

Machine Learning and Computational Statistics

Homework 4: Kernel Methods and Lagrangian Duality

Due: Monday, March 27, 2017, at 10pm (Submit via Gradescope)

Instructions: Your answers to the questions below, including plots and mathematical work, should be submitted as a single PDF file. It's preferred that you write your answers using software that typesets mathematics (e.g. L^AT_EX, L^AT_EX, or MathJax via iPython), though if you need to you may scan handwritten work. You may find the [minted](#) package convenient for including source code in your L^AT_EX document. If you are using L^AT_EX, then the [listings](#) package tends to work better.

1 Introduction

The problem set begins with a review of some important linear algebra concepts that we routinely use in machine learning and statistics. The solutions to each of these problems is at most a few lines long, and we've tried to give helpful hints. These aren't meant to be very challenging problems – in fact, we'd like this material to become second nature to you. We next have a couple problems on kernel methods: the first explores what geometric information about the data is stored in the kernel matrix, and the second revisits kernel ridge regression with a direct approach, rather than using the Representer Theorem. Next we have an optional problem that explores an interesting way to re-express the Pegasos-style SSGD on any ℓ_2 -regularized empirical risk objective function. The new expression also happens to allow efficient updates in the sparse feature setting. In the next required problem, we take a direct approach to kernelizing Pegasos. Finally we get to our coding problem, in which you'll have the opportunity to see how kernel ridge regression works with different kernels on a one-dimensional, highly non-linear regression problem. Optionally, we can code a kernelized SVM and see how it works on a classification problem with a two-dimensional input space. The problem set ends with optional more problems. The first of these reviews the proof of the Representer Theorem. The second applies Lagrangian duality to show the equivalence of Tikhonov and Ivanov regularization. And the third introduces an approach to “novelty” or “anomaly” detection as an exercise in the machinery of Lagrangian duality.

2 Positive Semidefinite Matrices

In statistics and machine learning, we use positive semidefinite matrices a lot. Let's recall some definitions from linear algebra that will be useful here:

Definition. A set of vectors $\{x_1, \dots, x_n\}$ is **orthonormal** if $\langle x_i, x_i \rangle = 1$ for any $i \in \{1, \dots, n\}$ (i.e. x_i has unit norm), and for any $i, j \in \{1, \dots, n\}$ with $i \neq j$ we have $\langle x_i, x_j \rangle = 0$ (i.e. x_i and x_j are orthogonal).

Note that if the vectors are column vectors in a Euclidean space, we can write this as $x_i^T x_j = 1$ ($i \neq j$) for all $i, j \in \{1, \dots, n\}$.

Definition. A matrix is **orthogonal** if it is a square matrix with orthonormal columns.

It follows from the definition that if a matrix $M \in \mathbf{R}^{n \times n}$ is orthogonal, then $M^T M = I$, where I is the $n \times n$ identity matrix. Thus $M^T = M^{-1}$, and so $MM^T = I$ as well.

Definition. A matrix M is **symmetric** if $M = M^T$.

Definition. For a square matrix M , if $Mv = \lambda v$ for some column vector v and scalar λ , then v is called an **eigenvector** of M and λ is the corresponding **eigenvalue**.

Theorem (Spectral Theorem). *A real, symmetric matrix $M \in \mathbf{R}^{n \times n}$ can be diagonalized as $M = Q\Sigma Q^T$, where $Q \in \mathbf{R}^{n \times n}$ is an orthogonal matrix whose columns are a set of orthonormal eigenvectors of M , and Σ is a diagonal matrix of the corresponding eigenvalues.*

Definition. A real, symmetric matrix $M \in \mathbf{R}^{n \times n}$ is **positive semidefinite (psd)** if for any $x \in \mathbf{R}^n$,

$$x^T M x \geq 0.$$

Note that unless otherwise specified, when a matrix is described as positive semidefinite, we are implicitly assuming it is real and symmetric (or complex and Hermitian in certain contexts, though not here).

As an exercise in matrix multiplication, note that for any matrix A with columns a_1, \dots, a_d , that is

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_d \\ | & & | \end{pmatrix} \in \mathbf{R}^{n \times d},$$

we have

$$A^T M A = \begin{pmatrix} a_1^T M a_1 & a_1^T M a_2 & \cdots & a_1^T M a_d \\ a_2^T M a_1 & a_2^T M a_2 & \cdots & a_2^T M a_d \\ \vdots & \vdots & \cdots & \vdots \\ a_d^T M a_1 & a_d^T M a_2 & \cdots & a_d^T M a_d \end{pmatrix}.$$

So M is psd if and only if for any $A \in \mathbf{R}^{n \times d}$, we have $\text{diag}(A^T M A) = (a_1^T M a_1, \dots, a_d^T M a_d)^T \succeq 0$, where \succeq is elementwise inequality, and 0 is a $d \times 1$ column vector of 0's.

1. Give an example of an orthogonal matrix that is not symmetric. (Hint: You can use a 2×2 matrix with only the entries -1, 0, and 1.

Solution:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

2. Use the definition of a psd matrix and the spectral theorem to show that all eigenvalues of a positive semidefinite matrix M are non-negative. [Hint: By Spectral theorem, $\Sigma = Q^T M Q$ for some Q . What if you take $A = Q$ in the “exercise in matrix multiplication” described above?]

Solution: Let $M \in \mathbf{R}^{n \times n}$ be a psd matrix. By the spectral theorem, we can write $M = Q\Sigma Q^T$. Then by the argument above,

$$\text{diag}(Q^T M Q) = \text{diag}(\Sigma) \succeq 0.$$

3. In this problem we show that a psd matrix is a matrix version of a non-negative scalar, in that they both have a “square root”. Show that a symmetric matrix M can be expressed as $M = BB^T$ for some matrix B , if and only if M is psd. [Hint: To show $M = BB^T$ implies M is psd, use the fact that for any vector v , $v^T v \geq 0$. To show that M psd implies $M = BB^T$ for some B , use the Spectral Theorem.]

Solution:

If we can write $M = BB^T$, then M is symmetric and $x^T M x = x^T B B^T x = (B^T x)^T (B^T x) = \|B^T x\|^2 \geq 0$, for any x . Thus M is symmetric positive semidefinite. Conversely, if M is psd, then by the Spectral Theorem there exists Q orthogonal and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_d)$ such that $M = Q \Sigma Q^T$. By the previous problem $\sigma_i \geq 0$, so we can define $\Sigma^{1/2} := \text{diag}(\sigma_1^{1/2}, \dots, \sigma_d^{1/2})$ and write

$$\begin{aligned} M &= Q \Sigma^{1/2} \Sigma^{1/2} Q^T \\ &= B B^T \end{aligned}$$

where $B := Q \Sigma^{1/2}$. Note that we can also write

$$M = Q \Sigma^{1/2} Q^T Q \Sigma^{1/2} Q^T = C^2,$$

where $C = Q \Sigma^{1/2} Q^T$. So M has a “square root”, just like non-negative real numbers.

3 Positive Definite Matrices

Definition. A real, symmetric matrix $M \in \mathbf{R}^{n \times n}$ is **positive definite** (spd) if for any $x \in \mathbf{R}^n$ with $x \neq 0$,

$$x^T M x > 0.$$

1. Show that all eigenvalues of a symmetric positive definite matrix are positive. [Hint: You can use the same method as you used for psd matrices above.]
2. Let M be a symmetric positive definite matrix. By the spectral theorem, $M = Q \Sigma Q^T$, where Σ is a diagonal matrix of the eigenvalues of M . By the previous problem, all diagonal entries of Σ are positive. If $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, then $\Sigma^{-1} = \text{diag}(\sigma_1^{-1}, \dots, \sigma_n^{-1})$. Show that the matrix $Q \Sigma^{-1} Q^T$ is the inverse of M .
3. Since positive semidefinite matrices may have eigenvalues that are zero, we see by the previous problem that not all psd matrices are invertible. Show that if M is a psd matrix and I is the identity matrix, then $M + \lambda I$ is symmetric positive definite for any $\lambda > 0$, and give an expression for the inverse of $M + \lambda I$.
4. Let M and N be symmetric matrices, with M positive semidefinite and N positive definite. Use the definitions of psd and spd to show that $M + N$ is symmetric positive definite. Thus $M + N$ is invertible. (Hint: For any $x \neq 0$, show that $x^T (M + N) x > 0$. Also note that $x^T (M + N) x = x^T M x + x^T N x$.)

4 [Optional] Kernel Matrices

The following problem will give us some additional insight into what information is encoded in the kernel matrix.

1. [Optional] Consider a set of vectors $S = \{x_1, \dots, x_m\}$. Let X denote the matrix whose rows are these vectors. Form the Gram matrix $K = XX^T$. Show that knowing K is equivalent to knowing the set of pairwise distances among the vectors in S as well as the vector lengths. [Hint: The distance between x and y is given by $d(x, y) = \|x - y\|$, and the norm of a vector x is defined as $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^T x}$.]

Solution: We use the distance corresponding to the standard inner product:

$$\begin{aligned} d(x, y) &:= \sqrt{\langle x - y, x - y \rangle} \\ &= \sqrt{\langle x, x \rangle + \langle y, y \rangle - 2\langle x, y \rangle} \end{aligned}$$

It is clear that $K(i, j) = \langle x_i, x_j \rangle$, so we can use the formula above to get all pairwise distances from the Gram matrix K . The length of vector x_i is just $\sqrt{K(i, i)}$.

Conversely, we can write

$$\langle x, y \rangle = \frac{1}{2} [\langle x, x \rangle + \langle y, y \rangle - d^2(x, y)]$$

Noting that $\langle x, x \rangle$ is the square of the length of x , we see that we can recover the Gram matrix from the distances and the lengths.

5 Kernel Ridge Regression

In lecture, we discussed how to kernelize ridge regression using the representer theorem. Here we pursue a bare-hands approach.

Suppose our input space is $\mathcal{X} = \mathbf{R}^d$ and our output space is $\mathcal{Y} = \mathbf{R}$. Let $\mathcal{D} = \{(x_1, y_1), \dots, (x_n, y_n)\}$ be a training set from $\mathcal{X} \times \mathcal{Y}$. We'll use the "design matrix" $X \in \mathbf{R}^{n \times d}$, which has the input vectors as rows:

$$X = \begin{pmatrix} -x_1 - \\ \vdots \\ -x_n - \end{pmatrix}.$$

Recall the ridge regression objective function:

$$J(w) = \|Xw - y\|^2 + \lambda \|w\|^2,$$

for $\lambda > 0$.

1. Show that for w to be a minimizer of $J(w)$, we must have $X^T X w + \lambda I w = X^T y$. Show that the minimizer of $J(w)$ is $w = (X^T X + \lambda I)^{-1} X^T y$. Justify that the matrix $X^T X + \lambda I$ is invertible, for $\lambda > 0$. (The last part should follow easily from the earlier exercises on psd and spd matrices.)

Solution:

$$\begin{aligned}
J(w) &= (Xw - y)^T (Xw - y) + \lambda w^T w \\
\partial_w J(w) &= 2X^T (Xw - y) + 2\lambda w \\
\partial_w J(w) = 0 &\iff 2X^T Xw + 2\lambda w - 2X^T y = 0 \\
&\iff (X^T X + \lambda I)w = X^T y \\
&\iff w = (X^T X + \lambda I)^{-1} X^T y
\end{aligned}$$

$X^T X$ is PSD, and adding λI to it makes it spd, by an earlier problem. Thus it's invertible for $\lambda > 0$.

2. Rewrite $X^T Xw + \lambda Iw = X^T y$ as $w = \frac{1}{\lambda}(X^T y - X^T Xw)$. Based on this, show that we can write $w = X^T \alpha$ for some α , and give an expression for α .

Solution:

$$\begin{aligned}
w &= X^T \left[\frac{1}{\lambda} (y - Xw) \right] \\
\implies \alpha &= \frac{1}{\lambda} (y - Xw)
\end{aligned}$$

3. Based on the fact that $w = X^T \alpha$, explain why we say w is “in the span of the data.”

Solution: $X^T \alpha$ is a linear combination of the columns of X^T , which contain the input vectors x_1, \dots, x_n :

$$w = \begin{pmatrix} | & \dots & | \\ x_1 & \dots & x_n \\ | & \dots & | \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 x_1 + \dots + \alpha_n x_n$$

4. Show that $\alpha = (\lambda I + XX^T)^{-1} y$. Note that XX^T is the kernel matrix for the standard vector dot product. (Hint: Replace w by $X^T \alpha$ in the expression for α , and then solve for α .)

Solution:

$$\begin{aligned}
\alpha &= \lambda^{-1} (y - Xw) \\
\lambda \alpha &= y - XX^T \alpha \\
XX^T \alpha + \lambda \alpha &= y \\
(XX^T + \lambda I) \alpha &= y \\
\alpha &= (\lambda I + XX^T)^{-1} y
\end{aligned}$$

5. Give a kernelized expression for the Xw , the predicted values on the training points. (Hint: Replace w by $X^T \alpha$ and α by its expression in terms of the kernel matrix XX^T .)

Solution:

$$\begin{aligned} Xw &= X(X^T\alpha) \\ &= (XX^T)(\lambda I + XX^T)^{-1}y \end{aligned}$$

6. Give an expression for the prediction $f(x) = x^T w^*$ for a new point x , not in the training set. The expression should only involve x via inner products with other x 's. [Hint: It is often convenient to define the column vector

$$k_x = \begin{pmatrix} x^T x_1 \\ \vdots \\ x^T x_n \end{pmatrix}$$

to simplify the expression.]

Solution: We have

$$\begin{aligned} f(x) &= x^T w^* = x^T X^T \alpha^* \\ &= x^T X^T (\lambda I + XX^T)^{-1} y \\ &= k_x^T (\lambda I + XX^T)^{-1} y \\ &= k_x^T \alpha \end{aligned}$$

6 [Optional] Pegasos and SSGD for ℓ_2 -regularized ERM¹

Consider the objective function

$$J(w) = \frac{\lambda}{2} \|w\|_2^2 + \frac{1}{n} \sum_{i=1}^n \ell_i(w),$$

where $\ell_i(w)$ represents the loss on the i th training point (x_i, y_i) . Suppose $\ell_i(w) : \mathbf{R}^d \rightarrow \mathbf{R}$ is a convex function. Let's write

$$J_i(w) = \frac{\lambda}{2} \|w\|_2^2 + \ell_i(w),$$

for the one-point approximation to $J(w)$ using the i th training point. $J_i(w)$ is probably a very poor approximation of $J(w)$. However, if we choose i uniformly at random from $1, \dots, n$, then we do have $\mathbb{E}J_i(w) = J(w)$. We'll now show that subgradients of $J_i(w)$ are unbiased estimators of some subgradient of $J(w)$, which is our justification for using SSGD methods.

In the problems below, you may use the following facts about subdifferentials without proof (as in Homework #3): 1) If $f_1, \dots, f_m : \mathbf{R}^d \rightarrow \mathbf{R}$ are convex functions and $f = f_1 + \dots + f_m$, then $\partial f(x) = \partial f_1(x) + \dots + \partial f_m(x)$ [**additivity**]. 2) For $\alpha \geq 0$, $\partial(\alpha f)(x) = \alpha \partial f(x)$ [**positive homogeneity**].

¹This problem is based on Shalev-Shwartz and Ben-David's book [Understanding Machine Learning: From Theory to Algorithms](#), Sections 14.5.3, 15.5, and 16.3).

1. [Optional] For each $i = 1, \dots, n$, let $g_i(w)$ be a subgradient of $J_i(w)$ at $w \in \mathbf{R}^d$. Let $v_i(w)$ be a subgradient of $\ell_i(w)$ at w . Give an expression for $g_i(w)$ in terms of w and $v_i(w)$

Solution: Since the regularization term is convex and $\ell_i(w)$ is convex, we can just add a subgradient of each. So we can take

$$g_i(w) = \lambda w + v_i(w).$$

2. [Optional] Show that $\mathbb{E}g_i(w) \in \partial J(w)$, where the expectation is over the randomly selected $i \in 1, \dots, n$. (In words, the expectation of our subgradient of a randomly chosen $J_i(w)$ is in the subdifferential of J .)

Solution: We have

$$\begin{aligned} \mathbb{E}g_i(w) &= \frac{1}{n} \sum_{i=1}^n [\lambda w + v_i(w)] \\ &= \lambda w + \frac{1}{n} \sum_{i=1}^n v_i(w) \end{aligned}$$

By the additivity of subdifferentials and the positive homogeneity, this is a subgradient for $J(w)$.

3. [Optional] Now suppose we are carrying out SSGD with the Pegasos step-size $\eta^{(t)} = 1/(\lambda t)$, $t = 1, 2, \dots$. In the t 'th step, suppose we select the i th point and thus take the step $w^{(t+1)} = w^{(t)} - \eta^{(t)}g_i(w^{(t)})$. Let's write $v^{(t)} = v_i(w^{(t)})$, which is the subgradient of the loss part of $J_i(w^{(t)})$ that is used in step t . Show that

$$w^{(t+1)} = -\frac{1}{\lambda t} \sum_{\tau=1}^t v^{(\tau)}$$

[Hint: One approach is proof by induction. First show it's true for $w^{(2)}$. Then assume it's true for $w^{(t)}$ and prove it's true for $w^{(t+1)}$. This will prove that it's true for all $t = 2, 3, \dots$ by induction.

Solution: Let's start with $t = 1$. Suppose we choose the i 'th point for the first step. Then since $w^{(1)} = 0$, we have

$$w^{(2)} = -\eta^{(1)}g_i(0) = -\frac{1}{\lambda t} [v_i(0)] = -\frac{1}{\lambda t} v^{(1)}.$$

Now assume the claim is true for $w^{(t)}$ and we'll prove it for $w^{(t+1)}$. We have

$$\begin{aligned}
w^{(t+1)} &= w^{(t)} - \eta^{(t)} \left[\lambda w^{(t)} + v_i(w^{(t)}) \right] \\
&= w^{(t)} - \left[\frac{1}{t} w^{(t)} + \frac{1}{\lambda t} v^{(t)} \right] \\
&= -\frac{1}{\lambda(t-1)} \sum_{\tau=1}^{t-1} v^{(\tau)} - \left[-\frac{1}{t\lambda(t-1)} \sum_{\tau=1}^{t-1} v^{(\tau)} + \frac{1}{\lambda t} v^{(t)} \right] \\
&= \sum_{\tau=1}^{t-1} \left[\frac{1}{\lambda t(t-1)} v^{(\tau)} - \frac{1}{\lambda(t-1)} v^{(\tau)} \right] - \frac{1}{\lambda t} v^{(t)} \\
&= \sum_{\tau=1}^{t-1} \left[\frac{1-t}{\lambda t(t-1)} v^{(\tau)} \right] - \frac{1}{\lambda t} v^{(t)} \\
&= -\frac{1}{\lambda t} \sum_{\tau=1}^{t-1} \left[v^{(\tau)} \right] - \frac{1}{\lambda t} v^{(t)} \\
&= -\frac{1}{\lambda t} \sum_{\tau=1}^t \left[v^{(\tau)} \right]
\end{aligned}$$

So we have proved the claim by induction.

4. [Optional] We can use the previous result to get a nice equivalent formulation of Pegasos. Let $\theta^{(t)} = \sum_{\tau=1}^{t-1} v^{(\tau)}$. Then $w^{(t+1)} = -\frac{1}{\lambda t} \theta^{(t+1)}$. Then Pegasos from Homework #3 is equivalent to Algorithm 1. Similar to the $w = sW$ decomposition from homework #3, this decomposition

Algorithm 1: Pegasos Algorithm Reformulation

```

input: Training set  $(x_1, y_1), \dots, (x_n, y_n) \in \mathbf{R}^d \times \{-1, 1\}$  and  $\lambda > 0$ .
 $\theta^{(1)} = (0, \dots, 0) \in \mathbf{R}^d$ 
 $w^{(1)} = (0, \dots, 0) \in \mathbf{R}^d$ 
 $t = 1$  # step number
repeat
    randomly choose  $j$  in  $1, \dots, n$ 
    if  $y_j \langle w^{(t)}, x_j \rangle < 1$ 
         $\theta^{(t+1)} = \theta^{(t)} + y_j x_j$ 
    else
         $\theta^{(t+1)} = \theta^{(t)}$ 
    endif
     $w^{(t+1)} = -\frac{1}{\lambda t} \theta^{(t+1)}$  # need not be explicitly computed
     $t = t + 1$ 
until bored
return  $w^{(t)} = -\frac{1}{\lambda(t-1)} \theta^{(t)}$ 

```

gives the opportunity for significant speedup. Explain how Algorithm 1 can be implemented

so that, if x_j has s nonzero entries, then we only need to do $O(s)$ accesses in every pass through the loop.

Solution: If we store θ in a hash table (e.g. a dict in Python), we can compute $\langle w^{(t)}, x_j \rangle$ on the fly by extracting only the s entries of θ that correspond to the s nonzero entries of x_j . Each of these entries can be scaled by $1/[(t-1)\lambda]$ to get the corresponding entries of $w^{(t)}$. The update of θ accesses s of its elements at most.

7 Kernelized Pegasos

Recall the SVM objective function

$$\min_{w \in \mathbf{R}^n} \frac{\lambda}{2} \|w\|^2 + \frac{1}{m} \sum_{i=1}^m \max \{0, 1 - y_i w^T x_i\}$$

and the Pegasos algorithm on the training set $(x_1, y_1), \dots, (x_n, y_n) \in \mathbf{R}^d \times \{-1, 1\}$ (Algorithm 2).

Algorithm 2: Pegasos Algorithm

```

input: Training set  $(x_1, y_1), \dots, (x_n, y_n) \in \mathbf{R}^d \times \{-1, 1\}$  and  $\lambda > 0$ .
 $w^{(1)} = (0, \dots, 0) \in \mathbf{R}^d$ 
 $t = 0$  # step number
repeat
   $t = t + 1$ 
   $\eta^{(t)} = 1/(t\lambda)$  # step multiplier
  randomly choose  $j$  in  $1, \dots, n$ 
  if  $y_j \langle w^{(t)}, x_j \rangle < 1$ 
     $w^{(t+1)} = (1 - \eta^{(t)}\lambda)w^{(t)} + \eta^{(t)}y_jx_j$ 
  else
     $w^{(t+1)} = (1 - \eta^{(t)}\lambda)w^{(t)}$ 
until bored
return  $w^{(t)}$ 

```

Note that in every step of Pegasos, we rescale $w^{(t)}$ by $(1 - \eta^{(t)}\lambda) = (1 - \frac{1}{t}) \in (0, 1)$. This “shrinks” the entries of $w^{(t)}$ towards 0, and it’s due to the regularization term $\frac{\lambda}{2}\|w\|_2^2$ in the SVM objective function. Also note that if the example in a particular step, say (x_j, y_j) , is not classified with the required margin (i.e. if we don’t have margin $y_j w_i^T x_j \geq 1$), then we also add a multiple of x_j to $w^{(t)}$ to end up with $w^{(t+1)}$. This part of the adjustment comes from the empirical risk. Since

we initialize with $w^{(1)} = 0$, we are guaranteed that we can always write²

$$w^{(t)} = \sum_{i=1}^n \alpha_i^{(t)} x_i$$

after any number of steps t . When we kernelize Pegasos, we'll be tracking $\alpha^{(t)} = (\alpha_1^{(t)}, \dots, \alpha_n^{(t)})^T$ directly, rather than w .

1. Kernelize the expression for the margin. That is, show that $y_j \langle w^{(t)}, x_j \rangle = y_j K_{j.} \alpha^{(t)}$, where $k(x_i, x_j) = \langle x_i, x_j \rangle$ and $K_{j.}$ denotes the j th row of the kernel matrix K corresponding to kernel k .

$$y_j \langle w^{(t)}, x_j \rangle = y_j \left\langle \sum_{i=1}^n \alpha_i^{(t)} x_i, x_j \right\rangle = y_j \sum_{i=1}^n \alpha_i^{(t)} k(x_i, x_j) = y_j K_{j.} \alpha^{(t)}$$

2. Suppose that $w^{(t)} = \sum_{i=1}^n \alpha_i^{(t)} x_i$ and for the next step we have selected a point (x_j, y_j) that does not have a margin violation. Give an update expression for $\alpha^{(t+1)}$ so that $w^{(t+1)} = \sum_{i=1}^n \alpha_i^{(t+1)} x_i$.

Note that if we set $\alpha^{(t+1)} = (1 - \eta^{(t)} \lambda) \alpha^{(t)}$, then $w^{(t+1)} = \sum_{i=1}^n \alpha_i^{(t+1)} x_i = (1 - \eta^{(t)} \lambda) \sum_{i=1}^n \alpha_i^{(t)} x_i = (1 - \eta^{(t)} \lambda) w^{(t)}$.

3. Repeat the previous problem, but for the case that (x_j, y_j) has a margin violation. Then give the full pseudocode for kernelized Pegasos. You may assume that you receive the kernel matrix K as input, along with the labels $y_1, \dots, y_n \in \{-1, 1\}$.
Note that if we set $\alpha_i^{(t+1)} = (1 - \eta^{(t)} \lambda) \alpha_i^{(t)}$ for all $i \neq j$ and set $\alpha_j^{(t+1)} = (1 - \eta^{(t)} \lambda) \alpha_j^{(t)} + \eta^{(t)} y_j$, then

$$w^{(t+1)} = (1 - \eta^{(t)} \lambda) w^{(t)} + \eta^{(t)} y_j x_j$$

So the kernelized Pegasos algorithm is [3](#).

4. [Optional] Above we've directly kernelized. While the direct implementation of the original Pegasos required updating all entries of w in every step, a direct kernelization of Algorithm [2](#), as we have done above, leads to updating all entries of α in every step. Give a version of the kernelized Pegasos algorithm that does not suffer from this inefficiency. You may try splitting the scale and direction similar to the approach of the previous problem set, or you may use a decomposition based on Algorithm [1](#) from the optional problem [6](#) above.

Solution: Hehehe

²Note: This resembles the conclusion of the representer theorem, but it's saying something different. Here, we are saying that the $w^{(t)}$ after every step of the Pegasos algorithm lives in the span of the data. The representer theorem says that a mathematical minimizer of the SVM objective function (i.e. what the Pegasos algorithm would converge to after infinitely many steps) lies in the span of the data. If, for example, we had chosen an initial $w^{(1)}$ that is NOT in the span of the data, then none of the $w^{(t)}$'s from Pegasos would be in the span of the data. However, we know Pegasos converges to a minimum of the SVM objective. Thus after a very large number of steps, $w^{(t)}$ would be very close to being in the span of the data. It's the gradient of the regularization term that pulls us back towards the span of the data. This is basically because the regularization is driving all components towards 0, while the empirical risk updates are only pushing things away from 0 in directions in the span of the data.

Algorithm 3: Pegasos Algorithm

```
input: Kernel matrix  $K \in \mathbf{R}^{n \times n}$ , labels  $y_1, \dots, y_n \in \{-1, 1\}$ , and  $\lambda > 0$ .  
 $\alpha^{(1)} = (0, \dots, 0) \in \mathbf{R}^n$   
 $t = 0$  # step number  
repeat  
   $t = t + 1$   
   $\eta^{(t)} = 1/(t\lambda)$  # step multiplier  
  randomly choose  $j$  in  $1, \dots, n$   
  if  $y_j (\sum_{i=1}^n \alpha_i \langle x_i, x_j \rangle)$   
     $\alpha_j^{(t+1)} = (1 - \eta^{(t)}\lambda) \alpha_j^{(t)} + \eta^{(t)} y_j$   
  else  
     $\alpha_j^{(t+1)} = (1 - \eta^{(t)}\lambda) \alpha_j^{(t)}$   
until bored  
return  $\alpha^{(t)}$  or  $w^{(t)} = \sum_{i=1}^n \alpha_i^{(t)} x_i$ 
```

8 Kernel Methods: Let's Implement

In this section you will get the opportunity to code kernel ridge regression and, optionally, kernelized SVM. To speed things along, we've written a great deal of support code for you, which you can find in the Jupyter notebooks in the homework zip file.

8.1 One more review of kernelization can't hurt (no problems)

Consider the following optimization problem on a data set $(x_1, y_1), \dots, (x_n, y_n) \in \mathbf{R}^d \times \mathcal{Y}$:

$$\min_{w \in \mathbf{R}^d} R\left(\sqrt{\langle w, w \rangle}\right) + L(\langle w, x_1 \rangle, \dots, \langle w, x_n \rangle),$$

where $w, x_1, \dots, x_n \in \mathbf{R}^d$, and $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbf{R}^d . The function $R : \mathbf{R}^{\geq 0} \rightarrow \mathbf{R}$ is nondecreasing and gives us our regularization term, while $L : \mathbf{R}^n \rightarrow \mathbf{R}$ is arbitrary³ and gives us our loss term. We noted in lecture that this general form includes soft-margin SVM and ridge regression, though not lasso regression. Using the representer theorem, we showed if the optimization problem has a solution, there is always a solution of the form $w = \sum_{i=1}^n \alpha_i x_i$, for some $\alpha \in \mathbf{R}^n$. Plugging this into the our original problem, we get the following “kernelized” optimization problem:

$$\min_{\alpha \in \mathbf{R}^n} R\left(\sqrt{\alpha^T K \alpha}\right) + L(K\alpha),$$

where $K \in \mathbf{R}^{n \times n}$ is the Gram matrix (or “kernel matrix”) defined by $K_{ij} = k(x_i, x_j) = \langle x_i, x_j \rangle$. Predictions are given by

$$f(x) = \sum_{i=1}^n \alpha_i k(x_i, x),$$

³You may be wondering “Where are the y_i 's?”. They're built into the function L . For example, a square loss on a training set of size 3 could be represented as $L(p_1, p_2, p_3) = \frac{1}{3} \left[(p_1 - y_1)^2 + (p_2 - y_2)^2 + (p_3 - y_3)^2 \right]$, where each p_i stands for the i th prediction $\langle w, x_i \rangle$.

and we can recover the original $w \in \mathbf{R}^d$ by $w = \sum_{i=1}^n \alpha_i x_i$.

The “**kernel trick**” is to swap out occurrences of the kernel k (and the corresponding Gram matrix K) with another kernel. For example, we could replace $k(x_i, x_j) = \langle x_i, x_j \rangle$ by $k'(x_i, x_j) = \langle \psi(x_i), \psi(x_j) \rangle$ for an arbitrary feature mapping $\psi : \mathbf{R}^d \rightarrow \mathbf{R}^D$. In this case, the recovered $w \in \mathbf{R}^D$ would be $w = \sum_{i=1}^n \alpha_i \psi(x_i)$ and predictions would be $\langle w, \psi(x_i) \rangle$.

More interestingly, we can replace k by another kernel $k''(x_i, x_j)$ for which we do not even know or cannot explicitly write down a corresponding feature map ψ . Our main example of this is the RBF kernel

$$k(x, x') = \exp \left(-\frac{\|x - x'\|^2}{2\sigma^2} \right),$$

for which the corresponding feature map ψ is infinite dimensional. In this case, we cannot recover w since it would be infinite dimensional. Predictions must be done using $\alpha \in \mathbf{R}^n$, with $f(x) = \sum_{i=1}^n \alpha_i k(x_i, x)$.

Your implementation of kernelized methods below should not make any reference to w or to a feature map ψ . Your “learning” routine should return α , rather than w , and your prediction function should also use α rather than w . This will allow us to work with kernels that correspond to infinite-dimensional feature vectors.

8.2 Kernels and Kernel Machines

There are many different families of kernels. So far we’ve spoken about linear kernels, RBF/Gaussian kernels, and polynomial kernels. The last two kernel types have parameters. In this section we’ll implement these kernels in a way that will be convenient for implementing our kernelized ML methods later on. For simplicity, and because it is by far the most common situation⁴, we will assume that our input space is $\mathcal{X} = \mathbf{R}^d$. This allows us to represent a collection of n inputs in a matrix $X \in \mathbf{R}^{n \times d}$, as usual.

1. Write functions that compute the RBF kernel $k_{\text{RBF}(\sigma)}(x, x') = \exp(-\|x - x'\|^2 / (2\sigma^2))$ and the polynomial kernel $k_{\text{poly}(a,d)}(x, x') = (a + \langle x, x' \rangle)^d$. The linear kernel $k_{\text{linear}}(x, x') = \langle x, x' \rangle$, has been done for you in the support code. Your functions should take as input two matrices $W \in \mathbf{R}^{n_1 \times d}$ and $X \in \mathbf{R}^{n_2 \times d}$ and should return a matrix $M \in \mathbf{R}^{n_1 \times n_2}$ where $M_{ij} = k(W_i, X_j)$. In words, the (i, j) ’th entry of M should be kernel evaluation between w_i (the i th row of W) and x_j (the j th row of X). The matrix M could be called the “cross-kernel” matrix, by analogy to the **cross-covariance matrix**. For the RBF kernel, you may use the `scipy` function `cdist(X1, X2, 'sqeuclidean')` in the package `scipy.spatial.distance` or (with some more work) write it in terms the linear kernel ([Bauckhage’s article](#) on calculating Euclidean distance matrices may be helpful).
2. Use the linear kernel function defined in the code to compute the kernel matrix on the set of points $x_0 \in \mathcal{D}_X = \{-4, -1, 0, 2\}$. Include both the code and the output.
3. Suppose we have the data set $\mathcal{D} = \{(-4, 2), (-1, 0), (0, 3), (2, 5)\}$. Then by the representer theorem, the final prediction function will be in the span of the functions $x \mapsto k(x_0, x)$ for $x_0 \in \mathcal{D}_X = \{-4, -1, 0, 2\}$. This set of functions will look quite different depending on the kernel function we use.

⁴We are noting this because one interesting aspect of kernel methods is that they can act directly on an arbitrary input space \mathcal{X} (e.g. text files, music files, etc.), so long as you can define a kernel function $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbf{R}$. But we’ll not consider that case here.

- (a) Plot the set of functions $x \mapsto k_{\text{linear}}(x_0, x)$ for $x_0 \in \mathcal{D}_X$ and for $x \in [-6, 6]$.
- (b) Plot the set of functions $x \mapsto k_{\text{poly}(1,3)}(x_0, x)$ for $x_0 \in \mathcal{D}_X$ and for $x \in [-6, 6]$.
- (c) Plot the set of functions $x \mapsto k_{\text{RBF}(1)}(x_0, x)$ for $x_0 \in \mathcal{D}_X$ and for $x \in [-6, 6]$.
- (d) By the representer theorem, the final prediction function will be of the form $f(x) = \sum_{i=1}^n \alpha_i k(x_i, x)$, where $x_1, \dots, x_n \in \mathcal{X}$ are the inputs in the training set. This is a special case of what is sometimes called a **kernel machine**, which is a function of the form $f(x) = \sum_{i=1}^r \alpha_i k(\mu_i, x)$, where $\mu_1, \dots, \mu_r \in \mathcal{X}$ are called **prototypes** or **centroids** (Murphy's book Section 14.3.1). In the special case that the kernel is an RBF kernel, we get what's called an **RBF Network** (proposed by **Broomhead and Lowe in 1988**). We can see that the prediction functions we get from our kernel methods will be kernel machines in which each input in the training set x_1, \dots, x_n serves as a prototype point. Complete the `predict` function of the class `Kernel_Machine` in the skeleton code. Construct a `Kernel_Machine` object with the RBF kernel (`sigma=1`), with prototype points at $-1, 0, 1$ and corresponding weights $1, -1, 1$. Plot the resulting function.

Note: For this problem, and for other problems below, it may be helpful to use **partial application** on your kernel functions. For example, if your polynomial kernel function has signature `polynomial_kernel(W, X, offset, degree)`, you can write `k = functools.partial(polynomial_kernel, offset=2, degree=2)`, and then a call to `k(W, X)` is equivalent to `polynomial_kernel(W, X, offset=2, degree=2)`, the advantage being that the extra parameter settings are built into `k(W, X)`. This can be convenient so that you can have a function that just takes a kernel function `k(W, X)` and doesn't have to worry about the parameter settings for the kernel.

8.3 Kernel Ridge Regression

In the zip file for this assignment, you'll find a training and test set, along with some skeleton code. We're considering a one-dimensional regression problem, in which $\mathcal{X} = \mathcal{Y} = \mathcal{A} = \mathbf{R}$. We'll fit this data using kernelized ridge regression, and we'll compare the results using several different kernel functions. Because the input space is one-dimensional, we can easily visualize the results.

1. Plot the training data. You should note that while there is a clear relationship between x and y , the relationship is not linear.
2. In a previous problem, we showed that in kernelized ridge regression, the final prediction function is $f(x) = \sum_{i=1}^n \alpha_i k(x_i, x)$, where $\alpha = (\lambda I + K)^{-1} y$ and $K \in \mathbf{R}^{n \times n}$ is the Gram matrix of the training data: $K_{ij} = k(x_i, x_j)$, for x_1, \dots, x_n . In terms of kernel machines, α_i is the weight on the kernel function evaluated at the prototype point x_i . Complete the function `train_kernel_ridge_regression` so that it performs kernel ridge regression and returns a `Kernel_Machine` object that can be used for predicting on new points.
3. Use the code provided to plot your fits to the training data for the RBF kernel with a fixed regularization parameter of 0.0001 for 3 different values of `sigma`: 0.01, 0.1, and 1.0. With values of `sigma` do you think would be more likely to over fit, and which less?
4. Use the code provided to plot your fits to the training data for the RBF kernel with a fixed `sigma` of 0.02 and 4 different values of the regularization parameter λ : 0.0001, 0.01, 0.1, and 2.0. What happens to the prediction function as $\lambda \rightarrow \infty$?

5. Find the best hyperparameter settings (including kernel parameters and the regularization parameter) for each of the kernel types. Summarize your results in a table, which gives training error and test error for each setting. Include in your table the best settings for each kernel type, as well as nearby settings that show that making small change in any one of the hyperparameters in either direction will cause the performance to get worse. You should use average square loss on the test set to rank the parameter settings. To make things easier for you, we have provided an sklearn wrapper for the kernel ridge regression function we have created so that you can use sklearn's GridSearchCV. Note: Because of the small dataset size, these models can be fit extremely fast, so there is no excuse for not doing extensive hyperparameter tuning.
6. Plot your best fitting prediction functions using the polynomial kernel and the RBF kernel. Use the domain $x \in (-0.5, 1.5)$. Comment on the results.
7. The data for this problem was generated as follows: A function $f : \mathbf{R} \rightarrow \mathbf{R}$ was chosen. Then to generate a point (x, y) , we sampled x uniformly from $(0, 1)$ and we sampled $\varepsilon \sim \mathcal{N}(0, 0.1^2)$ (so $\text{Var}(\varepsilon) = 0.1^2$). The final point is $(x, f(x) + \varepsilon)$. What is the Bayes decision function and the Bayes risk for the loss function $\ell(\hat{y}, y) = (\hat{y} - y)^2$.
8. [Optional] Attempt to improve performance by using different kernel functions. [Chapter 4](#) from Rasmussen and Williams' book *Gaussian Processes for Machine Learning* describes many kernel functions, though they are called **covariance functions** in that book (but they have exactly the same definition). Note that you may also create a kernel function by first explicitly creating feature vectors, if you are so inspired.
9. [Optional] Use any machine learning model you like to get the best performance you can.

8.4 [Optional] Kernelized Support Vector Machines with Kernelized Pegasos

1. Load the SVM training and test data from the zip file. Plot the training data using the code supplied. Is the data linearly separable? Quadratically separable? What if we used some RBF kernel?
2. Unlike for kernel ridge regression, there is no closed-form solution for SVM classification (kernelized or not). Implement kernelized Pegasos. Because we are not using a sparse representation for this data, you will probably not see much gain by implementing the "optimized" versions described in the problems above.
3. Find the best hyperparameter settings (including kernel parameters and the regularization parameter) for each of the kernel types. Summarize your results in a table, which gives training error and test error (i.e. average 0/1 loss) for each setting. Include in your table the best settings for each kernel type, as well as nearby settings that show that making small change in any one of the hyperparameters in either direction will cause the performance to get worse. You should use the 0/1 loss on the test set to rank the parameter settings.
4. Plot your best fitting prediction functions using the linear, polynomial, and the RBF kernel. The code provided may help.

9 Representer Theorem [Optional]

Recall the following theorem from lecture:

Theorem (Representer Theorem). *Let*

$$J(w) = R(\|w\|) + L(\langle w, \psi(x_1) \rangle, \dots, \langle w, \psi(x_n) \rangle),$$

where $R : \mathbf{R}^{\geq 0} \rightarrow \mathbf{R}$ is nondecreasing (the **regularization** term) and $L : \mathbf{R}^n \rightarrow \mathbf{R}$ is arbitrary (the **loss** term). If $J(w)$ has a minimizer, then it has a minimizer of the form

$$w^* = \sum_{i=1}^n \alpha_i \psi(x_i).$$

Furthermore, if R is strictly increasing, then all minimizers have this form.

Note: There is nothing in this theorem that guarantees $J(w)$ has a minimizer at all. If there is no minimizer, then this theorem does not tell us anything.

In this problem, we will prove the part of the Representer theorem for the case that R is strictly increasing.

1. Let M be a closed subspace of a Hilbert space \mathcal{H} . For any $x \in \mathcal{H}$, let $m_0 = \text{Proj}_M x$ be the projection of x onto M . By the Projection Theorem, we know that $(x - m_0) \perp M$. Then by the Pythagorean Theorem, we know $\|x\|^2 = \|m_0\|^2 + \|x - m_0\|^2$. From this we concluded in lecture that $\|m_0\| \leq \|x\|$. Show that we have $\|m_0\| = \|x\|$ only when $m_0 = x$. (Hint: Use the postive-definiteness of the inner product: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$, and the fact that we're using the norm derived from such an inner product.)

Solution: If $\|m_0\| = \|x\|$, then we must have $\|x - m_0\|^2 = \langle x - m_0, x - m_0 \rangle = 0$, which happens only when $x - m_0 = 0$, by definition of inner product.

2. Give the proof of the Representer Theorem in the case that R is strictly increasing. That is, show that if R is strictly increasing, then all minimizers have this form claimed. (Hint: Consider separately the cases that $\|w\| < \|w^*\|$ and the case $\|w\| = \|w^*\|$.)

Solution: Let's repeat the proof of the Representer Theorem: Let w^* be a minimizer. Let $M = \text{span}(\psi(x_1), \dots, \psi(x_n))$ [the "span of the data"]. Let $w = \text{Proj}_M w^*$. So $\exists \alpha$ s.t. $w = \sum_{i=1}^n \alpha_i \psi(x_i)$. Then $w^\perp := w^* - w$ is orthogonal to M . Since projections decrease norms, $\|w\| \leq \|w^*\|$. If $\|w\| = \|w^*\|$, then by the previous problem, we must have $w = w^*$. So w^* is in the span of the data (i.e. has the form given). If $\|w\| < \|w^*\|$ then since R is strictly increasing, we must have $R(\|w\|) < R(\|w^*\|)$. We also have $L(\langle w^*, \psi(x_1) \rangle, \dots, \langle w^*, \psi(x_n) \rangle) = L(\langle w, \psi(x_1) \rangle, \dots, \langle w, \psi(x_n) \rangle)$, since $\langle w^*, \psi(x_i) \rangle = \langle w + w^\perp, \psi(x_i) \rangle = \langle w, \psi(x_i) \rangle$. Thus we must have $J(w) < J(w^*)$, which is a contradiction to our assumption that w^* is a minimizer.

3. Suppose that $R : \mathbf{R}^{\geq 0} \rightarrow \mathbf{R}$ and $L : \mathbf{R}^n \rightarrow \mathbf{R}$ are both convex functions. Use properties of convex functions to show that $w \mapsto L(\langle w, \psi(x_1) \rangle, \dots, \langle w, \psi(x_n) \rangle)$ is a convex function of w , and then that $J(w)$ is also convex function of w . For simplicity, you may assume that our feature space is \mathbf{R}^d , rather than a generic Hilbert space. You may also use the fact that the composition of a convex function and an affine function is convex. That is, suppose $f : \mathbf{R}^n \rightarrow \mathbf{R}$, $A \in \mathbf{R}^{n \times m}$ and $b \in \mathbf{R}^n$. Define $g : \mathbf{R}^m \rightarrow \mathbf{R}$ by $g(x) = f(Ax + b)$. Then if f

is convex, then so is g . From this exercise, we can conclude that if L and R are convex, then J does have a minimizer of the form $w^* = \sum_{i=1}^n \alpha_i \psi(x_i)$, and if R is also strictly increasing, then all minimizers of J have this form.

Solution: We can take A to be a matrix whose i th row is $\psi(x_i)$. Then Aw is a column vector with the i th entry $\langle w, \psi(x_i) \rangle$. We can write $w \mapsto L(\langle w, \psi(x_1) \rangle, \dots, \langle w, \psi(x_n) \rangle)$ as $w \mapsto L(Aw)$, which is convex since L is convex and the fact given in the problem statement. Since all norms on \mathbf{R}^d are convex $\|w\|$ is convex. Since R is nondecreasing and convex, we know that $w \mapsto R(\|w\|)$ is convex, by the statement in BV p. 84 (3.11). Thus $J(w)$ is convex.

10 Ivanov and Tikhonov Regularization [Optional]

In lecture there was a claim that the Ivanov and Tikhonov forms of ridge and lasso regression are equivalent. We will now prove a more general result.

10.1 Tikhonov optimal implies Ivanov optimal

Let $\phi : \mathcal{F} \rightarrow \mathbf{R}$ be any performance measure of $f \in \mathcal{F}$, and let $\Omega : \mathcal{F} \rightarrow \mathbf{R}$ be any complexity measure. For example, for ridge regression over the linear hypothesis space $\mathcal{F} = \{f_w(x) = w^T x \mid w \in \mathbf{R}^d\}$, we would have $\phi(f_w) = \frac{1}{n} \sum_{i=1}^n (w^T x_i - y_i)^2$ and $\Omega(f_w) = w^T w$.

1. Suppose that for some $\lambda > 0$ we have the Tikhonov regularization solution

$$f_* = \arg \min_{f \in \mathcal{F}} [\phi(f) + \lambda \Omega(f)]. \quad (1)$$

Show that f_* is also an Ivanov solution. That is, $\exists r > 0$ such that

$$f_* = \arg \min_{f \in \mathcal{F}} \phi(f) \text{ subject to } \Omega(f) \leq r. \quad (2)$$

(Hint: Start by figuring out what r should be. If you're stuck on this, ask for help. Then one approach is proof by contradiction: suppose f_* is not the optimum in (2) and show that contradicts the fact that f_* solves (1).)

Solution: Take $r = \Omega(f_*)$. Suppose there exists some g with $\Omega(g) \leq r$ and $\phi(g) < \phi(f_*)$. Then

$$\phi(g) + \Omega(g) < \phi(f_*) + \lambda \Omega(f_*),$$

which means that f_* would not be the minimizer in (1).

10.2 Ivanov optimal implies Tikhonov optimal (when we have Strong Duality)

For the converse, we will restrict our hypothesis space to a parametric set. That is,

$$\mathcal{F} = \{f_w(x) : \mathcal{X} \rightarrow \mathbf{R} \mid w \in \mathbf{R}^d\}.$$

So we will now write ϕ and Ω as functions of $w \in \mathbf{R}^d$.

Let w^* be a solution to the following Ivanov optimization problem:

$$\begin{array}{ll} \text{minimize} & \phi(w) \\ \text{subject to} & \Omega(w) \leq r. \end{array}$$

Assume that strong duality holds for this optimization problem and that the dual solution is attained. Then we will show that there exists a $\lambda \geq 0$ such that $w_* = \arg \min_{w \in \mathbf{R}^d} [\phi(w) + \lambda \Omega(w)]$.

1. Write the Lagrangian $L(w, \lambda)$ for the Ivanov optimization problem.

Solution: The Lagrangian is

$$L(w, \lambda) = \phi(w) + \lambda [\Omega(w) - r].$$

2. Write the dual optimization problem in terms of the dual objective function $g(\lambda)$, and give an expression for $g(\lambda)$. [Writing $g(\lambda)$ as an optimization problem is expected - don't try to solve it.]

Solution:

$$\max_{\lambda \geq 0} g(\lambda)$$

where $g(\lambda) = \min_w (\phi(w) + \lambda [\Omega(w) - r])$.

3. We assumed that the dual solution is attained, so let $\lambda^* = \arg \max_{\lambda \geq 0} g(\lambda)$. We also assumed strong duality, which implies $\phi(w^*) = g(\lambda^*)$. Show that the minimum in the expression for $g(\lambda^*)$ is attained at w^* . [Hint: You can use the same approach we used when we derived that strong duality implies complementary slackness⁵.] **Conclude the proof** by showing that for the choice of $\lambda = \lambda^*$, we have $w_* = \arg \min_{w \in \mathbf{R}^d} [\phi(w) + \lambda \Omega(w)]$.

Solution:

$$\begin{aligned} \phi(w^*) &= g(\lambda^*) \\ &= \min_w \phi(w) + \lambda^* [\Omega(w) - r] \\ &\leq \phi(w^*) + \underbrace{\lambda^* [\Omega(w^*) - r]}_{\leq 0} \\ &\leq \phi(w^*). \end{aligned}$$

So all the inequalities are equalities. Therefore

$$\begin{aligned} w^* &= \arg \min_w \phi(w) + \lambda^* [\Omega(w) - r] \\ &= \arg \min_w \phi(w) + \lambda^* \Omega(w) \end{aligned}$$

⁵See <https://davidrosenberg.github.io/mlcourse/Archive/2016/Lectures/3b.convex-optimization.pdf#page=30> slide 30.

4. The conclusion of the previous problem allows $\lambda = 0$, which means we're not actually regularizing at all. To ensure we get a proper Ivanov regularization problem, we need an additional assumption. The one below is taken from [?]:

$$\inf_{w \in \mathbf{R}^d} \phi(w) < \inf_{\substack{w \in \mathbf{R}^d \\ \Omega(w) \leq r}} \phi(w)$$

Note that this is a rather intuitive condition: it is simply saying that we can fit the training data better [strictly better] if we don't use any regularization. With this additional condition, show that $w_* = \arg \min_{w \in \mathbf{R}^d} [\phi(w) + \lambda \Omega(w)]$ for some $\lambda > 0$.

Solution: Recall the dual function $g(\lambda) = \min_w (\phi(w) + \lambda [\Omega(w) - r])$. With the given condition, we know that

$$g(0) = \min_w \phi(w) < \inf_{\substack{w \in \mathbf{R}^d \\ \Omega(w) \leq r}} \phi(w) = g(\lambda^*).$$

Thus $\lambda^* \neq 0$. Since $\lambda \geq 0$, we conclude that $\lambda > 0$.

10.3 Ivanov implies Tikhonov for Ridge Regression.

To show that Ivanov implies Tikhonov for the ridge regression problem (square loss with ℓ_2 regularization), we need to demonstrate strong duality and that the dual optimum is attained. Both of these things are implied by Slater's constraint qualifications.

1. Show that the Ivanov form of ridge regression is a convex optimization problem with a strictly feasible point.

Solution: The Ivanov form of ridge regression is

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n (y_i - w^T x_i)^2 \\ & \text{subject to} && w^T w \leq r. \end{aligned}$$

First we show this is a convex optimization problem: The objective is convex in w since we have a nonnegative combination of convex functions $(y_i - w^T x_i)^2$. Each of these expressions is convex since the square is a convex function, and we're applying it to an affine transformation of w . The constraint function is also a convex function of w . Slater's constraint qualification is satisfied since we can take $w = 0$, so long as $r > 0$. Thus we have strong duality.

11 Novelty Detection [Optional]

(Problem derived from Michael Jordan's Stat 241b Problem Set #2, Spring 2004)

A novelty detection algorithm can be based on an algorithm that finds the smallest possible sphere containing the data in feature space.

1. Let $\phi : \mathcal{X} \rightarrow \mathcal{H}$ be our feature map, mapping elements of the input space into our “feature space” \mathcal{H} , which is a Hilbert space (and thus has an inner product). Formulate the novelty detection algorithm described above as an optimization problem.

Solution: The problem we wish to solve is

$$\min_{c \in F, r \in \mathbf{R}} r^2$$

subject to

$$\|\varphi(x_i) - c\|^2 \leq r^2 \text{ all } i$$

Note that this solution will be attained at $\pm r$, but it is clear that the one we want is $|r|$.

2. Give the Lagrangian for this problem, and write an equivalent, unconstrained “inf sup” version of the optimization problem.

Solution:

$$\inf_{c \in F, r \in \mathbf{R}} \sup_{\lambda \geq 0} r^2 + \sum \lambda_i (\|\varphi(x_i) - c\|^2 - r^2)$$

3. Show that we have strong duality and thus we will have an equivalent optimization problem if we swap the inf and the sup. [Hint: Use Slater’s qualification conditions.]

Solution: Find any point that satisfies the constraints strictly.

4. Solve the inner minimization problem and give the dual optimization problem. [Note: You may find it convenient to define the kernel function $k(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle$ and to write your final problem in terms of the corresponding kernel matrix K to simplify notation.]

Solution:

$$\sup_{\lambda \geq 0} \inf_{c \in F, r \in \mathbf{R}} r^2 + \sum \lambda_i (\|\varphi(x_i) - c\|^2 - r^2)$$

We can expand the inner expression (which is the Lagrangian) as

$$\begin{aligned} L(r, c, \lambda) &:= r^2 + \sum \lambda_i (\|\varphi(x_i) - c\|^2 - r^2) = r^2 (1 - \sum \lambda_i) + \sum \lambda_i \langle \varphi(x_i) - c, \varphi(x_i) - c \rangle \\ &= r^2 (1 - \sum \lambda_i) + \sum \lambda_i (K(i, i) - 2 \langle \varphi(x_i), c \rangle + \langle c, c \rangle) \\ &= r^2 (1 - \sum \lambda_i) + \sum \lambda_i K(i, i) - \left\langle 2 \sum \lambda_i \varphi(x_i), c \right\rangle + \langle c, c \rangle \sum \lambda_i \end{aligned}$$

The problem looks easier now that r and c are separated. Define $\theta(\lambda) := \inf_{c \in F, r \in \mathbf{R}} L(r, c, \lambda)$.

If $1 - \sum \lambda_i < 0$, then $\lim_{r \rightarrow \infty} L(r, c, \lambda) = -\infty$, so $\theta(\lambda) = -\infty$.

If $1 - \sum \lambda_i > 0$, then the best we can do is to set $r = 0$ (boundary point is minimal).

If $1 - \sum \lambda_i = 0$, then r is arbitrary.

Notice that $L(r, c, \lambda)$ is quadratic in the entries of c , thus the minimum w.r.t. c is attained at the unique critical point. Differentiating with respect to the column vector c and equating to

zero, we get the vector equation

$$\begin{aligned} -2 \sum \lambda_i \varphi(x_i) + 2c \sum \lambda_i \\ c &= \frac{\sum \lambda_i \varphi(x_i)}{\sum \lambda_i} \end{aligned}$$

BELOW HERE MAY BE BROKEN!

Thus if $1 - \sum \lambda_i \geq 0$, the minimum is attained at $c = \sum \lambda_i \varphi(x_i)$ and $r^2(1 - \sum \lambda_i) = 0$, and the value is

$$\begin{aligned} \theta(\lambda) &= \sum \lambda_i K(i, i) - 2 \sum_{ij} \lambda_i \lambda_j K(i, j) + \sum_{ij} \lambda_i \lambda_j K(i, j) \\ &= \text{diag}(K)^T \lambda - \lambda^T K \lambda \end{aligned}$$

So our final result for $\theta(\lambda)$ is

$$\theta(\lambda) = \begin{cases} -\infty & \text{for } 1 - \sum \lambda_i < 0 \\ \text{diag}(K)^T \lambda - \lambda^T K \lambda & \text{for } 1 - \sum \lambda_i \geq 0 \end{cases}$$

Since we clearly have points for which $\lambda \geq 0$ and for $1 - \sum \lambda_i \geq 0$, we clearly have $\sup_{\lambda \geq 0} \theta(\lambda) = \sup_{\lambda \geq 0, \sum \lambda_j \leq 1} \theta(\lambda)$. In fact, we can restrict our search to $\sum \lambda_j = 1$, because for $\sum \lambda_i \leq 1$,

$$\theta(\lambda) = \sum \lambda_i \|\varphi(x_i) - c\|^2$$

and, written in this way, it is clear that increasing any λ_i always increases $\theta(\lambda)$. Thus the value of the dual problem is

$$\sup_{\lambda \geq 0, \sum \lambda_i = 1} \text{diag}(K)^T \lambda - \lambda^T K \lambda$$

5. Write an expression for the optimal sphere in terms of the solution to the dual problem.

Solution: Note that our constraint set is now compact, thus the sup is attained at some λ^* satisfying the constraints. Since there is no duality gap, we must have

$$\begin{aligned} r^* &= \sqrt{\theta(\lambda^*)} \\ &= \sqrt{\text{diag}(K)^T \lambda^* - (\lambda^*)^T K \lambda^*} \end{aligned}$$

Now the optimal center is

$$c^* = \sum \lambda_i^* \varphi(x_i)$$

6. Write down the complementary slackness conditions for this problem, and characterize the points that are the “support vectors”.

Solution: By the complementary slackness conditions, we must have

$$\lambda_i^* [||\varphi(x_i) - c^*||^2 - (r^*)^2] = 0$$

We call those x_i for which $\lambda_i^* > 0$ the support vectors. Note that we can represent c^* using only the support vectors.

7. Briefly explain how you would apply this algorithm in practice to detect “novel” instances.
8. [More Optional] Redo this problem allowing some of the data to lie outside of the sphere, where the number of points outside the sphere can be increased or decreased by adjusting a parameter. (Hint: Use slack variables).

(a) **Solution:**

Again assume we have the standard inner product in feature space with feature map $x \mapsto \varphi(x) = (\varphi_1(x), \varphi_2(x), \dots) \in F$. We wish to solve

$$\min_{c, r, \xi} r^2 + b \sum \xi_i$$

subject to

$$\begin{aligned} ||\varphi(x_i) - c||^2 &\leq r^2 + \xi_i \quad \text{all } i \\ \xi_i &\geq 0 \end{aligned}$$

This is equivalent to the Lagrangian formulation:

$$\inf_{c, r, \xi} \sup_{\lambda \geq 0, \alpha \geq 0} r^2 + b \sum \xi_i + \sum_i \lambda_i (||\varphi(x_i) - c||^2 - r^2 - \xi_i) - \sum_i \alpha_i \xi_i$$

By Slater's conditions, this is equivalent to the dual problem

$$\sup_{\lambda \geq 0, \alpha \geq 0} \inf_{c, r, \xi} r^2 + b \sum \xi_i + \sum_i \lambda_i (||\varphi(x_i) - c||^2 - r^2 - \xi_i) - \sum_i \alpha_i \xi_i$$

Expanding the inner expression (which is the Lagrangian), we get

$$\begin{aligned} L(r, c, \xi, \lambda, \alpha) &:= r^2 + b \sum \xi_i + \sum_i \lambda_i (||\varphi(x_i) - c||^2 - r^2 - \xi_i) - \sum_i \alpha_i \xi_i \\ &= r^2(1 - \sum \lambda_i) - \sum_i \xi_i (\lambda_i + \alpha_i - b) + \sum_i \lambda_i ||\varphi(x_i) - c||^2 \end{aligned}$$

Define $\theta(\lambda, \alpha) := \inf_{c \in F, r \in \mathbf{R}, \xi \in \mathbf{R}} L(r, c, \xi, \lambda, \alpha)$.

If $1 - \sum \lambda_i < 0$, then $\lim_{r \rightarrow \inf} L(r, c, \lambda) = -\infty$, so $\theta(\lambda) = -\infty$.

If $1 - \sum \lambda_i > 0$, then the best we can do is to set $r = 0$ (boundary point is minimal).

If $1 - \sum \lambda_i = 0$, then r is arbitrary.

If $\lambda_i + \alpha_i - b \neq 0$, then by sending $\xi_i \rightarrow \infty$ or $\xi_i \rightarrow -\infty$, we get $\theta(\lambda) = -\infty$.

If $\lambda_i + \alpha_i - b = 0$, then ξ_i is arbitrary.

Notice that $L(r, c, \xi, \lambda, \alpha)$ is quadratic in the entries of c , thus the minimum w.r.t. c is attained at the unique critical point. Differentiating with respect to the column vector c and equating to zero, we get the vector equation

$$\begin{aligned} -2 \sum \lambda_i \varphi(x_i) + 2c &= 0 \\ c &= \sum \lambda_i \varphi(x_i) \end{aligned}$$

Thus if $1 - \sum \lambda_i \geq 0$ and $\lambda_i + \alpha_i - b = 0$, then minimum is attained at $c = \sum \lambda_i \varphi(x_i)$ and $r^2(1 - \sum \lambda_i) = 0$, and the value is

$$\begin{aligned} \theta(\lambda, \alpha) &= \sum \lambda_i K(i, i) - 2 \sum_{ij} \lambda_i \lambda_j K(i, j) + \sum_{ij} \lambda_i \lambda_j K(i, j) \\ &= \text{diag}(K)^T \lambda - \lambda^T K \lambda \end{aligned}$$

So our final result for $\theta(\lambda)$ is

$$\theta(\lambda, \alpha) = \begin{cases} -\infty & \text{for } 1 - \sum \lambda_i < 0 \text{ or } \lambda_i + \alpha_i - b \neq 0 \\ \text{diag}(K)^T \lambda - \lambda^T K \lambda & \text{for } 1 - \sum \lambda_i \geq 0 \text{ and } \lambda_i + \alpha_i - b = 0 \end{cases}$$

Since we clearly have points for which $\lambda \geq 0$ and $\alpha \geq 0$ satisfying the second set of constraints, $\sup_{\lambda \geq 0, \alpha \geq 0} \theta(\lambda) = \sup_{\lambda \geq 0, \alpha \geq 0, \sum \lambda_j \leq 1, \lambda_i + \alpha_i = b} \theta(\lambda)$. In fact, we can restrict our search to $\sum \lambda_j = 1$, because for $\sum \lambda_i \leq 1$ and $\lambda_i + \alpha_i = b$,

$$\theta(\lambda, \alpha) = \sum \lambda_i \|\varphi(x_i) - c\|^2$$

and, written in this way, it is clear that increasing any λ_i always increases $\theta(\lambda)$. Thus the value of the dual problem is

$$\sup_{\lambda \geq 0, \alpha \geq 0, \sum \lambda_i = 1, \lambda_i + \alpha_i = b} \text{diag}(K)^T \lambda - \lambda^T K \lambda$$

The constraint set under the sup is equivalent to $\sum \lambda_i = 1, 0 \leq \lambda_i \leq b$. Note that our constraint set is now compact, thus the sup is attained at some λ^* satisfying the constraints. So the optimal center is

$$c^* = \sum \lambda_i^* \varphi(x_i)$$

We work a little more to get the optimal radius: By the KKT conditions, we must have

$$\begin{aligned} \lambda_i^* [\|\varphi(x_i) - c^*\|^2 - (r^*)^2 - \xi_i^*] &= 0 \\ \alpha_i^* \xi_i^* = (b - \lambda_i^*) \xi_i^* &= 0 \end{aligned}$$

Assuming $b > 0$, for each i , either $\lambda_i^* = 0$ and $\xi_i^* = 0$. If $\xi_i^* > 0$, then $\lambda_i^* = b$ and $\xi_i^* = \|\varphi(x_i) - c^*\|^2 - (r^*)^2$. If $0 < \lambda_i^* < b$, then we must have $\xi_i^* = 0 = \|\varphi(x_i) - c^*\|^2 - (r^*)^2$. In other words, x_i must lie on the circle.

If there is some λ_i^* for which $0 < \lambda_i^* < b$, then

$$\begin{aligned}
(r^*)^2 &= \|\varphi(x_i) - c^*\|^2 \\
&= \left\langle \varphi(x_i) - \sum_j \lambda_j^* \varphi(x_j), \varphi(x_i) - \sum_j \lambda_j^* \varphi(x_j) \right\rangle \\
&= K(i, i) - 2 \sum_j \lambda_j^* K(j, i) + \sum_{jk} \lambda_j^* \lambda_k^* K(j, k) \\
&= K(i, i) - 2(\lambda^*)^T K(:, i) + (\lambda^*)^T K \lambda^*
\end{aligned}$$

The “support vectors” are those vectors x_i corresponding to $\lambda_i > 0$. By the results below, these points are the points either on the circle (c^*, r^*) or outside of it.

What if all nonzero $\lambda_i^* = b$? Recall that $\sum_i \lambda_i^* = 1$. So this amounts to $\sum_I \lambda_i^* = |I|b = 1$. So $|I| = 1/b$. So this can only happen if $1/b$ is a positive integer.

Below are some results derived from the original constraints and the KKT conditions above, possibly assuming $\lambda_i \in \{0, b\}$.

$$\begin{aligned}
\lambda_i^* [\|\varphi(x_i) - c^*\|^2 - (r^*)^2 - \xi_i^*] &= 0 \\
\alpha_i^* \xi_i^* = (b - \lambda_i^*) \xi_i^* &= 0
\end{aligned}$$

$\lambda_i^* = b$ implies $\xi_i^* = \|\varphi(x_i) - c^*\|^2 - (r^*)^2 \geq 0$ (≥ 0 follows from $\xi_i \geq 0$).

$\xi_i^* = \|\varphi(x_i) - c^*\|^2 - (r^*)^2 > 0$ implies $\lambda_i^* = b$.

$\xi_i^* = \|\varphi(x_i) - c^*\|^2 - (r^*)^2 = 0$ implies that x_i lies on the sphere of radius r .

$\lambda_i^* = 0$ implies $\xi_i^* = 0$.

$\xi_i^* = 0$ implies that $\lambda_i^* [\|\varphi(x_i) - c^*\|^2 - (r^*)^2] = 0$.

From the basic constraints we know that

$\|\varphi(x_i) - c^*\|^2 - (r^*)^2 > 0$ implies $\xi_i^* > 0$ and $\lambda_i = b$. (second implication from first and KKT)

$\|\varphi(x_i) - c^*\|^2 - (r^*)^2 = 0$ implies $\xi_i^* = 0$. (show by contradiction)

$\|\varphi(x_i) - c^*\|^2 - (r^*)^2 < 0$ implies $\lambda_i^* = 0$ and $\xi_i^* = 0$ (since $\xi_i^* \geq 0$).

If $\lambda_i^* = b$ and $\|\varphi(x_i) - c^*\|^2 - (r^*)^2 = 0$ then $\xi_i^* = 0$.

If $\lambda_i^* = b$ and $\xi_i^* = 0$, then $\|\varphi(x_i) - c^*\|^2 - (r^*)^2 = 0$.

In general, if $0 < \lambda_i^* < b$, then $\xi_i^* = \|\varphi(x_i) - c^*\|^2 - (r^*)^2 = 0$.