计算机辅助几何设计
2019秋学期

B-Splines

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Basis

B-Splines

~~Bezier~~

Mathematical view: spline functions

Graphics view: spline curves (created using spline functions)

Motivation

- Back to the algebraic approach for Bezier curves

  →Bernstein polynomials

- Problem: global influence of the Bezier points

- Introduction of new basis function

  →B-spline functions

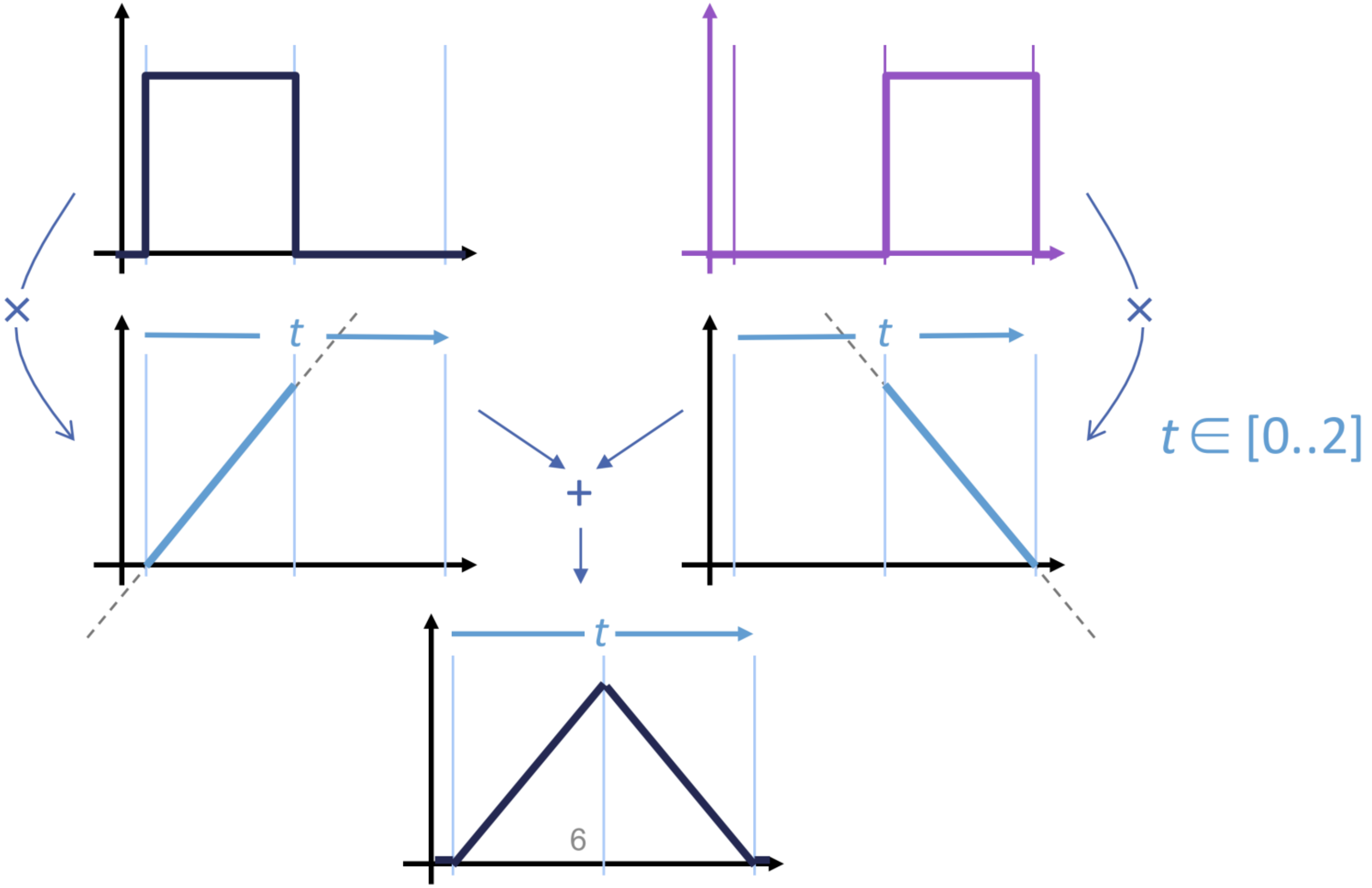
Some history

- **Early use of splines on computers for data interpolation**
  - Ferguson at Boeing, 1963
  - Gordon and de Boor at General Motors
  - B-splines, de Boor 1972

- **Free form curve design**
  - Gordon and Riesenfeld, 1974 → B-splines as a generalization of Bezier curves

Repeated linear interpolation

- **Another way to increase smoothness:**



Repeated linear interpolation

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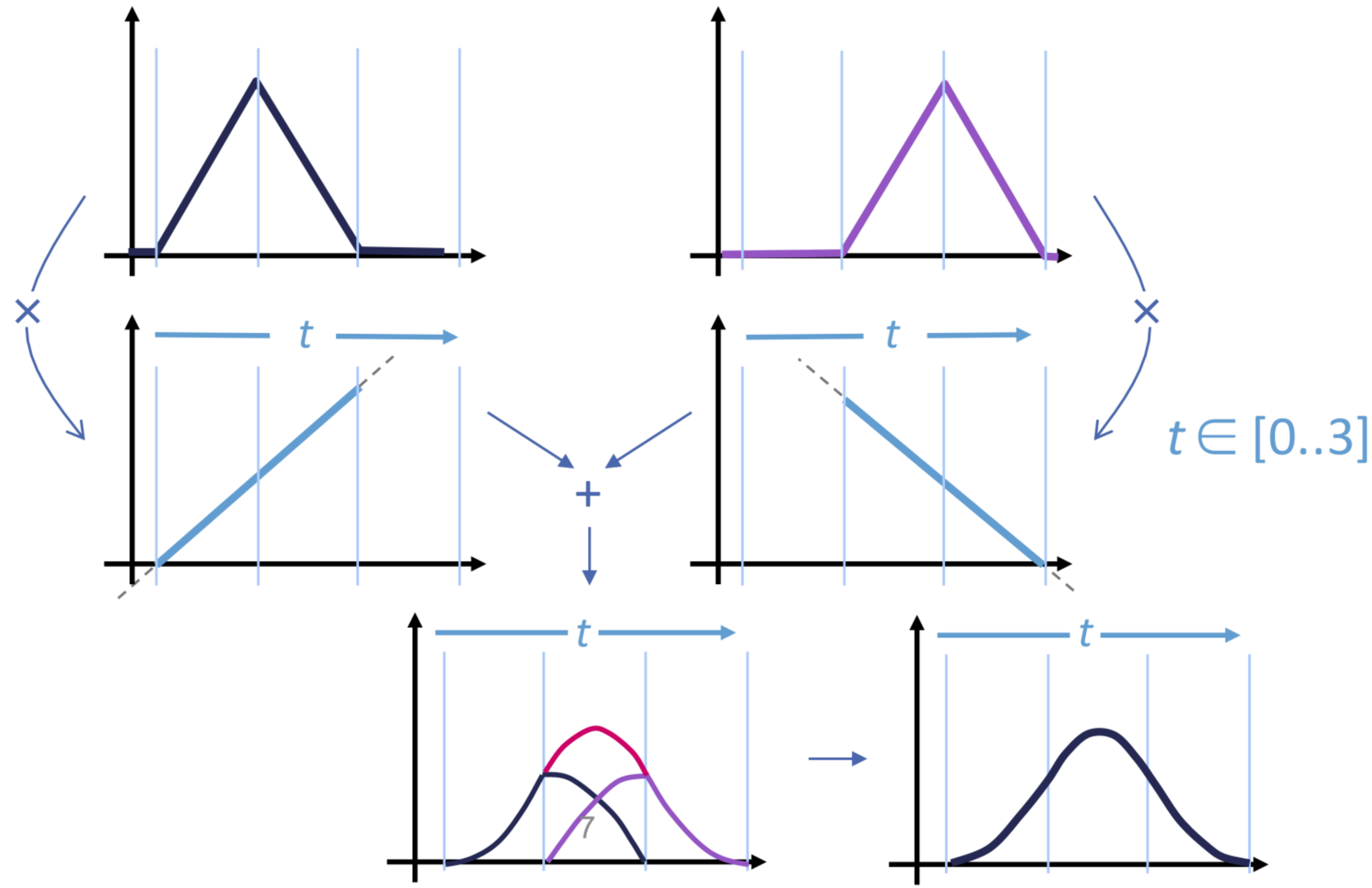
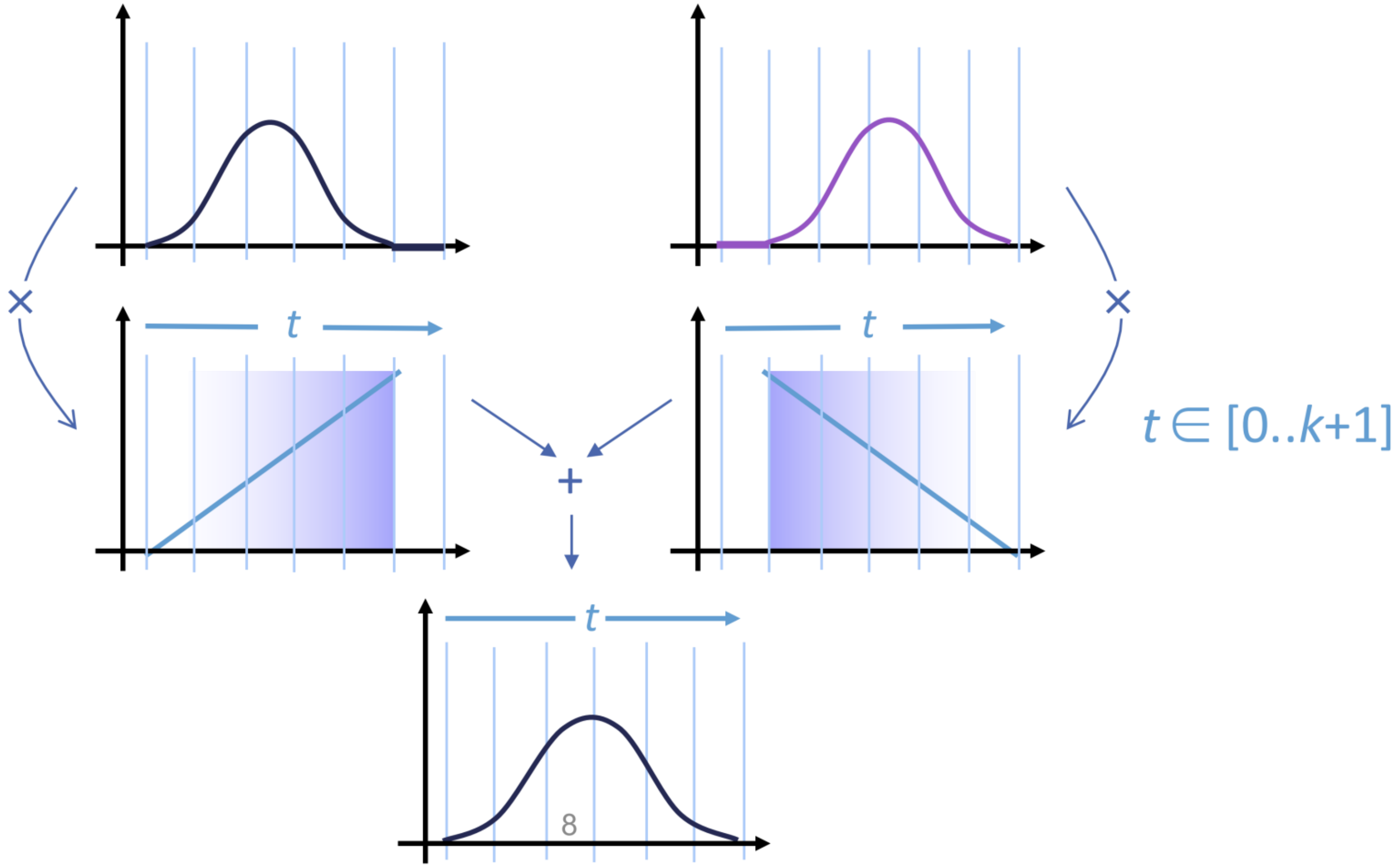


Repeated linear interpolation

- Another way to increase smoothness

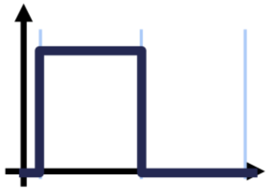


N.B.: we will use N_i^k or $N_{i,k}$ interchangeably

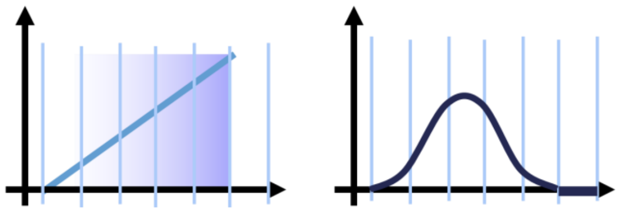
De Boor Recursion: uniform case

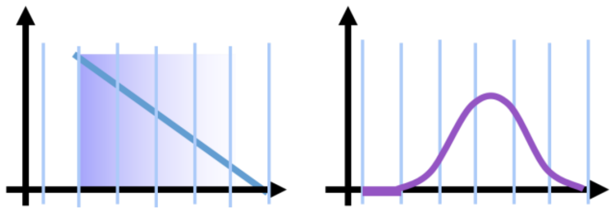
- **The uniform B-spline basis of order $\boldsymbol{k}$ (degree $\boldsymbol{k-1}$) is given as**

$$N_i^1(t) = \begin{cases} 1, \text{if } i \le t < i+1 \\ 0, \quad \text{otherwise} \end{cases}$$



$$N_i^k(t) = \frac{t-i}{(i+k-1)-i} N_i^{k-1}(t) \qquad + \qquad \frac{(i+k)-t}{(i+k)-(i+1)} N_{i+1}^{k-1}(t)$$





$$= \frac{t-i}{k-1} N_i^{k-1}(t) + \frac{i+k-t}{k-1} N_{i+1}^{k-1}(t)$$

B-spline curves: general case

- Given: knot sequence $t_0 < t_1 < \cdots < t_n < \cdots < t_{n+k}$
  ($(t_0, t_1, \dots, t_{n+k})$ is called knot vector)
- Normalized B-spline functions $N_{i,k}$ of the order k (degree $k-1$) are defined as:

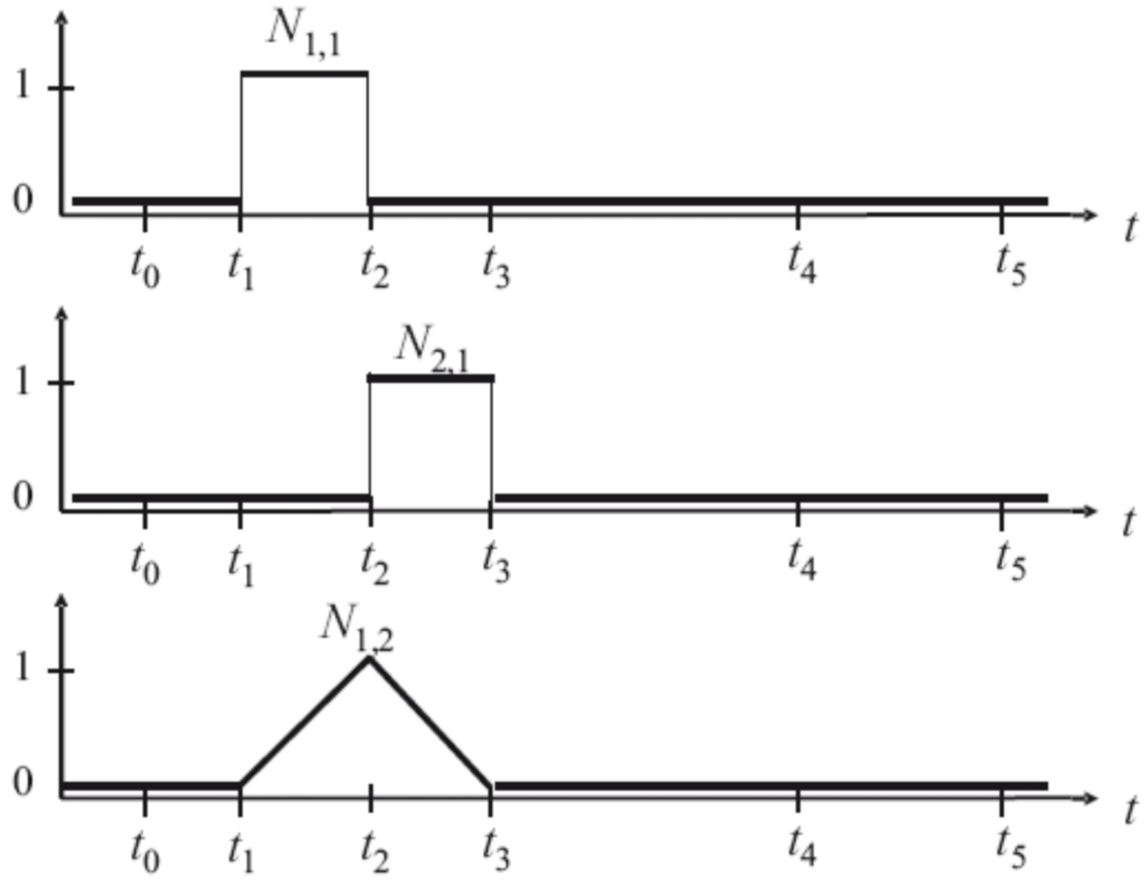
$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

for $k > 1$ and $i = 0, \dots, n$

- Remark:
  - If a knot value is repeated k times, the denominator may vanish
  - In this case: The fraction is treated as a zero

Example

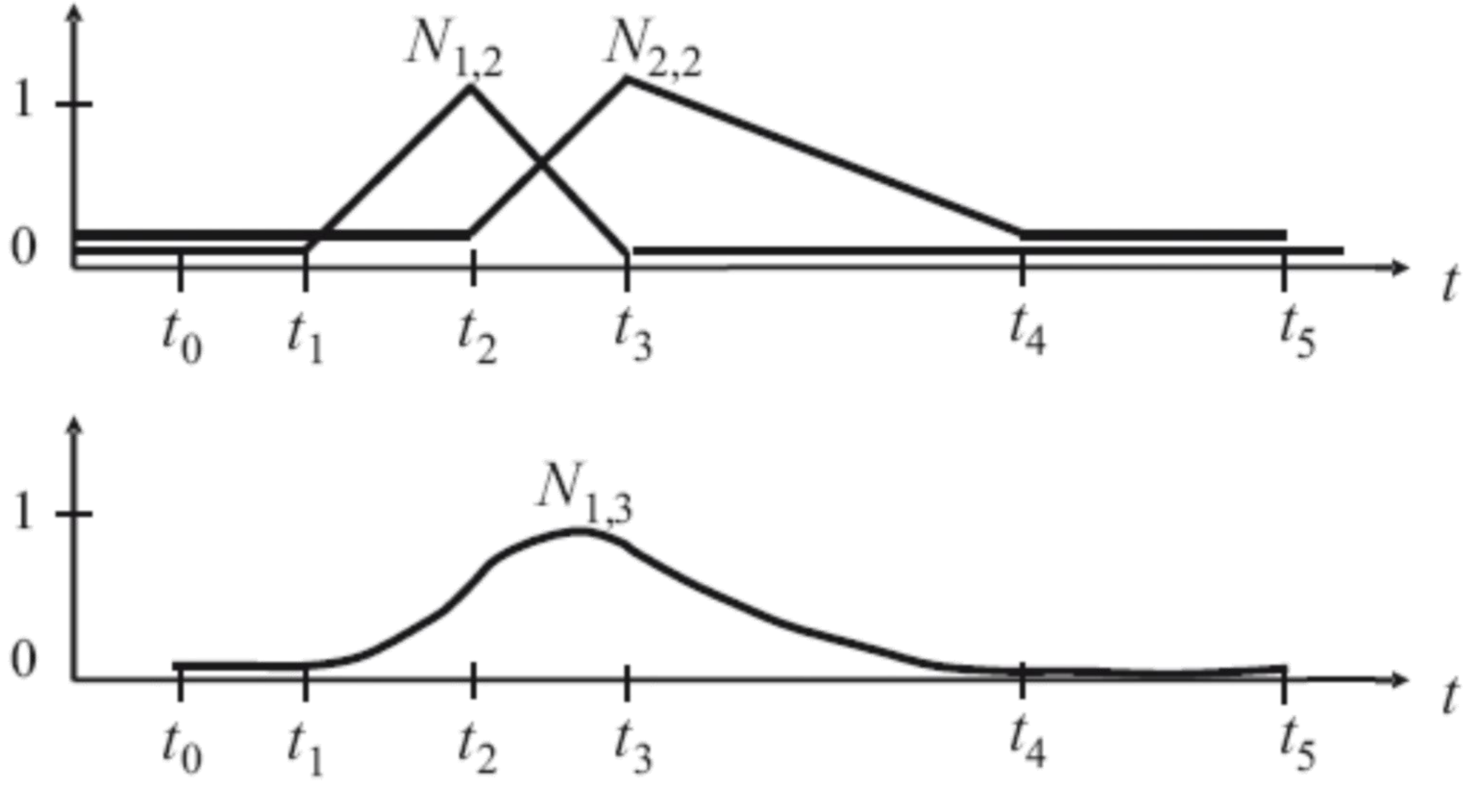


$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

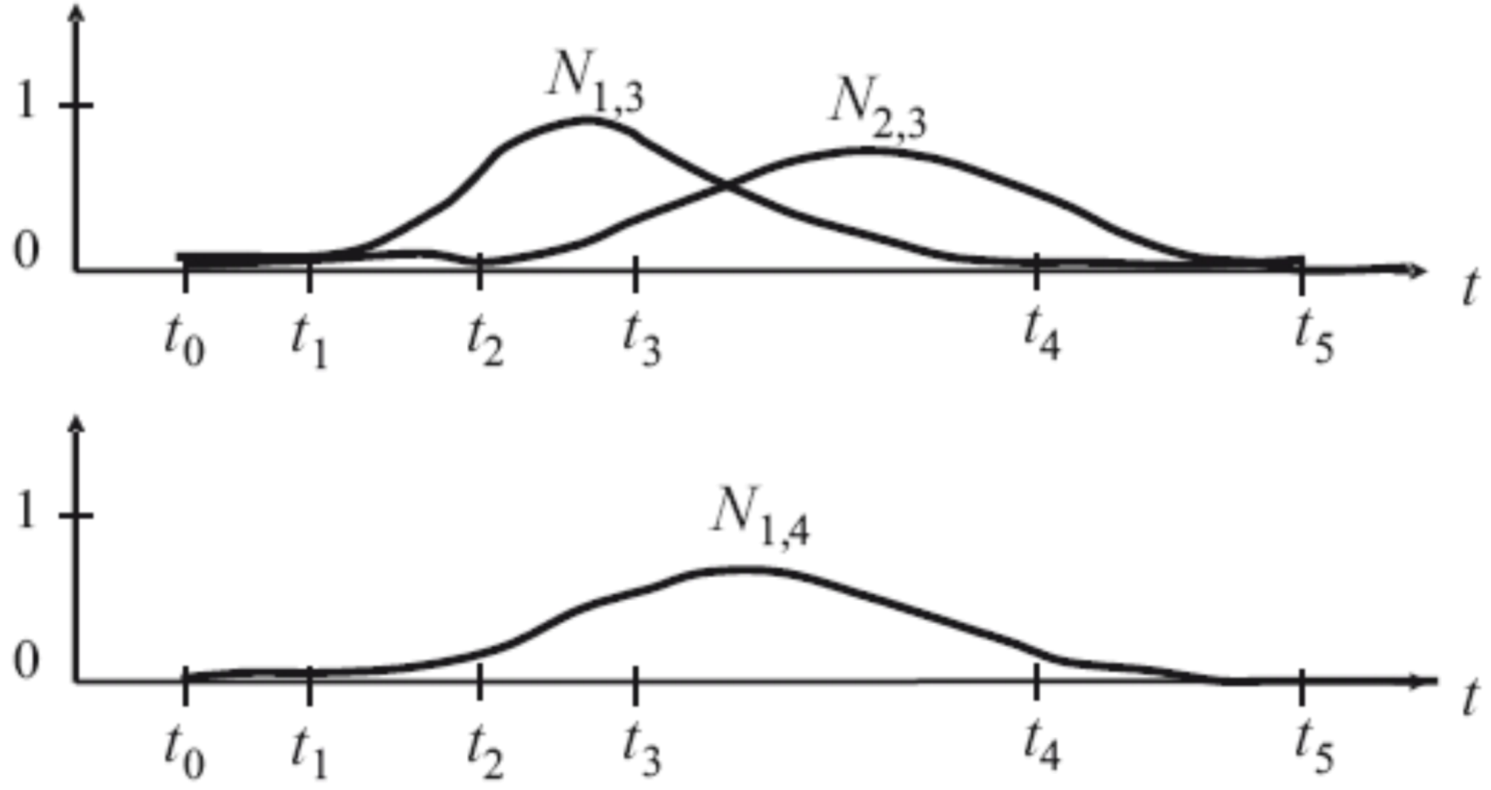
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

for $k > 1$ and $i = 0, \dots, n$

Example



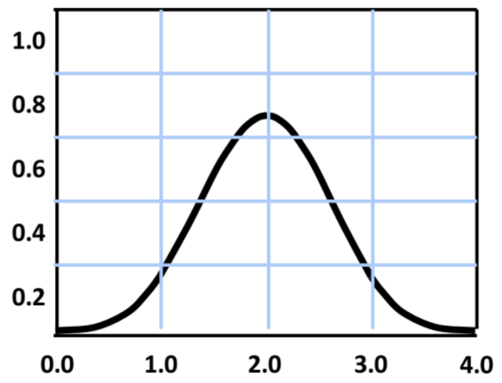
Example



Key Ideas

- **Key Ideas**
  - We design one basis function $b(t)$
  - Properties:
    - $b(t)$is C^2 continuous
    - $b(t)$is piecewise polynomial, degree 3 (cubic)
    - $b(t)$has local support
    - Overlaying shifted $b(t+i)$ forms a partition of unity
    - $b(t) \geq 0$ for all t
  - In short:
    - All desirable properties build into the basis
    - Linear combinations will inherit these

illustration only



Shifted Basis Functions

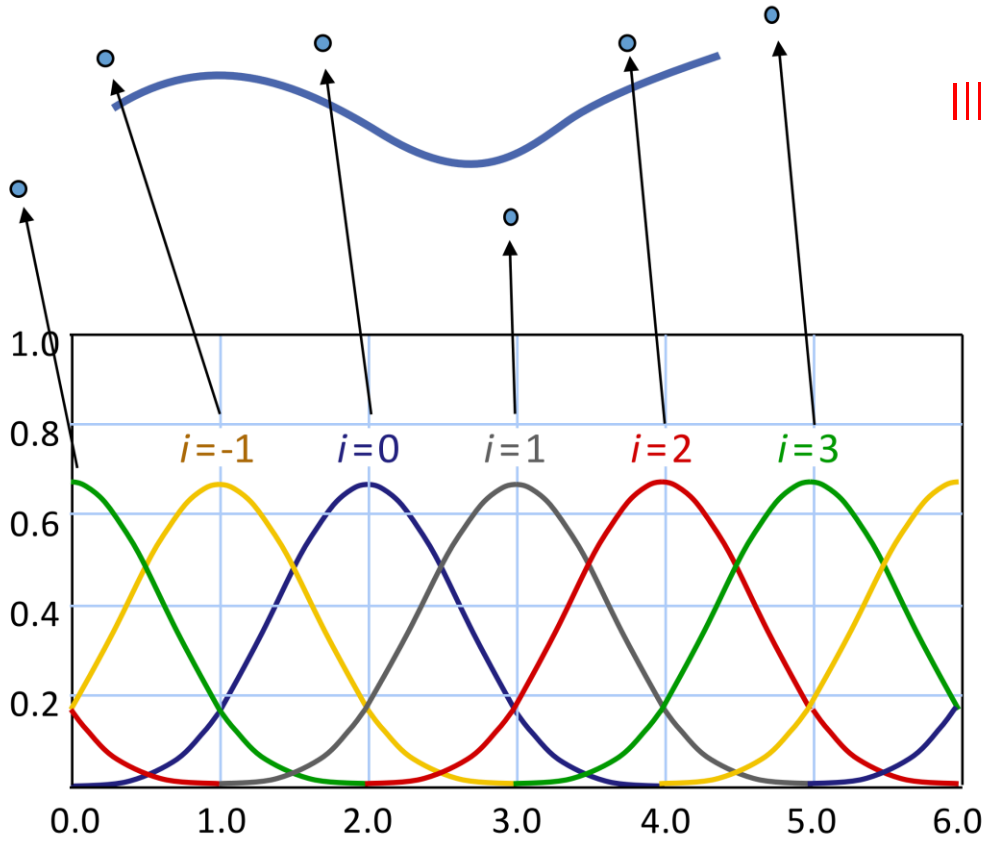


Illustration only

Shifted basis function $b(t)$

Basis properties

- For the so defined basis functions, the following properties can be shown:
  - $N_{i,k}(t) > 0$ for $t_i < t < t_{i+k}$
  - $N_{i,k}(t) = 0$ for $t_0 < t < t_i$ or $t_{i+k} < t < t_{n+k}$
  - $\sum_{i=0}^{n} N_{i,k}(t) = 1$ for $t_{k-1} \leq t \leq t_{n+1}$
- For $t_i \leq t_j \leq t_{i+k}$, the basis functions $N_{i,k}(t)$ are C^{k-2} at the knots t_j
- The interval $[t_i, t_{i+k}]$ is called support of $N_{i,k}$

B-spline curves

- **B-spline curves**
  - Given: $n+1$ control points $\boldsymbol{d}_0, \dots, \boldsymbol{d}_n \in \mathbb{R}^3$
    knot vector $T = (t_0, \dots, t_n, \dots t_{n+k})$

  - Then, the B-spline curve $\boldsymbol{x}(t)$ of the order k is defined as

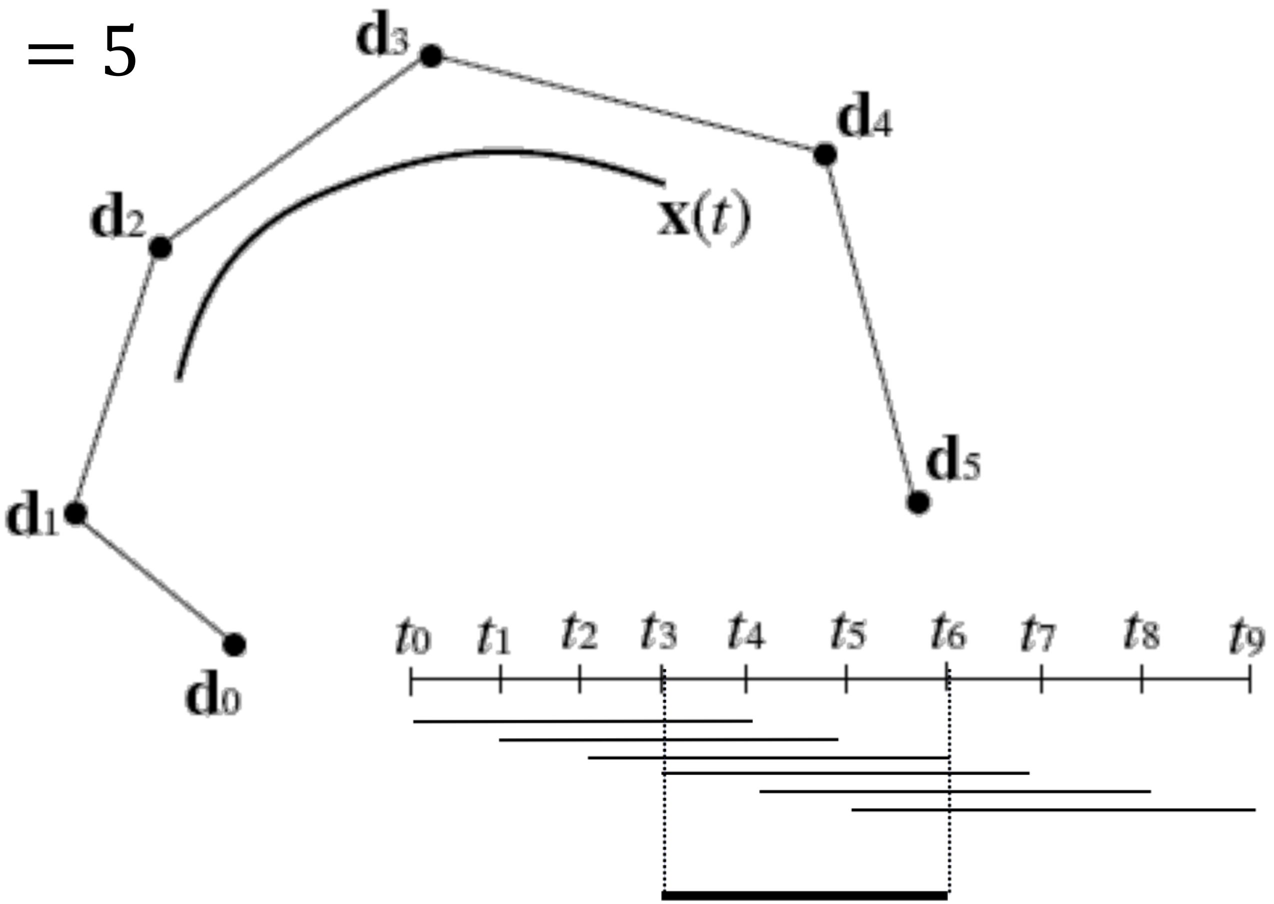
$$\boldsymbol{x}(t) = \sum_{i=0}^{n} N_{i,k}(t) \cdot \boldsymbol{d}_i$$

  - The points $\boldsymbol{d}_i$ are called *de Boor points*

Carl R. de Boor
German-American mathematician
University of Wisconsin-Madison

Example

- $k = 4, n = 5$



Support intervals of $N_{i,k}$

Curve defined in interval $t_3 \le t \le t_6$

B-spline curves

- **Multiple weighted knot vectors**
  - So far: $T = (t_0, \ldots, t_n, \ldots, t_{n+k})$ with $t_0 < t_1 < \cdots < t_{n+k}$
  - Now: also multiple knots allowed, i.e. with $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$
  - The recursive definition of the B spline function $N_{i,k}$ $(i = 0, \ldots, n)$ works nonetheless, as long as no more than k knots coincide

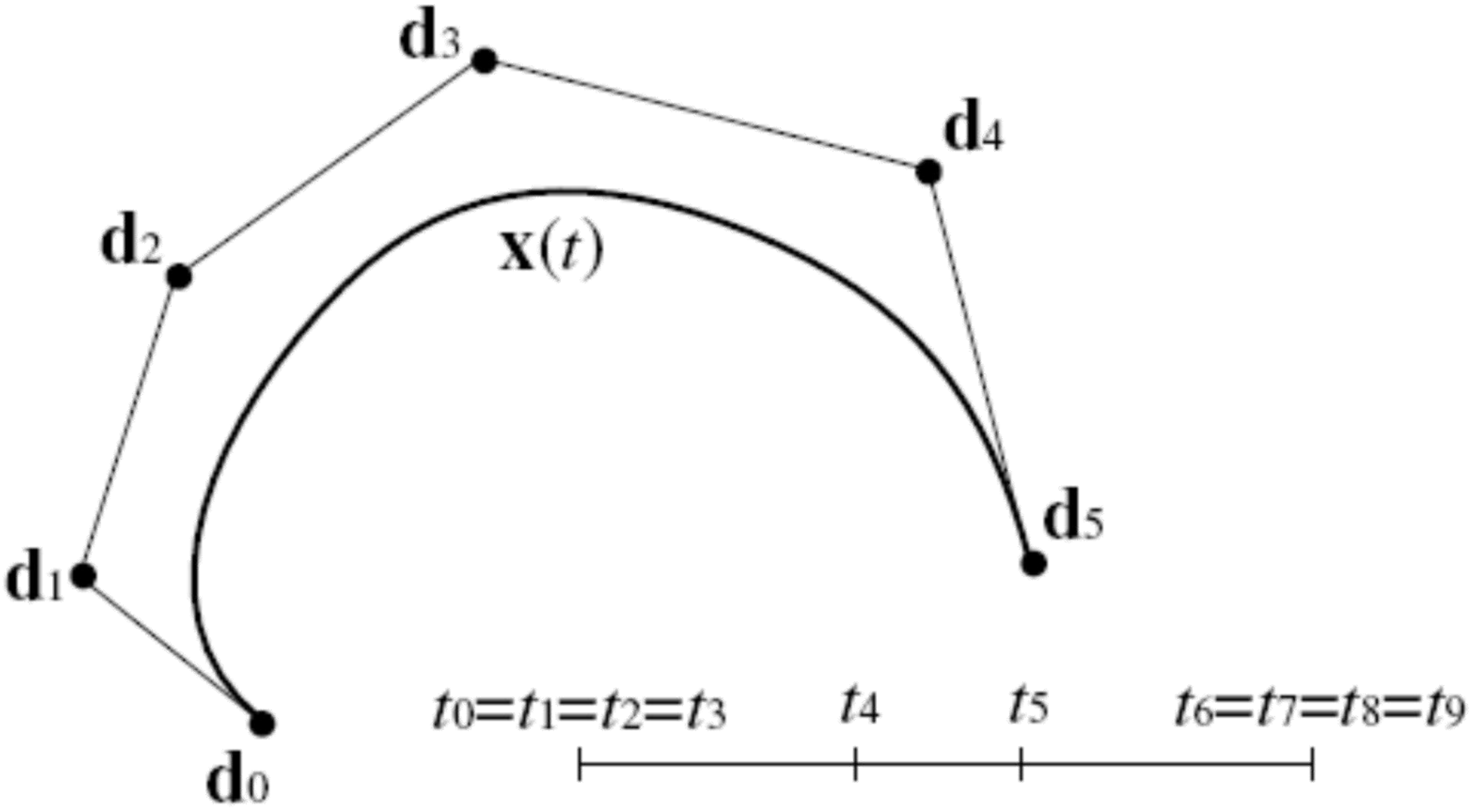
B-spline curves

- **Effect of multiple knots:**
  - set: $t_0 = t_1 = \cdots = t_{k-1}$
  - and $t_{n+1} = t_{n+2} = \cdots = t_{n+k}$

$\boldsymbol{d}_0$ and $\boldsymbol{d}_n$ are interpolated

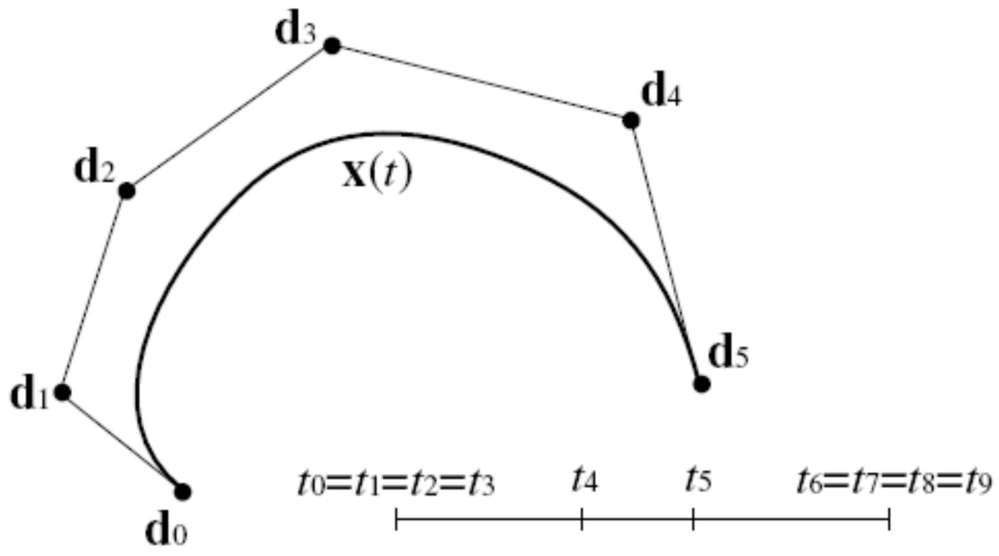
B-spline curves

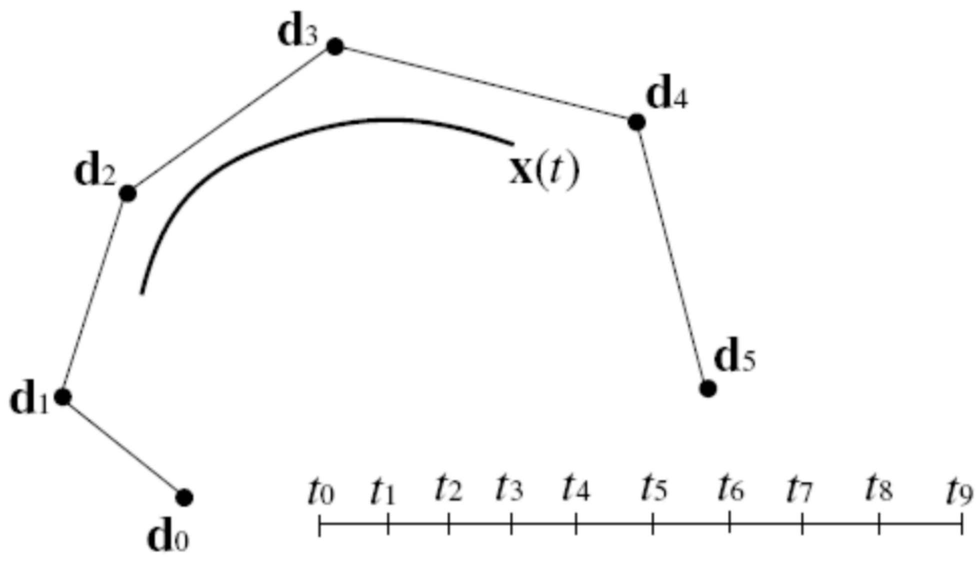
- Example: $k = 4, n = 5$



B-spline curves

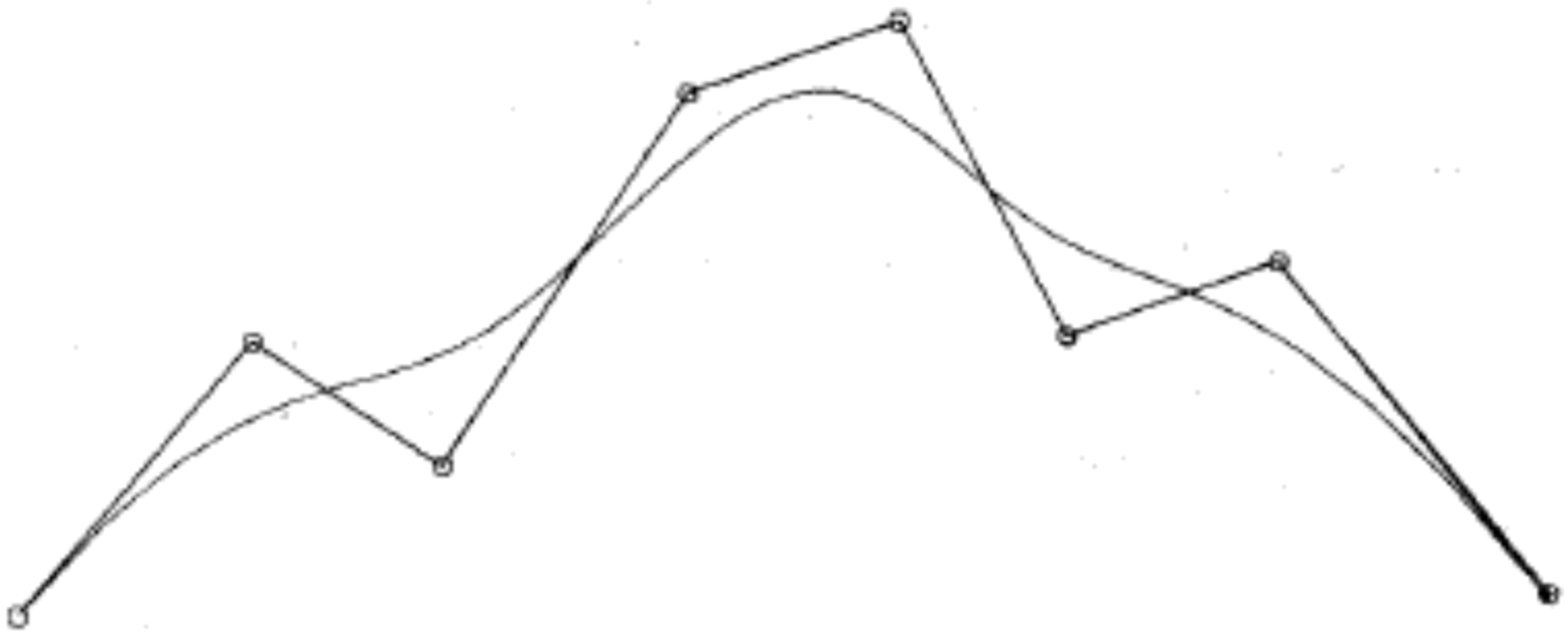
- Example: $k = 4, n = 5$





B-spline curves

- Further example



B-spline curves

- **Interesting property:**
  - B-spline functions $N_{i,k}$ $(i = 0, \dots, k-1)$ of the order k over the knot vector $T = (t_0, t_1, \dots, t_{2k-1}) = (\underbrace{0, \dots, 0}_{k \text{ times}}, \underbrace{1, \dots, 1}_{k \text{ times}})$

    are Bernstein polynomials B_i^{k-1} of degree $k-1$

B-spline curves properties

- Given:
  - $T = (\underbrace{t_0, \ldots, t_0}_{\textbf{\textit{k}}\ \textbf{times}}, t_k, \ldots, t_n, \underbrace{t_{n+1}, \ldots, t_{n+1}}_{\textbf{\textit{k}}\ \textbf{times}})$
  - de Boor polygon $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n$
- Then, the following applies for the related B-spline curve $\boldsymbol{x}(t)$:

B-spline curves properties

- $\boldsymbol{x}(t_0) = \boldsymbol{d}_0, \boldsymbol{x}(t_{n+1}) = \boldsymbol{d}_n$ (end point interpolation)
- $\dot{\boldsymbol{x}}(t_0) = \frac{k-1}{t_k - t_0}(\boldsymbol{d}_1 - \boldsymbol{d}_0)$ (tangent direction at $\boldsymbol{d}_0$, similar in $\boldsymbol{d}_n$)
- $\boldsymbol{x}(t)$ consists of $n-k+2$ polynomial curve segments of degree $k-1$ (assuming no multiple inner knots)

B-spline curves properties

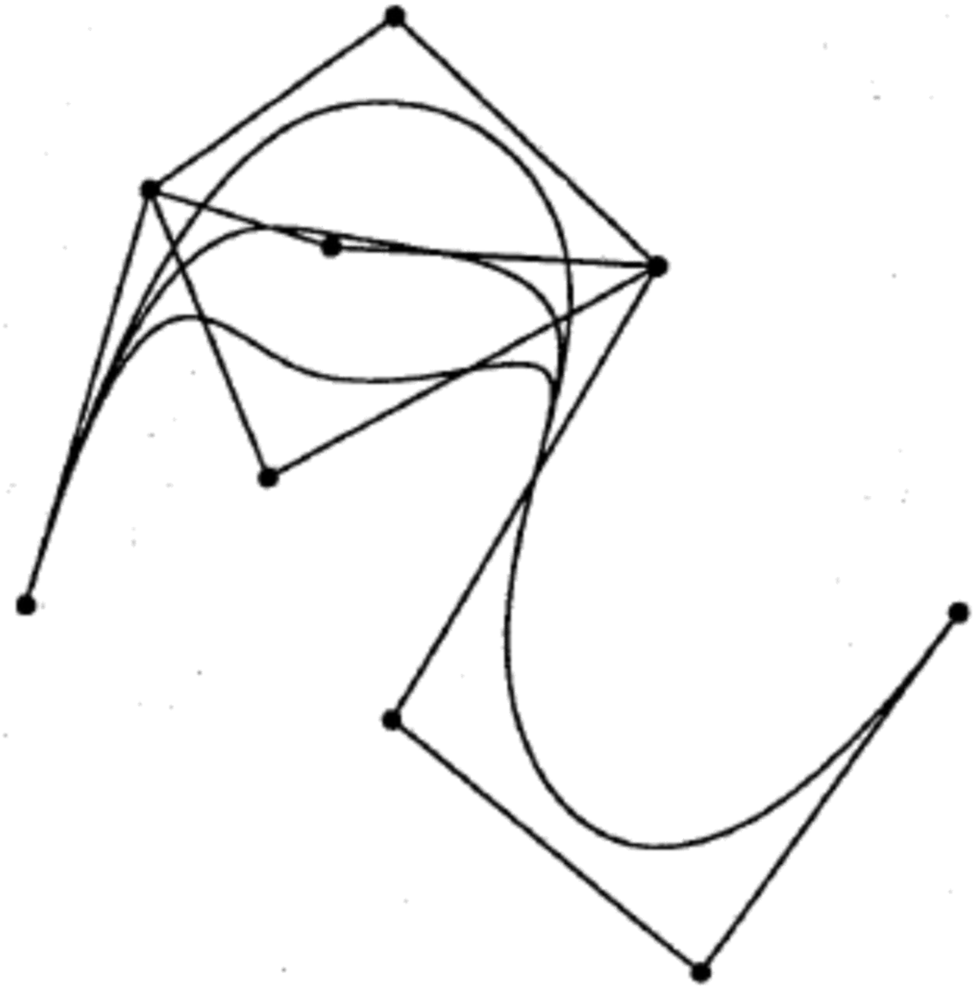
- Multiple inner knots ⇒ reduction of continuity of $x(t)$.

  l-times inner knot ($1 \leq l < k$) means

  C^{k-l-1}-continuity
- Local impact of the de Boor points: moving of d_i only changes the curve in the region $[t_i, t_{i+k}]$
- The insertion of new de Boor points does not change the polynomial degree of the curve segments

B-spline curves properties

- **Locality of B-spline curves**



B-spline curves

- **Evaluation of B-spline curves**
  - Using B-spline functions

  - Using the de Boor algorithm

    Similar algorithm to the de Casteljau algorithm for Bezier curves;

    consists of a number of linear interpolations on the de Boor polygon

The de Boor algorithm

- Given:

  $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n$: de Boor points

  $(t_0, \ldots, t_{k-1} = t_0, t_k, t_{k+1}, \ldots, t_n, t_{n+1}, \ldots, t_{n+k} = t_{n+1})$:

  Knot vector

- wanted:

  Curve point $\boldsymbol{x}(t)$ of the B-spline curve of the order k

The de Boor algorithm

1. Search index r with $t_r \leq t < t_{r+1}$
2. for $i = r - k + 1, \dots, r$

   $d_i^0 = d_i$

- for $j = 1, \dots, k - 1$

  for $i = r - k + 1 + j, \dots, r$

  $$d_i^j = \left(1 - \alpha_i^j\right) \cdot d_{i-1}^{j-1} + \alpha_i^j \cdot d_i^{j-1}$$

  with $\alpha_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$

Then: $d_r^{k-1} = x(t)$

B-spline curves

- The intermediate coefficients $d_i^j(t)$ can be placed into a triangular shaped matrix of points – the de Boor scheme:

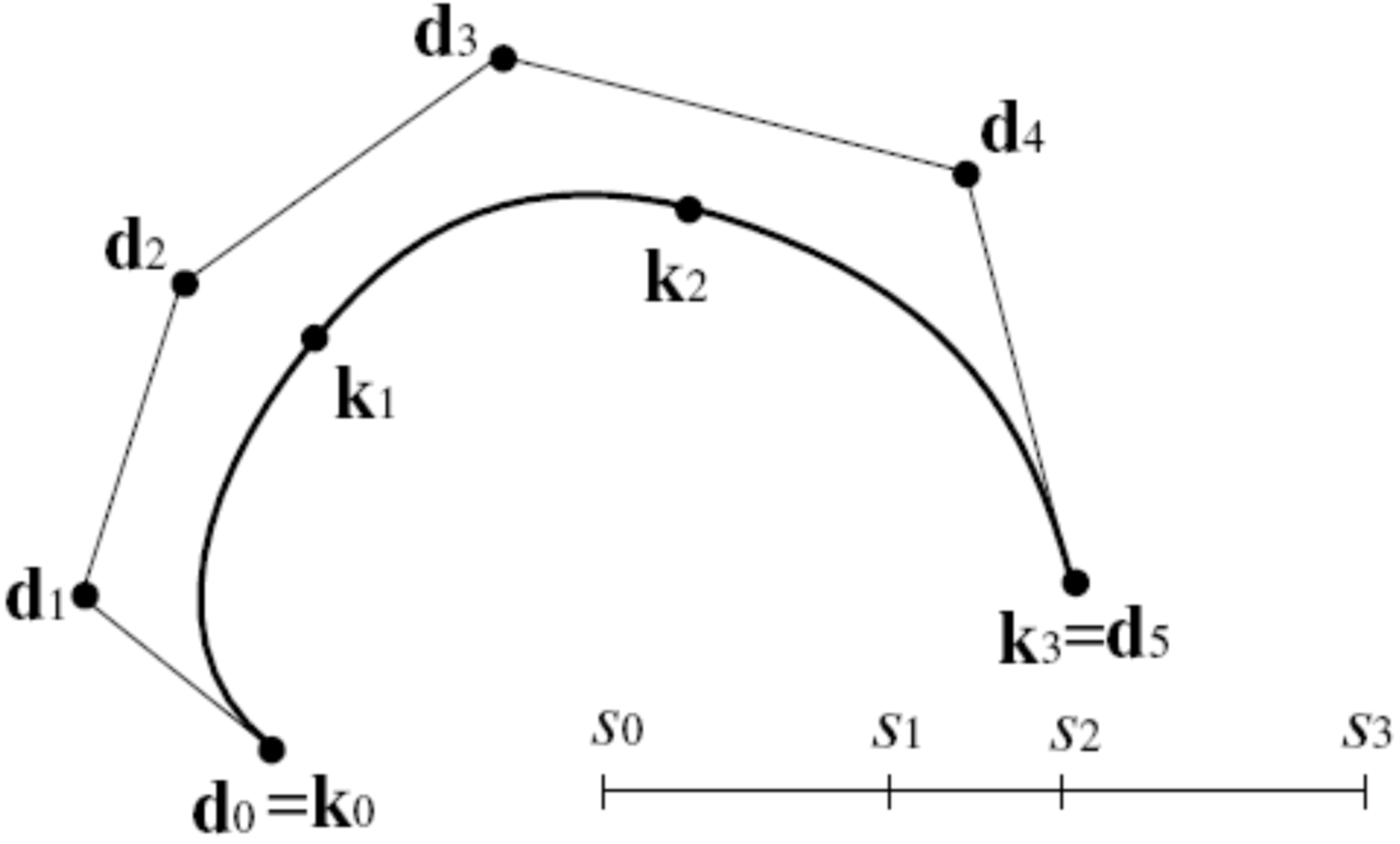
$$
\begin{array}{llll}
d_{r-k+1} = d_{r-k+1}^0 & & & \\
d_{r-k+2} = d_{r-k+2}^0 & d_{r-k+2}^1 & & \\
\quad \cdots & & & \\
d_{r-1} = d_{r-1}^0 & d_{r-1}^1 & \cdots \quad d_{r-1}^{k-2} & \\
d_r = d_r^0 & d_r^1 & \cdots \quad d_r^{k-2} & d_r^{k-1} = x(t)
\end{array}
$$

B-spline curves: interpolation

- **Interpolating B-spline curves**
  - Given: $n+1$ control points $\boldsymbol{k}_0,\cdots,\boldsymbol{k}_n$
    knot sequence $s_0,\cdots,s_n$
  - Wanted: piecewise cubic interpolating B-spline curve $\boldsymbol{x}$
    i.e., $\boldsymbol{x}(s_i) = \boldsymbol{k}_i$ for $i = 0, \ldots, n$
  - Approach: piecewise cubic $\Rightarrow k = 4$
    - $\boldsymbol{x}(t)$ consists of n segments $\Rightarrow n+3$ de Boor points

B-spline curves: interpolation

- Example: $n = 3$



B-spline curves: interpolation

- We choose the knot vector
  - $T = (t_0, t_1, t_2, t_3, t_4, \dots, t_{n+2}, t_{n+3}, t_{n+4}, t_{n+5}, t_{n+6})$
    $= (s_0, s_0, s_0, s_0, s_1, \dots, s_{n-1}, s_n, \quad s_n, \quad s_n, \quad s_n)$

- Then, the following conditions arise:

$$\boldsymbol{x}(s_0) = \boldsymbol{k}_0 = \boldsymbol{d}_0$$

$$\boldsymbol{x}(s_i) = \boldsymbol{k}_i = N_{i,4}(s_i)\boldsymbol{d}_i + N_{i+1,4}(s_i)\boldsymbol{d}_{i+1} + N_{i+2,4}(s_i)\boldsymbol{d}_{i+2}$$

for $i = 1, \dots, n-1$

$$\boldsymbol{x}(s_n) = \boldsymbol{k}_n = \boldsymbol{d}_{n+2}$$

- Total: $n + 1$ conditions for $n + 3$ unknown de Boor points

$\rightarrow$ 2 end conditions

B-spline curves: interpolation

- Here as example: natural end conditions

$$\ddot{x}(s_0) = 0 \Leftrightarrow \frac{d_2 - d_1}{s_2 - s_0} = \frac{d_1 - d_0}{s_1 - s_0}$$

$$\ddot{x}(s_n) = 0 \Leftrightarrow \frac{d_{n+2} - d_{n+1}}{s_n - s_{n-1}} = \frac{d_{n+1} - d_n}{s_n - s_{n-2}}$$

B-spline curves: interpolation

- This results in the following tridiagonal system of equations:

$$
\begin{pmatrix}
1 & & & & & & & \\
\alpha_0 & \beta_0 & \gamma_0 & & & & & \\
 & \alpha_1 & \beta_1 & \gamma_1 & & & & \\
 & & & \cdot & & & & \\
 & & & & \cdot & & & \\
 & & & & & \cdot & & \\
 & & & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} & \\
 & & & & & \alpha_n & \beta_n & \gamma_n \\
 & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
d_0 \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \\ d_{n+1} \\ d_{n+2}
\end{pmatrix}
=
\begin{pmatrix}
k_0 \\ 0 \\ k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_{n-1} \\ 0 \\ k_n
\end{pmatrix}
$$

B-spline curves: interpolation

- with

$$\alpha_0 = s_2 - s_0$$
$$\beta_0 = -(s_2 - s_0) - (s_1 - s_0)$$
$$\gamma_0 = s_1 - s_0$$

$$\alpha_n = s_n - s_{n-1}$$
$$\beta_n = -(s_n - s_{n-1}) - (s_n - s_{n-2})$$
$$\gamma_n = s_n - s_{n-2}$$

Natural end conditions

$$\alpha_i = N_{i,4}(s_i)$$
$$\beta_i = N_{i+1,4}(s_i)$$
$$\gamma_i = N_{i+2,4}(s_i)$$

for $i = 1, \ldots, n-1$

B-spline curves: interpolation

- Solving a tridiagonal system of equations: Thomas-algorithm!
- O(n)
- Only for diagonally dominant matrices

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & . & \\ & & . & . & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ d_n \end{bmatrix}$$

B-spline curves: interpolation

- Solving a tridiagonal system of equation: Thomas-algorithm!

Forward elimination phase

for $k = 2 : n$

$$m = \frac{a_k}{b_{k-1}}$$

$$b_k = b_k - mc_{k-1}$$

$$d_k = d_k - md_{k-1}$$

end

Backward substitution phase

$$x_n = \frac{d_n}{b_n}$$

for $k = 2 : n$

$$x_k = \frac{d_k - c_k x_{k+1}}{b_k}$$

end

Bezier curves to B-splines

- **Conversion between cubic Bezier and B-spline curves**
  - Given:

    $\boldsymbol{k}_0, \dots, \boldsymbol{k}_n$: control points

    $t_0,\cdots,t_n$: knot sequence

    2 end conditions

    $b_0,\cdots,b_{3n}$: Bezier points for C^2-continuous interpolating cubic Bezier spline curve
  - Wanted: same curve in B-spline form

Bezier curves to B-splines

- Knot vector $T = (t_0, t_0, t_0, t_0, t_1, \dots, t_{n-1}, t_n, t_n, t_n, t_n)$
- $\boldsymbol{d}_0, \dots, \boldsymbol{d}_{n+2}$ are determined by

$$d_0 = b_0$$
$$d_1 = b_1$$
$$d_i = b_{3i-4} + \frac{\Delta_{i-1}}{\Delta_{i-2}}(b_{3i-4} - b_{3i-5}) \text{ for } i = 2, \dots, n$$
$$d_{n+1} = b_{3n-1}$$
$$d_{n+2} = b_{3n}$$

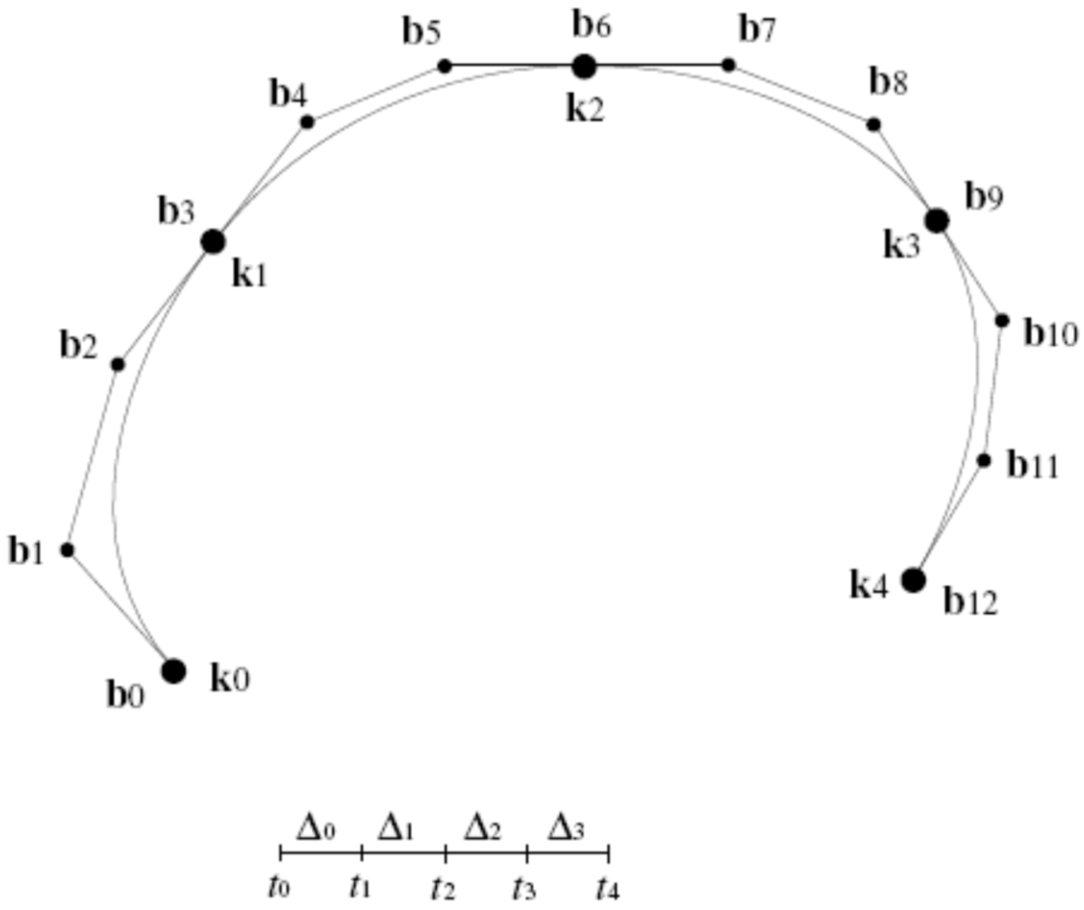
where $\Delta_i = t_{i+1} - t_i$ for $i = 0, \dots, n-1$

- The inverse problem is solvable as well

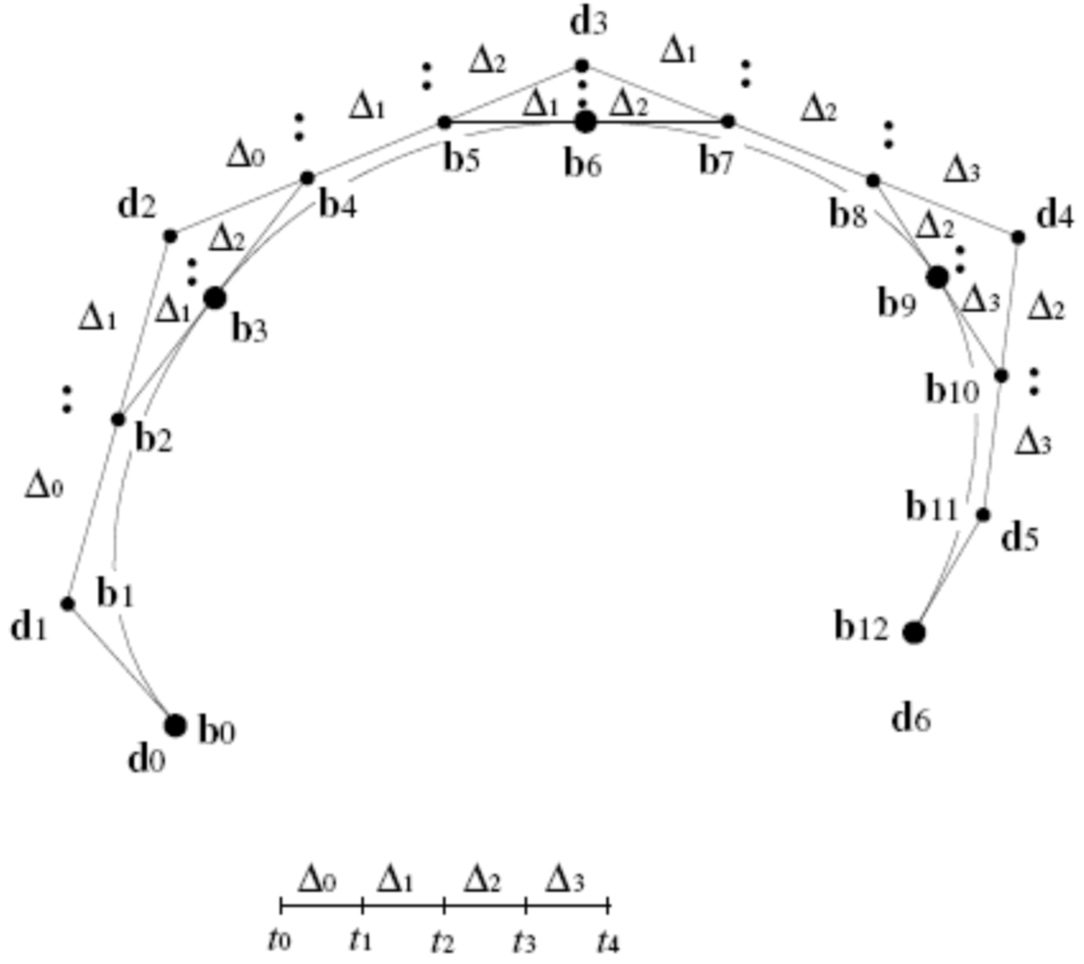
Remember the condition on $\boldsymbol{d}^-$ and $\boldsymbol{d}^+$ for C^2 continuity of Bezier splines

Bezier curves to B-splines

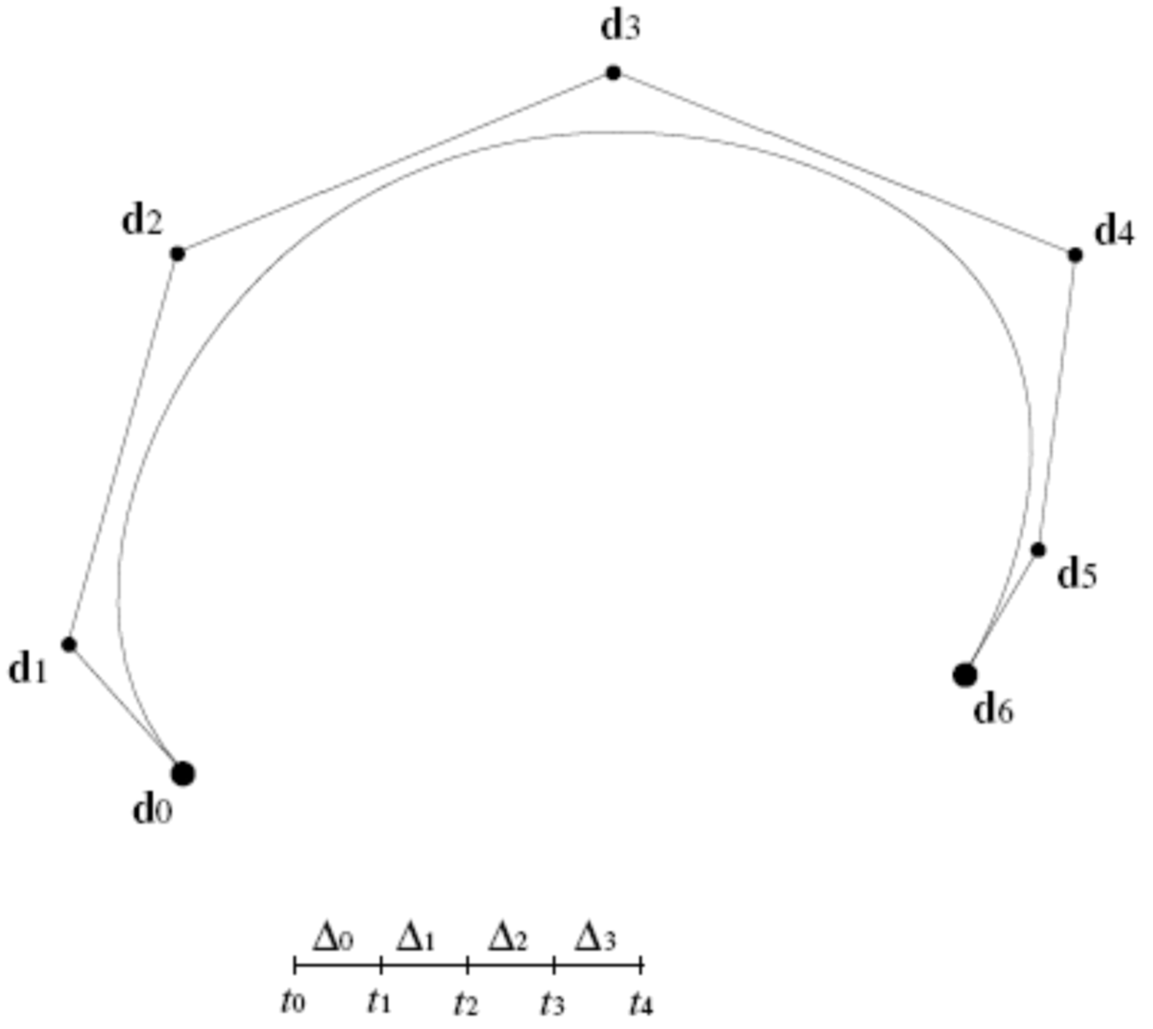
- Examples: $n = 4$



Bezier curves to B-splines



Bezier curves to B-splines



Summary of Bezier and B-spline curves

1. Bezier curve for $n+1$ control points $b_0, \cdots, b_n$:
    - Polynomial curve of degree n
    - Uniquely defined by control points
    - End point interpolation, remaining points are approximated
    - Pseudo-local impact of control points

Summary of Bezier and B-spline curves

2. Interpolating cubic Bezier-spline curves by the control points $k_0, \cdots, k_n$
   - Consists of n piecewise cubic curve segments
   - C^2-continuous at the control points
   - Uniquely defined by parameterization (i.e. knot sequence) and two end conditions
   - Interpolates all control points
   - Pseudo-local impact of the control points

Summary of Bezier and B-spline curves

3. Piecewise cubic B-spline curve for control points $d_0, \ldots, d_n$ and knot vector $T = (t_0, t_0, t_0, t_0, t_1, \ldots, t_{n-1}, t_n, t_n, t_n, t_n)$
   - Consists of $n - 2$ piecewise cubic curve segments which are C^2-conditions at the knots
   - Uniquely defined by d_i and T
   - End point interpolation, the remaining points are approximated
   - Local impact of the de Boor points

Summary of Bezier and B-spline curves

4. Interpolating cubic B-spline through the control points $k_0, \ldots, k_n$
   - Possible to formulate like (3) using 2 end conditions and solution of a tridiagonal system of equations for each x, y- and z- component
   - Identical curve to (2)

B-splines

detailed examples

B-spline curves: general case (reminder)

- Given: knot sequence $t_0 < t_1 < \cdots < t_n < \cdots < t_{n+k}$

  ($(t_0, t_1, \dots, t_{n+k})$ is called knot vector)
- Normalized B-spline functions $N_{i,k}$ of the order k (degree $k-1$) are defined as:

$$N_{i,1}(t) = \begin{cases} 1, & t_i \leq t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

for $k > 1$ and $i = 0, \dots, n$

- Remark:
  - If a knot value is repeated k times, the denominator may vanish
  - In this case: The fraction is treated as a zero

B-spline basis evaluation: ex. 1

- For order 4 and knot sequence

$$T = [t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7] = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1]$$

  Evaluate the B-spline function $N_{0,4}(t)$, $N_{1,4}(t)$, $N_{2,4}(t)$, $N_{3,4}(t)$

B-spline basis evaluation: ex. 1

$$N_{0,1}(t) = N_{1,1}(t) = N_{2,1}(t) = N_{4,1}(t) = N_{6,1}(t) = 0$$

$$N_{3,1}(t) = 1 \qquad (0 \le t < 1)$$

$$N_{0,2}(t) = \frac{t-t_0}{t_1-t_0} N_{0,1}(t) + \frac{t_2-t}{t_2-t_1} N_{1,1}(t) = 0$$

$$N_{1,2}(t) = \frac{t-t_1}{t_2-t_1} N_{1,1}(t) + \frac{t_3-t}{t_3-t_2} N_{2,1}(t) = 0$$

$$N_{2,2}(t) = \frac{t-t_2}{t_3-t_2} N_{2,1}(t) + \frac{t_4-t}{t_4-t_3} N_{3,1}(t) = (1-t)N_{3,1}(t)$$

$$N_{3,2}(t) = \frac{t-t_3}{t_4-t_3} N_{3,1}(t) + \frac{t_5-t}{t_5-t_4} N_{4,1}(t) = tN_{3,1}(t)$$

$$N_{4,2}(t) = \frac{t-t_4}{t_5-t_4} N_{4,1}(t) + \frac{t_6-t}{t_6-t_5} N_{5,1}(t) = 0$$

$$N_{5,2}(t) = \frac{t-t_5}{t_6-t_5} N_{5,1}(t) + \frac{t_7-t}{t_7-t_6} N_{6,1}(t) = 0$$

B-spline basis evaluation: ex. 1

$$N_{0,3}(t) = \frac{t-t_0}{t_2-t_0} N_{0,2}(t) + \frac{t_3-t}{t_3-t_1} N_{1,2}(t) = 0$$

$$N_{1,3}(t) = \frac{t-t_1}{t_3-t_1} N_{1,2}(t) + \frac{t_4-t}{t_4-t_2} N_{2,2}(t) = (1-t)^2 N_{3,1}(t)$$

$$N_{2,3}(t) = \frac{t-t_2}{t_4-t_2} N_{2,2}(t) + \frac{t_5-t}{t_5-t_3} N_{3,2}(t) = 2t(1-t) N_{3,1}(t)$$

$$N_{3,3}(t) = \frac{t-t_3}{t_5-t_3} N_{3,2}(t) + \frac{t_6-t}{t_6-t_4} N_{4,2}(t) = t^2 N_{3,1}(t)$$

$$N_{4,3}(t) = \frac{t-t_4}{t_6-t_4} N_{4,2}(t) + \frac{t_7-t}{t_7-t_5} N_{5,2}(t) = 0$$

B-spline basis evaluation: ex. 1

- Finally

$$N_{0,4}(t) = \frac{t-t_0}{t_3-t_0} N_{0,3}(t) + \frac{t_4-t}{t_4-t_1} N_{1,3}(t) = (1-t)^3 N_{3,1}(t)$$

$$N_{1,4}(t) = \frac{t-t_1}{t_4-t_1} N_{1,3}(t) + \frac{t_5-t}{t_5-t_2} N_{2,3}(t) = 3(1-t)^2 t N_{3,1}(t)$$

$$N_{2,4}(t) = \frac{t-t_2}{t_5-t_2} N_{2,3}(t) + \frac{t_6-t}{t_6-t_3} N_{3,3}(t) = 3(1-t)t^2 N_{3,1}(t)$$

$$N_{3,4}(t) = \frac{t-t_3}{t_6-t_3} N_{3,3}(t) + \frac{t_7-t}{t_7-t_4} N_{4,3}(t) = t^3 N_{3,1}(t)$$

B-spline basis evaluation: ex. 1

- Finally

$$N_{0,4}(t) = \frac{t-t_0}{t_3-t_0} N_{0,3}(t) + \frac{t_4-t}{t_4-t_1} N_{1,3}(t) = (1-t)^3 N_{3,1}(t)$$

$$N_{1,4}(t) = \frac{t-t_1}{t_4-t_1} N_{1,3}(t) + \frac{t_5-t}{t_5-t_2} N_{2,3}(t) = 3(1-t)^2 t N_{3,1}(t)$$

$$N_{2,4}(t) = \frac{t-t_2}{t_5-t_2} N_{2,3}(t) + \frac{t_6-t}{t_6-t_3} N_{3,3}(t) = 3(1-t) t^2 N_{3,1}(t)$$

$$N_{3,4}(t) = \frac{t-t_3}{t_6-t_3} N_{3,3}(t) + \frac{t_7-t}{t_7-t_4} N_{4,3}(t) = t^3 N_{3,1}(t)$$

- We clearly get the Bernstein basis function as mentioned earlier

B-spline basis evaluation: ex. 2

- For order 4 and not sequence

$$T = [t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7] = [-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4]$$

Evaluate the corresponding basis

B-spline basis evaluation: ex. 2

$$N_{0,2}(t) = (t+3)N_{0,1}(t) + (-1-t)N_{1,1}(t)$$
$$N_{1,2}(t) = (t+2)N_{1,1}(t) + (-t)N_{2,1}(t)$$
$$N_{2,2}(t) = (t+1)N_{2,1}(t) + (1-t)N_{3,1}(t)$$
$$N_{3,2}(t) = tN_{3,1}(t) + (2-t)N_{4,1}(t)$$
$$N_{4,2}(t) = (t-1)N_{4,1}(t) + (3-t)N_{5,1}(t)$$
$$N_{5,2}(t) = (t-2)N_{5,1}(t) + (4-t)N_{6,1}(t)$$

B-spline basis evaluation: ex. 2

$$N_{0,3}(t) = \frac{1}{2}(t+2)N_{0,2}(t) + \frac{1}{2}(0-t)N_{1,2}(t)$$

$$N_{1,3}(t) = \frac{1}{2}(t+1)N_{1,2}(t) + \frac{1}{2}(1-t)N_{2,2}(t)$$

$$N_{2,3}(t) = \frac{1}{2}(t+0)N_{2,2}(t) + \frac{1}{2}(2-t)N_{3,2}(t)$$

$$N_{3,3}(t) = \frac{1}{2}(t-1)N_{3,2}(t) + \frac{1}{2}(3-t)N_{4,2}(t)$$

B-spline basis evaluation: ex. 2

- Finally

$$N_{0,4}(t) = \frac{1}{3}(t+3)N_{0,3}(t) + \frac{1}{3}(1-t)N_{1,3}(t)$$

$$N_{1,4}(t) = \frac{1}{3}(t+2)N_{1,3}(t) + \frac{1}{3}(2-t)N_{2,3}(t)$$

$$N_{2,4}(t) = \frac{1}{3}(t+1)N_{2,3}(t) + \frac{1}{3}(3-t)N_{3,3}(t)$$

$$N_{3,4}(t) = \frac{1}{3}tN_{3,3}(t) + \frac{1}{3}(4-t)N_{4,3}(t)$$

B-spline basis evaluation: ex. 2

- Then substituting

$$N_{0,4}(t) = \frac{1}{6}(t+3)^3 N_{0,1}(t) + \left\{-(t+1)^3 + \frac{2}{3}t^3 - \frac{1}{6}(t-1)^3\right\} N_{1,1}(t) + \left\{\frac{2}{3}t^3 - \frac{1}{t}(t-1)^3\right\} N_{2,1}(t) - \frac{1}{6}(t-1)^3 N_{3,1}(t)$$

$$N_{1,4}(t) = \frac{1}{6}(t+2)^3 N_{1,1}(t) + \left\{-t^3 + \frac{2}{3}(t-1)^3 - \frac{1}{6}(t-2)^3\right\} N_{2,1}(t) + \left\{\frac{2}{3}(t-1)^3 - \frac{1}{t}(t-2)^3\right\} N_{3,1}(t) - \frac{1}{6}(t-2)^3 N_{4,1}(t)$$

$$N_{2,4}(t) = \frac{1}{6}(t+1)^3 N_{2,1}(t) + \left\{-(t-1)^3 + \frac{2}{3}(t-2)^3 - \frac{1}{6}(t-3)^3\right\} N_{3,1}(t) + \left\{\frac{2}{3}(t-2)^3 - \frac{1}{t}(t-3)^3\right\} N_{4,1}(t) - \frac{1}{6}(t-3)^3 N_{5,1}(t)$$

$$N_{3,4}(t) = \frac{1}{6}t^3 N_{3,1}(t) + \left\{-(t-2)^3 + \frac{2}{3}(t-3)^3 - \frac{1}{6}(t-4)^3\right\} N_{4,1}(t) + \left\{\frac{2}{3}(t-3)^3 - \frac{1}{t}(t-4)^3\right\} N_{5,1}(t) - \frac{1}{6}(t-4)^3 N_{6,1}(t)$$

de Boor algorithm (reminder)

1. Search index r with $t_r \leq t < t_{r+1}$
2. for $i = r - k + 1, \ldots, r$

   $d_i^0 = d_i$ sometimes noted as $d_i^0(t) = d_i$

- for $j = 1, \ldots, k - 1$

  for $i = r - k + 1 + j, \ldots, k$

  $$d_i^j = \left(1 - \alpha_i^j\right) \cdot d_{i-1}^{j-1} + \alpha_i^j \cdot d_i^{j-1}$$

  with $\alpha_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$

Then: $d_r^{k-1} = x(t)$

Order $\boldsymbol{k}$
$\boldsymbol{n+1}$ points
$\boldsymbol{n+k+1}$ knots

de boor algorithm: ex. 1

- For order 4, de Boor points $Q_0, Q_1, \dots, Q_8$ and knot sequence
  $$T = [t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7 \quad t_8 \quad t_9 \quad t_{10} \quad t_{11} \quad t_{12}]$$
  $$= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 6 \quad 6 \quad 6]$$
  Evaluate the B-spline curve at $t = 4.75$

de Boor algorithm: ex. 1

- Since $t_7 \leq 4.75 < t_8$, $r = 7$, therefore $i = 7 - 4 + 1 = 4$

$$\begin{aligned} Q_5^{[1]}(4.75) &= (1-\lambda)Q_4^{[0]}(4.75) + \lambda Q_5^{[0]}(4.75) \\ &= (1-\lambda)Q_4 + \lambda Q_5 = 0.083Q_4 + 0.917Q_5 \end{aligned}$$

$$\left(\lambda = \frac{4.75 - t_5}{t_8 - t_5} = 0.917\right)$$

$$\begin{aligned} Q_6^{[1]}(4.75) &= (1-\lambda)Q_5^{[0]}(4.75) + \lambda Q_6^{[0]}(4.75) \\ &= (1-\lambda)Q_5 + \lambda Q_6 = 0.417Q_5 + 0.583Q_6 \end{aligned}$$

$$\left(\lambda = \frac{4.75 - t_6}{t_9 - t_6} = 0.583\right)$$

$$\begin{aligned} Q_7^{[1]}(4.75) &= (1-\lambda)Q_6^{[0]}(4.75) + \lambda Q_7^{[0]}(4.75) \\ &= (1-\lambda)Q_6 + \lambda Q_7 = 0.625Q_6 + 0.375Q_7 \end{aligned}$$

$$\left(\lambda = \frac{4.75 - t_7}{t_{10} - t_7} = 0.375\right)$$

de Boor algorithm: ex. 1

- Then

$$Q_6^{[2]}(4.75) = (1-\lambda)Q_5^{[1]}(4.75) + \lambda Q_6^{[1]}(4.75) \qquad \left(\lambda = \frac{4.75 - t_6}{t_8 - t_6} = 0.875\right)$$

$$= 0.125(0.083Q_4 + 0.917Q_5) + 0.875(0.417Q_5 + 0.583Q_6)$$

$$= 0.01Q_4 + 0.479Q_5 + 0.510Q_6$$

$$Q_7^{[1]}(4.75) = (1-\lambda)Q_6^{[1]}(4.75) + \lambda Q_7^{[1]}(4.75) \qquad \left(\lambda = \frac{4.75 - t_7}{t_9 - t_7} = 0.375\right)$$

$$= 0.625(0.417Q_5 + 0.583Q_6) + 0.375(0.625Q_6 + 0.375Q_7)$$

$$= 0.261Q_5 + 0.598Q_6 + 0.141Q_7$$

de Boor algorithm: ex. 1

- Then

$$Q_7^{[3]}(4.75) = (1-\lambda)Q_6^{[2]}(4.75) + \lambda Q_7^{[2]}(4.75)$$

$$\left(\lambda = \frac{4.75 - t_7}{t_8 - t_7} = 0.75\right)$$

$$= 0.25(0.01Q_4 + 0.479Q_5 + 0.510Q_6)$$

$$+ 0.75(0.261Q_5 + 0.598Q_6 + 0.141Q_7)$$

$$= 0.0025Q_4 + 0.316Q_5 + 0.576Q_6 + 0.106Q_7$$

Order $\boldsymbol{k}$
$\boldsymbol{n+1}$ points
$\boldsymbol{n+k+1}$ knots

de boor algorithm: ex. 2

- For order 4, de Boor points $Q_0, Q_1, \dots, Q_6$ and knot sequence

$$T = [t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7 \quad t_8 \quad t_9 \quad t_{10}]$$
$$= [-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7]$$

Evaluate the B-spline curve at $t = 3.5$

de Boor algorithm: ex. 1

- Since $t_6 \leq 3.5 < t_7$, $r = 7$, therefore $i = 6 - 4 + 1 = 3$

$$Q_4^{[1]}(3.5) = (1-\lambda)Q_3^{[0]} + \lambda Q_4^{[0]} \qquad \left(\lambda = \frac{3.5 - t_4}{4-1} = 0.833\right)$$
$$= 0.167Q_3 + 0.833Q_0$$

$$Q_5^{[1]}(3.5) = (1-\lambda)Q_4^{[0]} + \lambda Q_5^{[0]} \qquad \left(\lambda = \frac{3.5 - t_5}{4-1} = 0.5\right)$$
$$= 0.167Q_0 + 0.833Q_1$$

$$Q_6^{[1]}(3.5) = (1-\lambda)Q_5^{[0]} + \lambda Q_6^{[0]} \qquad \left(\lambda = \frac{3.5 - t_6}{4-1} = 0.167\right)$$
$$= 0.167Q_1 + 0.833Q_2$$

$$Q_5^{[2]}(3.5) = (1-\lambda)Q_4^{[1]} + \lambda Q_5^{[1]} \qquad \left(\lambda = \frac{3.5 - t_5}{4-2} = 0.75\right)$$
$$= 0.25(0.167Q_3 + 0.833Q_0) + 0.75(0.5Q_0 + 0.5Q_1)$$
$$= 0.583Q_0 + 0.375Q_1 + 0.042Q_3$$

$$Q_5^{[1]}(3.5) = (1-\lambda)Q_4^{[0]} + \lambda Q_5^{[0]} \qquad \left(\lambda = \frac{3.5 - t_6}{4-2} = 0.25\right)$$
$$= 0.75(0.5Q_0 + 0.5Q_1) + 0.25(0.833Q_1 + 0.167Q_2)$$
$$= 0.375Q_0 + 0.583Q_1 + 0.042Q_2$$

$$Q_6^{[3]}(3.5) = (1-\lambda)Q_5^{[2]} + \lambda Q_6^{[2]} \qquad \left(\lambda = \frac{3.5 - t_6}{4-3} = 0.5\right)$$
$$= 0.5(0.583Q_0 + 0.375Q_1 + 0.042Q_3) + 0.5(0.375Q_0 + 0.583Q_1 + 0.042Q_2)$$
$$= 0.479Q_0 + 0.479Q_1 + 0.021Q_2 + 0.021Q_3$$