计算机辅助几何设计
2019秋学期

Bezier Curves (continue)

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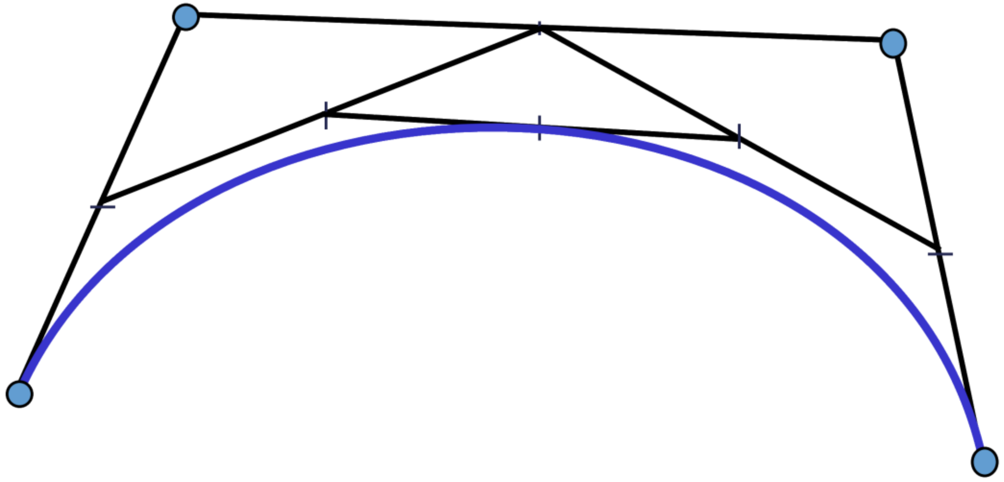
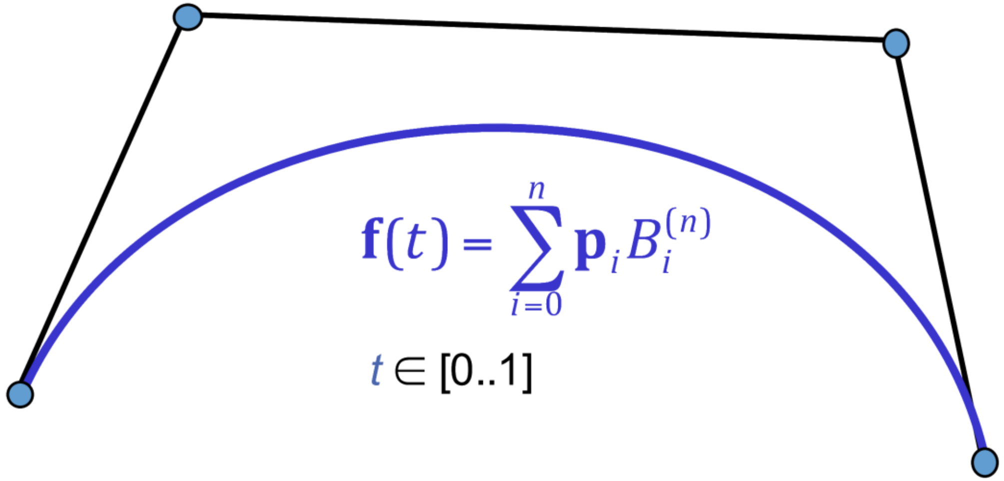
Recap

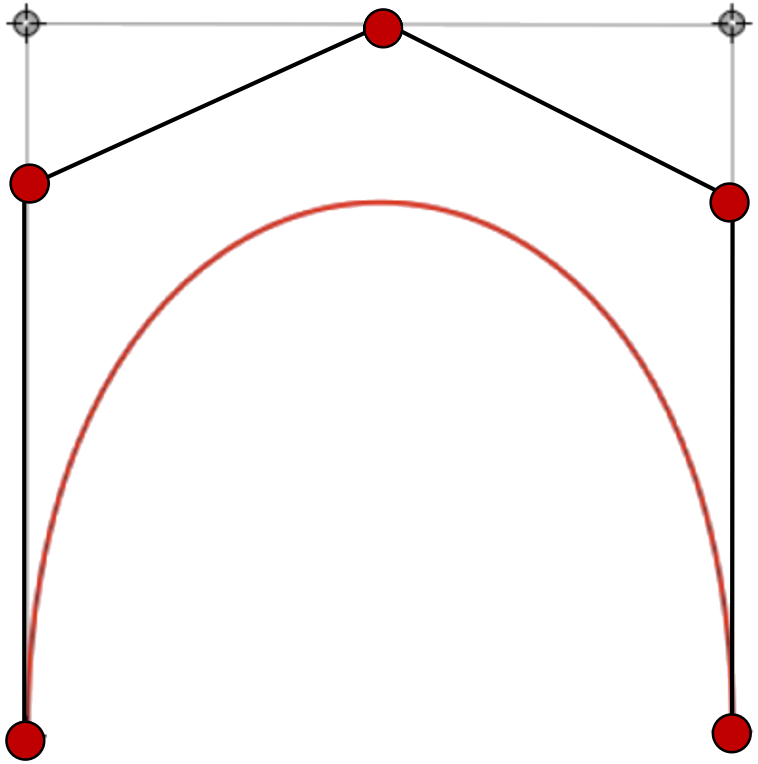


de Casteljau algorithm



Bernstein form

Degree elevation



- $\bar{\boldsymbol{b}}_0 = \boldsymbol{b}_0$
- $\bar{\boldsymbol{b}}_{n+1} = \boldsymbol{b}_n$

$$\bar{\boldsymbol{b}}_j = \frac{j}{n+1}\boldsymbol{b}_{j-1} + \left(1 - \frac{j}{n+1}\right)\boldsymbol{b}_j$$

$$j = 1, \dots, n$$

Bezier Curves

Subdivision

Subdivision

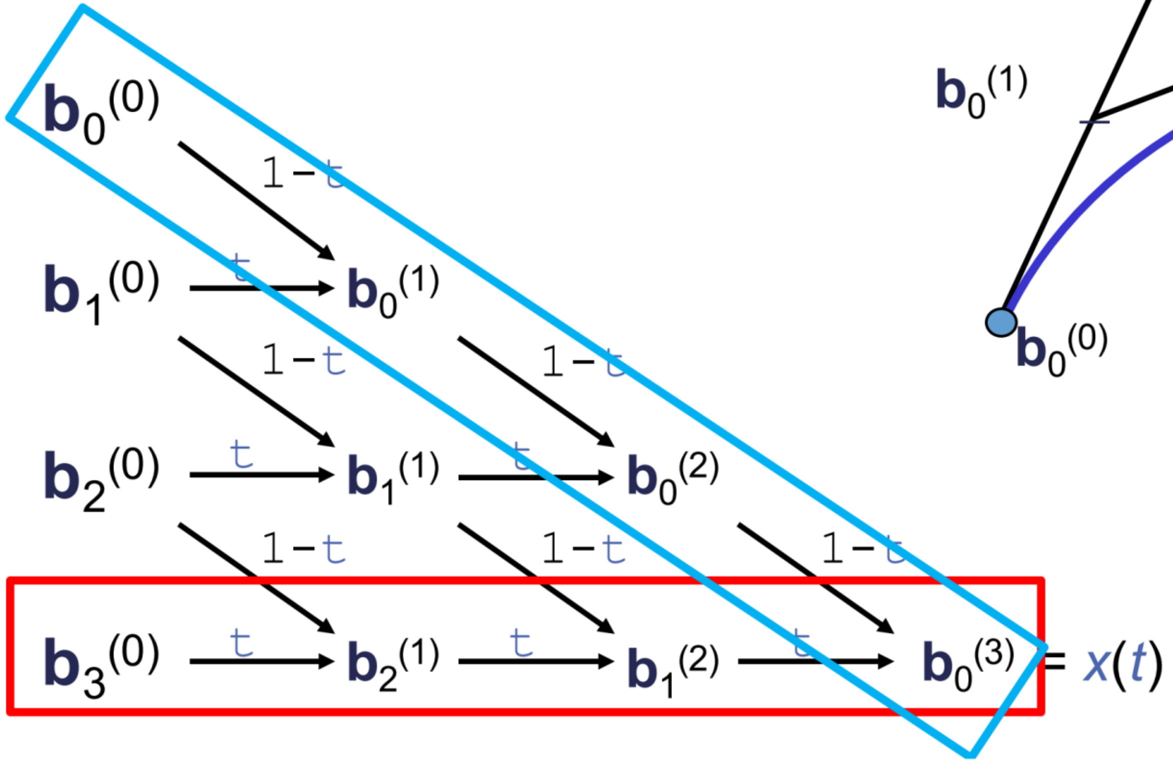
- **Given:** $b_0, \dots, b_n \to x(t), t \in [0,1]$

- **Wanted:** $b_0^{(1)}, \dots, b_n^{(1)} \to x^{(1)}(t),$

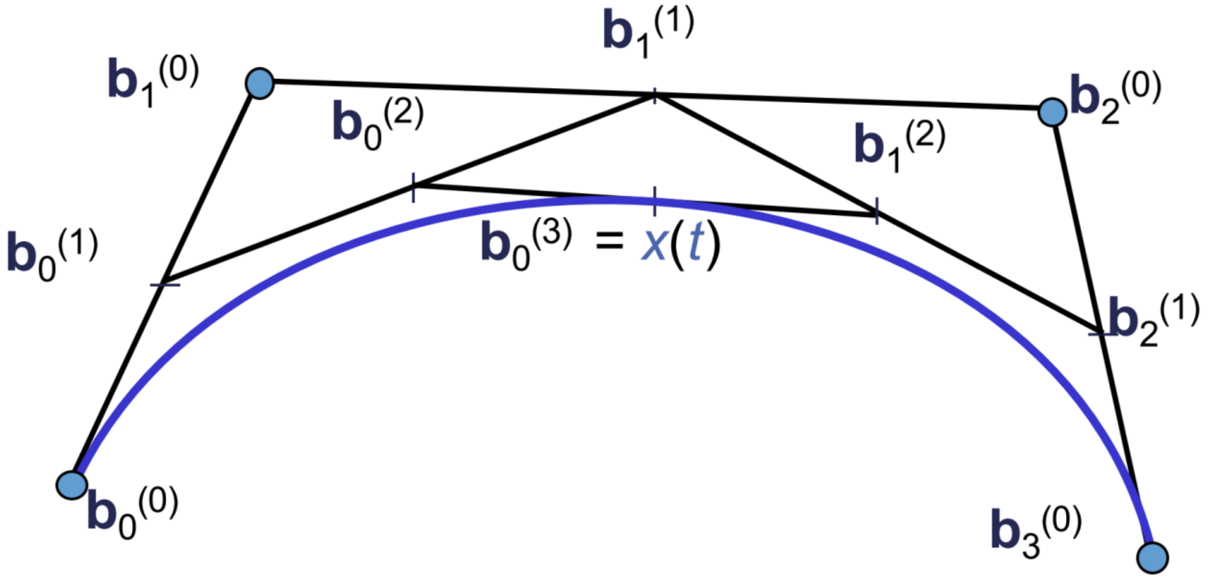
  $b_0^{(2)}, \dots, b_n^{(2)} \to x^{(2)}(t),$

  with $x = x^{(1)} \cup x^{(2)}$

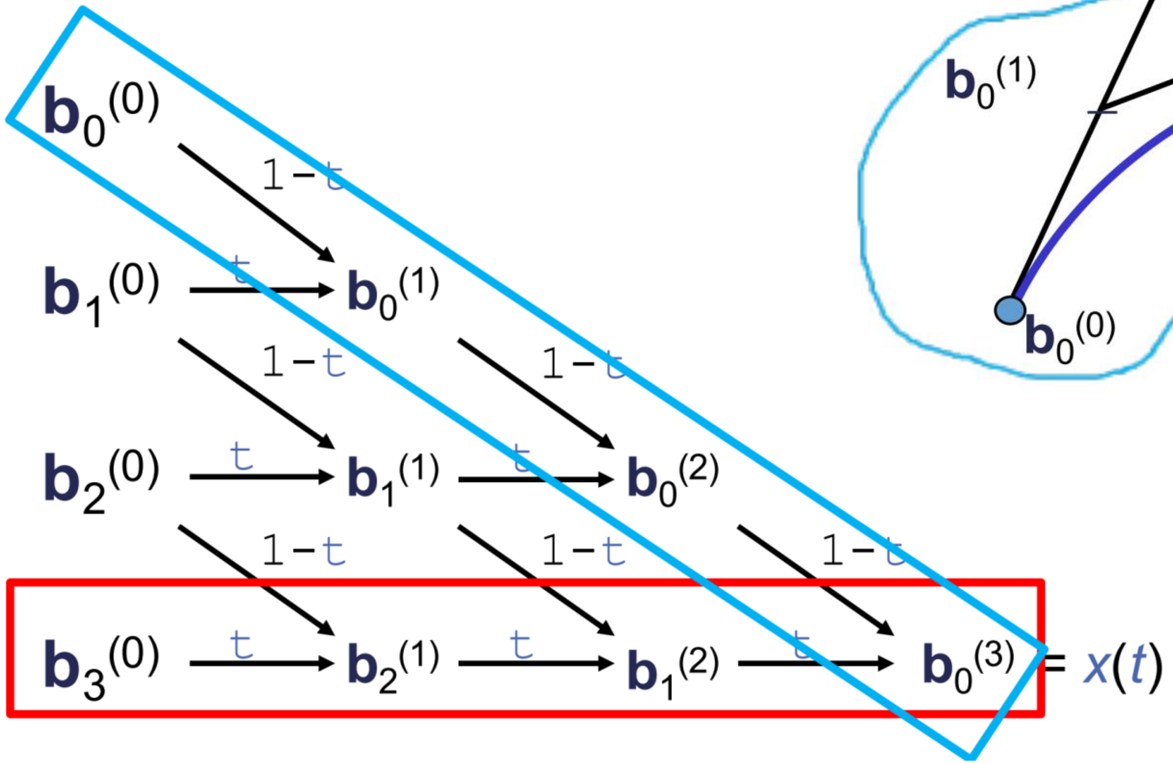
Subdivision: Example



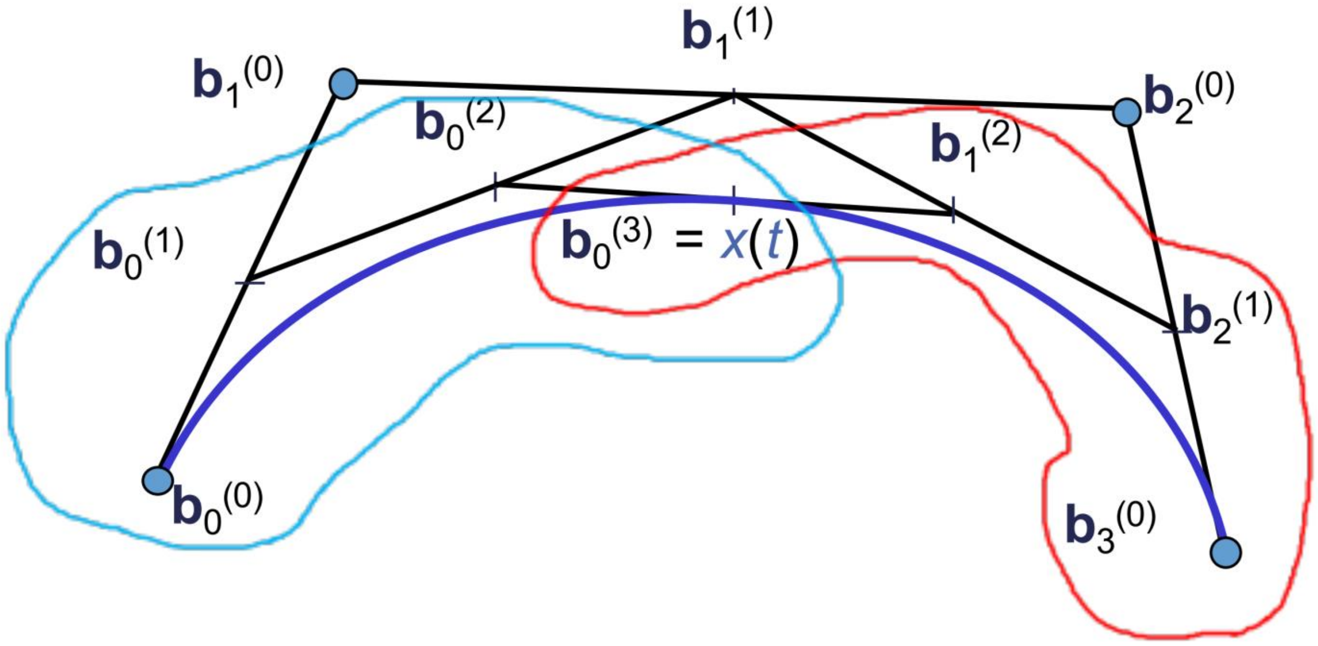
de Casteljau scheme



Subdivision: Example



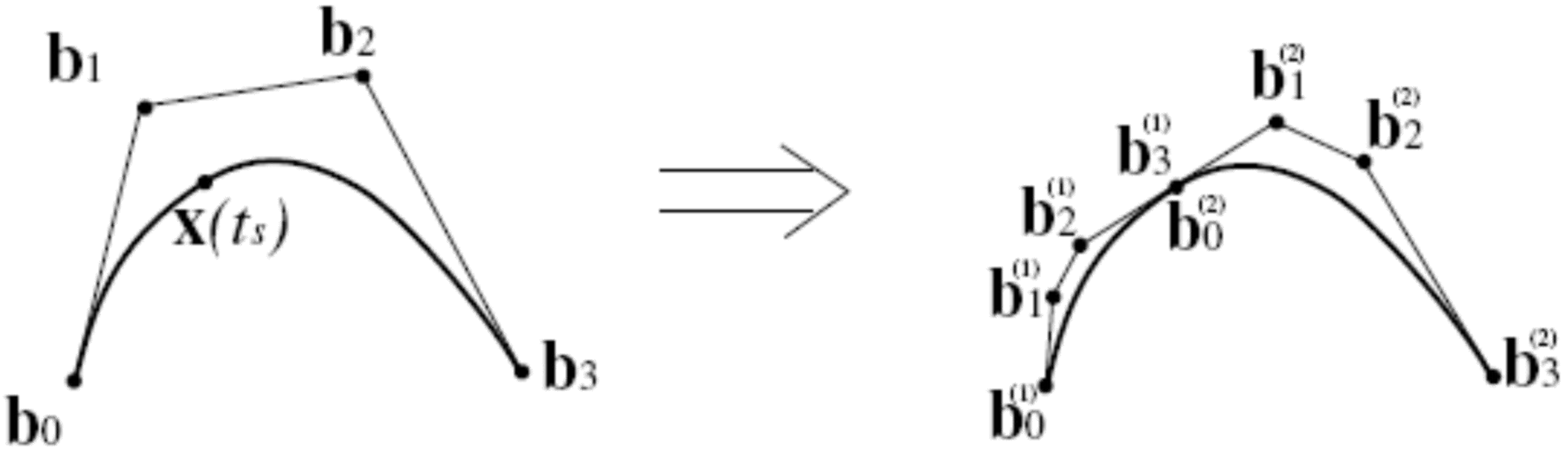
de Casteljau scheme



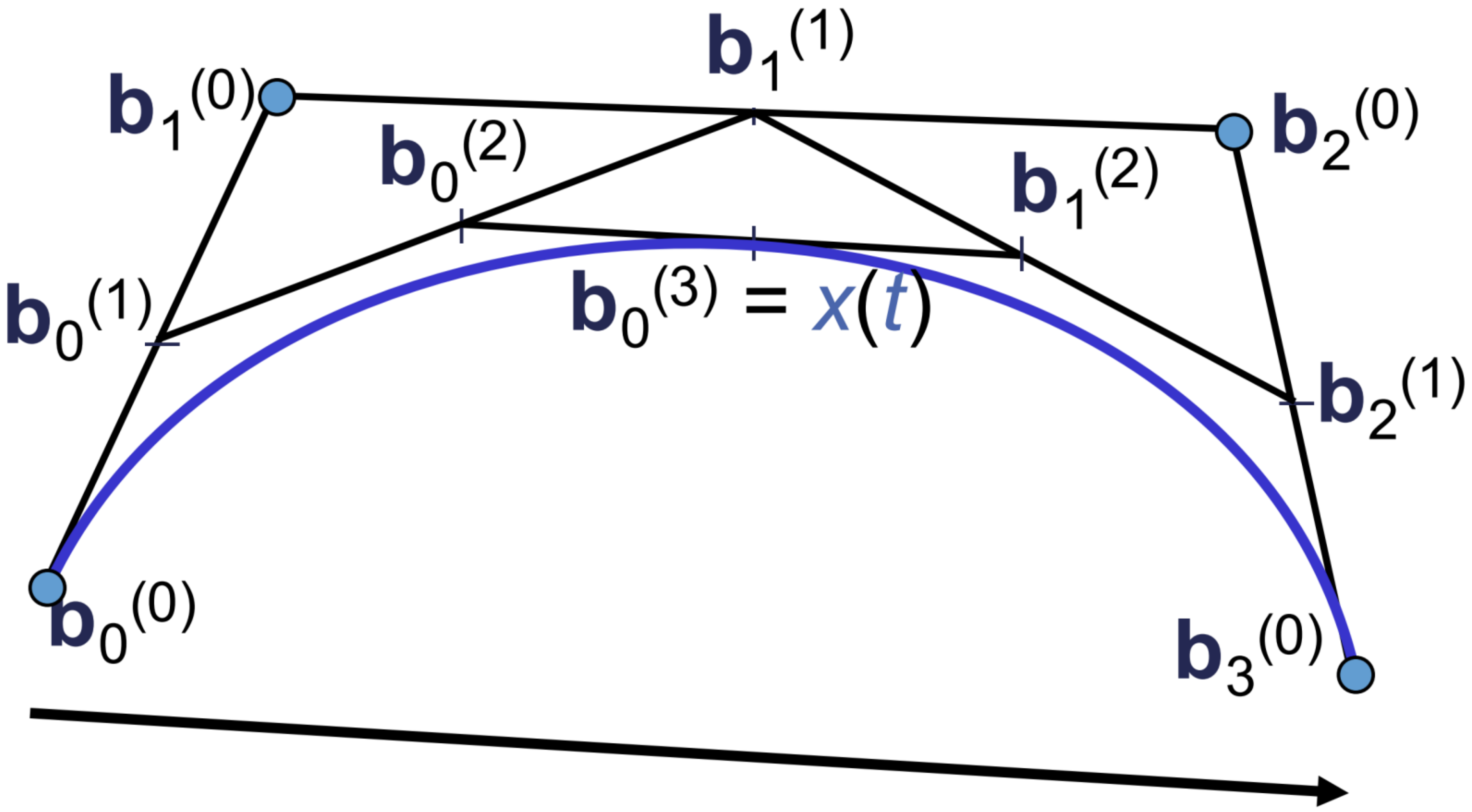
Subdivision

Solution: $b_i^{(1)} = b_0^i,\, b_i^{(2)} = b_0^{n-i}$ for $i = 0, \ldots, n$

That means that the new points are intermediate points of the de Casteljau algorithm!

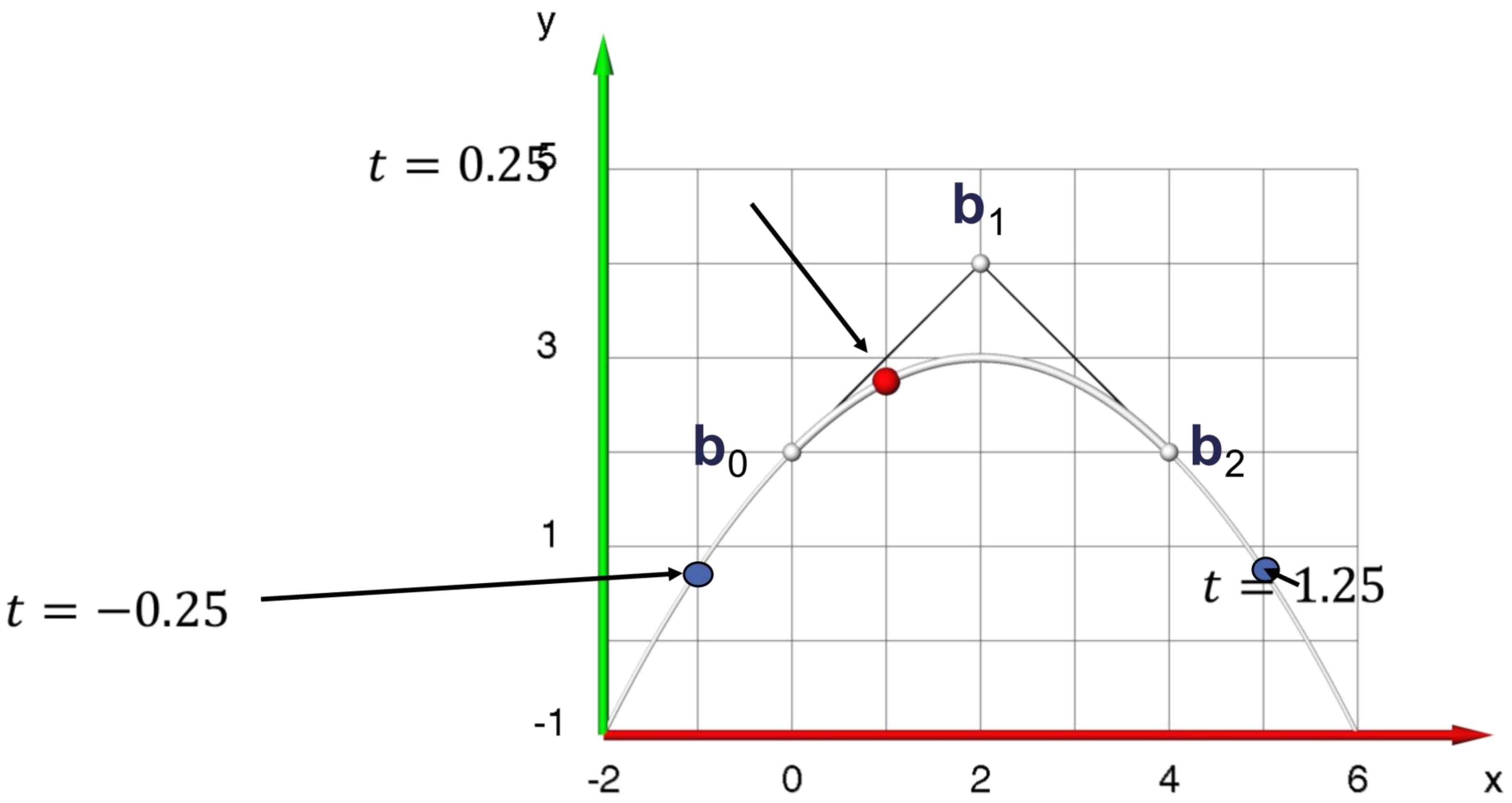


Curve range



parameterization: $t \in [0,1]$

Curve range



Matrix representations
(common in software implementations)

Homogeneous coordinates

$$[P] = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & y_n & z_n & 1 \end{pmatrix}$$

Transformations

- Basic representation $[P^*] = [P][T]$

  $[P^*]$ is the new coordinates matrix

  $[P]$ is the original coordinates matrix, or points matrix

  $[T]$ is the transformation matrix

Transformations

- Translation (2D example)

$$[T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$[P^*] = [P][T_t]$$

Transformations

- Basic representation $[P^*] = [P][T]$

  $[P^*]$ is the new coordinates matrix

  $[P]$ is the original coordinates matrix, or points matrix

  $[T]$ is the transformation matrix

$$[P] = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \\ \dots & \dots & \dots \\ x_n & y_n & 0 \end{pmatrix}$$

Transformations

- Uniform scaling

$$[T] = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Non-uniform scaling

$$[T] = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation 2D

$$\left.\begin{array}{l} x = r\cos\alpha \\ y = r\sin\alpha \end{array}\right\} \textit{Original coordinates of point } P$$

$$\left.\begin{array}{l} x^* = r\cos(\alpha+\theta) \\ y^* = r\sin(\alpha+\theta) \end{array}\right\} \textit{The new coordinates}$$

$$[x^* \quad y^* \quad 0 \quad 1] = [x \quad y \quad 0 \quad 1]\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation about an arbitrary axis (2D)
  - Translate the fixed axis so it coincides with z-axis

    → apply to object
  - Rotate object about the axis
  - Translate object back

$$[P^*] = [P][T_t][T_r][T_{-t}]$$

Transformations

- Rotation about an arbitrary axis (2D)

Step 1: Translate the fixed axis so it coincides with z-axis

Step 2: Rotate object about the axis

Step 3: Translate the fixed axis back to the original position

$$[P^*] = [P][T_t][T_r][T_{-t}]$$

Transformations

- Scaling with an arbitrary point (x, y)

$$[P^*] = [P][T_t][T_s][T_{-t}]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ x - sx & y - sy & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation about an arbitrary point (x, y)

$$[T_{\text{cond}}] = [T_t][T_s][T_{-t}]$$

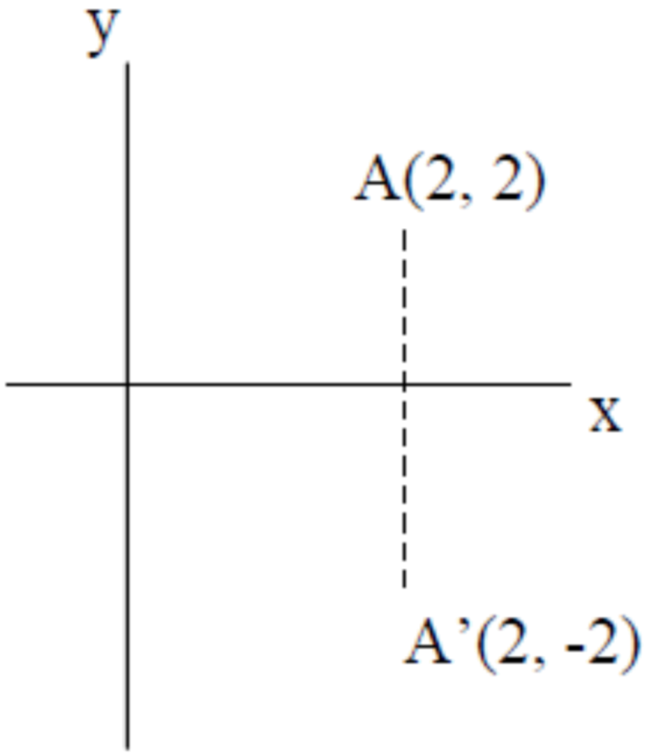
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

Transformations

- Mirroring about x-axis (negative scaling along y-axis)

$$[P^*] = [2 \quad 2 \quad 0 \quad 1]\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= [2 \quad -2 \quad 0 \quad 1]$$



Transformations

- Mirroring about arbitrary axis

  - Translate line to pass through origin
  - Rotate axis to coincide with x-axis
  - Mirror about x-axis
  - Rotate back
  - Translate back to original position

$$[P^*] = [P][T_t][T_r][T_m][T_{-r}][T_{-t}]$$

Transformations

- Rotation about coordinates axes (3D)

$$[T_{rz}] = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[T_{rx}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[T_{ry}] = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation θ about an arbitrary axis (3D)
  1. Translate the given line so that it will pass through the origin
  2. Rotate about the x-axis so that the line lies in the xz-plane (angle α)
  3. Rotate about the y-axis so that the line coincides with the z-axis (angle ϕ)
  4. Rotate the geometric object about the z-axis (angle θ – given rotation angle)
  5. Reverse of step 3
  6. Reverse of step 2
  7. Reverse of step 1

$$[P^*] = [P][T_t][T_r]_\alpha[T_r]_\phi[T_r]_\theta[T_r]_{-\phi}[T_r]_{-\alpha}[T_{-t}]$$

Alternatively you can use Quaternions!

Bezier Curves

- Cubic Bezier curves $P(t) = V_0 B_{0,3} + V_1 B_{1,3} + V_2 B_{2,3} + V_3 B_{3,3}$

$$B_{0,3} = \frac{3!}{0!\,3!} t^0 (1-t)^3 = (1-t)^3$$

$$B_{1,3} = \frac{3!}{1!\,2!} t^1 (1-t)^2 = 3t(1-t)^2$$

$$B_{2,3} = \frac{3!}{2!\,1!} t^2 (1-t)^1 = 3t^2(1-t)$$

$$B_{3,3} = \frac{3!}{3!\,0!} t^3 (1-t)^0 = t^3$$

Bezier Curves

- Cubic Bezier curves $$P(t) = V_0B_{0,3} + V_1B_{1,3} + V_2B_{2,3} + V_3B_{3,3}$$
- In Matrix form:
- -The curve

$$P(t) = [t^3 \quad t^2 \quad t \quad 1]\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

- -The tangent

$$P'(t) = [3t^2 \quad 2t \quad 1 \quad 0]\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

B-spline Curves (to be covered later)

- Uniform cubic B-Spline curve

$$P_i(t) = \frac{1}{6}\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{bmatrix} V_{i-1} \\ V_i \\ V_{i+1} \\ V_{i+2} \end{bmatrix}$$

B-spline Curves (to be covered later)

- Other splines:
  - Catmull-Rom

$$P_i(t) = [t^3 \quad t^2 \quad t \quad 1]\frac{1}{2}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{bmatrix} V_{i-1} \\ V_i \\ V_{i+1} \\ V_{i+2} \end{bmatrix}$$

  - Cardinal splines

$$\begin{pmatrix} -a & 2-a & a-2 & a \\ 2a & a-3 & 3-2a & -a \\ -a & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

  - Tensioned splines

$$\frac{1}{6}\begin{pmatrix} -a & 12-9a & 9a-12 & a \\ 2a & a-3 & 18-15a & -a \\ -3a & 0 & 3a & 0 \\ 0 & 6-2a & a & 0 \end{pmatrix}$$