathsf{T} J_f)Y$$

$$\Rightarrow \mathbf{I} = J_f^\mathsf{T} J_f$$

- Note: depends on the point p—could write $g_p(X, Y) = \langle df_p(X), df_p(Y)\rangle$

Induced Metric—Example

Can use the differential to obtain the induced metric:

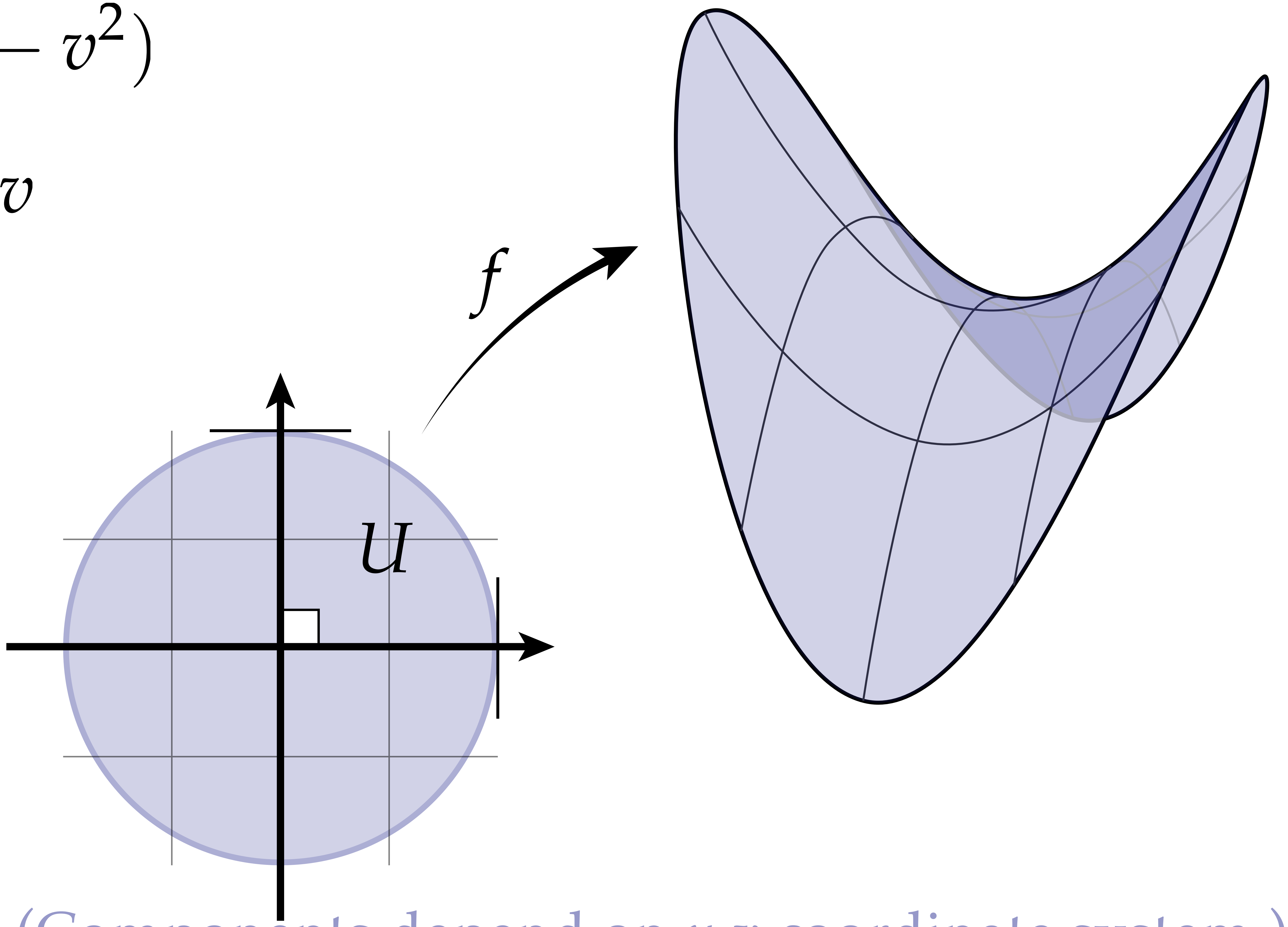
$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u,v,u^2-v^2)$$

$$df = (1,0,2u)du + (0,1,-2v)dv$$

$$J_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

$$\mathbf{I} = J_f^{\mathsf{T}} J_f$$

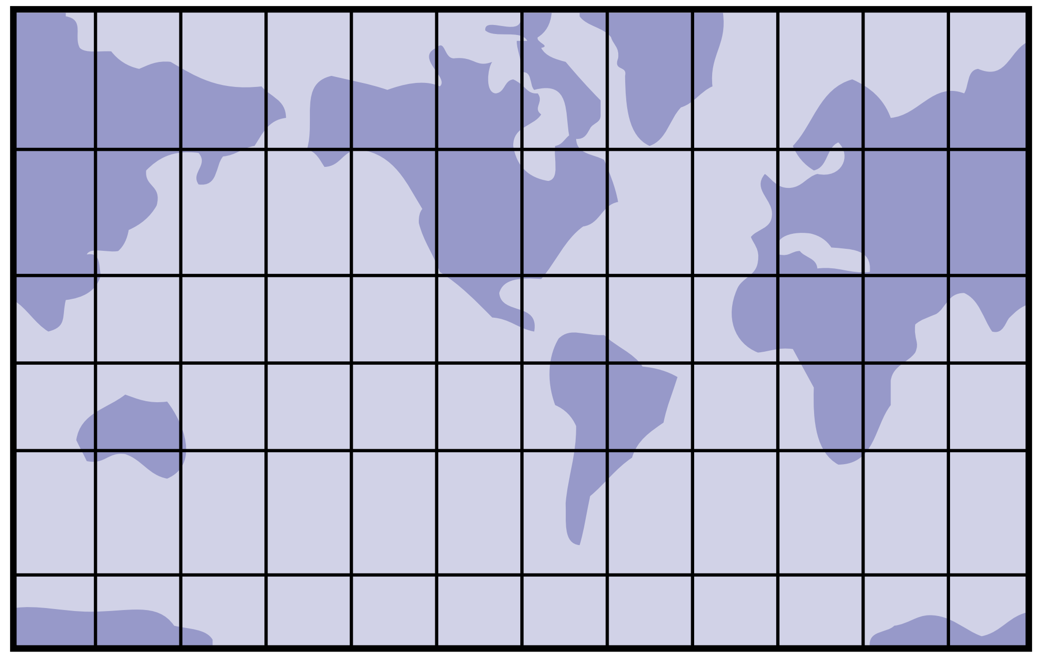
$$= \begin{array}{cc} \begin{array}{cc} \qquad\quad E & \qquad\qquad F \end{array} & \\ \begin{bmatrix} 1+4u^2 & -4uv \\ -4uv & 1+4v^2 \end{bmatrix} & G \end{array}$$

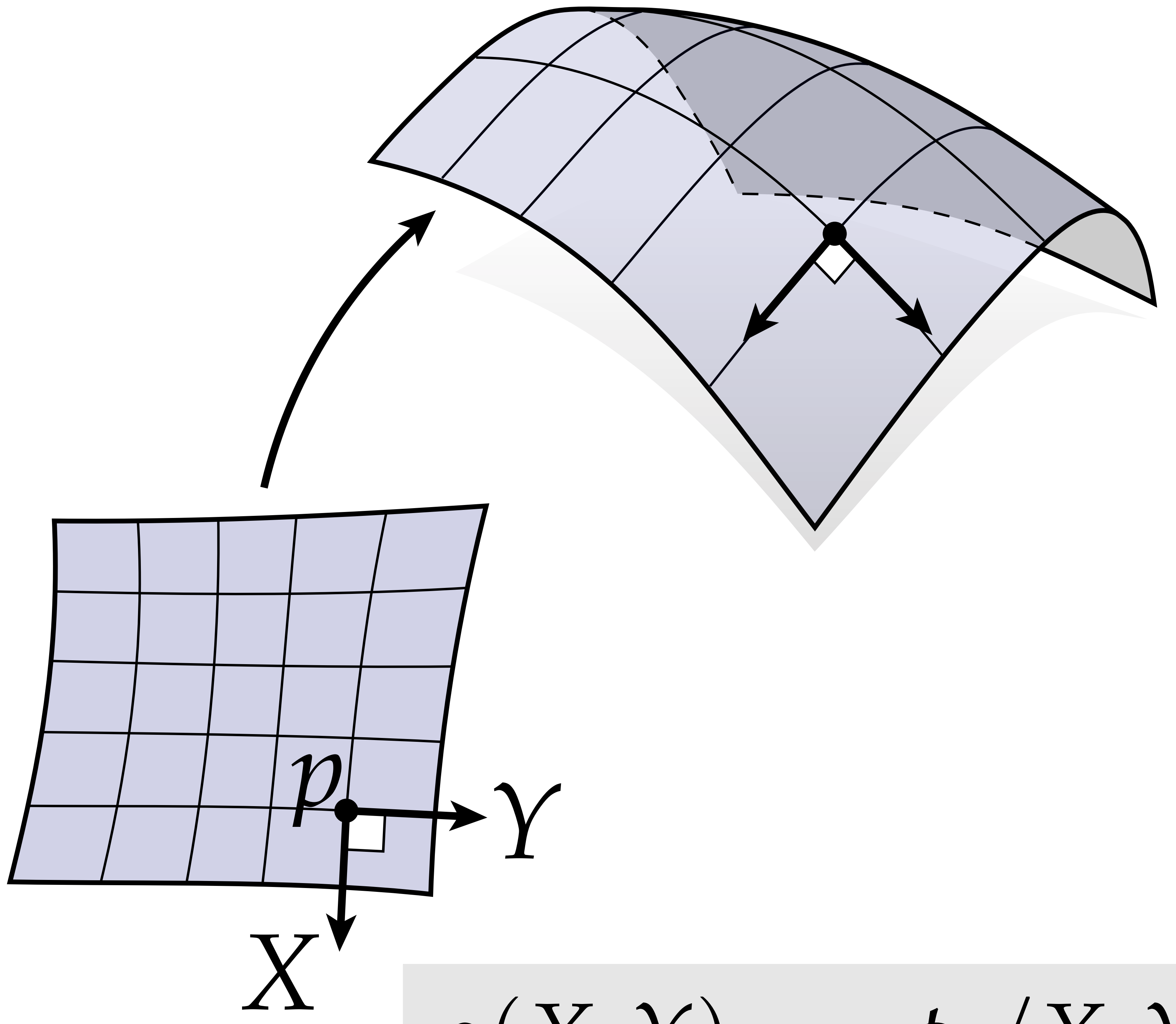


(Components depend on u,v coordinate system.)

Conformal Coordinates

- As we’ve just seen, there can be a complicated relationship between length & angle on the domain (2D) and the image (3D)
- For curves, we simplified life by using an *arc-length* or *isometric* parameterization: lengths on domain are identical to lengths along curve
- For surfaces, usually not possible to preserve all *lengths* (e.g., globe). Remarkably, however, can always preserve *angles* (**conformal**)
- Equivalently, a parameterization *f* is *conformal* if at each point the induced metric is simply a positive rescaling of the 2D Euclidean metric
  - one *coordinate-invariant* number, rather than three



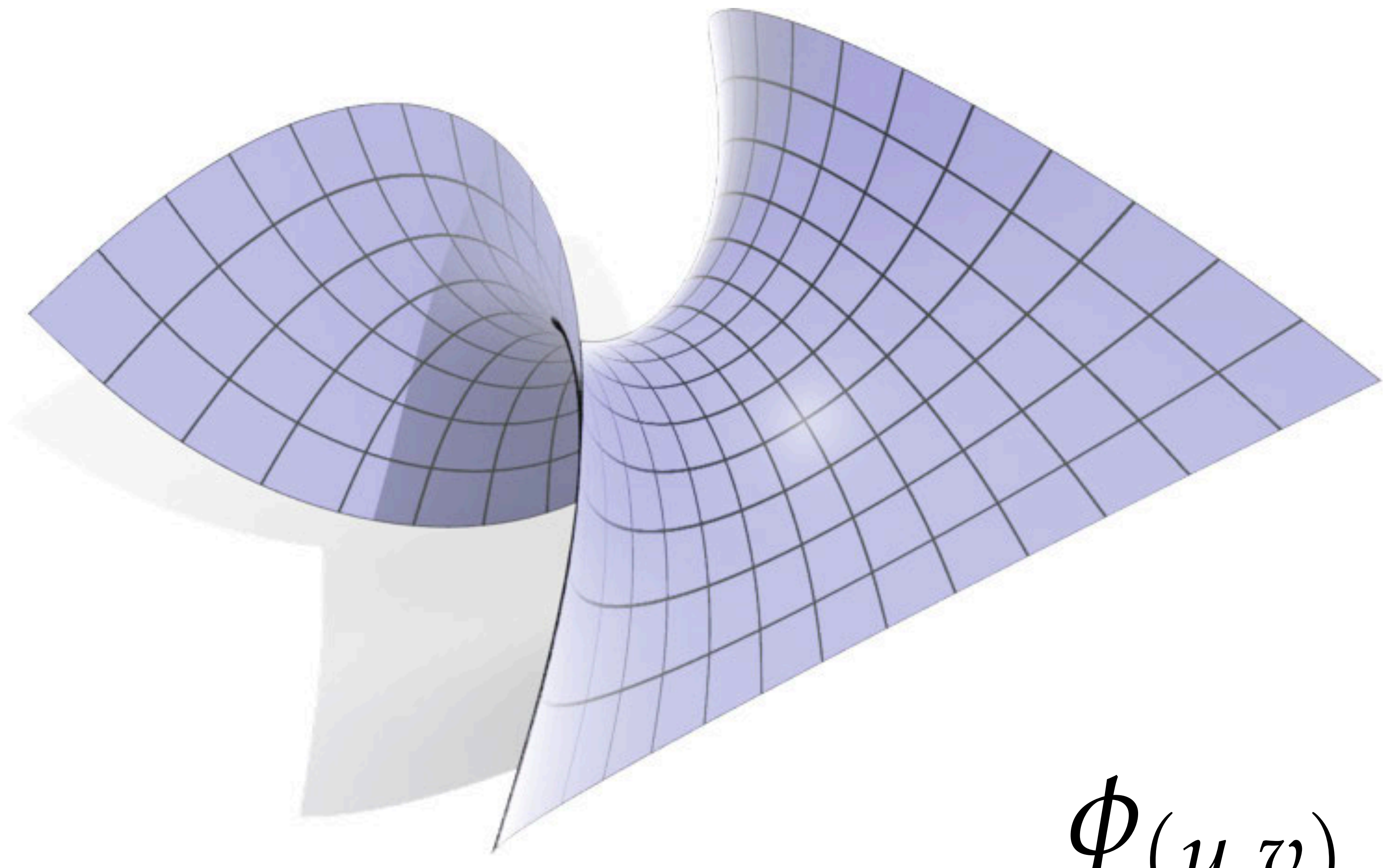


$$g(X,Y)_p = \phi_p\langle X,Y\rangle$$

Example (Enneper Surface)

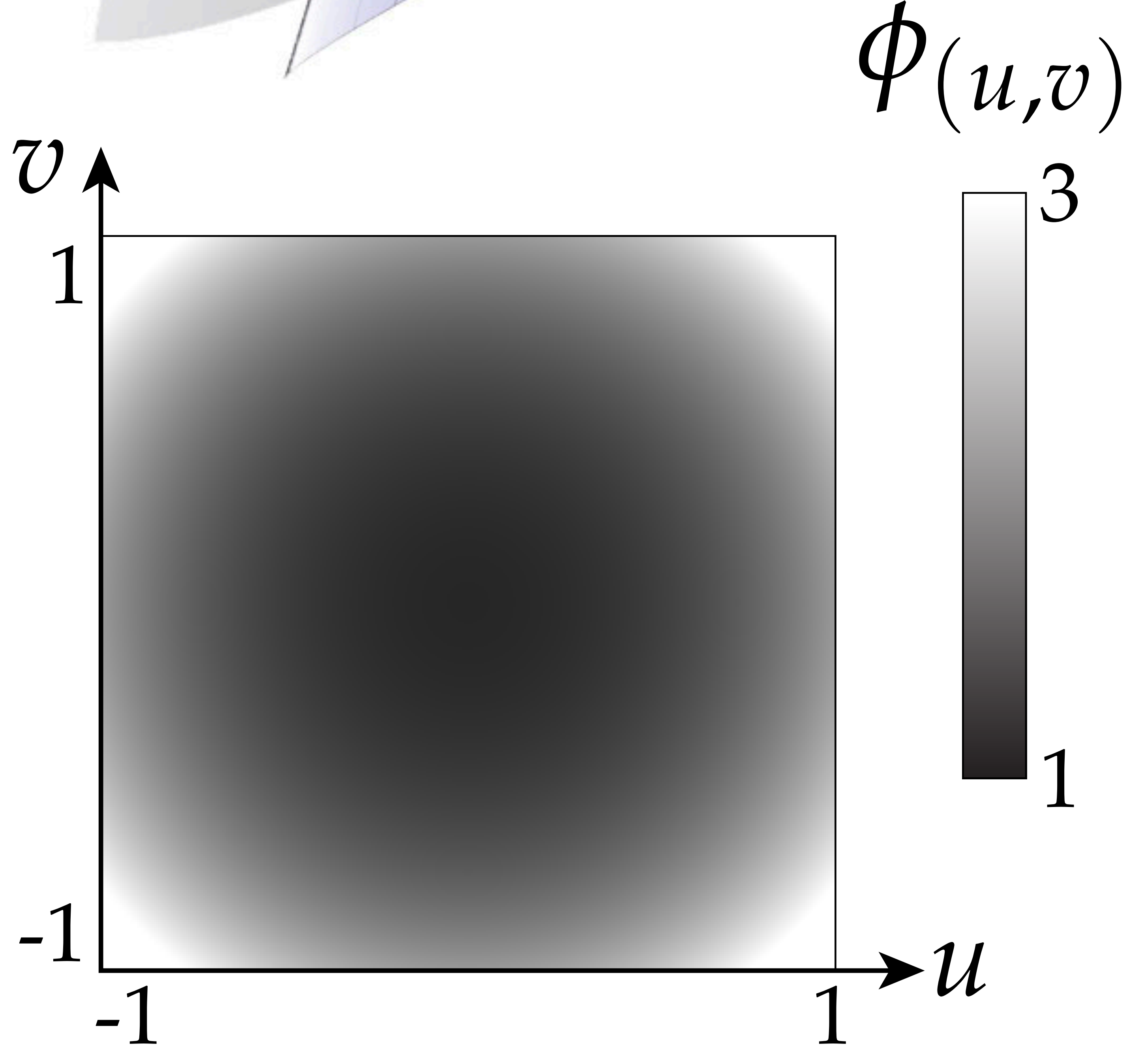
Consider the surface

$$f(u,v) := \begin{bmatrix} uv^2 + u - \frac{1}{3}u^3 \\ \frac{1}{3}v\left(v^2 - 3u^2 - 3\right) \\ (u-v)(u+v) \end{bmatrix}$$



Its Jacobian matrix is

$$J_f = \begin{bmatrix} -u^2 + v^2 + 1 & 2uv \\ -2uv & -u^2 + v^2 - 1 \\ 2u & -2v \end{bmatrix}$$



Its metric then works out to be just a scalar function times the usual metric of the Euclidean plane:

$$\mathbf{I} = J_f^T J_f = \left(u^2 + v^2 + 1\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This function is called the *conformal scale factor*.

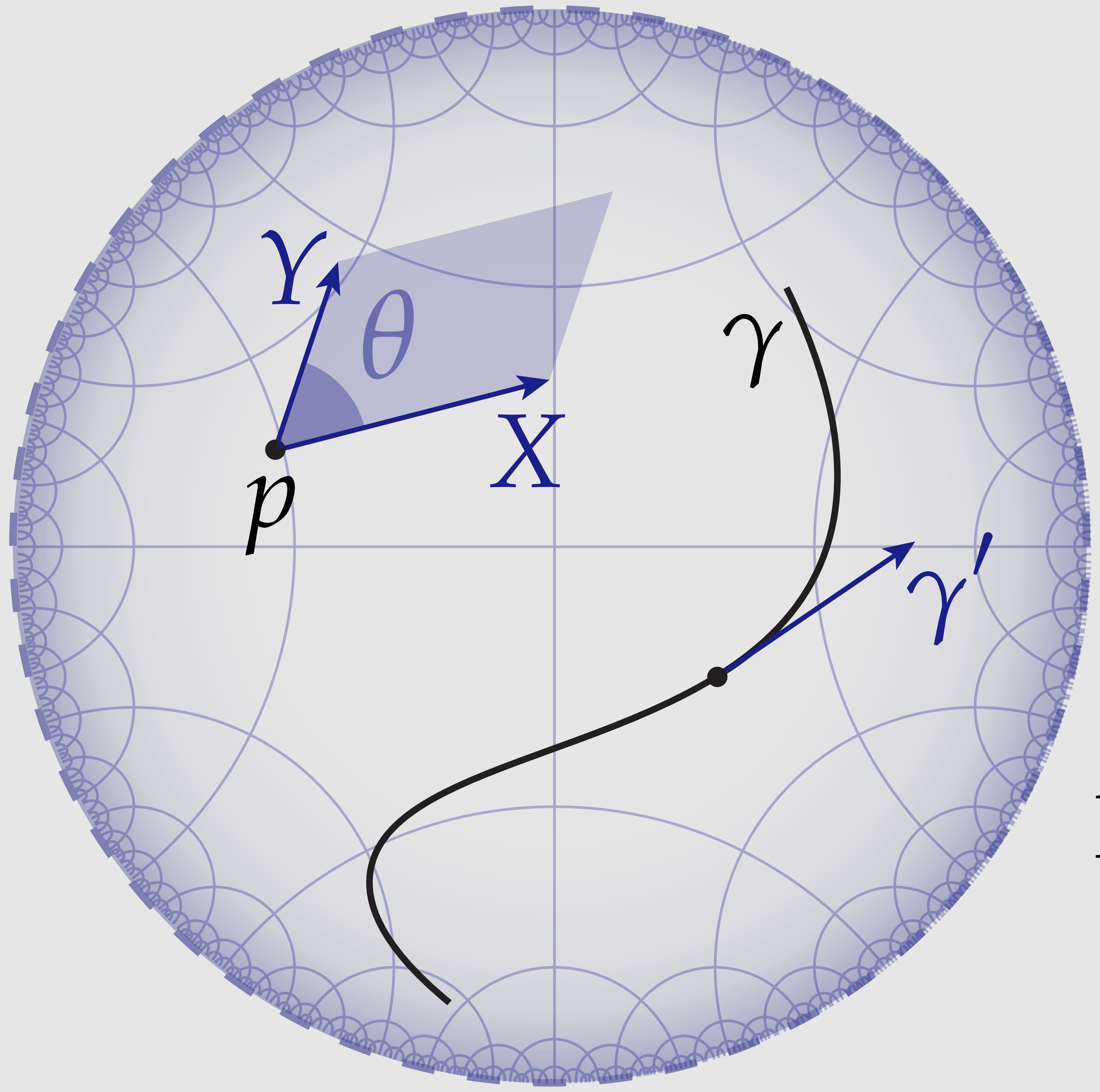
Abstract Riemannian Metric

- Ultimately, induced Riemannian metric is just a (smoothly-varying) inner product at each point
- Suppose we just write down some arbitrary smoothly-varying inner product (*Riemannian metric*)
- **(Intrinsic viewpoint)** Key idea in differential geometry: don't need to know "where this metric came from"/how it sits in space
  - given only an inner product, can still measure angles, lengths, areas, distances, … via the usual formulas

Example: *hyperbolic metric* on unit disk.

$$U := \{p \in \mathbb{R}^2 : |p| < 1\}$$

$$g_p(X, Y) = \frac{4}{(1 - |p|^2)^2} \langle X, Y \rangle$$



$$|X| = \sqrt{g_p(X, X)}$$

$$\theta = \arccos\left(g_p(X/|X|, Y/|Y|)\right)$$

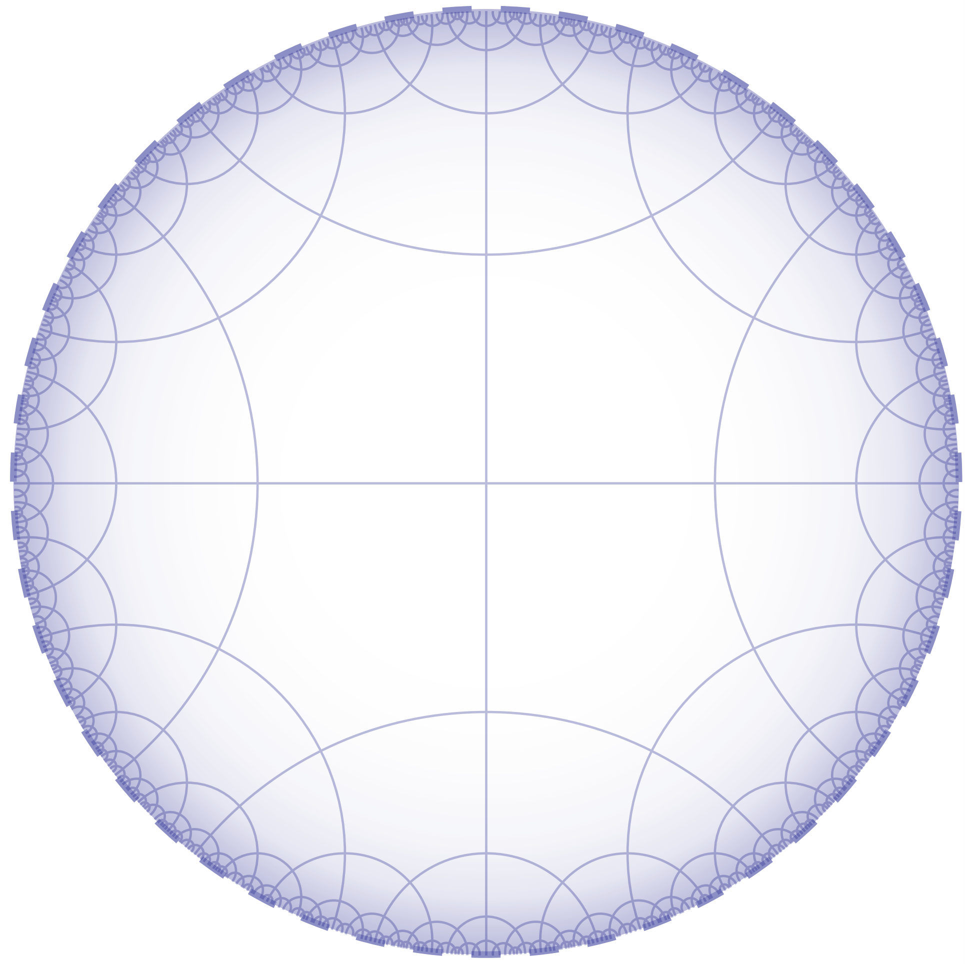
$$\mathrm{area}(X, Y) = \sqrt{\det(g_p)}(X \times Y)$$

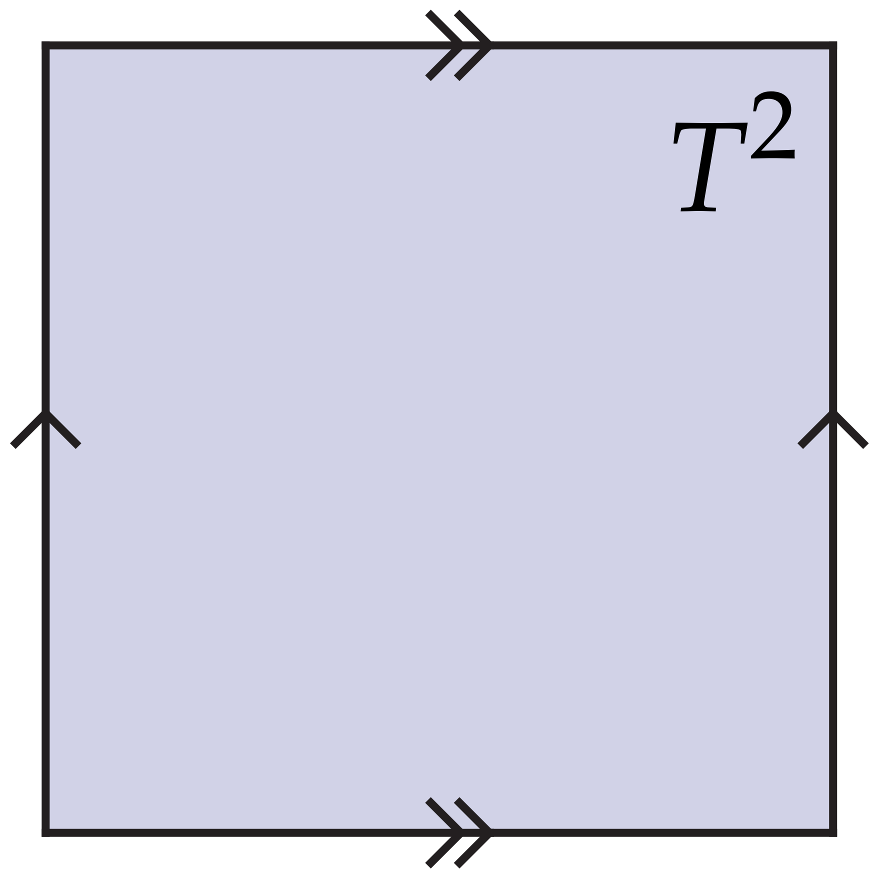
$$\mathrm{length}(\gamma) = \int_0^L g_{\gamma(s)}(\gamma', \gamma')^{1/2}\, ds$$

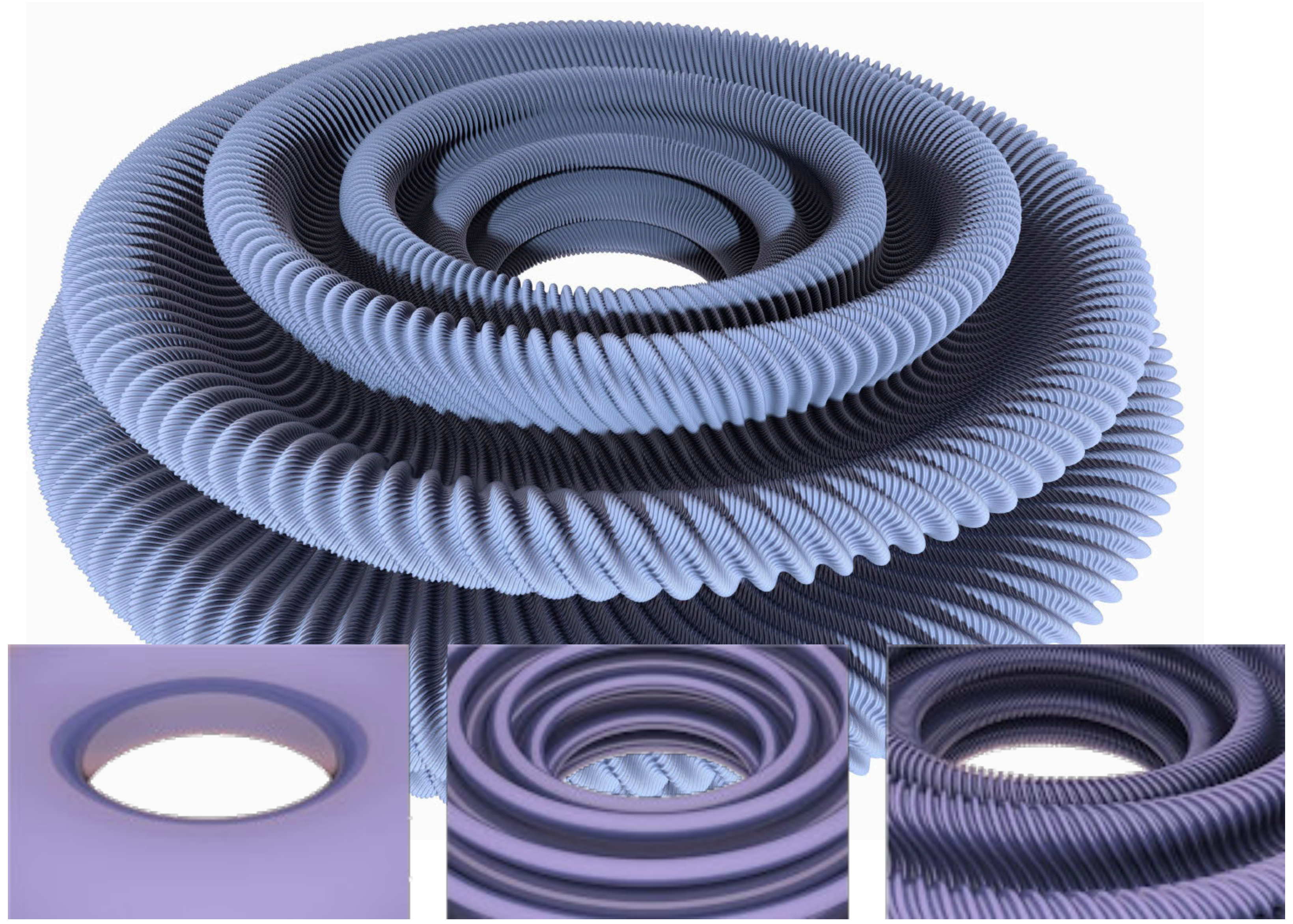
$$\ldots$$

Embedding Theorems

- Still natural to ask: given a Riemannian metric g, can we find an embedding f such that $g(X, Y) = \langle df(X), df(Y)\rangle$?
- Lots of theorems about this topic—depends on things like the continuity of f
- **(Hilbert.)** For instance, *can't* find smooth embedding of hyperbolic metric into $\mathbb{R}^3$
- Positive result: *Nash embedding theorems*
  - always have global C^k embedding in sufficiently high dimension
  - given a "short" embedding (doesn't increase distance), exists a C^1 embedding of n-manifold in $\mathbb{R}^{n+1}$







torus images: Borrelli, Jabrane, Lazarus, Thibert, *"Flat Tori in Three-Dimensional Space and Convex Integration"*

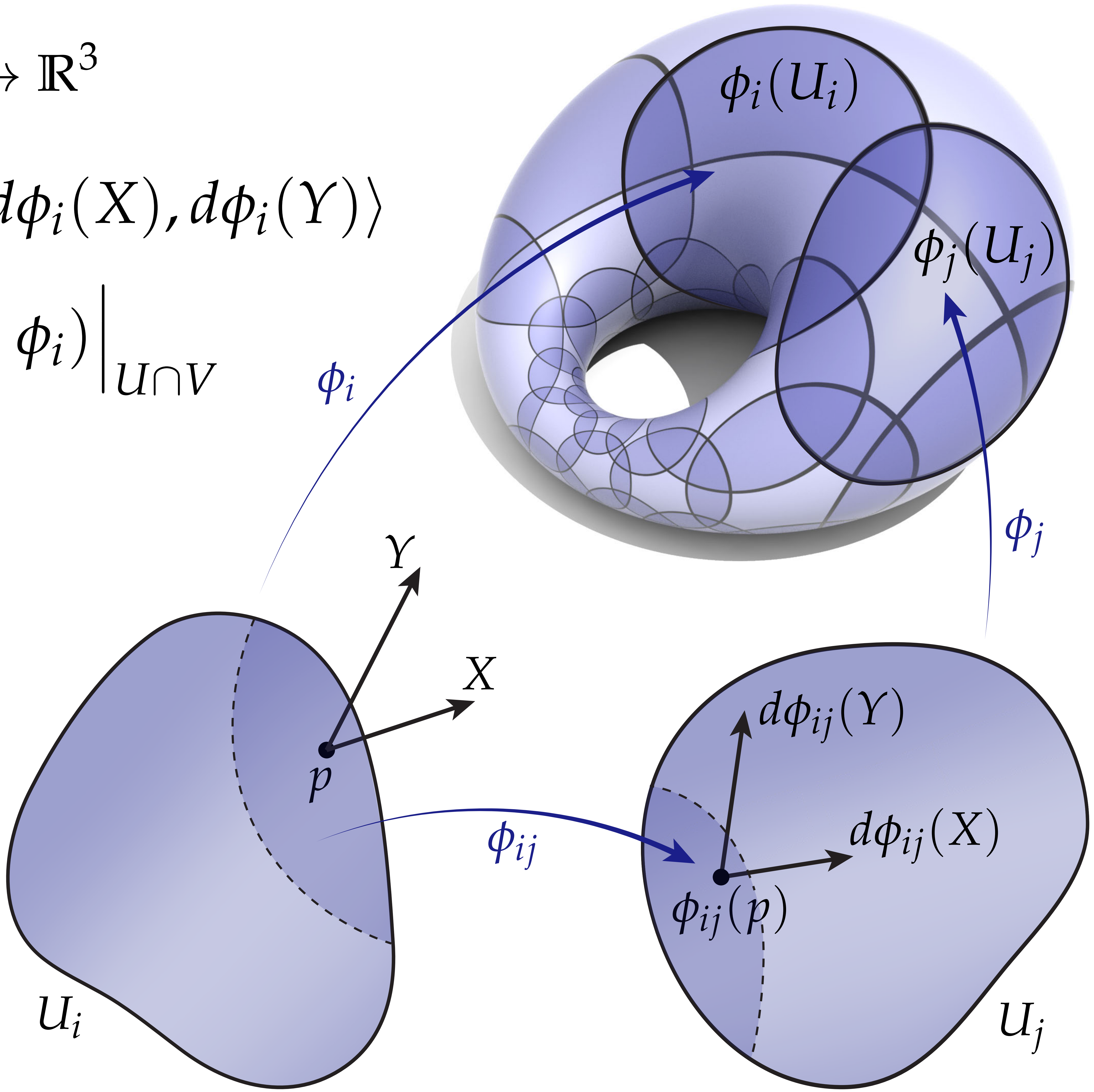
Atlas & Charts

- Most surfaces aren't easily expressed as the image of one parameterized "patch"
- Instead, cover a surface with overlapping patches ("charts")
- As usual, each chart ϕ_i defines an induced Riemannian metric g_i
- Though things look different in local charts, the induced metrics give identical measurements (by definition)

$$\phi_i : \mathbb{R}^2 \supset U_i \to \mathbb{R}^3$$

$$g_i(X,Y) = \langle d\phi_i(X), d\phi_i(Y)\rangle$$

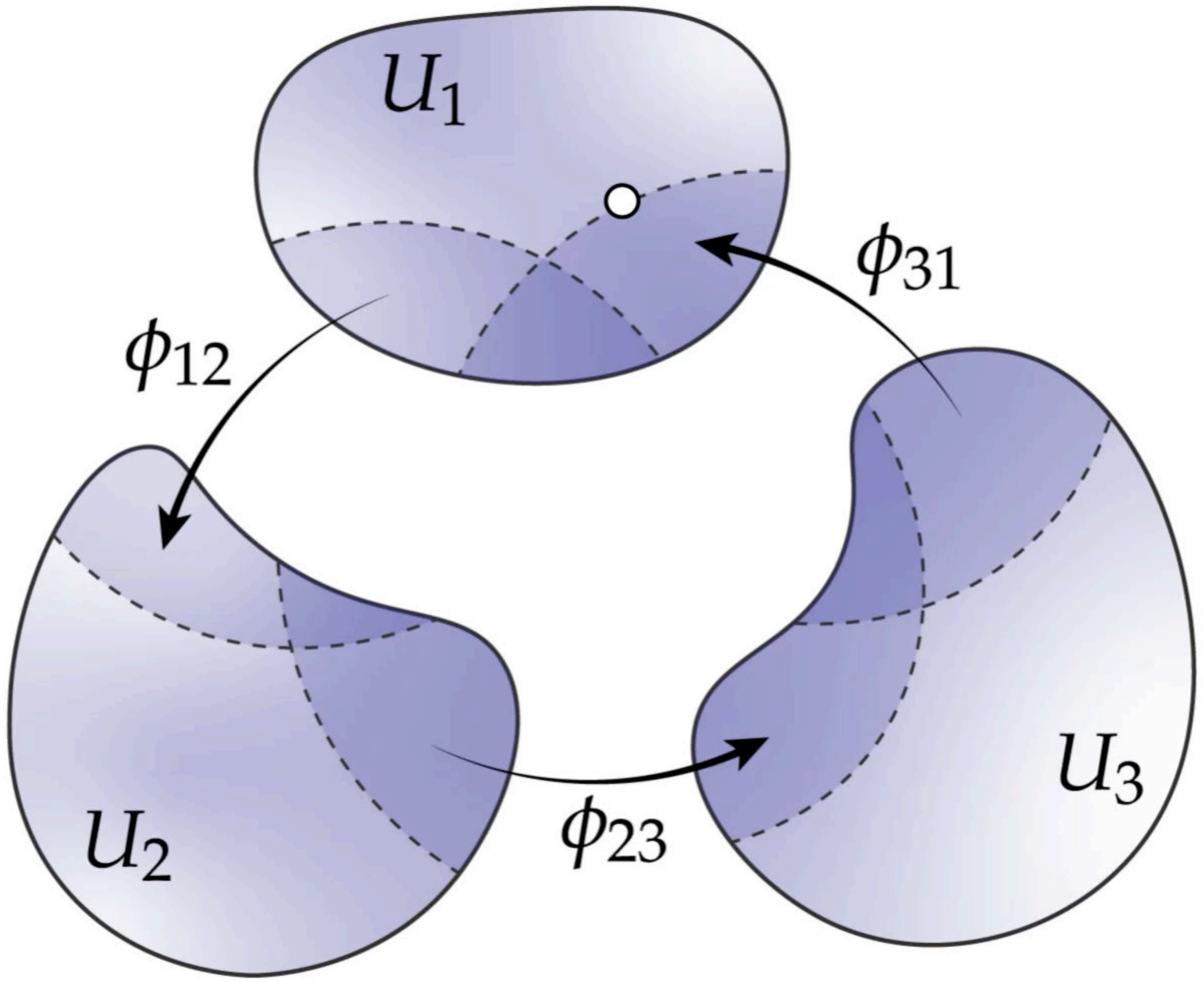
$$\phi_{ij} := \left.(\phi_j^{-1} \circ \phi_i)\right|_{U \cap V}$$



$$\boxed{g_j(d\phi_{ij}(X), d\phi_{ij}(Y)) = g_i(X,Y)}$$

Abstract Riemannian Manifold

- Can again adopt *intrinsic* viewpoint: don't need to know where metric came from, as long as agreement across charts
- Leads to notion of *Riemannian manifold*. Roughly speaking*:
  - collection of open sets $U_i \subset \mathbb{R}^n$
  - transition maps ϕ_{ij} on overlaps (differentiable both ways)
  - local metric g_i per patch, compatible on overlaps
  - Riemannian manifold M is "union" of all these pieces
  - <u>do not</u> need embeddings $\phi_i : U_i \to \mathbb{R}^m$

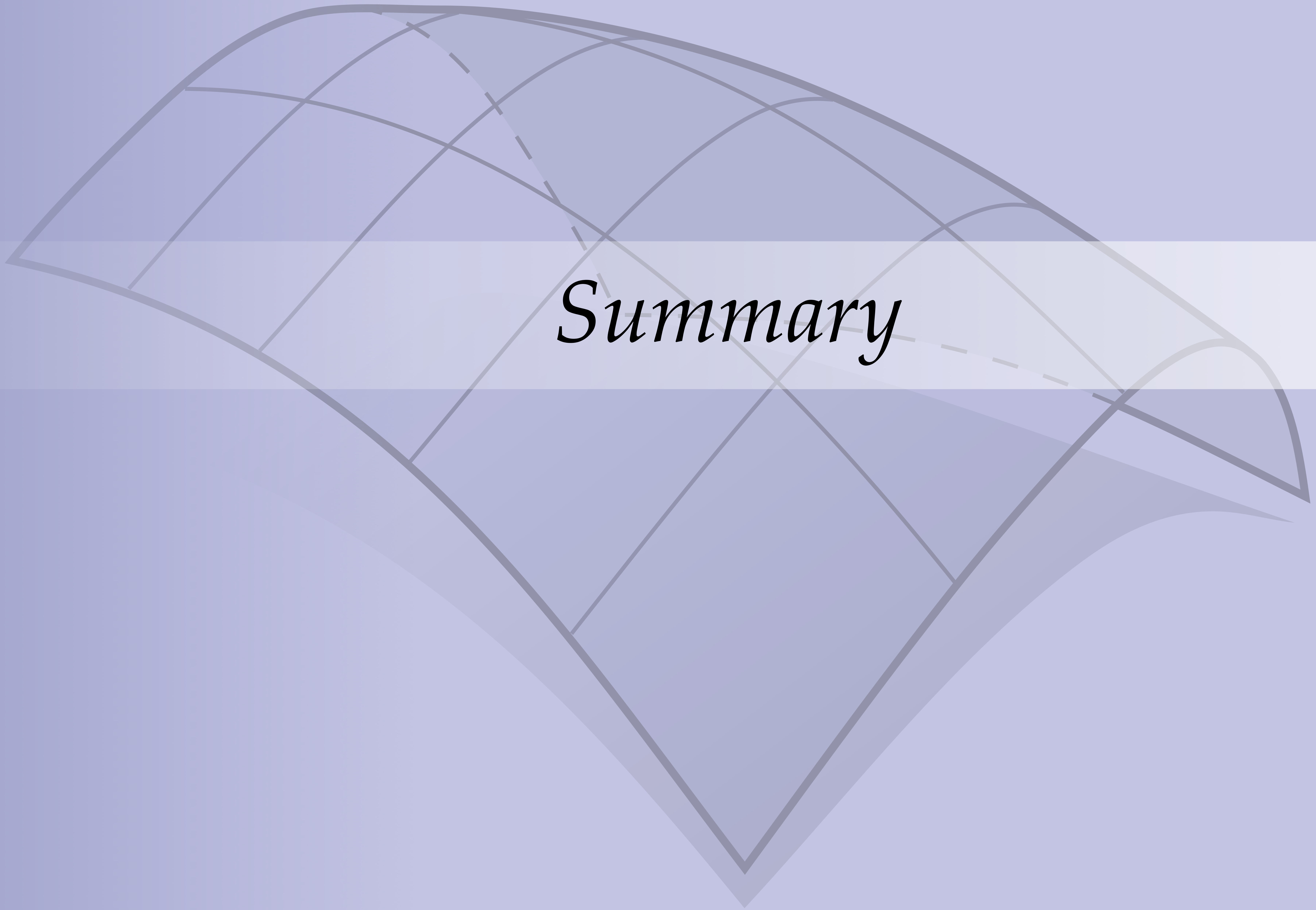


$$\boxed{g_j(d\phi_{ij}(X), d\phi_{ij}(Y)) = g_i(X, Y)}$$

- This information is again often enough to "do geometry" (measure lengths, angles, areas, distances, …)
- **Key idea:** works even when geometry is not—or cannot—be embedded in low dimensions

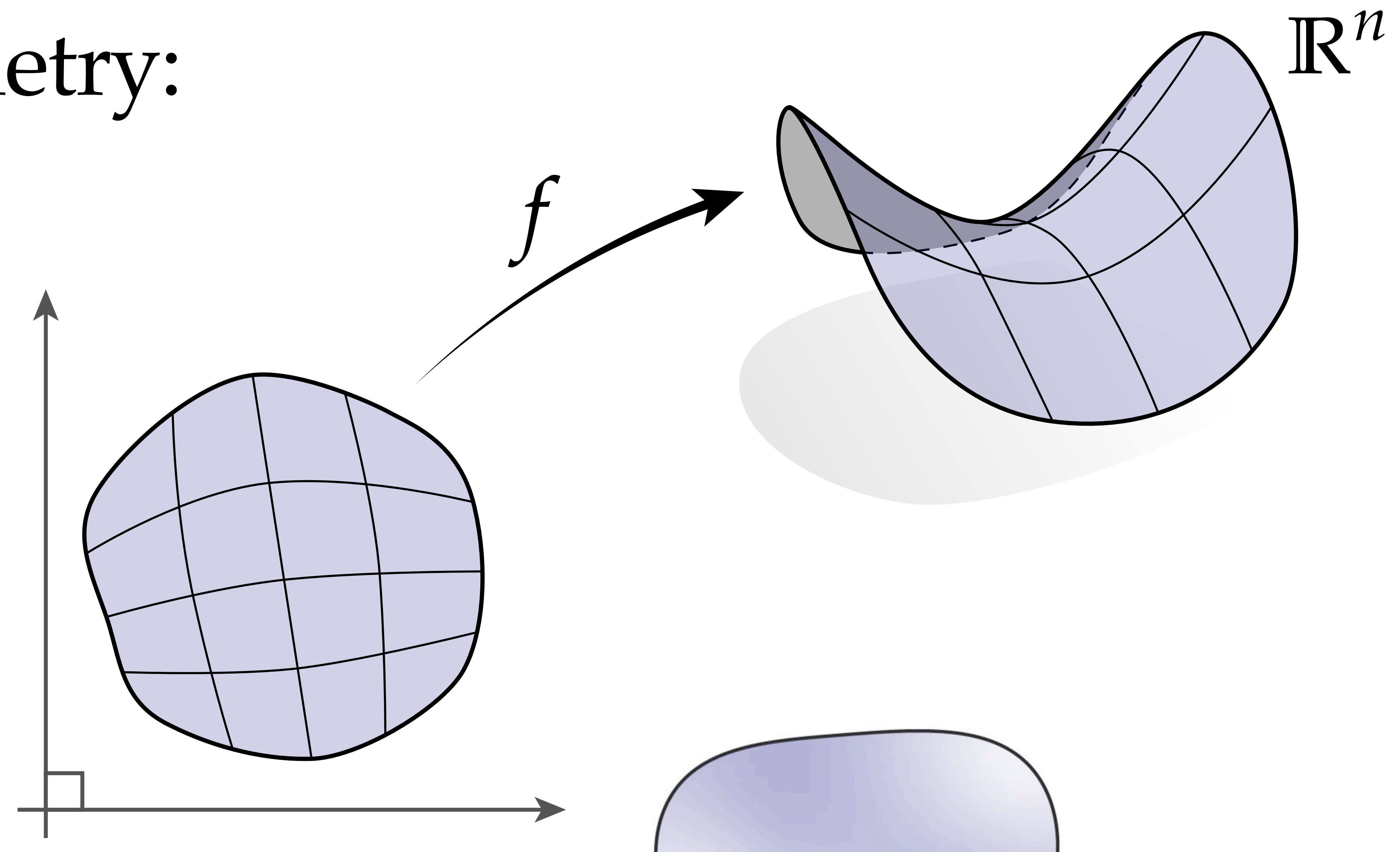
*To make this more precise, need notion of *topological spaces* (takes some work to define…)

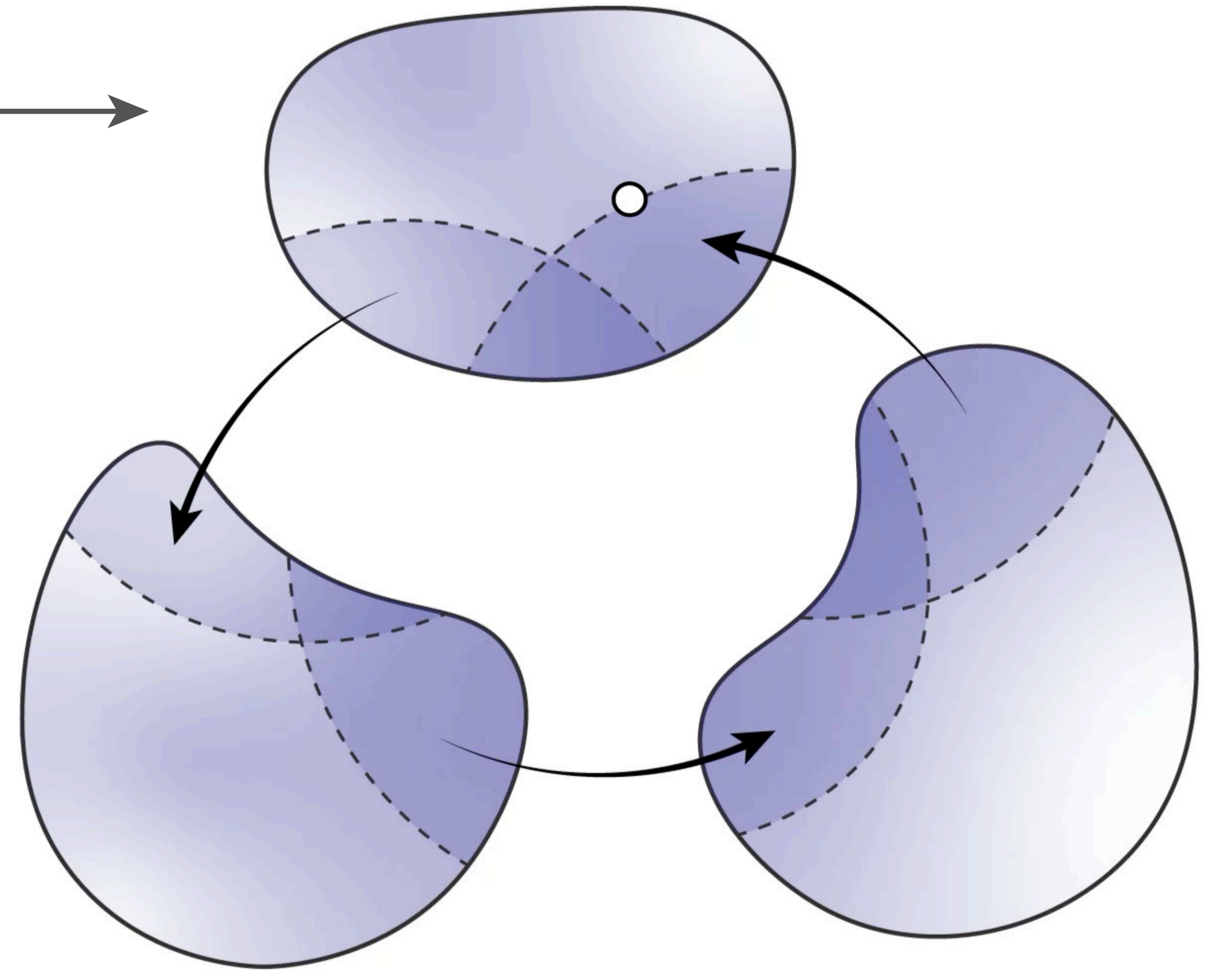
Summary



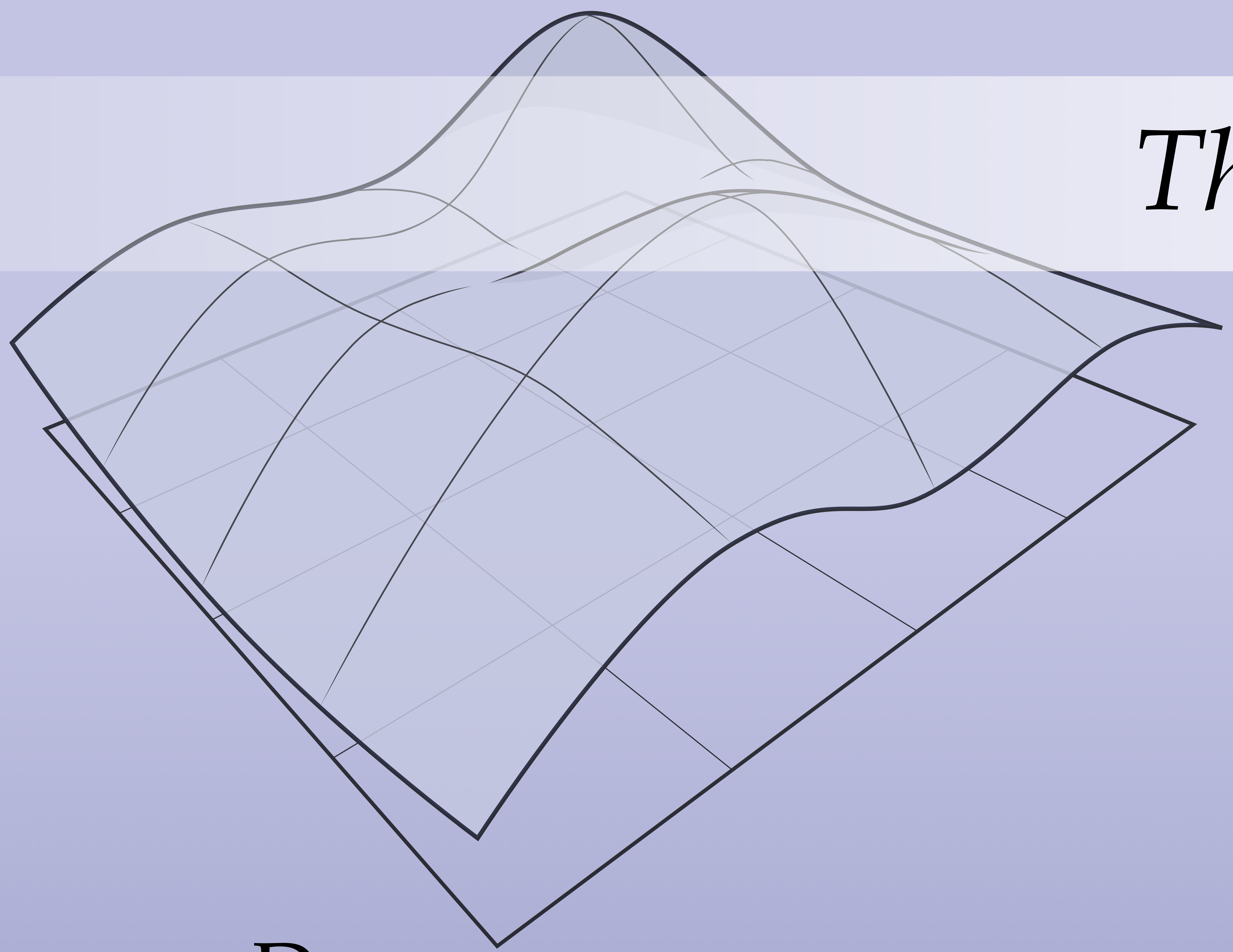
Smooth Surfaces I—Summary

- Introduced two very important descriptions of geometry:
- **(Extrinsic)** Parameterization
  - encode geometry as map into $\mathbb{R}^n$
  - describes where points are in space
  - patch together local parameterizations
- **(Intrinsic)** Riemannian metric
  - encode geometry via inner product on local domains
  - lets you measure angles, lengths, areas, distances, …
  - to be meaningful, metrics must agree on overlaps
- Next time: more extrinsic geometry, connect to exterior calculus





Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858