计算机辅助几何设计
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Differential Geometry of Curves

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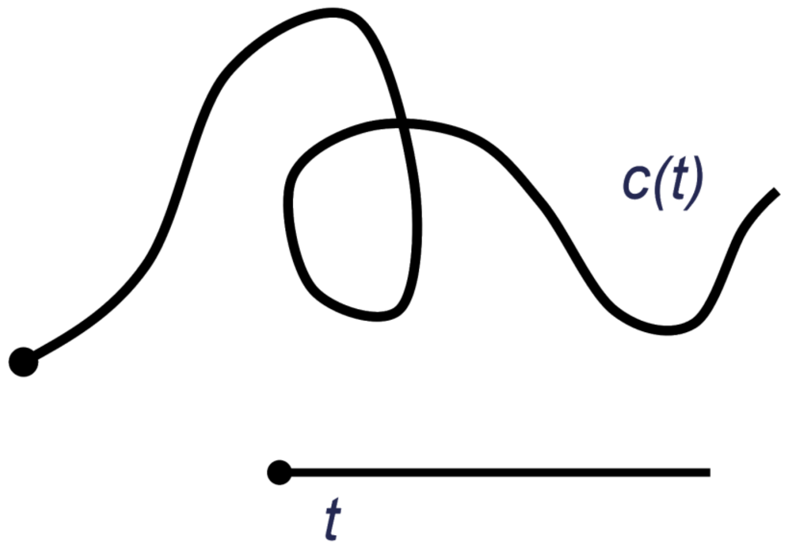
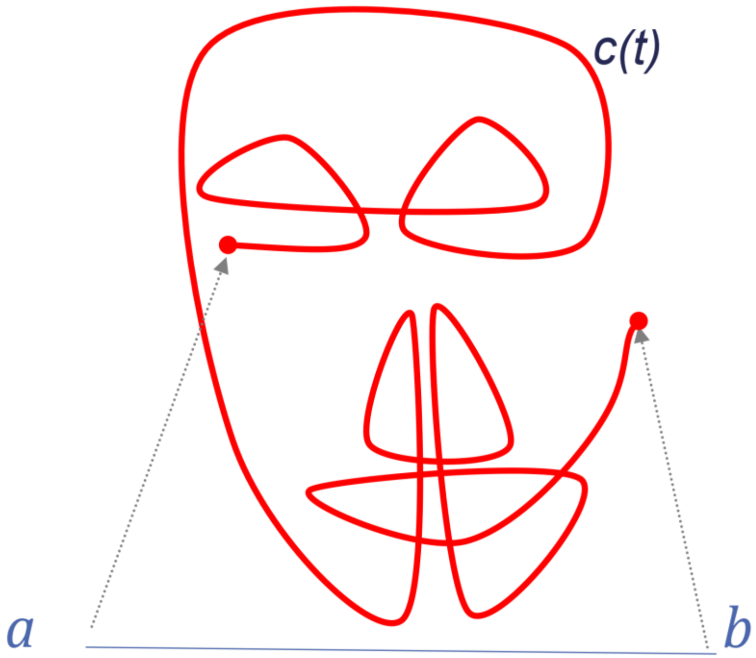
Parametric Curves

- **Parametric Curves:**
  - Think of a curve c as the path of a moving particle
  - Not always enough to know where a particle went – we also want to know when it got there $\rightarrow c(t)$
  - Parameter t is often thought of as time



Parametric Curves

- **Parametric Curves:**
  - A *parameterization of class* $C^k(k \geq 1)$ of a curve in $\mathbb{R}^n$ is a smooth map $c: I = [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^n$, where c is of class C^k



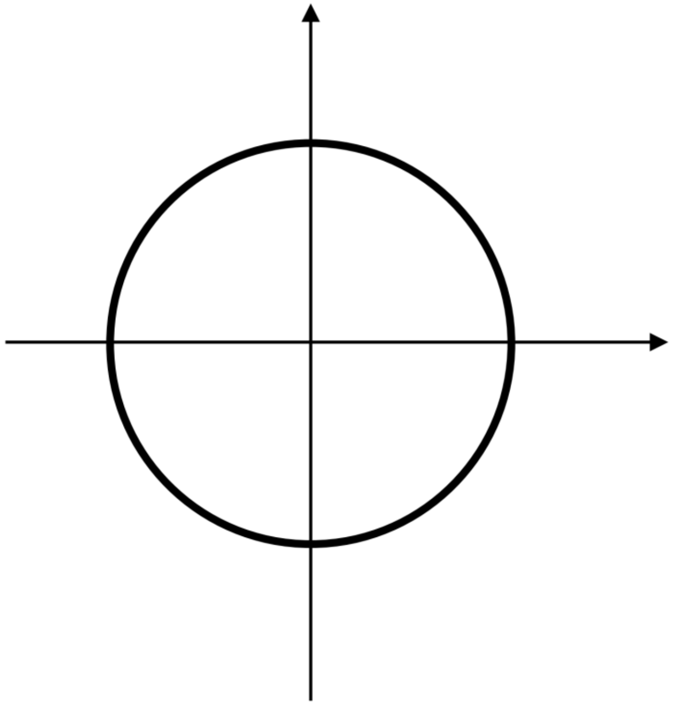
Parametric Curves

- **Parametric Curves:**
  - The image set $c(I)$ is called the ***trace*** of the curve
    - Different parameterizations can have the same trace.
  - A point in the trace, which corresponds to more than one parameter value t, is called ***self-intersection*** of the curve

Parametric Curves: Examples

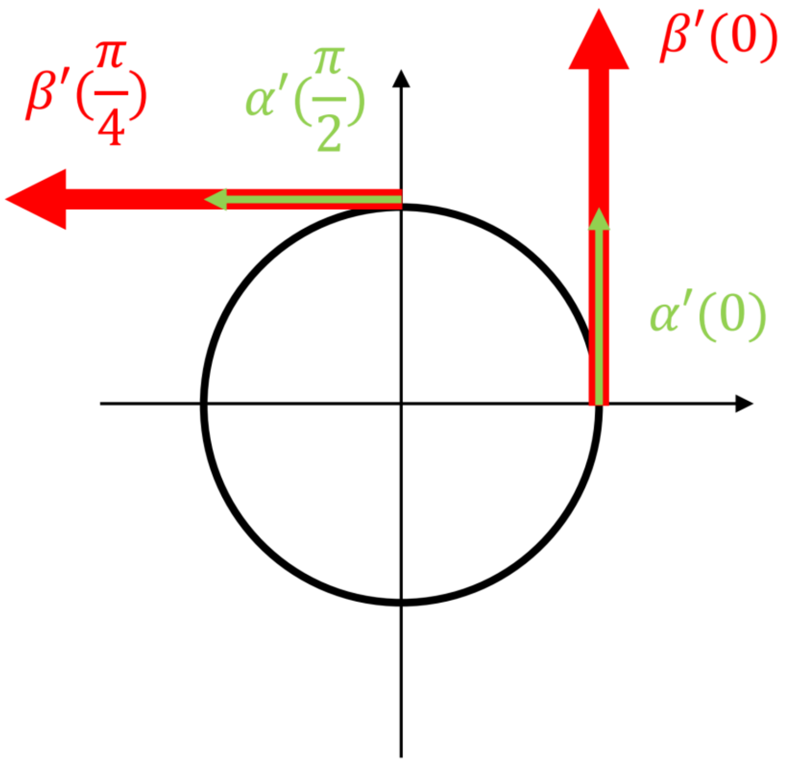
- $c(t) = (t, 0), t \in]0, \infty[$, and also $c(t) = (e^t, 0), t \in \mathbb{R}$ (both represent the *positive* x-axis

- **Circle**
  - $c(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$
  - $c(t) = (\cos 2t, \sin 2t), \quad t \in [0, \pi]$
  - $c(t) = (\cos t, \sin t), \quad t \in \mathbb{R}$



The velocity vector

- The derivative $c'(t)$ is called the velocity vector to the curve c at time t
  - $c'(t)$ gives the direction of the movement
  - $|c'(t)|$ gives the speed
- Example
  - $\alpha(t) = (\cos t, \sin t), \qquad t \in [0, 2\pi]$
  - $\beta(t) = (\cos 2t, \sin 2t), \qquad t \in [0, \pi]$



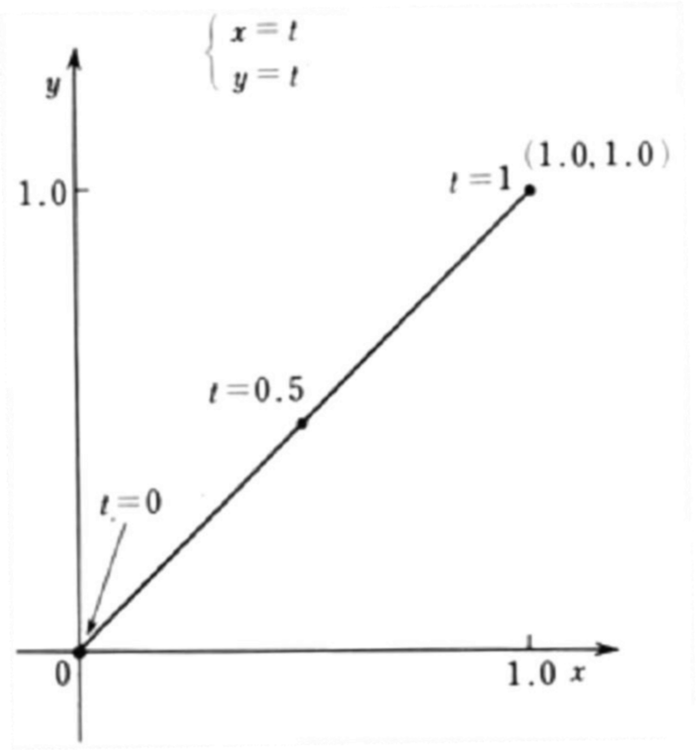
Regular parametric curves

- **Regular parametrization**
  - A parameterization is called *regular* if $c'(t) \neq 0$ for all t
  - A point at which a curve is regular is called an *ordinary* point
  - A point at which a curve is non-regular is called an *singular* point

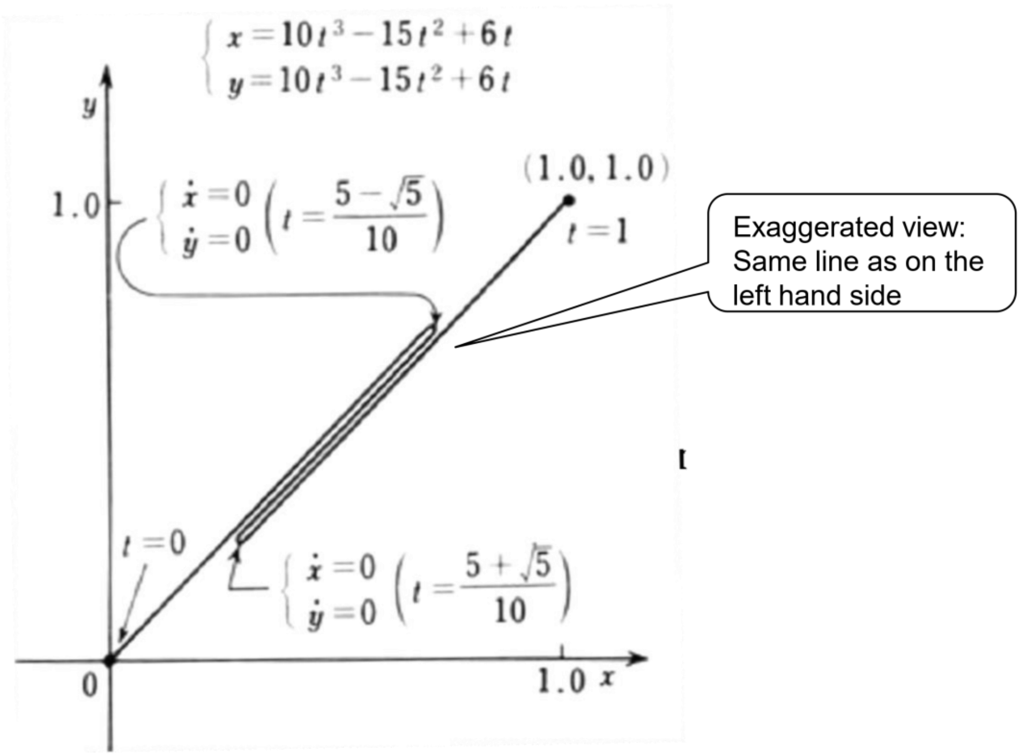
Examples: regularity

- **Examples: issues with non-regular parameterization**

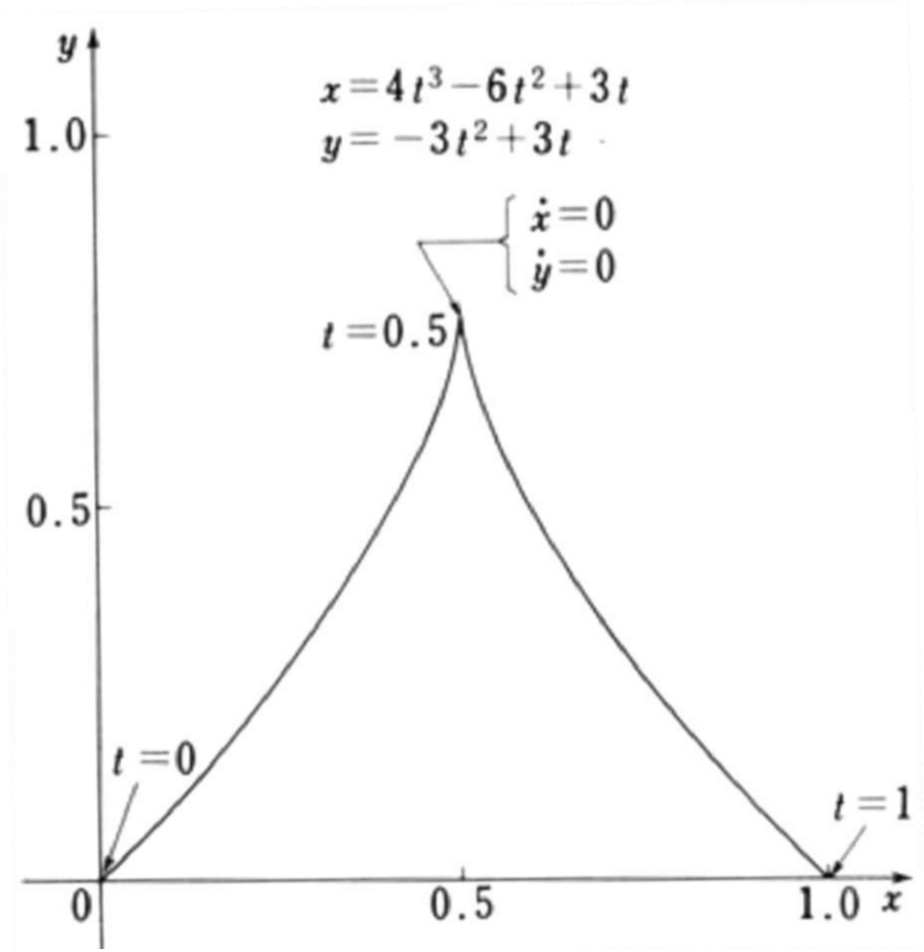
Regular parametrization



Non-regular parametrization

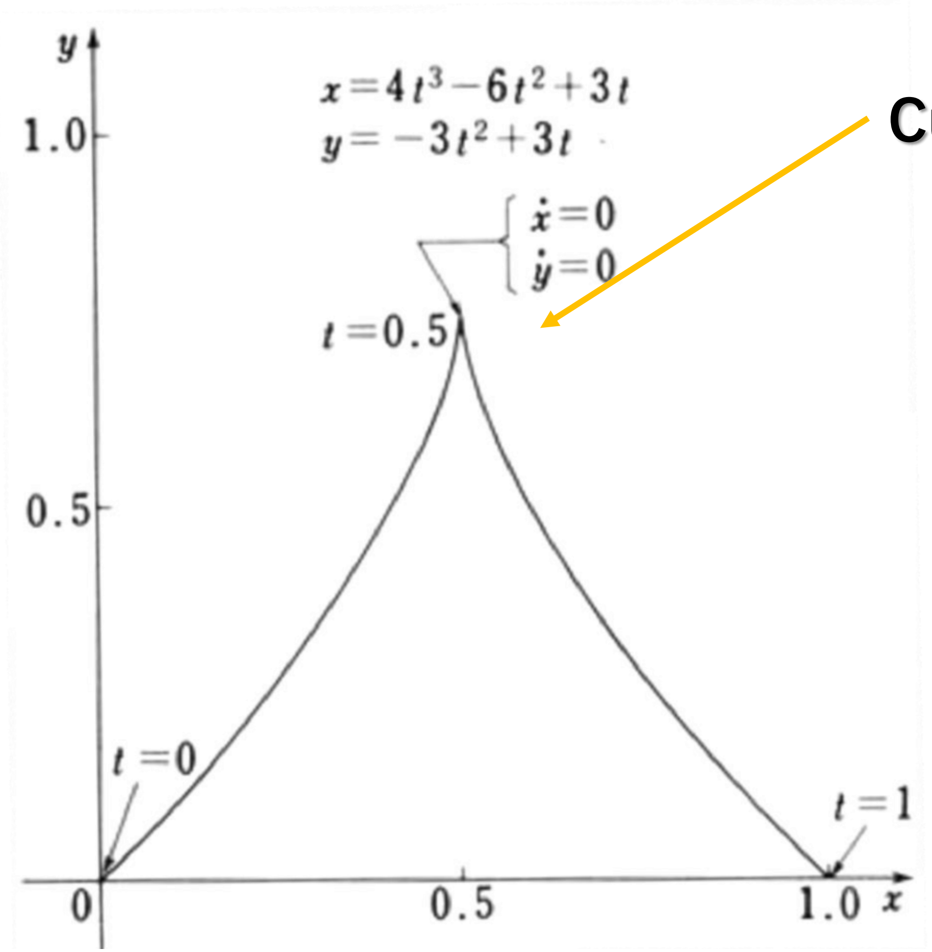


Examples: cusps



Singularities can be desired design features

Examples: cusps



Cusp

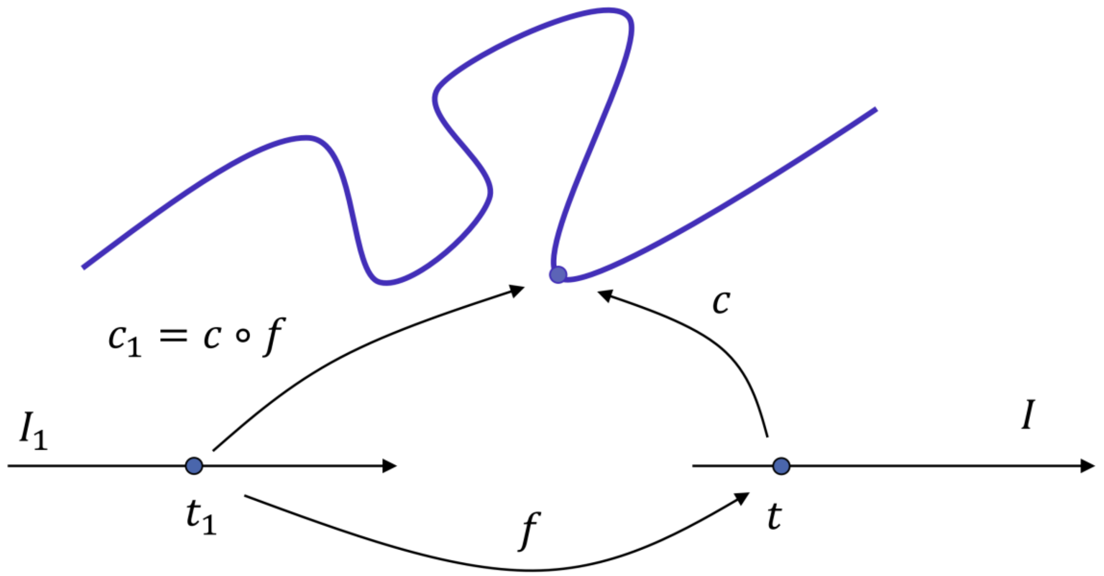
Singularities can be desired design features

Change of parameterization

- **Given a smooth regular parametrization, an allowable change of parameter is any real smooth (differentiable) function**

$$f: I_1 \to I \text{ such that } f' \neq 0 \text{ on } I_1$$

  - It is orientation preserving when $f' > 0$

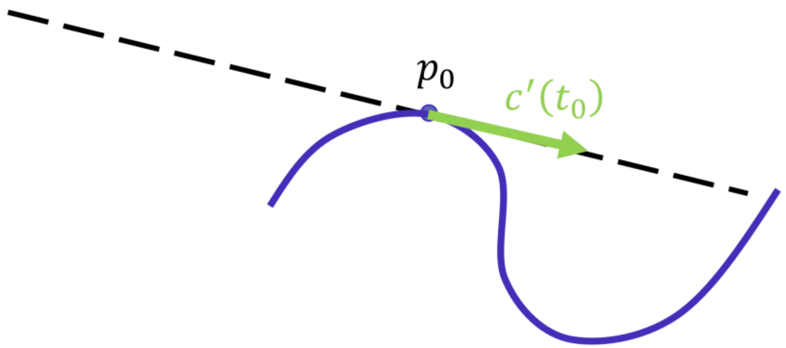


Change of parameterization

- **Parameter Transformations:**
  - We can regard a *regular curve* as a collection of regular parameterizations, any two of which are reparameterizations of each other (equivalence class)
  - We are interested in properties that are invariant under parameter transformations

Geometric observations

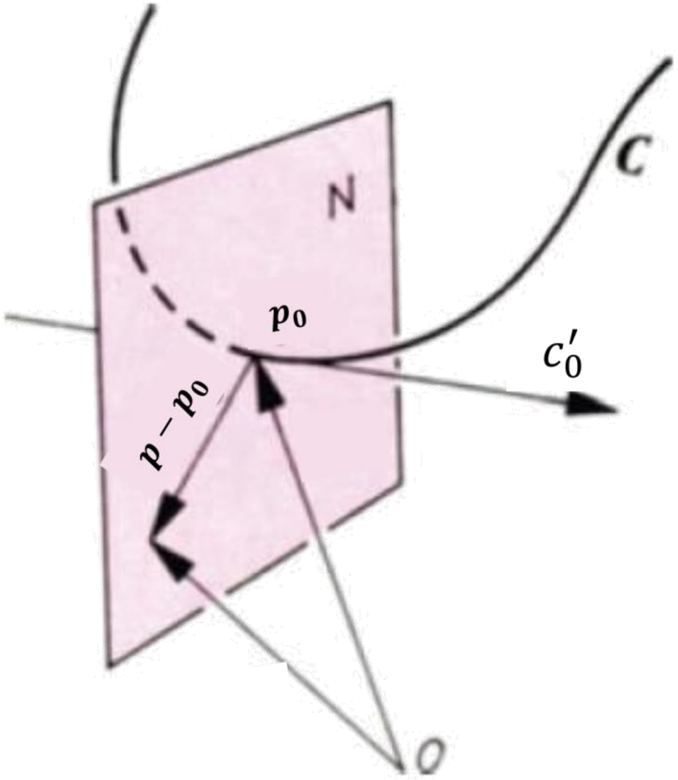
- **Tangent vector:**
  - The tangent line to a regular curve $c(t)$ at $p_0 = c(t_0)$ can be defined as points p which satisfy $p - p_0 \parallel c'_0$, where $c'_0 = c'(t_0)$
  - The normalized vector $t = \frac{c'}{|c'|}$ is called the tangent vector



Geometric observations

- **The normal plane:**
  - The normal plane can be obtained as points p whose coordinates satisfy $p - p_0 \perp c_0'$

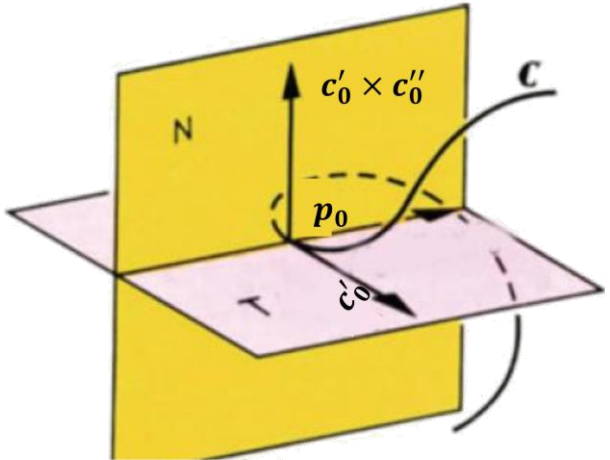
    $\Leftrightarrow (p - p_0) \cdot c_0' = 0$



Geometric observations

- **Osculating plane: 密切平面**
  - Assume the curve $c(t)$ is not a straight line. Any three arbitrary non-collinear points p_1, p_2, p_3 determine a plane
  - If p_1, p_2, p_3 tend to the same points p_0 of c, then their plane converges to a plane called the osculating plane T of c at p_0
  - The osculating plane is well defined if the first two derivatives c_0' and c_0'' at p_0 are linearly independent and is give as:

$$(c_0' \times c_0'') \cdot (p - p_0) = 0$$



Geometric observations

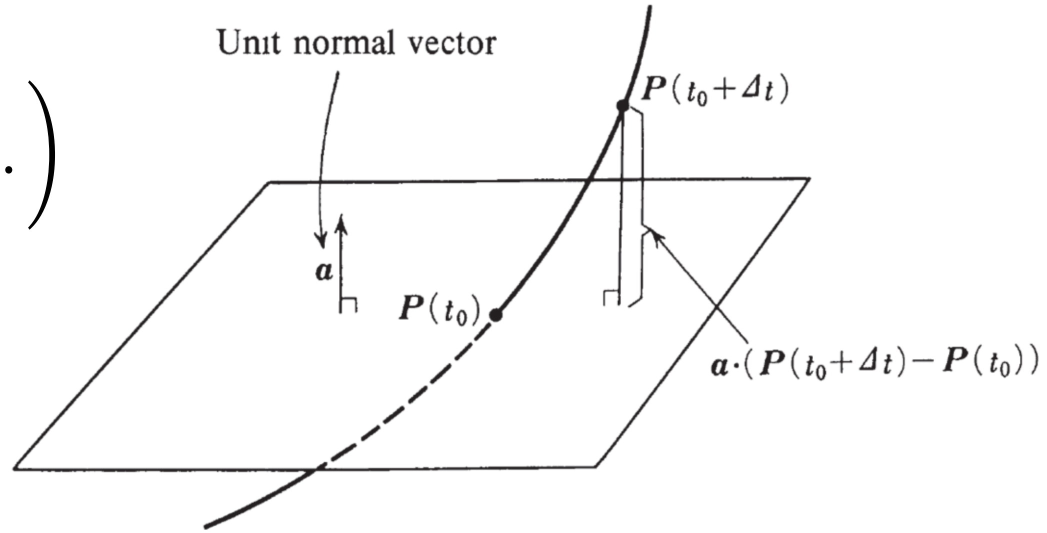
Observe the distance between $p(t_0 + \Delta t)$ and a given plane passing through $p(t_0)$ with normal vector a

$$a \cdot \left(p(t_0 + \Delta t) - p(t_0)\right) = a \cdot \left(\dot{p}(t_0)\Delta t + \frac{\ddot{p}(t_0)}{2!}\Delta t^2 + \cdots\right)$$

The distance is minimal when

$$a \cdot \dot{P}(t_0) = 0, a \cdot \ddot{P}(t_0) = 0$$

That is when the plane is osculating

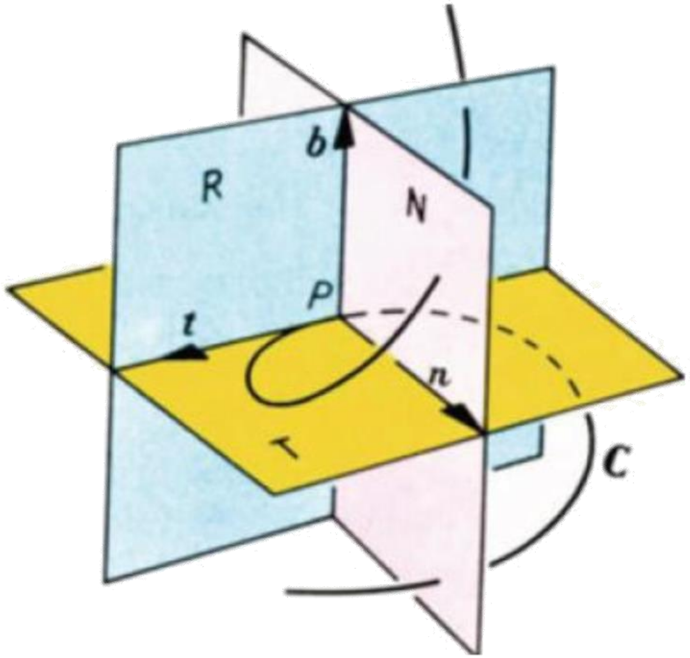


→ The osculating plane is the plane that best fits the curve at $p(t_0)$

Geometric observations

- **The rectifying plane: 从切平面**
  - The plane normal to both, the osculating plane and the normal plane, is called the rectifying plane R and can be obtained as points p whose coordinates satisfy

$$\left(c_0' \times (c_0' \times c_0'')\right) \cdot (p - p_0) = 0$$



Geometric observations

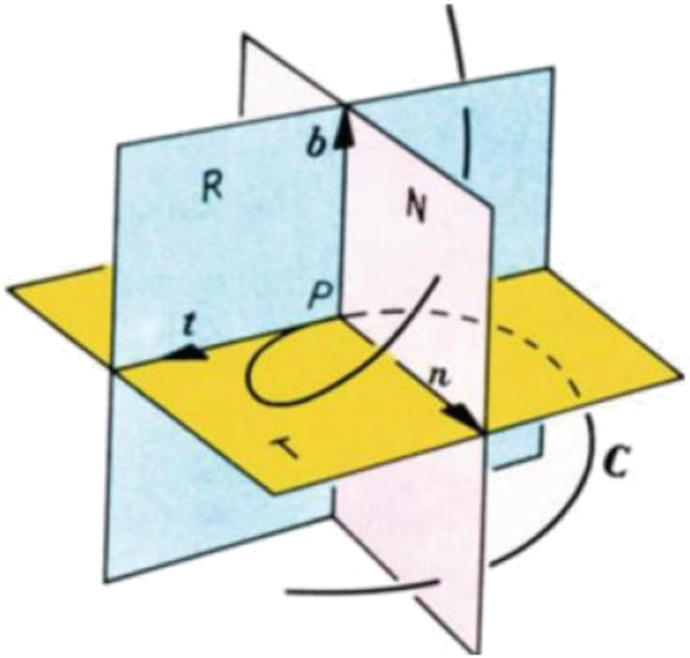
Normals: any vector in the normal plane is normal to the curve, in particular:

- The normal n lying in the osculating plane is called the principal normal at p_0.

  It has a direction $(c_0' \times c_0'') \times c_0'$

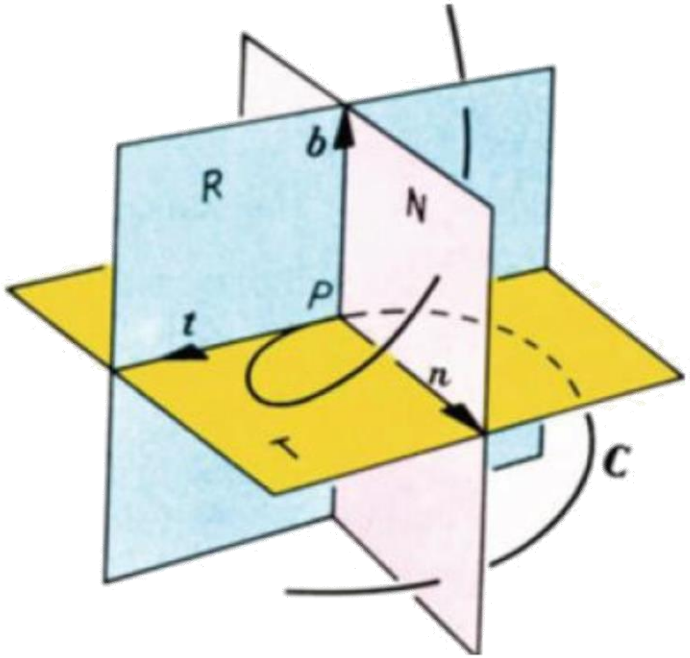
- The normal b lying in the rectifying plane is called the binormal. 副法向

  It has a direction $c_0' \times c_0''$



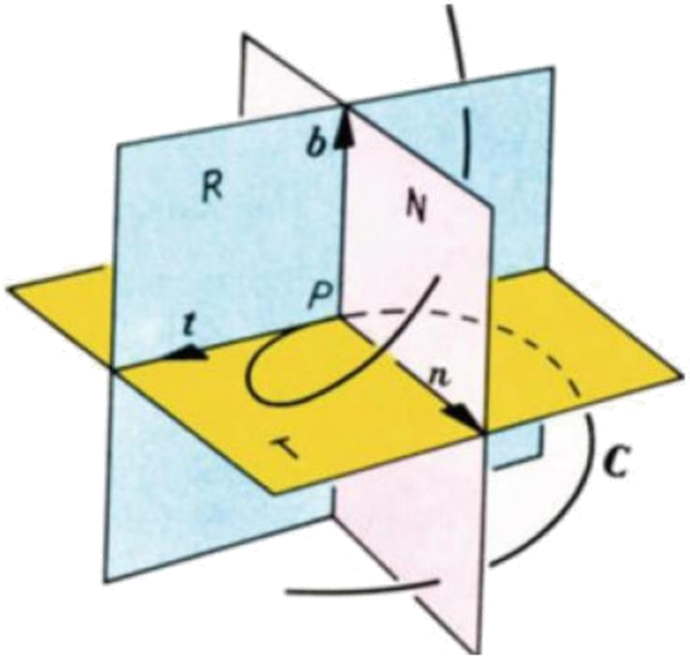
The Frenet frame

- **We can define a local coordinates system on the curve by three vectors**
  - The tangent $t = \frac{c'}{\|c_0'\|}$
  - The binormal $b = \frac{c_0' \times c_0''}{\|c_0' \times c_0''\|}$
  - The principal normal $n = b \times t$



The Frenet frame and associated planes

- The tangent $t = \frac{c'}{\|c_0'\|}$
  - the normal plane $(p - p_0) \cdot t = 0$
- The binormal $b = \frac{c_0' \times c_0''}{\|c_0' \times c_0''\|}$
  - the osculating plane $(p - p_0) \cdot b = 0$
- The principal normal $n = b \times t$
  - the rectifying plane $(p - p_0) \cdot n = 0$

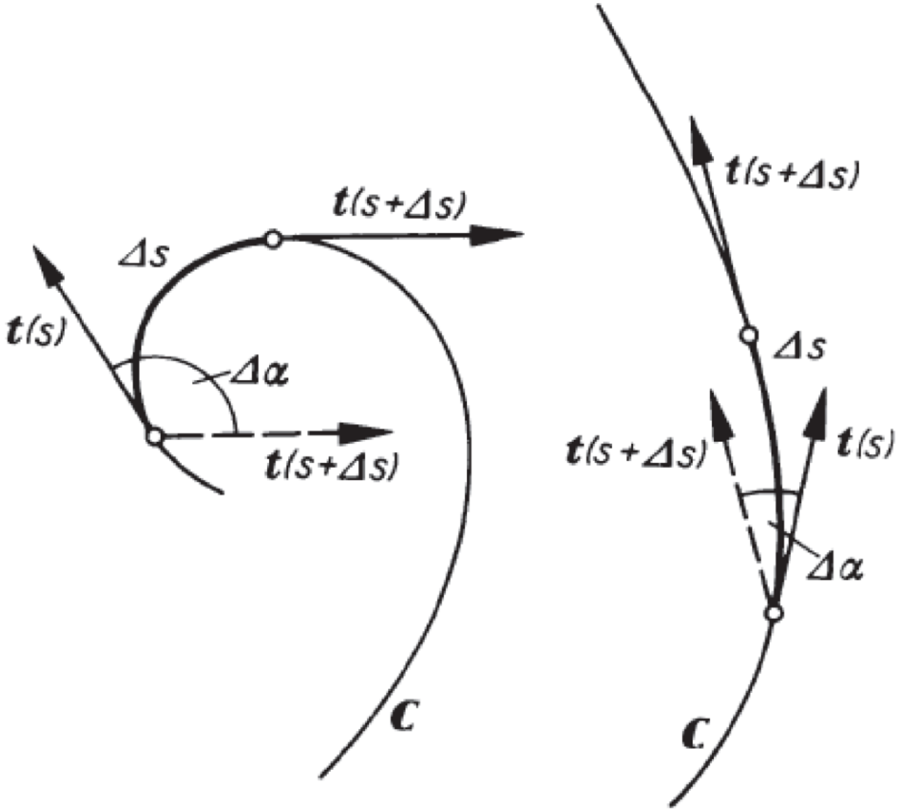


Curvature

- Common conceptions of curvature
  - Measures bending of a curve
  - A straight line does not bend → 0 curvature
  - A circle has constant bending → constant curvature

Curvature

- **Euler's heuristic approach for planar curves**
  - Variation of the tangent angle: how much does the curve differ from a straight line



Curvature for regular parameterization

- The curvature is denoted by κ and defined as

$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

Examples:

- Consider the circle $c(t) = (r\cos t\,, r\sin t\,, 0)$
- The curvature is given by

$$\kappa(t) = \frac{\|(-r\,sin\,t\,, r\cos t\,, 0) \times (-r\,cos\,t\,, -r\,sin\,t\,, 0)\|}{r^3} = \frac{\|(0{,}0, r^2)\|}{r^3} = \frac{1}{r}$$

- Consider the helix $c(t) = (r\cos t\,, r\sin t\,, at)$, the ***curvature*** reads

$$\kappa(t) = \frac{r}{r^2 + a^2}$$

Special case: planar curves

- For a regular planar curve $c(t) = \big(x(t), y(t)\big)$

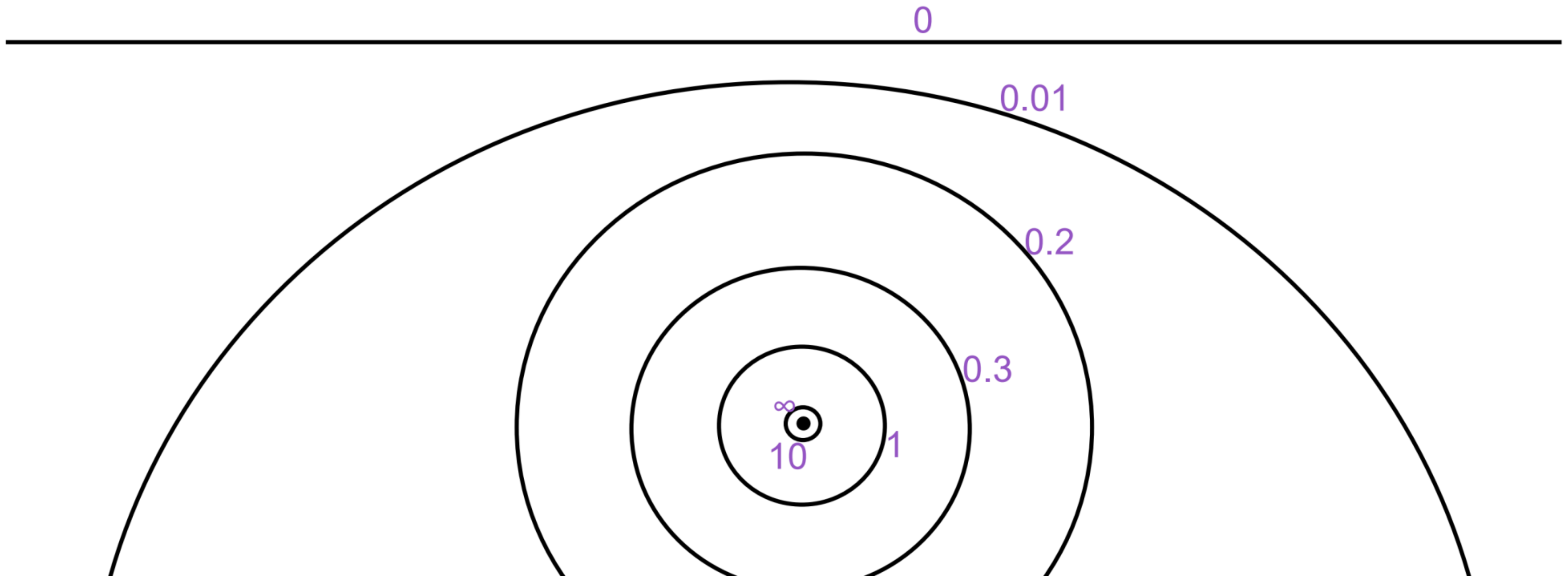
$$\kappa(t) = \frac{|x'y'' - x''y'|}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

- Sometimes we talk about signed curvature, and then curvature can be allowed to be signed (negative, zero, or positive)

$$\kappa(t) = \frac{x'y'' - x''y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

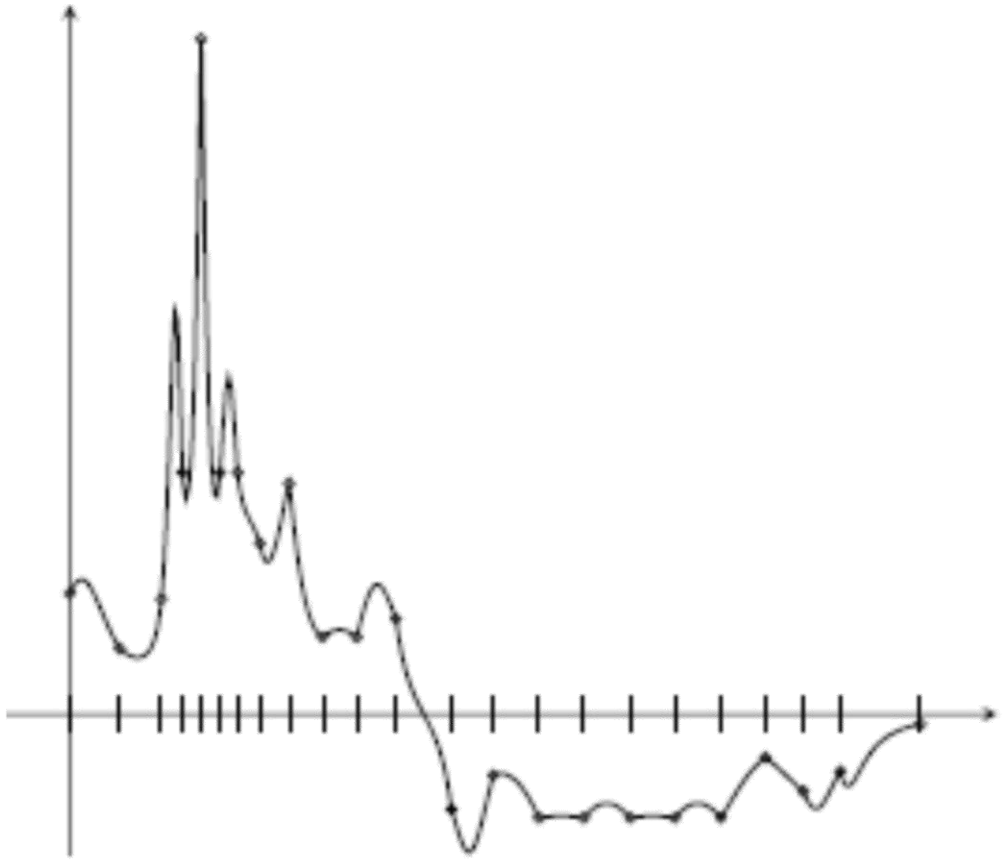
Examples

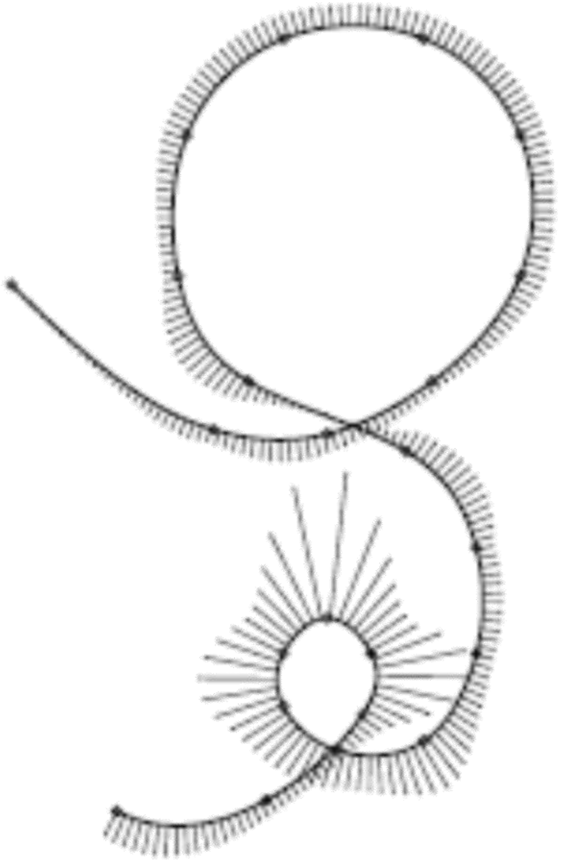
- **Curvature of circles**
  - Curvature of a circle is constant, $\kappa \equiv \frac{1}{r}$ (r = radius)
  - Accordingly: define radius of curvature as $\frac{1}{\kappa}$



Curvature in practice

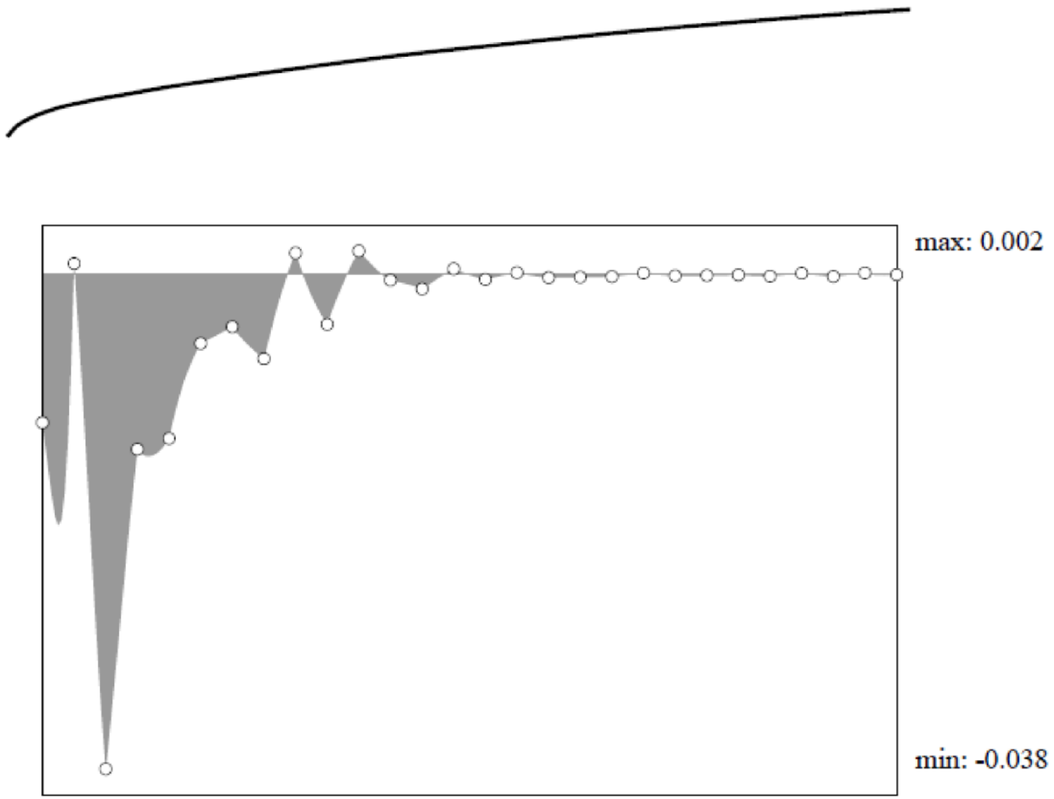
- Most of commercial package allow inspecting the quality of the curvature

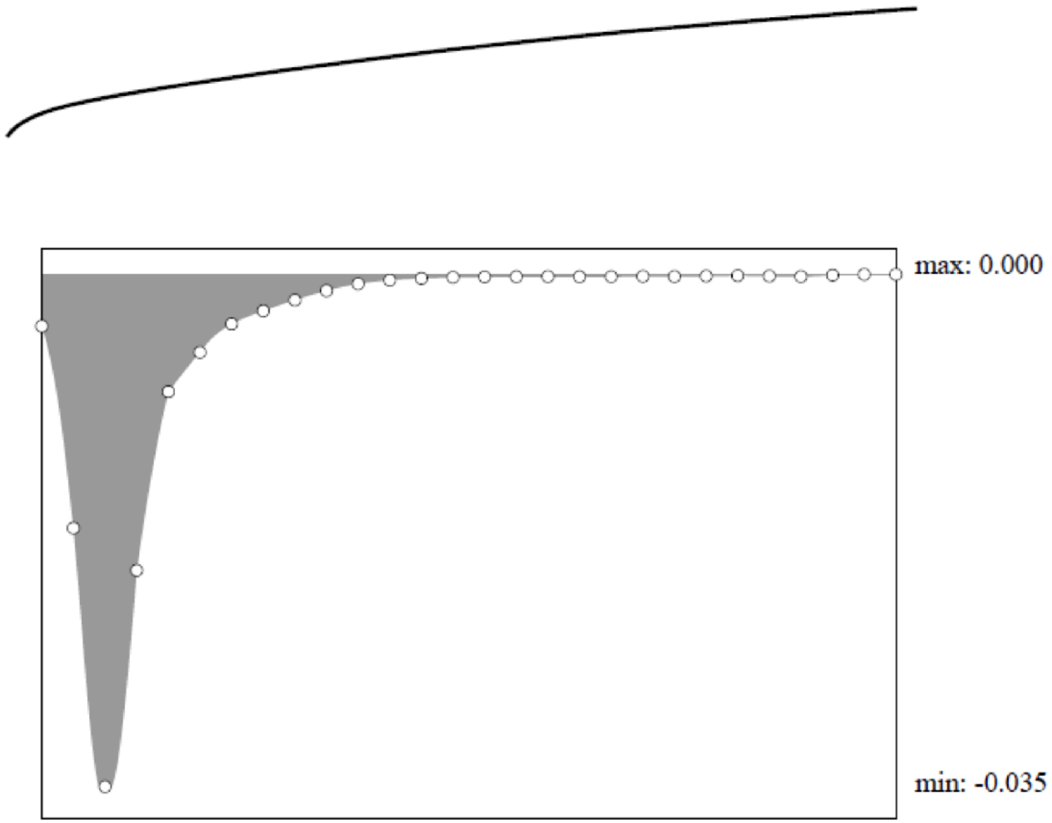




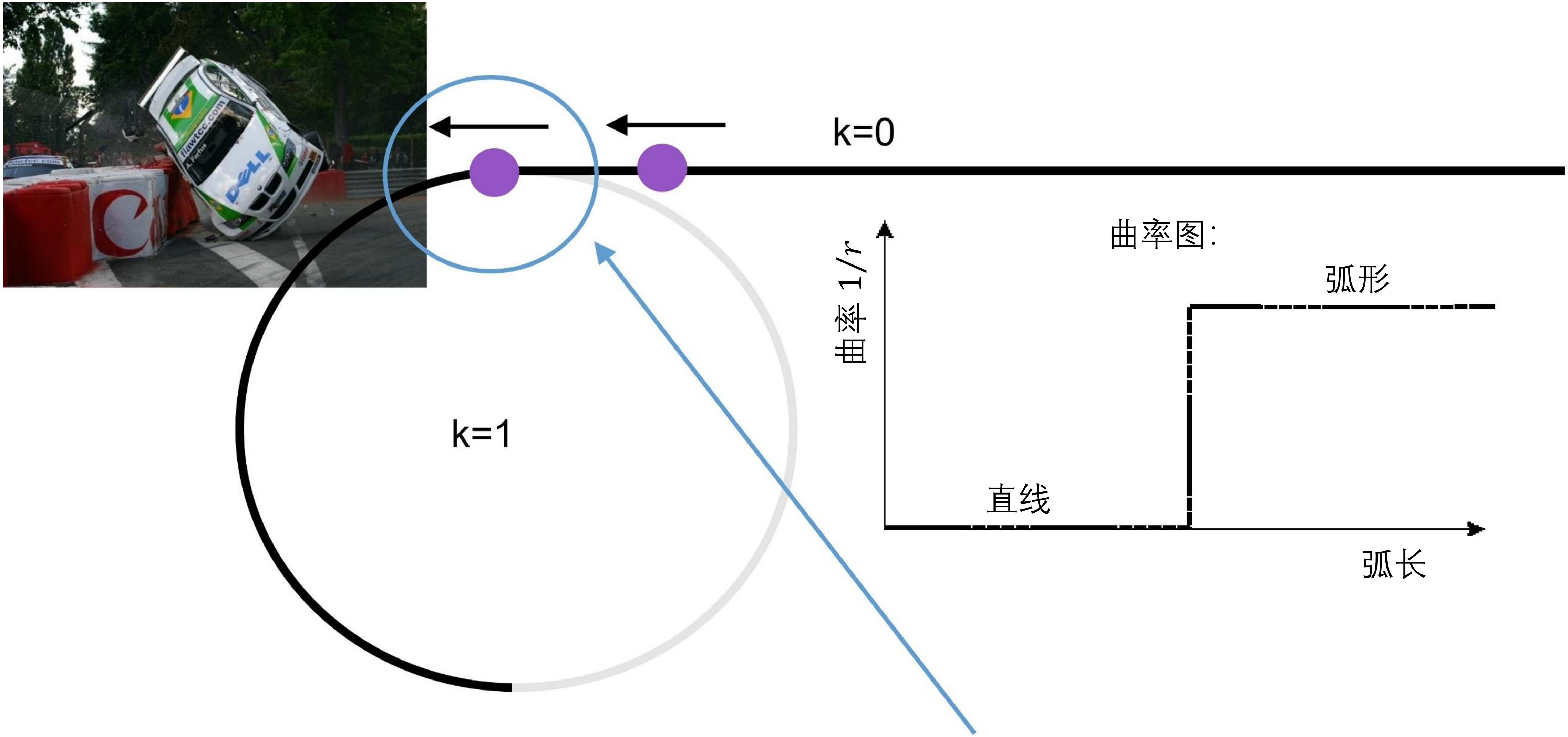
Curvature in practice

- Most commercial package allow checking the quality of the curvature even meticulously!

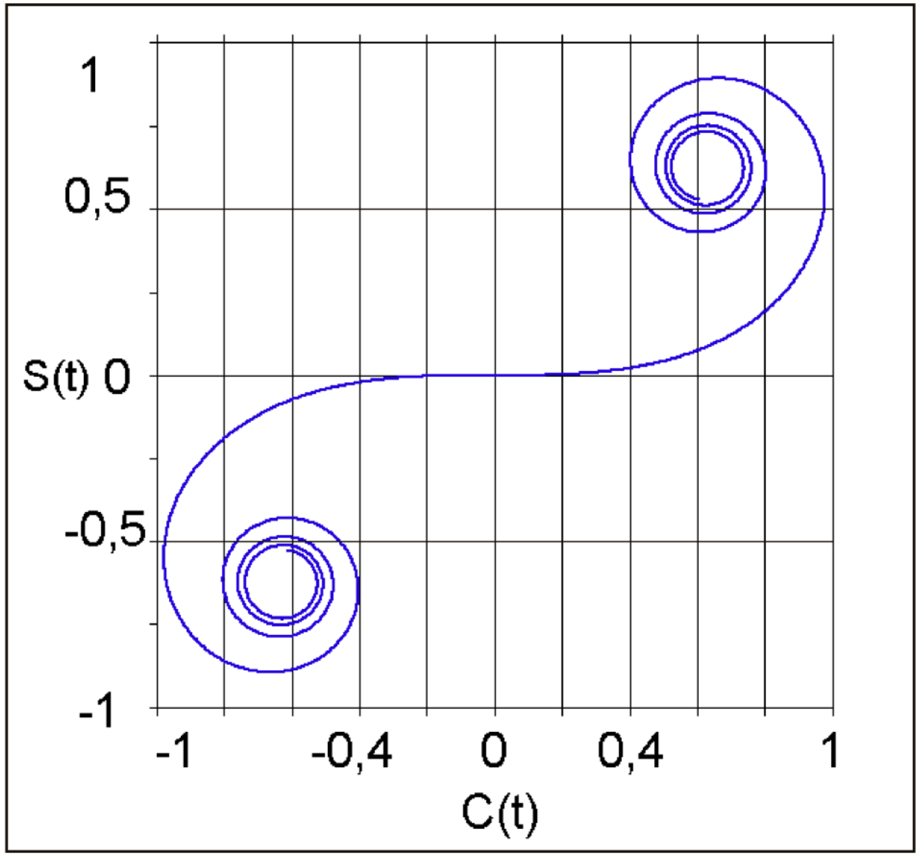


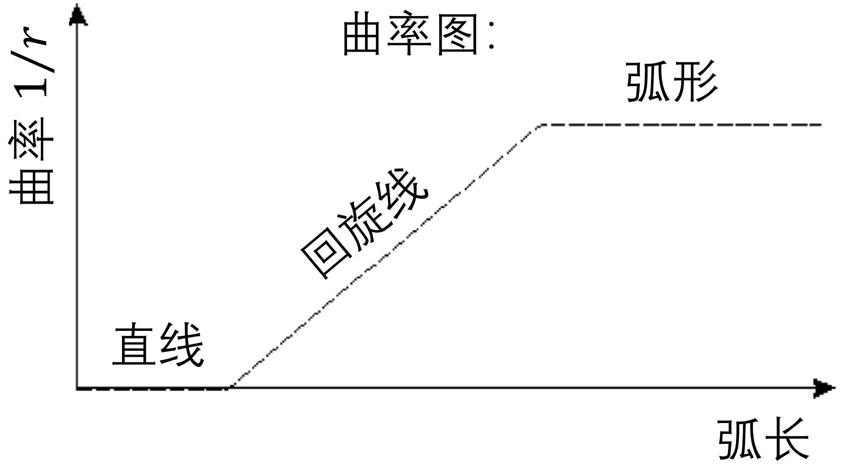


Curvature and Road Construction



Clothoide, Euler Spiral　羊角螺线





$$c(t) = \begin{pmatrix} \int_0^t \cos\frac{\pi}{2}u^2\,du \\ \int_0^t \sin\frac{\pi}{2}u^2\,du \end{pmatrix}$$

Torsion for regular parameterization

- **Definition**
  - The torsion τ measures the variation of the binormal vector
  - (deviation of the curve from its projection on the osculating plane, can be regarded as how far is the curve is from being a planar curve) and is given by

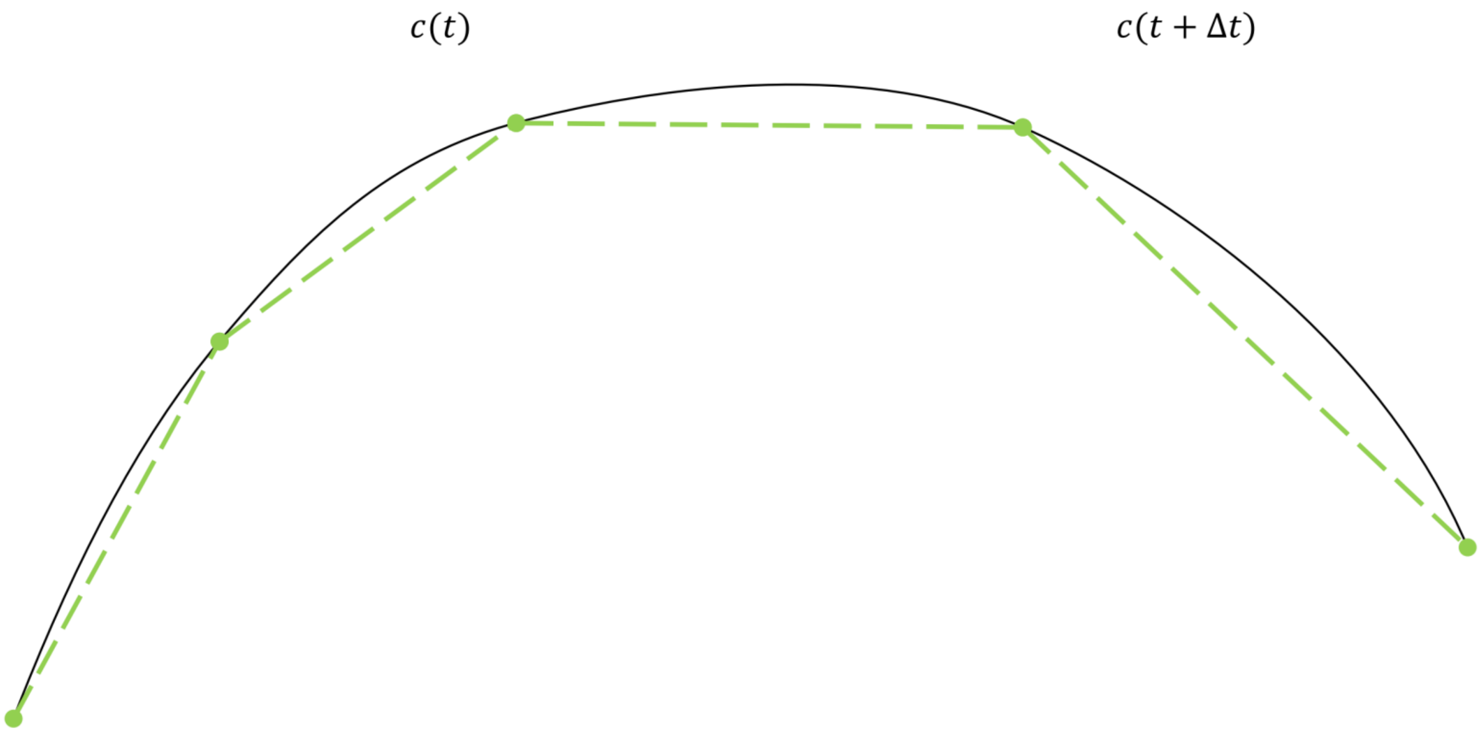
$$\tau(t) = \frac{(c' \times c'') \cdot c'''}{\|c' \times c''\|^2}$$

Torsion

- **Examples:**
  - Torsion for a planar curve
  - Torsion for a quadratic curve

Measuring lengths on curves

- **The arc length of a curve**
  - Can be regarded as the limit of the sum of infinitesimal segments along the curve



Measuring lengths on curves

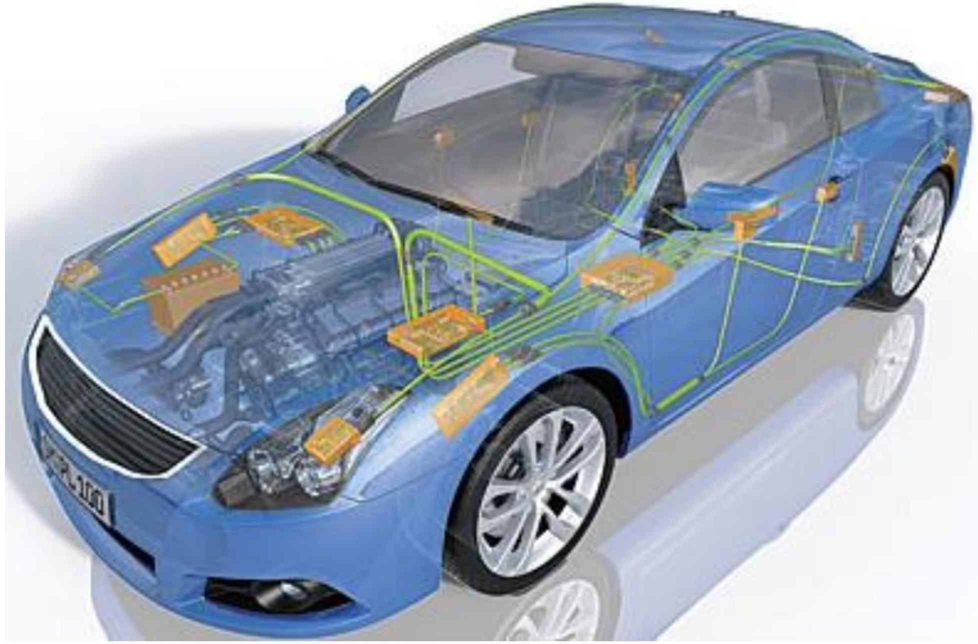
- **The arc length of a curve**
  - The arc length of a regular curve C is defined as :

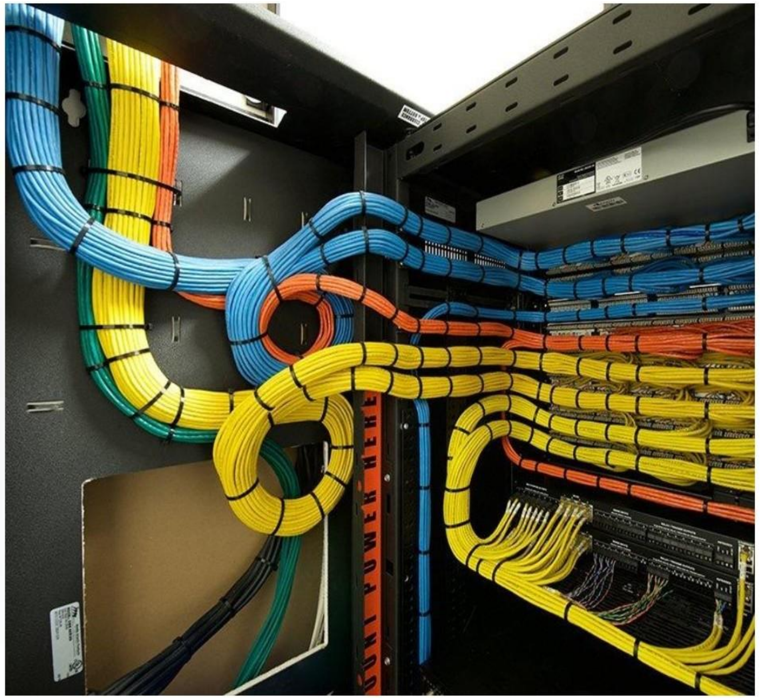
$$\text{length}_c = \int_a^b ||\dot{c}|| dt$$

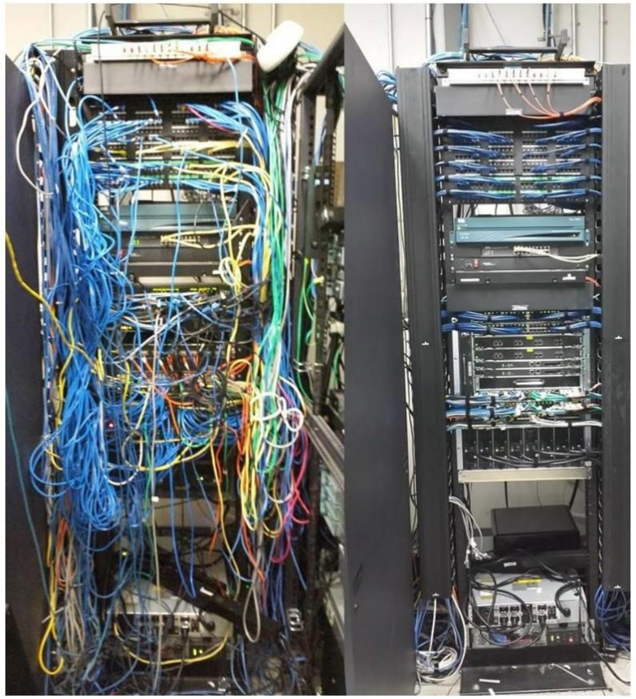
  - Independent of the parameterization (to prove this use integration by substitution)

Measuring lengths on curves

- Curve arc length matters in practice (e.g., cable routing problems)







Arc-length parametrized curves

Arc length parametrization

- Consider the portion of $c(t)$ spanned from 0 to t, the length s of this arc is a function of t:

$$s(t) = \int_0^t \|c'(u)\| dt$$

- Since $\frac{ds}{dt} = \|c'(u)\| > 0$ (why?) → s can be introduced as a new parameterization

Arc length parametrization

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$$s(t) = \int_0^t \|c'(u)\| dt$$

- Since $\frac{ds}{dt} = \|c'(u)\| > 0$ (why?) → s can be introduced as a new parameterization
- We have $c'(s) = \frac{dc}{ds} = \frac{dc/_{dt}}{ds/_{dt}} \Rightarrow \|c'(s)\|$=1
- $c(s)$ is called an *arc-length* (or *unit-speed*) *parametrized curve*, the parameter s is called the *arc length* of c or the *natural parameter*

Reparameterization by arc length

- Arc-length (or unit-speed) parameterization:
  - Any regular curve admits an arc-length parameterization
  - This does not mean that the arc-length parameterization can be computed

$$s(t) = \int_0^t \|c'(u)\| dt$$

Examples

- Find an arc-length parameterization for the Helix: $\begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$

Examples

$$s(t) = \int_0^t \|c'(u)\| dt$$

- Find an arc-length parameterization for the Helix: $\begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$

$$s(t) = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2} du = t\sqrt{2} \Rightarrow t = \frac{s}{\sqrt{2}}$$

The arc-length parameterized Helix: $\begin{pmatrix} \cos \frac{s}{\sqrt{2}} \\ \sin \frac{s}{\sqrt{2}} \\ \frac{s}{\sqrt{2}} \end{pmatrix}$

$$s(t) = \int_0^t \|c'(u)\| dt$$

Examples

- How about the ellipse $\alpha(t) = \begin{pmatrix} 2\cos t \\ \sin t \\ 0 \end{pmatrix}$?

$$s(t) = \int_0^t \|c'(u)\| dt$$

Examples

- How about the ellipse $\alpha(t) = \begin{pmatrix} 2\cos t \\ \sin t \\ 0 \end{pmatrix}$?

$$s(t) = \int_0^t \sqrt{4(-\sin u)^2 + (\cos u)^2} du = \int_0^t \sqrt{4 - 3\cos^2 u}\, du$$

Does not admit any closed form antiderivative

$$s(t) = \int_0^t \|c'(u)\| dt$$

Examples

- How about $\alpha(t) = \begin{pmatrix} t \\ \frac{t^2}{2} \\ 0 \end{pmatrix}$?

$$s(t) = \int_0^t \|c'(u)\| dt$$

Examples

- How about $\alpha(t) = \begin{pmatrix} t \\ \frac{t^2}{2} \\ 0 \end{pmatrix}$?

$$s(t) = \int_0^t \sqrt{1+u^2}du = t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)$$

- No straightforward way to write t as a function of s!

Geometric consequences of Arc length parameterization

- Since $\|c'(u)\| = 1$

Geometric consequences of Arc length parameterization

- Since $||c'(u)|| = 1$, by noting that $c' \cdot c' = 1$ and taking the derivative, we have $c' \cdot c'' = 0$
- c'' is perpendicular to c' (both lives on the osculating plane)
- Therefore c'' is a direction vector of the principal normal (provided that $c'' \neq 0$)

$$\Rightarrow n = \frac{c''}{||c''||}$$

Curvature again

- The curvature of an arc-length parametrized curve (unit speed curve) $c(t)$ simplifies to

$$\kappa = \|c''(u)\|$$

Further mathematical Formulations: Frenet Curves

Frenet Curves

- Frenet curves
  - A *Frenet curve* is an arc-length parametrized curve c in $\mathbb{R}^n$ such that $c'(s), c''(s), \dots, c^{n-1}(s)$ are linearly independent

Frenet Curves

- Frenet curves
  - A *Frenet curve* is an arc-length parametrized curve c in $\mathbb{R}^n$ such that $c'(s), c''(s), \dots, c^{n-1}(s)$ are linearly independent
- Frenet frame
  - Every Frenet curve has a unique Frenet frame $e_1(s), e_2(s), \dots, e_n(s)$ that satisfies
    - $e_1(s), e_2(s), \dots, e_n(s)$ is orthonormal and positively oriented

Frenet Curves

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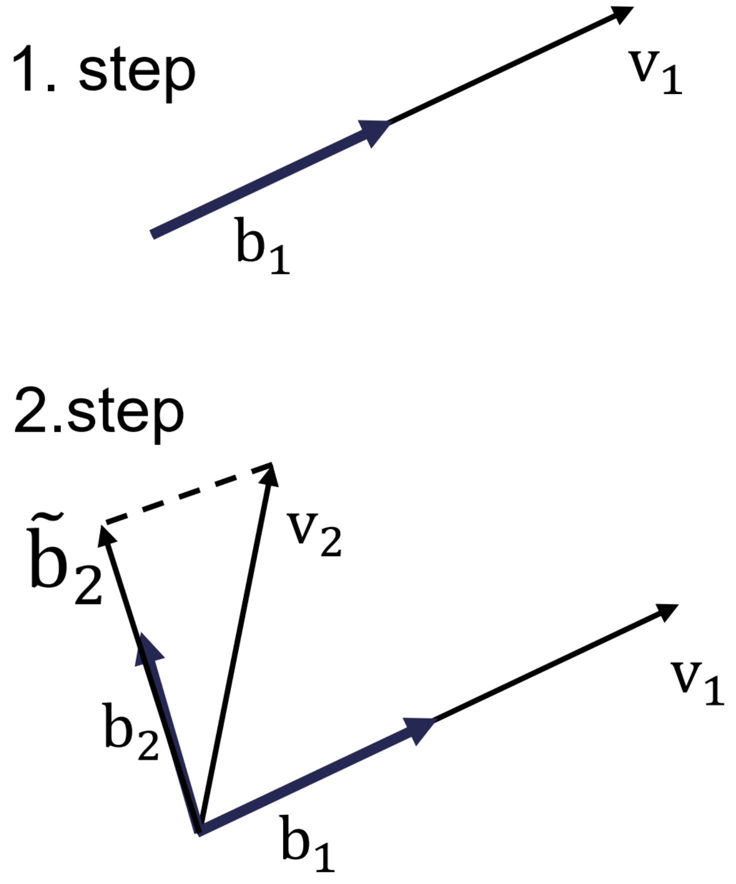
- Frenet frame
  - Every Frenet curve has a unique Frenet frame $e_1(s), e_2(s), \dots, e_n(s)$ that satisfies
    - $e_1(s), e_2(s), \dots, e_n(s)$ is orthonormal and positively oriented
  - Apply the Gram-Schmidt process to $\{c', c'', \dots, c^n\}$

Construction of Orthonormal Bases: Gram-Schmidt Process

- Input: Linear independent set $\{v_1, v_2, \dots, v_n\}$
- Output: Orthogonal set $\{b_1, b_2, \dots, b_n\}$
  - Set $b_1 = \frac{v_1}{\|v_1\|}$
  - For $k = 2, \dots, n$
    - $\widetilde{b_k} = v_k - \sum_{i=1}^{k-1} \langle v_k, b_i \rangle \ b_i$
    - $b_k = \frac{\widetilde{b_k}}{\|\widetilde{b_k}\|}$



Planar Curves

The Frenet Frame of an arc-length parametrized planar curve

Tangent vector

Normal vector

$$e_1(s) = c'(s) \qquad e_2(s) = R^{90^\circ} e_1(s)$$

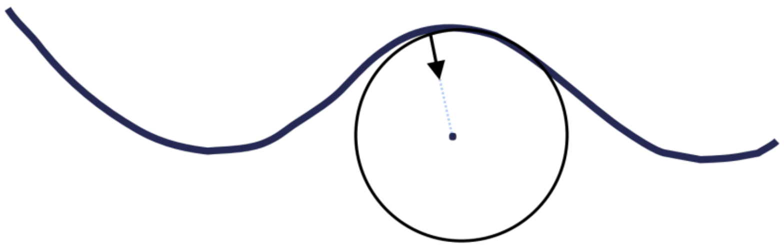
Frame equation

$$\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix}$$

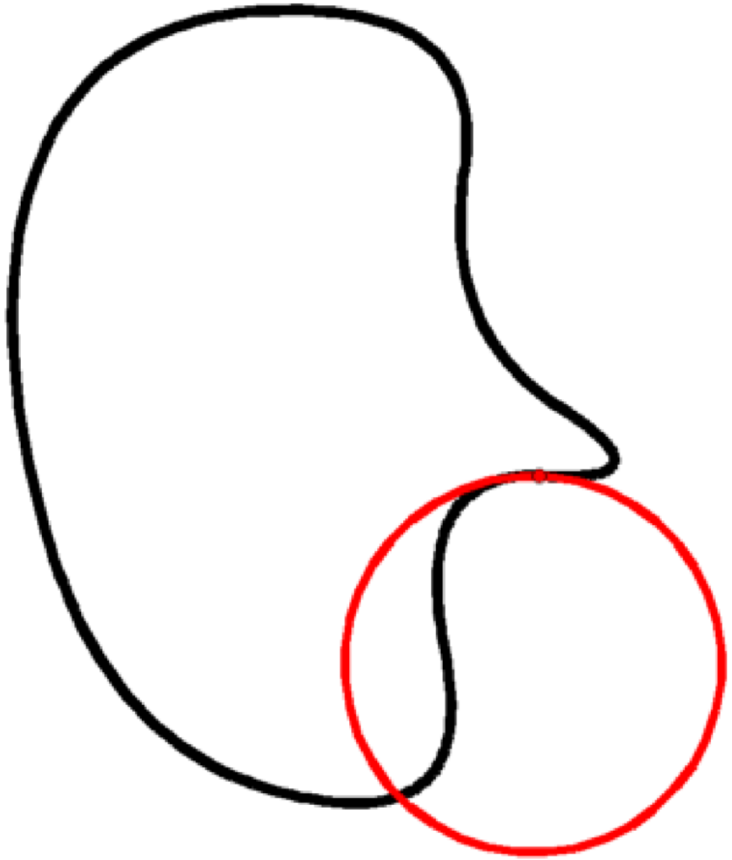
Signed Curvature

$\kappa(s) = \langle e_1'(s), e_2(s) \rangle$ is called the signed curvature of the curve

Osculating circle



- **Osculating circle**
  - Radius: $1/\kappa$
  - Center: $c(s) + \frac{1}{\kappa} e_2(s)$



Properties

- Rigid motions
  - Rigid motion: $x \to Ax + b$ with orthogonal A (in other words: affine maps that preserve distances)
    - Orientation preserving (no mirroring) if $\det A = +1$
    - Mirroring leads to $\det A = -1$
- Invariance under rigid motions for planar curves
  - Curvature is invariant under rigid motion
    - Absolute value is invariant
    - Signed value is invariant for orientation preserving rigid motion
- Rigidity of planar curves
  - Two Frenet curves with identical signed curvature function differ only by an orientation preserving rigid motion

Fundamental Theorem

- **Fundamental theorem for planar curves**
  - Let $\kappa: (a, b) \mapsto \mathbb{R}$ be a smooth function. For some $s_0 \in (a, b)$, suppose we are given a point p_0 and two orthonormal vectors t_0 and n_0. Then there exist a unique Frenet curve $c: (a, b) \mapsto \mathbb{R}^2$ such that
    - $c(s_0) = p_0$
    - $e_1(s_0) = t_0$
    - $e_2(s_0) = n_0$
    - The curvature of c equals the given function κ
  - In other words: for every smooth function there is a unique (up to rigid motion) curve that has this function as its curvature

Arc-length Derivative

- Arc-length parameterization
  - Finding an arc-length parameterization for a parameterized curve is usually difficult
  - Still one can compute the Frenet frame and its derivatives. For this we define the so called arc-length derivative
- Arc-length derivative
  - For a parameterized curve $c: [a, b] \mapsto \mathbb{R}^n$, we define the *arc-length derivative* of any differentiable function $f: [a, b] \mapsto \mathbb{R}$ as

$$f'(t) = \frac{1}{\|\dot{c}(t)\|} \dot{f}(t)$$

Compute the signed curvature

- Computing the Frenet frame
  - For $c: [a, b] \mapsto \mathbb{R}^2$, the Frenet frame at $c(t)$ can be computed as (using arc length derivative)

$$e_1(t) = c'(t) = \frac{\dot{c}(t)}{\|\dot{c}(t)\|}$$

$$e_2(t) = R^{90^\circ} e_1(t)$$

- Computing the signed curvature
  - The signed curvature is given by

$$\kappa(t) = \langle e_1'(t), e_2(t) \rangle = \frac{\left\langle \ddot{c}(t), R^{90^\circ} \dot{c}(t) \right\rangle}{\|\dot{c}(t)\|^3}$$

Space Curves

- Frenet frame of arc-length parametrized space curves
  - Frenet frame of a Frenet curve in $\mathbb{R}^3$
    - Tangent vector

$$e_1(s) = c'(s)$$

    - Normal vector

$$e_2(s) = \frac{1}{\|c''(t)\|} c''(t)$$

    - Binormal vector

$$e_3(s) = e_1(s) \times e_2(s)$$

Frenet Frame of Space Curves

- Frenet–Serret equations

$$\begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}$$

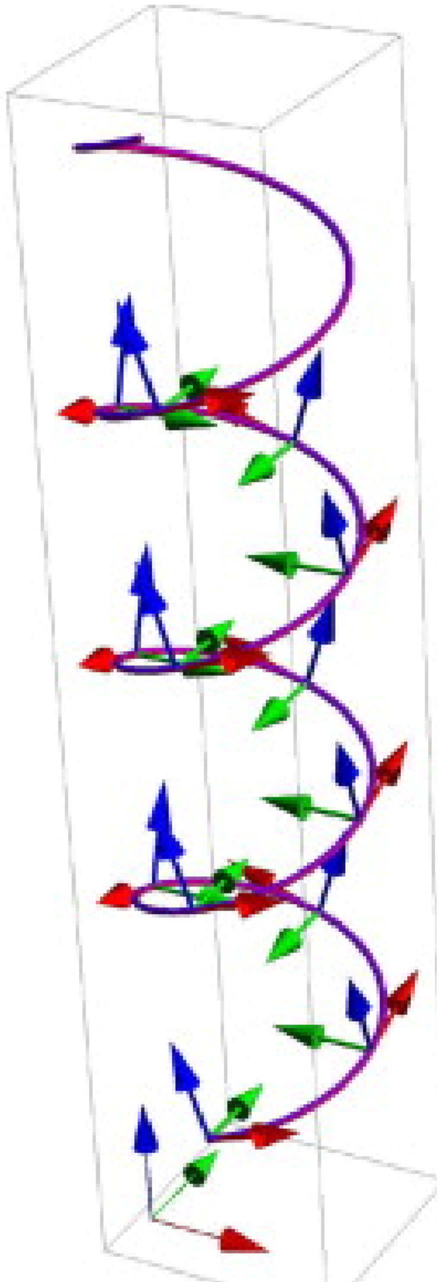
- The signed curvature still is $\kappa(s) = \langle e_1'(s), e_2(s) \rangle$

Frenet Frame of Space Curves

- Frenet–Serret equations

$$\begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}$$

- The torsion $\tau(s) = \langle e_2'(s), e_3(s) \rangle$ measures how the curve bends out of the plane spanned by e_1 and e_2



Frenet Frame of Space Curves

- Frenet equations for curves in $\mathbb{R}^n$

$$\begin{pmatrix} e_1(s) \\ e_2(s) \\ \dots \\ e_n(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1(s) & 0 & \dots & 0 \\ -\kappa_1(s) & 0 & \kappa_2(s) & \dots & 0 \\ 0 & -\kappa_2(s) & 0 & \dots & \\ & & & \dots & \kappa_{n-1}(s) \\ 0 & \dots & & -\kappa_{n-1}(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ \dots \\ e_n(s) \end{pmatrix}$$

- The function $\kappa_i(s)$ are called the i^{th} Frenet curvatures

Summary of relations

- **For regular curves:**
  - The tangent $t = \frac{c'}{\|c'\|}$, the normal plane $(p - p_0) \cdot t = 0$
  - The binormal $b = \frac{c' \times c''}{\|c' \times c''\|}$, the osculating plane $(p - p_0) \cdot b = 0$
  - The principal normal $n = b \times t$, the rectifying plane $(p - p_0) \cdot n = 0$
  - The curvature $\kappa(t) = \frac{c' \times c''}{\|c'\|^3}$
  - The torsion $\tau(t) = \frac{(c' \times c'') \cdot c'''}{\|c' \times c''\|^2}$

Summary of relations

- **For an arc-length parameterized (unit speed) curves** $c(s)$**:**
  - The tangent $t = c'$
  - The binormal $b = t \times n$
  - The principal normal $n = \frac{t'}{\|t'\|} = \frac{c''}{\|c''\|}$,
  - The curvature $\kappa(t) = \|t'\| = \|c''\|$
  - The signed curvature $\kappa(s) = t' = c''$
  - The torsion $\tau(t) = -b' \cdot n$

Special case: planar curves

- For a regular planar curve $c(t) = \big(x(t), y(t)\big)$, it is defined as
$$\kappa(t) = \frac{|x'y'' - x''y'|}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

- Sometimes we talk about signed curvature, and then curvature can be allowed to be signed (negative, zero, or positive)
$$\kappa(t) = \frac{x'y'' - x''y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$