GAMES103: Intro to Physics-Based Animation

Linear Finite Element Method

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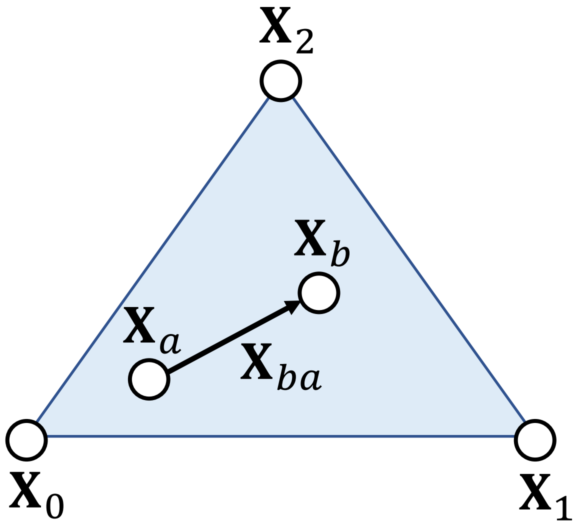
Topics for the Day

- The linear finite element method (FEM)
- The finite volume method (FVM)
- Hyperelastic models

Linear Finite Element Method

The Linear FEM Assumption

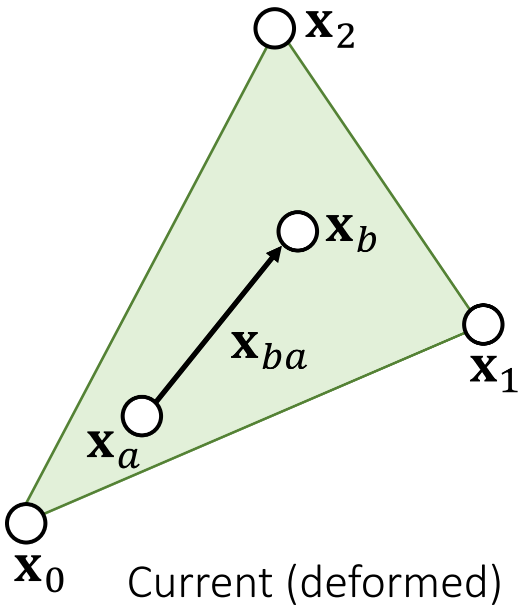
In a nutshell, linear FEM assumes that for any point $\mathbf{X}$ in the reference triangle, its deformed correspondence is: $\mathbf{x} = \mathbf{FX} + \mathbf{c}$.



Reference (undeformed)

$$\mathbf{x} = \underbrace{\mathbf{F}}_{\text{constant matrix}}\mathbf{X} + \overbrace{\mathbf{c}}^{\text{constant translational vector}}$$

$\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$, known as deformation gradient



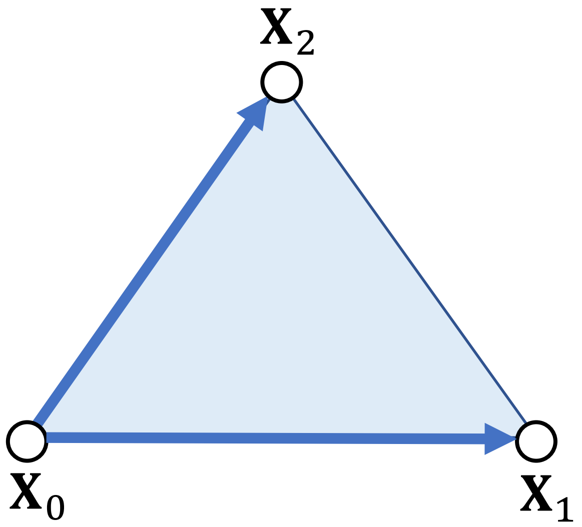
Current (deformed)

For any vector between two points, we can use $\mathbf{F}$ to convert it from reference to deformed:

$$\mathbf{x}_{ba} = \mathbf{FX}_b + \mathbf{c} - \mathbf{FX}_a - \mathbf{c} = \mathbf{FX}_{ba}.$$

Deformation Gradient

Therefore, we can calculate the deformation gradient by edge vectors.



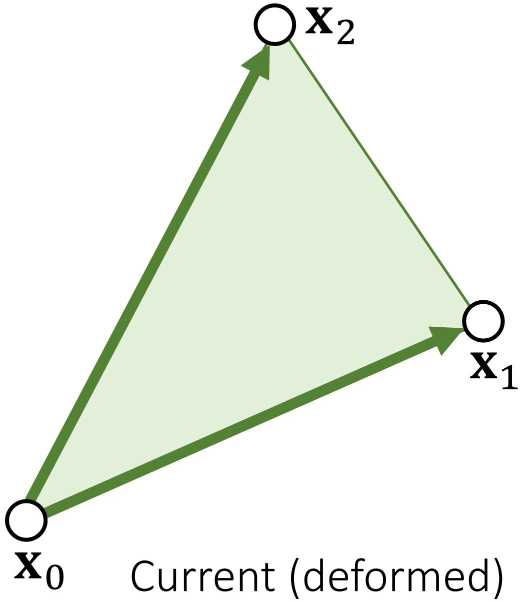
Reference (undeformed)

$$\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c}$$

$$\begin{cases} \mathbf{F}\mathbf{X}_{10} = \mathbf{x}_{10} \\ \mathbf{F}\mathbf{X}_{20} = \mathbf{x}_{20} \end{cases}$$

$$\mathbf{F}[\mathbf{X}_{10} \quad \mathbf{X}_{20}] = [\mathbf{x}_{10} \quad \mathbf{x}_{20}]$$

$$\mathbf{F} = [\mathbf{x}_{10} \quad \mathbf{x}_{20}][\mathbf{X}_{10} \quad \mathbf{X}_{20}]^{-1}$$

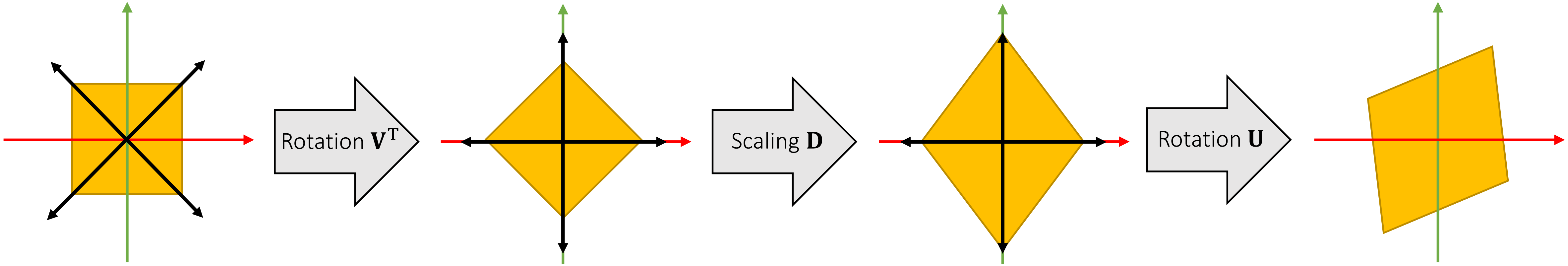


Current (deformed)

Problem: $\mathbf{F}$ is related to deformation, but it contains rotation.

Green Strain

Ideally, we need a tensor to describe shape deformation only. Recall that SVD gives $\mathbf{F}=\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}$, where only $\mathbf{V}^{\mathrm{T}}$ and $\mathbf{D}$ are relevant to deformation.

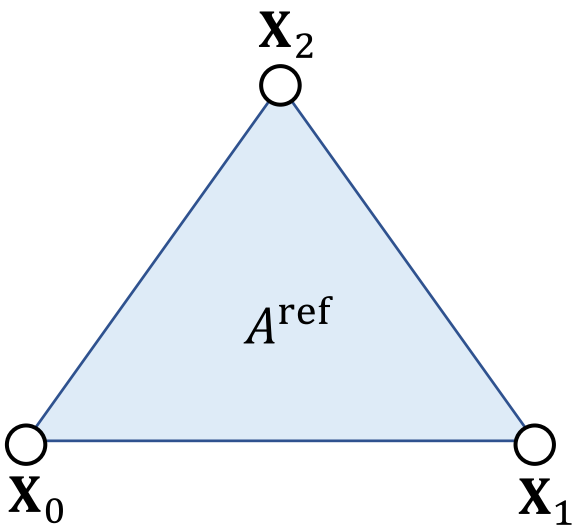


So we get rid of $\mathbf{U}$ as: $\mathbf{G}=\frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{I}\right)=\frac{1}{2}\left(\mathbf{V}\mathbf{D}^{2}\mathbf{V}^{\mathrm{T}}-\mathbf{I}\right)=\begin{bmatrix}\varepsilon_{uu} & \varepsilon_{uv}\\ \varepsilon_{uv} & \varepsilon_{vv}\end{bmatrix}$, *Green strain*.

- If no deformation, $\mathbf{G}=\mathbf{0}$; if deformation increases, $\|\mathbf{G}\|$ increases.
- Three deformation modes: ε_{uu}, ε_{vv} and ε_{uv}.
- $\mathbf{G}$ is rotation invariant: if additional rotation $\mathbf{R}$, then deformation gradient is $\mathbf{RF}$ but green strain is the same: $\mathbf{G}=\frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{F}-\mathbf{I}\right)=\frac{1}{2}\left(\mathbf{V}\mathbf{D}^{2}\mathbf{V}^{\mathrm{T}}-\mathbf{I}\right)$.

Strain Energy Density Function

Let $\mathbf{G}$ be the the green strain describing deformation. We consider the energy density per reference area as: $W(\mathbf{G})$.



Total energy:

Constant within the triangle

$$E = \int W(\mathbf{G})dA = A^{\text{ref}}W\boxed{(\varepsilon_{uu}, \varepsilon_{vv}, \varepsilon_{uv})}$$

The Saint Venant-Kirchhoff Model (StVK)

$$W(\varepsilon_{uu}, \varepsilon_{vv}, \varepsilon_{uv}) = \frac{\lambda}{2}(\varepsilon_{uu} + \varepsilon_{vv})^2 + \mu(\varepsilon_{uu}^2 + \varepsilon_{vv}^2 + 2\varepsilon_{uv}^2)$$

λ and μ are Lamé parameters.

$$\frac{\partial W}{\partial \mathbf{G}} = \begin{bmatrix} \frac{\partial W}{\partial \varepsilon_{uu}} & \frac{\partial W}{\partial \varepsilon_{uv}} \\ \frac{\partial W}{\partial \varepsilon_{uv}} & \frac{\partial W}{\partial \varepsilon_{vv}} \end{bmatrix} = \begin{bmatrix} 2\mu\varepsilon_{uu} + \lambda\varepsilon_{uu} & 2\mu\varepsilon_{uv} \\ 2\mu\varepsilon_{uv} & 2\mu\varepsilon_{vv} + \lambda\varepsilon_{vv} \end{bmatrix} = 2\mu\mathbf{G} + \lambda\,\mathrm{trace}(\mathbf{G})\mathbf{I} = \boxed{\mathbf{S}}$$

Second Piola-Kirchhoff stress tensor, something about force

Forces

Given everything we have, we can now calculate the forces.

$$\mathbf{f}_i = -\left(\frac{\partial E}{\partial \mathbf{x}_i}\right)^{\mathrm{T}} = -A^{\mathrm{ref}}\left(\frac{\partial W}{\partial \mathbf{x}_i}\right)^{\mathrm{T}} = -A^{\mathrm{ref}}\left(\boxed{\frac{\partial W}{\partial \varepsilon_{uu}}}\frac{\partial \varepsilon_{uu}}{\partial \mathbf{x}_i} + \boxed{\frac{\partial W}{\partial \varepsilon_{vv}}}\frac{\partial \varepsilon_{vv}}{\partial \mathbf{x}_i} + 2\boxed{\frac{\partial W}{\partial \varepsilon_{uv}}}\frac{\partial \varepsilon_{uv}}{\partial \mathbf{x}_i}\right)^{\mathrm{T}}$$

As in stress

Forces

Recall that,

$$\mathbf{F} = [\mathbf{x}_{10} \quad \mathbf{x}_{20}]\boxed{[\mathbf{r}_{10} \quad \mathbf{r}_{20}]^{-1}} = [\mathbf{x}_{10} \quad \mathbf{x}_{20}]\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = [a\mathbf{x}_{10} + c\mathbf{x}_{20} \quad b\mathbf{x}_{10} + d\mathbf{x}_{20}]$$

By definition,

$$\mathbf{G} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}) = \begin{bmatrix} \frac{1}{2}(a\mathbf{x}_{10} + c\mathbf{x}_{20})^{\mathrm{T}}(a\mathbf{x}_{10} + c\mathbf{x}_{20}) - \frac{1}{2} & \frac{1}{2}(a\mathbf{x}_{10} + c\mathbf{x}_{20})^{\mathrm{T}}(b\mathbf{x}_{10} + d\mathbf{x}_{20}) \\ \frac{1}{2}(a\mathbf{x}_{10} + c\mathbf{x}_{20})^{\mathrm{T}}(b\mathbf{x}_{10} + d\mathbf{x}_{20}) & \frac{1}{2}(b\mathbf{x}_{10} + d\mathbf{x}_{20})^{\mathrm{T}}(b\mathbf{x}_{10} + d\mathbf{x}_{20}) - \frac{1}{2} \end{bmatrix}$$

So:

$$\frac{\partial \varepsilon_{uu}}{\partial \mathbf{x}_1} = a(a\mathbf{x}_{10} + c\mathbf{x}_{20})^{\mathrm{T}} \qquad \frac{\partial \varepsilon_{vv}}{\partial \mathbf{x}_1} = b(b\mathbf{x}_{10} + d\mathbf{x}_{20})^{\mathrm{T}} \qquad \frac{\partial \varepsilon_{uv}}{\partial \mathbf{x}_1} = \frac{1}{2}a(b\mathbf{x}_{10} + d\mathbf{x}_{20})^{\mathrm{T}} + \frac{1}{2}b(a\mathbf{x}_{10} + c\mathbf{x}_{20})^{\mathrm{T}}$$

$$\frac{\partial \varepsilon_{uu}}{\partial \mathbf{x}_2} = c(a\mathbf{x}_{10} + c\mathbf{x}_{20})^{\mathrm{T}} \qquad \frac{\partial \varepsilon_{vv}}{\partial \mathbf{x}_2} = d(b\mathbf{x}_{10} + d\mathbf{x}_{20})^{\mathrm{T}} \qquad \frac{\partial \varepsilon_{uv}}{\partial \mathbf{x}_2} = \frac{1}{2}c(b\mathbf{x}_{10} + d\mathbf{x}_{20})^{\mathrm{T}} + \frac{1}{2}d(a\mathbf{x}_{10} + c\mathbf{x}_{20})^{\mathrm{T}}$$

Forces

Together, we get:

$$[\mathbf{f}_1 \quad \mathbf{f}_2] = -A^{\text{ref}} \underbrace{\mathbf{F}}_{\text{deformation gradient}} \overbrace{\mathbf{S}}^{\text{Second Piola-Kirchhoff Stress}} [\mathbf{r}_{10} \quad \mathbf{r}_{20}]^{-1}$$

$$= -A^{\text{ref}} [a\mathbf{x}_{10} + c\mathbf{x}_{20} \quad b\mathbf{x}_{10} + d\mathbf{x}_{20}] \begin{bmatrix} \dfrac{\partial W}{\partial \varepsilon_{uu}} & \dfrac{\partial W}{\partial \varepsilon_{uv}} \\ \dfrac{\partial W}{\partial \varepsilon_{uv}} & \dfrac{\partial W}{\partial \varepsilon_{vv}} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Implementations

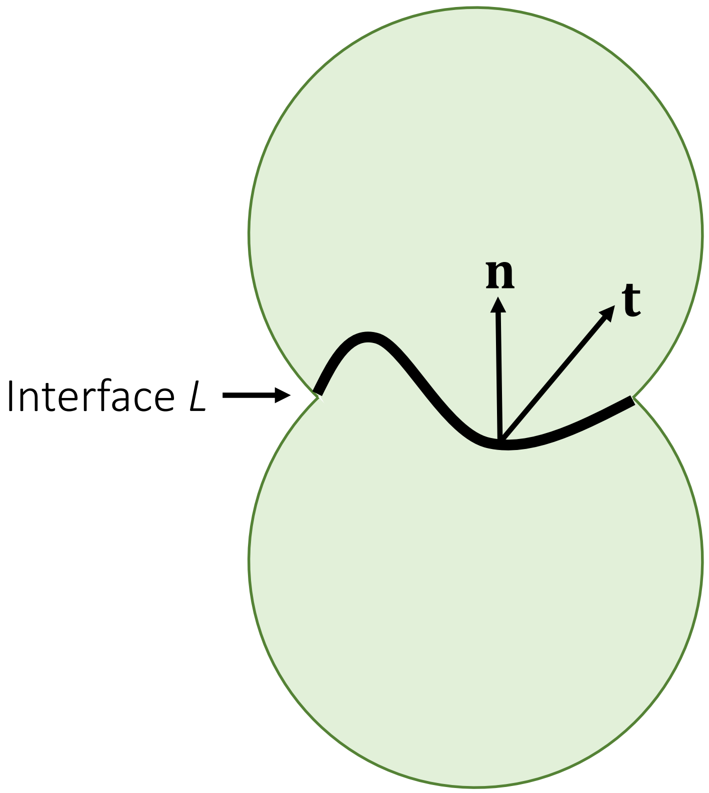
- More details?
  - Volino et al. 2009. *A simple approach to nonlinear tensile stiffness for accurate cloth simulation*. TOG
  - Only talks about cloth (2D reference -> 3D deformation)

- What about tetrahedron (3D reference -> 3D deformation)?
  - Same idea, but everything is now in 3D.
  - Deformation gradient $\mathbf{F} \in \mathbf{R}^{3\times 3}$
  - Green strain $\mathbf{G} \in \mathbf{R}^{3\times 3}$
  - Stress tensor $\mathbf{S} \in \mathbf{R}^{3\times 3}$
  - Forces $\mathbf{f}_i \in \mathbf{R}^3$
  - Messy code…

Finite Volume Method

Traction and Stress

First, let's consider traction $\mathbf{t}$: the internal force per unit length (area).



Total interface force:

$$\mathbf{f} = \oint_L \mathbf{t}\, dl$$

Stress tensor $\boldsymbol{\sigma}$ (interface normal -> traction):

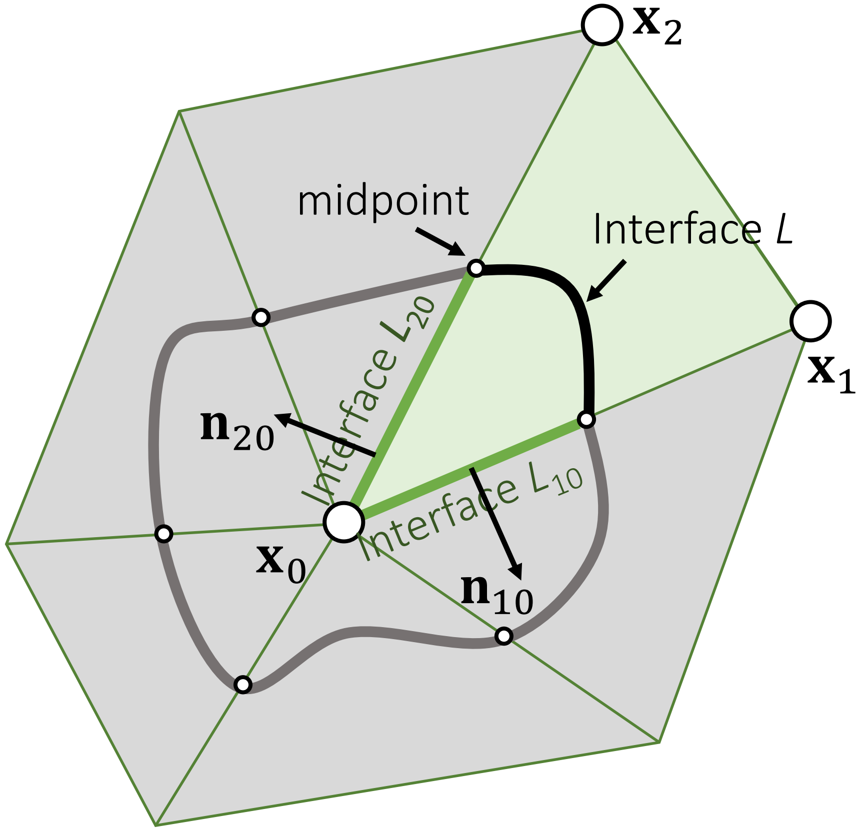
$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$$

So,

$$\mathbf{f} = \oint_L \boldsymbol{\sigma}\mathbf{n} dl$$

The Finite Volume Method

FVM considers force calculation in an integration perspective, not a differentiation perspective.



Force contributed by an element:

$$\mathbf{f}_0 = \oint_L \boldsymbol{\sigma}\mathbf{n}dl$$

Since $\boldsymbol{\sigma}$ is constant within the element,

$$\oint_L \boldsymbol{\sigma}\mathbf{n}dl + \oint_{L_{20}} \boldsymbol{\sigma}\mathbf{n}dl + \oint_{L_{10}} \boldsymbol{\sigma}\mathbf{n}dl = 0$$

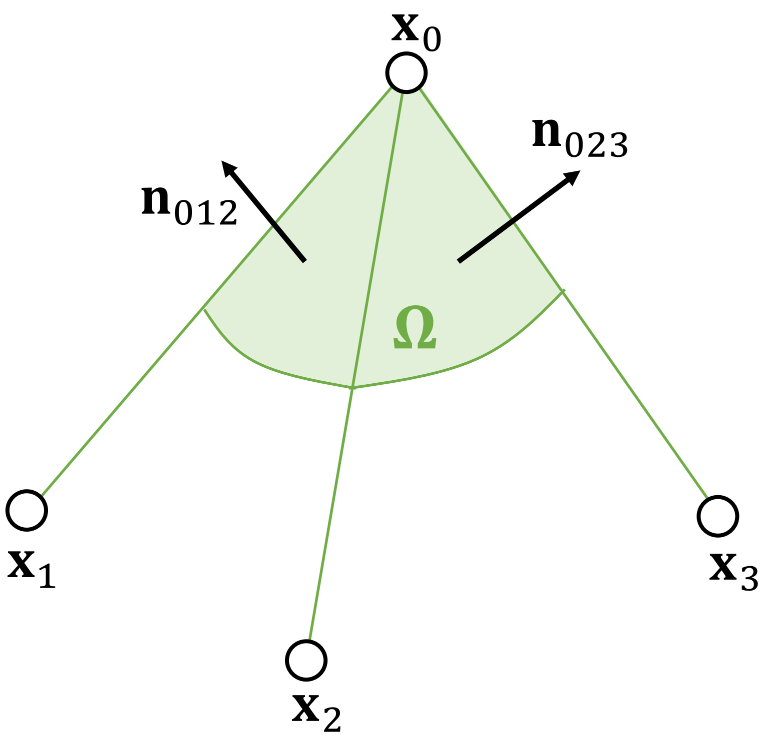
(Divergence Theorem)

We know the force is:

$$\mathbf{f}_0 = -\oint_{L_{20}} \boldsymbol{\sigma}\mathbf{n}dl - \oint_{L_{10}} \boldsymbol{\sigma}\mathbf{n}dl = -\boldsymbol{\sigma}\left(\frac{\|\mathbf{x}_{20}\|}{2}\mathbf{n}_{20} + \frac{\|\mathbf{x}_{10}\|}{2}\mathbf{n}_{10}\right)$$

The Finite Volume Method

In 3D, FVM works in the same way.



Force:

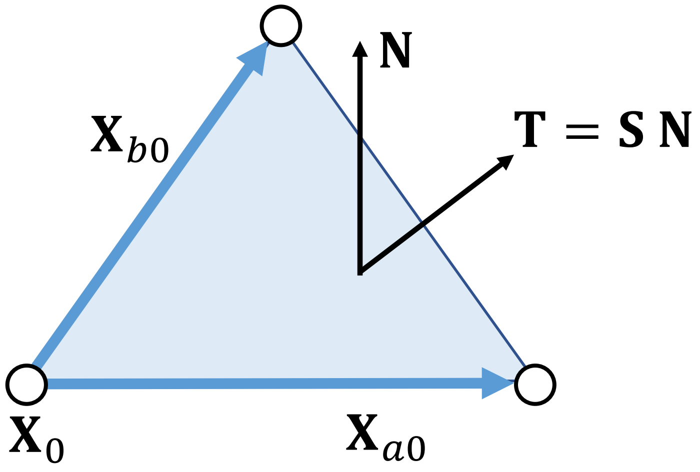
$$\mathbf{f}_0 = -\oint_{\Omega} \boldsymbol{\sigma}\mathbf{n}dA = -\boldsymbol{\sigma}\left(\frac{A_{012}}{3}\mathbf{n}_{012} + \frac{A_{023}}{3}\mathbf{n}_{023} + \frac{A_{031}}{3}\mathbf{n}_{031}\right)$$

$$= -\frac{\boldsymbol{\sigma}}{3}\left(\frac{\|\mathbf{x}_{10}\times\mathbf{x}_{20}\|}{2}\frac{\mathbf{x}_{10}\times\mathbf{x}_{20}}{\|\mathbf{x}_{10}\times\mathbf{x}_{20}\|} + \frac{\|\mathbf{x}_{20}\times\mathbf{x}_{30}\|}{2}\frac{\mathbf{x}_{20}\times\mathbf{x}_{30}}{\|\mathbf{x}_{20}\times\mathbf{x}_{30}\|} + \frac{\|\mathbf{x}_{30}\times\mathbf{x}_{10}\|}{2}\frac{\mathbf{x}_{30}\times\mathbf{x}_{10}}{\|\mathbf{x}_{30}\times\mathbf{x}_{10}\|}\right)$$

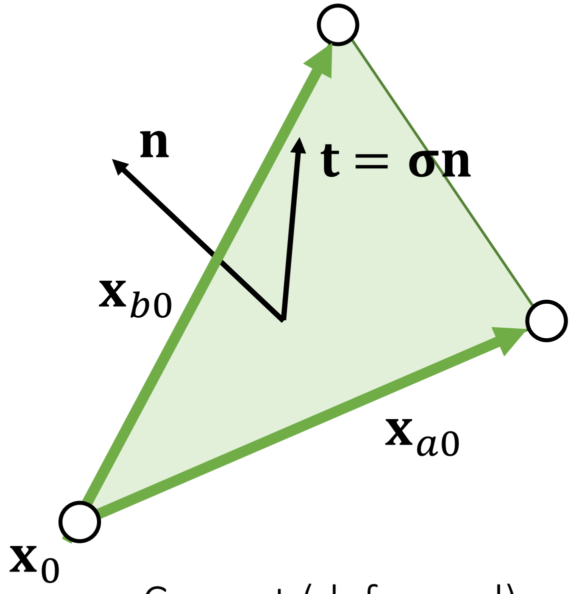
$$= -\frac{\boldsymbol{\sigma}}{6}(\mathbf{x}_{10}\times\mathbf{x}_{20} + \mathbf{x}_{20}\times\mathbf{x}_{30} + \mathbf{x}_{30}\times\mathbf{x}_{10})$$

This stress is not that stress

Although the use of stress tensor is the same: mapping from the interface normal to the traction, it can be defined by different configurations.



Reference (undeformed)



Current (deformed)

In FEM, we define the energy density W in the reference state. Therefore, this stress $\mathbf{S}$ is a mapping from the normal $\mathbf{N}$ to the traction $\mathbf{T}$, both in the reference state.

In FVM, we need $\boldsymbol{\sigma}$ to convert the normal into $\mathbf{t}$ for force calculation. Therefore, this stress assumes the normal $\mathbf{n}$ and the traction $\mathbf{t}$ are in the deformed state.

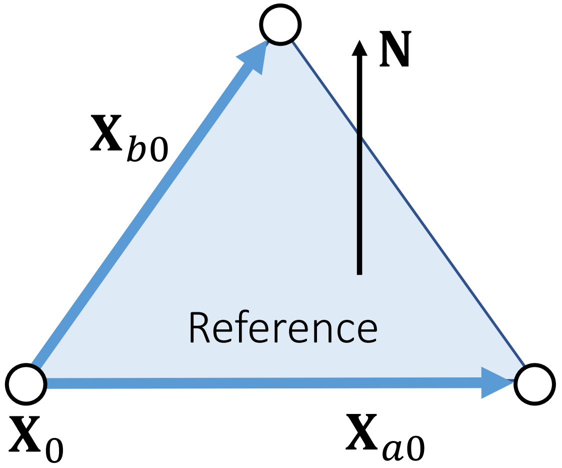
Different Stresses

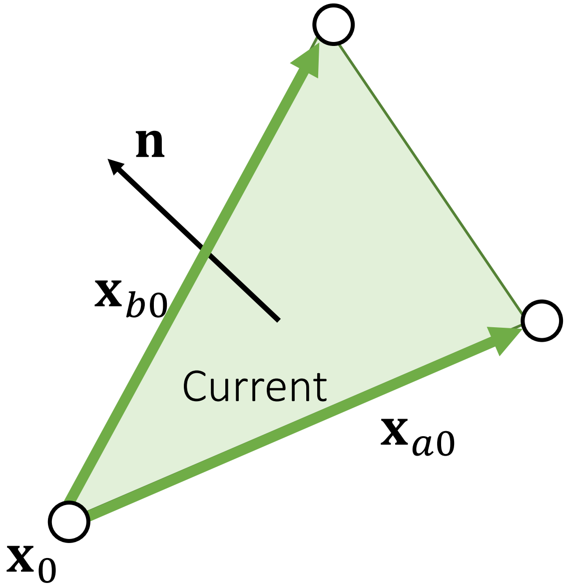
We can now have different stresses, serving the same purpose but in different forms.

Output \ Input	Interface normal $\mathbf{N}$ in the *reference* state (unformed)	Interface normal $\mathbf{n}$ in the *current* state (deformed)
Traction in the *reference* state (unformed)	Second Piola–Kirchhoff stress ($\boldsymbol{S}$)	
Traction in the *current* state (formed)	First Piola–Kirchhoff stress ($\mathbf{P}$)	Cauchy Stress ($\boldsymbol{\sigma}$)

$$\boxed{\mathbf{P} = \mathbf{FS}}$$

Area Weighted Normals





Now let's figure out the relationship between $A^{\mathrm{ref}}\mathbf{N}$ and $A\mathbf{n}$, the two area weighted normals.

$$\boxed{2A^{\mathrm{ref}}\mathbf{N} = \mathbf{X}_{a0} \times \mathbf{X}_{b0}}$$

$$
\begin{aligned}
2A\mathbf{n} = \mathbf{x}_{a0} \times \mathbf{x}_{b0} = \mathbf{F}\mathbf{x}_{a0} \times \mathbf{F}\mathbf{x}_{b0} &= \left(\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}\mathbf{X}_{a0}\right) \times \left(\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}\mathbf{X}_{b0}\right) \\
&= \mathbf{U}\left(\left(\mathbf{D}\mathbf{V}^{\mathrm{T}}\mathbf{X}_{a0}\right) \times \left(\mathbf{D}\mathbf{V}^{\mathrm{T}}\mathbf{X}_{b0}\right)\right) \\
&= \mathbf{U}\begin{bmatrix} d_1 d_2 & & \\ & d_0 d_2 & \\ & & d_0 d_1 \end{bmatrix}\left(\left(\mathbf{V}^{\mathrm{T}}\mathbf{X}_{a0}\right) \times \left(\mathbf{V}^{\mathrm{T}}\mathbf{X}_{b0}\right)\right) \\
&= \mathbf{U}\begin{bmatrix} d_1 d_2 & & \\ & d_0 d_2 & \\ & & d_0 d_1 \end{bmatrix}\mathbf{V}^{\mathrm{T}}\left(\mathbf{X}_{a0} \times \mathbf{X}_{b0}\right) \\
&= \det(\mathbf{F})\mathbf{F}^{-\mathrm{T}}\left(\mathbf{X}_{a0} \times \mathbf{X}_{b0}\right) = \det(\mathbf{F})\mathbf{F}^{-\mathrm{T}}\left(2A^{\mathrm{ref}}\mathbf{N}\right)
\end{aligned}
$$

Different Stresses

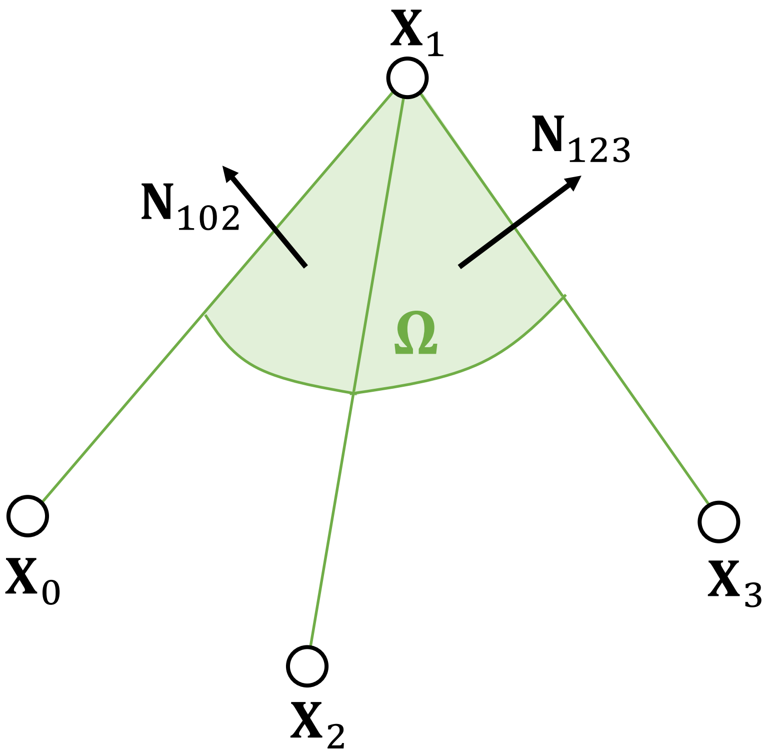
We can now have different stresses, serving the same purpose but in different forms.

Output \ Input	Interface normal $\mathbf{N}$ in the *reference* state (unformed)	Interface normal $\mathbf{n}$ in the *current* state (deformed)
Traction in the *reference* state (unformed)	Second Piola–Kirchhoff stress ($\mathbf{S}$)	
Traction in the *current* state (formed)	$\mathbf{P} = \mathbf{FS}$ First Piola–Kirchhoff stress ($\mathbf{P}$)	$\boldsymbol{\sigma} = \det^{-1}(\mathbf{F})\mathbf{FSF}^{\mathrm{T}}$ Cauchy Stress ($\boldsymbol{\sigma}$) $\boldsymbol{\sigma} = \det^{-1}(\mathbf{F})\mathbf{PF}^{\mathrm{T}}$
Force in the *current* state	$\mathbf{f} = -\frac{1}{3}\sum A^{\mathrm{ref}}\mathbf{PN}$	$\mathbf{f} = -\frac{1}{3}\sum A\boldsymbol{\sigma}\mathbf{n}$

$$A\mathbf{n} = \det(\mathbf{F})\mathbf{F}^{-\mathrm{T}}(A^{\mathrm{ref}}\mathbf{N})$$

The Finite Volume Method

Anyway, now we have the following formula for force calculation.



Force: $\mathbf{f}_i = -\frac{\mathbf{P}}{6}\boxed{\mathbf{b}_i}$

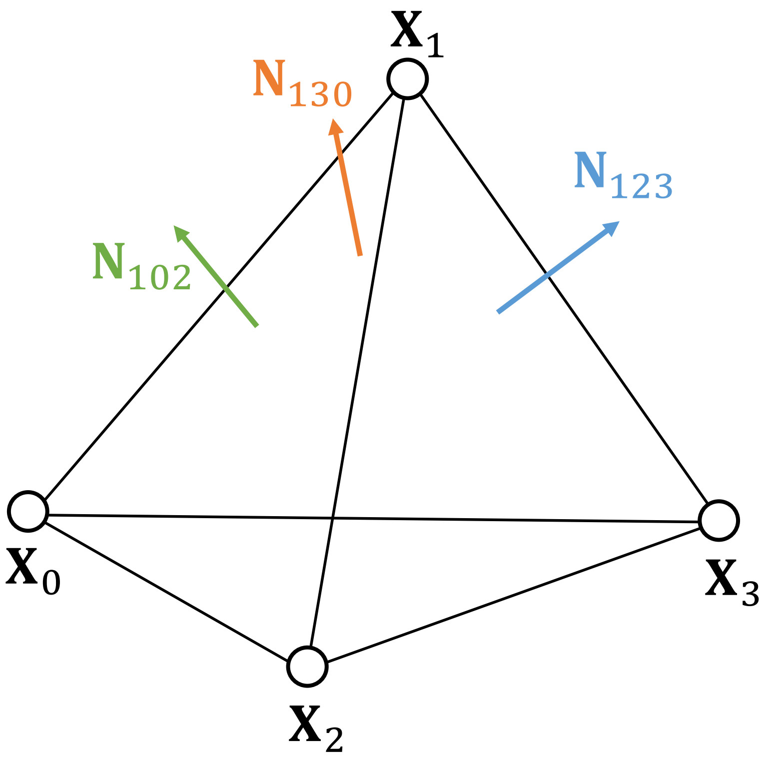
$$\mathbf{b}_1 = -\mathbf{FS}\,\frac{1}{6}\left(\boxed{\mathbf{X}_{01}\times\mathbf{X}_{21}+\mathbf{X}_{21}\times\mathbf{X}_{31}+\mathbf{X}_{31}\times\mathbf{X}_{01}}\right)$$

Constant, can be precomputed

Second Piola–Kirchhoff stress:

$\mathbf{S} = \frac{\partial W}{\partial \mathbf{G}}$, as in previous FEM formulation

The Finite Volume Method



Think about more…

$$\mathbf{X}_{10}^{\mathrm{T}}(\mathbf{X}_{01} \times \mathbf{X}_{21} + \mathbf{X}_{21} \times \mathbf{X}_{31} + \mathbf{X}_{31} \times \mathbf{X}_{01}) = \mathbf{X}_{10}^{\mathrm{T}}(\mathbf{X}_{21} \times \mathbf{X}_{31})$$

$$= \mathbf{X}_{01}^{\mathrm{T}}(\mathbf{X}_{31} \times \mathbf{X}_{21}) = 6Vol$$

Meanwhile,

$$\mathbf{X}_{20}^{\mathrm{T}}(\mathbf{X}_{01} \times \mathbf{X}_{21} + \mathbf{X}_{21} \times \mathbf{X}_{31} + \mathbf{X}_{31} \times \mathbf{X}_{01}) =$$

$$\mathbf{X}_{20}^{\mathrm{T}}(\mathbf{X}_{20} \times \mathbf{X}_{10} + \mathbf{X}_{20} \times \mathbf{X}_{31}) = 0$$

$$\mathbf{X}_{30}^{\mathrm{T}}(\mathbf{X}_{01} \times \mathbf{X}_{21} + \mathbf{X}_{21} \times \mathbf{X}_{31} + \mathbf{X}_{31} \times \mathbf{X}_{01}) =$$

$$\mathbf{X}_{30}^{\mathrm{T}}(\mathbf{X}_{21} \times \mathbf{X}_{30} + \mathbf{X}_{10} \times \mathbf{X}_{30}) = 0$$

Therefore,

We get:

$$[\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]^{\mathrm{T}}\mathbf{b}_1 = [\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]^{\mathrm{T}}(\mathbf{X}_{01} \times \mathbf{X}_{21} + \mathbf{X}_{21} \times \mathbf{X}_{31} + \mathbf{X}_{31} \times \mathbf{X}_{01}) = \begin{bmatrix} 6Vol \\ 0 \\ 0 \end{bmatrix}$$

$$[\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]^{\mathrm{T}}\mathbf{b}_2 = \begin{bmatrix} 0 \\ 6Vol \\ 0 \end{bmatrix}$$

$$[\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]^{\mathrm{T}}\mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 6Vol \end{bmatrix}$$

$$[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3] = 6Vol[\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]^{-1}$$

$$= \frac{1}{\det([\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]^{-1})}[\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]^{-1}$$

A Quick Summary (cont.)

A FEM/FVM Framework

$$\mathbf{D}_\mathrm{m} = [\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]$$

Deformation gradient → $$\mathbf{F} = [\mathbf{x}_{10} \quad \mathbf{x}_{20} \quad \mathbf{x}_{30}]\mathbf{D}_\mathrm{m}^{-1}$$

Green strain → $$\mathbf{G} = \frac{1}{2}\left(\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{I}\right)$$

Second PK Stress → $$\mathbf{P} = \mathbf{F}\frac{\partial W}{\partial \mathbf{G}}$$

Forces → $$[\mathbf{f}_1 \quad \mathbf{f}_2 \quad \mathbf{f}_3] = -\frac{1}{6\det(\mathbf{D}_\mathrm{m}^{-1})}\mathbf{P}\mathbf{D}_\mathrm{m}^{-\mathrm{T}}$$

$$\mathbf{f}_0 = -\mathbf{f}_1 - \mathbf{f}_2 - \mathbf{f}_3$$

After-Class Reading

Teran et al. 2003. *Finite Volume Methods for the Simulation of Skeleton Muscles*. SCA.

Eurographics/SIGGRAPH Symposium on Computer Animation (2003)
D. Breen, M. Lin (Editors)

Finite Volume Methods for the Simulation of Skeletal Muscle

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Abstract

Since it relies on a geometrical rather than a variational framework, many find the finite volume method (FVM) more intuitive than the finite element method (FEM). We show that the FVM allows one to interpret the stress inside a tetrahedron as a simple "multidimensional force" pushing on each face. Moreover, this interpretation leads to a heuristic method for calculating the force on each node, which is as simple to implement and comprehend as masses and springs. In the finite volume spirit, we also present a geometric rather than interpolating function definition of strain. We use the FVM and a quasi-incompressible, transversely isotropic, hyperelastic constitutive model to simulate contracting muscle tissue. B-spline solids are used to model fiber directions, and the muscle activation levels are derived from key frame animations.

Categories and Subject Descriptors (according to ACM CCS): I.3.7 [Computer Graphics]: Animation; I.3.5 [Computer Graphics]: Physically based modeling

1. Introduction

The pioneering work of Lasseter[18] on applying the principles of traditional animation to computer graphics emphasizes squash and stretch, timing, anticipation, follow through, arcs, and secondary action which all appeal to the use of physics based animation techniques. A variety of authors have worked to incorporate ideas such as these into their animations, e.g. Neff and Fiume[24] incorporated tension and relaxation into the animation of an articulated skeleton. Moreover, when considering the difficulties, such as collapsing elbows[19], associated with applying free form deformations[28] or related techniques to shape animation, one draws the conclusion that physics based simulation of muscle and fatty tissue should be the ultimate goal. Unfortunately, progress toward this goal has been rather slow due to the high cost of FEM and the poor quality of volumetric mass-spring models.

Significant effort has been placed into accelerating FEM calculations including recent approaches that precompute and cache various quantities[23], modal analysis[16], and approximations to local rotations[22]. In spite of significant effort into alternative (and related) methods for the robust simulation of deformable bodies, FVM has been largely ignored. Two aspects of FVM make it attractive. First, it has a firm basis in geometry as opposed to the FEM variational setting. This not only increases the level of intuition and the simplicity of implementation, but also increases the likelihood of aggressive (but robust) optimization and control. Second, there is a large community of researchers using these methods to model solid materials subject to very high deformations. For example, Caramana and Shashkov[5] used the FVM with subcell pressures to eliminate the artificial hour-glass type motions that can arise in materials with strongly diagonally dominant stress tensors, such as incompressible biological materials.

FEM has many attractive features that make it appealing to the engineering community, e.g. a solid theoretical framework for proving theorems and the ability to extend it to higher order accuracy. However, in graphics, visual realism is more important than the convergence rate, and thus the focus is on simulating a large number of cheap first order accurate elements rather than fewer more expensive higher order accurate elements. In fact, the same can be said for many engineering applications where quasi-static and other

After-Class Reading (Optional)

Volino et al. 2009. *A Simple Approach to Nonlinear Tensile Stiffness for Accurate Cloth Simulation*. TOG.

A simple approach to nonlinear tensile stiffness for accurate cloth simulation

Pascal Volino, Nadia Magnenat-Thalmann, François Faure

Hyperelastic Models

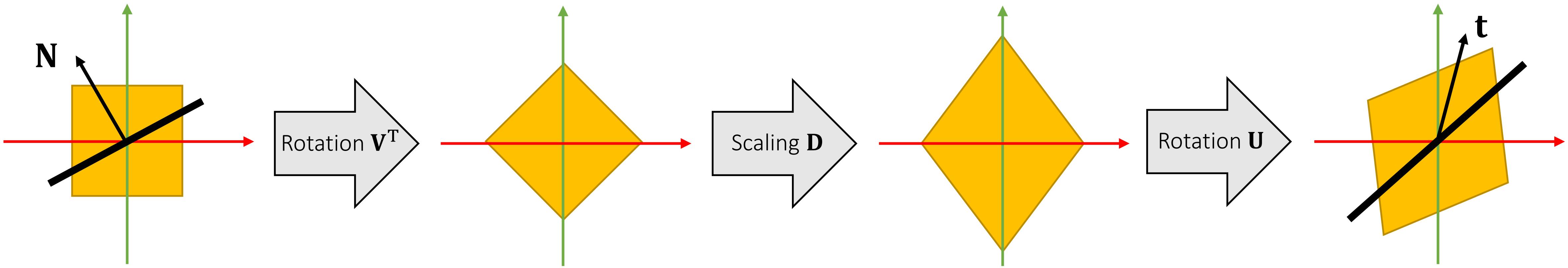
First Piola–Kirchhoff stress

We treat the first Piola–Kirchhoff stress tensor $\mathbf{P}$ as a function of deformation gradient $\mathbf{F}$:

$$\mathbf{f}_0 = -\frac{\mathbf{P}(\mathbf{F})}{6}(\mathbf{X}_{10} \times \mathbf{X}_{20} + \mathbf{X}_{20} \times \mathbf{X}_{30} + \mathbf{X}_{30} \times \mathbf{X}_{10})$$

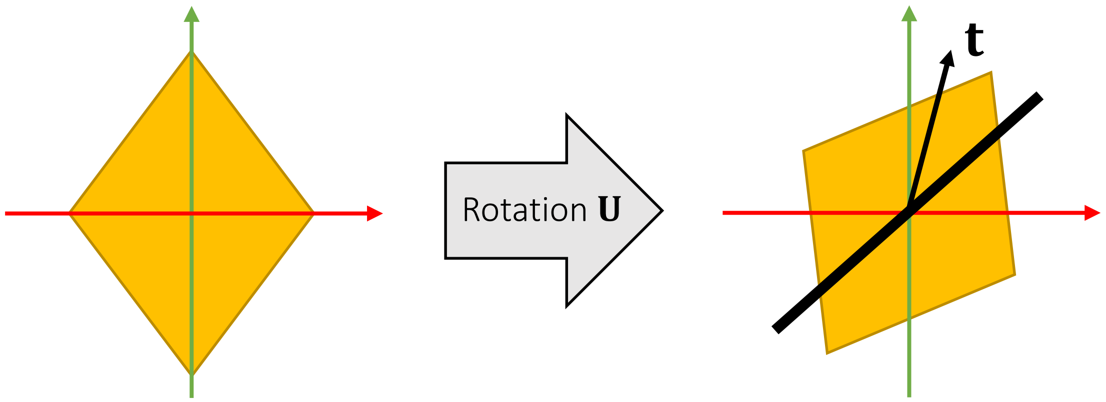
It converts an interface normal $\mathbf{N}$ in the reference state to a traction $\mathbf{t}$ in the deformed state.

$$\boxed{\mathbf{t} = \mathbf{P}(\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}})\mathbf{N}}$$



Rotation-Invariance

The stress tensor $\mathbf{P}$ is rotation-invariant to $\mathbf{U}$:

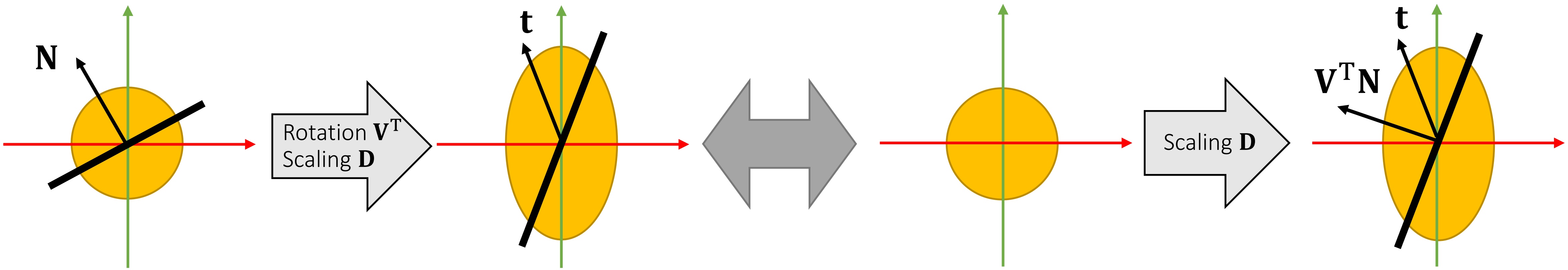


$$\mathbf{t} = \mathbf{U}\mathbf{P}(\mathbf{D}\mathbf{V}^{\mathrm{T}})\mathbf{N} \qquad \mathbf{t} = \mathbf{P}(\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}})\mathbf{N}$$

$$\boxed{\mathbf{P}\left(\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}\right) = \mathbf{U}\mathbf{P}(\mathbf{D}\mathbf{V}^{\mathrm{T}})}$$

Isotropic Materials

The stress tensor $\mathbf{P}$ is rotation-invariant to $\mathbf{U}$:



$$\mathbf{t} = \mathbf{P}\left(\mathbf{D}\mathbf{V}^{\mathrm{T}}\right)\mathbf{N} \qquad \mathbf{t} = \mathbf{P}(\mathbf{D})\mathbf{V}^{\mathrm{T}}\mathbf{N}$$

$$\boxed{\mathbf{P}\left(\mathbf{D}\mathbf{V}^{\mathrm{T}}\right) = \mathbf{P}(\mathbf{D})\mathbf{V}^{\mathrm{T}}}$$

Isotropic Materials

Together, we claim: $\mathbf{P}(\mathbf{F}) = \mathbf{P}(\mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}) = \mathbf{U}\mathbf{P}(\underbrace{\lambda_0, \lambda_1, \lambda_2}_{\text{Principal stretches: the singular values of } \mathbf{F}})\mathbf{V}^{\mathrm{T}}$

In many literatures, people parameterize $\mathbf{P}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}})$ by principal invariants, for:

$$\boxed{I_{\mathbf{C}} = \mathrm{trace}(\mathbf{C}) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2}$$

$$\boxed{III_{\mathbf{C}} = \det(\mathbf{C}^2) = \lambda_0^4 + \lambda_1^4 + \lambda_2^4}$$

$$\boxed{II_{\mathbf{C}} = \frac{1}{2}\left(\mathrm{trace}^2(\mathbf{C}) - \mathrm{trace}(\mathbf{C}^2)\right) = \lambda_0^2\lambda_1^2 + \lambda_0^2\lambda_2^2 + \lambda_1^2\lambda_2^2}$$

$\mathbf{C} = \mathbf{U}^{\mathrm{T}}\mathbf{U}$ is the right Cauchy-Green deformation tensor.

Isotropic Models

The Saint Venant-Kirchhoff model (StVK):

$$W = \frac{s_0}{2}(I_\mathbf{C} - 3)^2 + \frac{s_1}{4}(II_\mathbf{C} - 2I_\mathbf{C} + 3)$$

The neo-Hookean model:

$$W = s_0\left(III_\mathbf{C}^{-1/3} I_\mathbf{C} - 3\right) + s_1\left(III_\mathbf{C}^{-1/2} - 1\right)$$

Against shearing

Against bulky change (volume change)

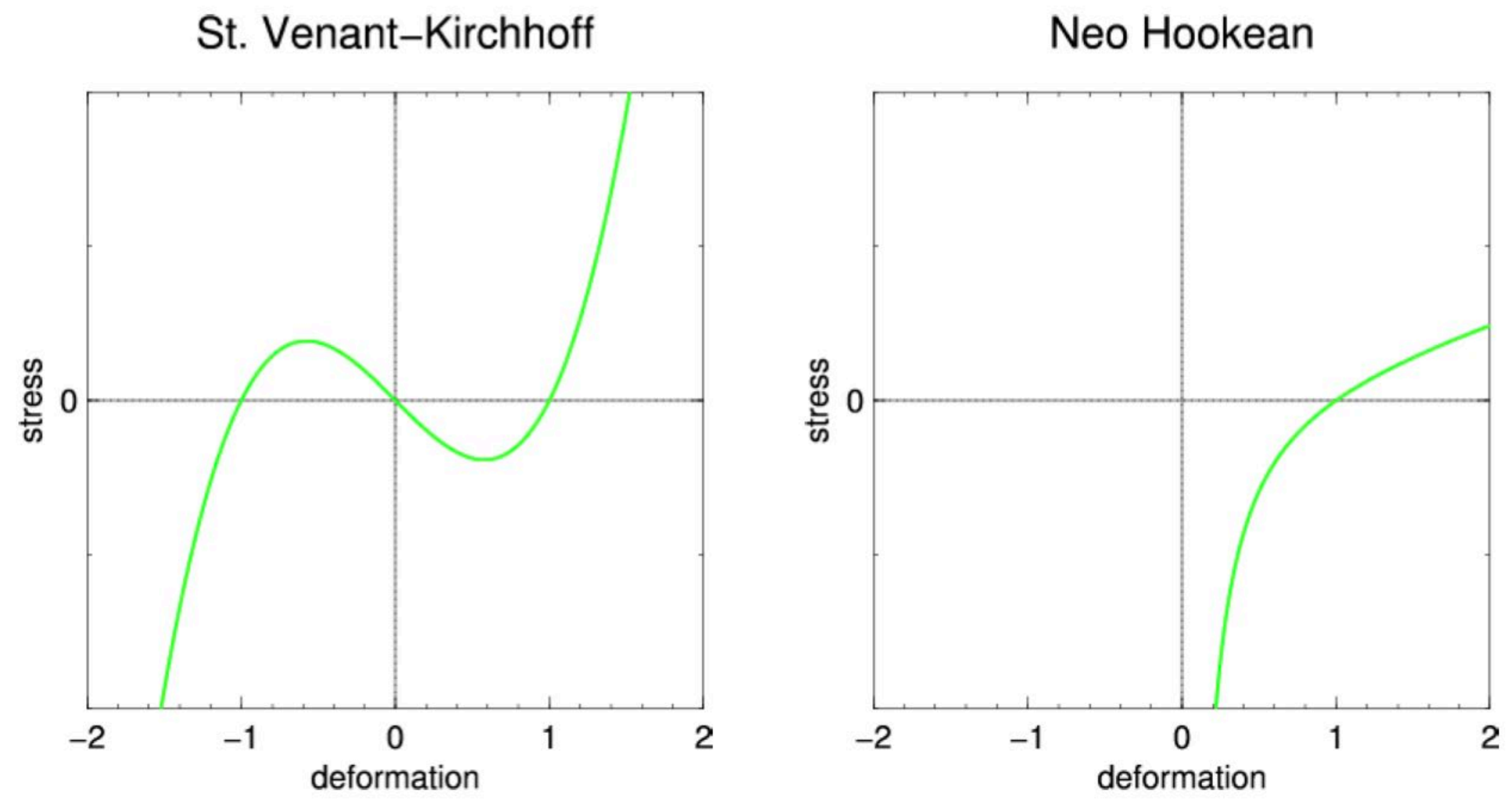
The Mooney-Rivlin model:

$$W = s_0\left(III_\mathbf{C}^{-1/3} I_\mathbf{C} - 3\right) + s_1\left(III_\mathbf{C}^{-1/2} - 1\right) + s_2\left(\frac{1}{2} III_\mathbf{C}^{-2/3}\left(I_\mathbf{C}^2 - II_\mathbf{C}\right) - 3\right)$$

The Fung model:

$$W = s_0\left(III_\mathbf{C}^{-1/3} I_\mathbf{C} - 3\right) + s_1\left(III_\mathbf{C}^{-1/2} - 1\right) + s_2\left(e^{s_3\left(III_\mathbf{C}^{-1/3} I_\mathbf{C} - 3\right)} - 1\right)$$

The Limitation of StVK



Irving et al. 2004. *Invertible Finite Elements For Robust Simulation of Large Deformation. SCA*

Poisson Effects

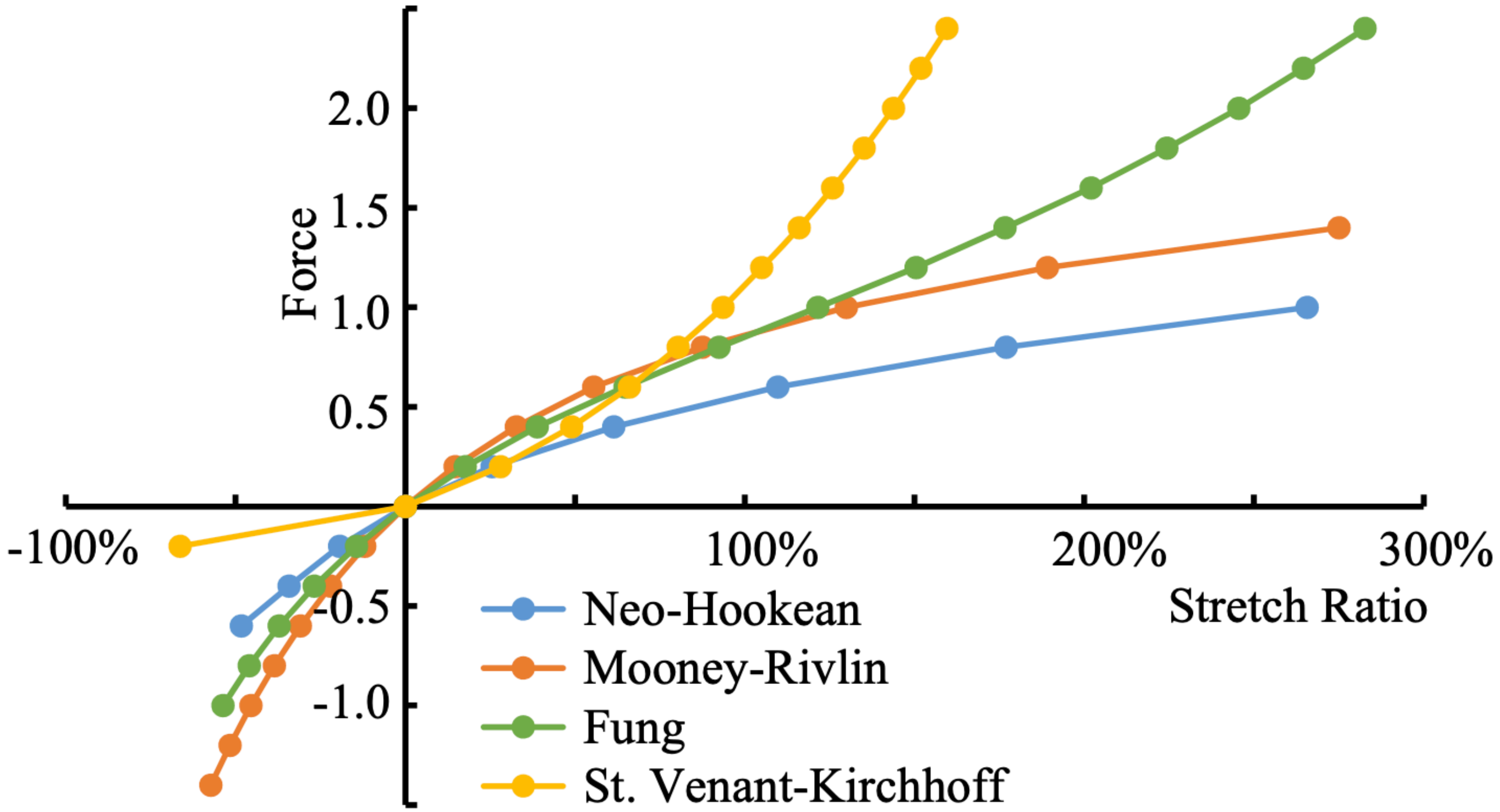


Figure 11: *The force-displacement curves generated by the box example. These curves are consistent with the stress-strain relationships of the underlying hyperelastic models.*

Model Behaviors

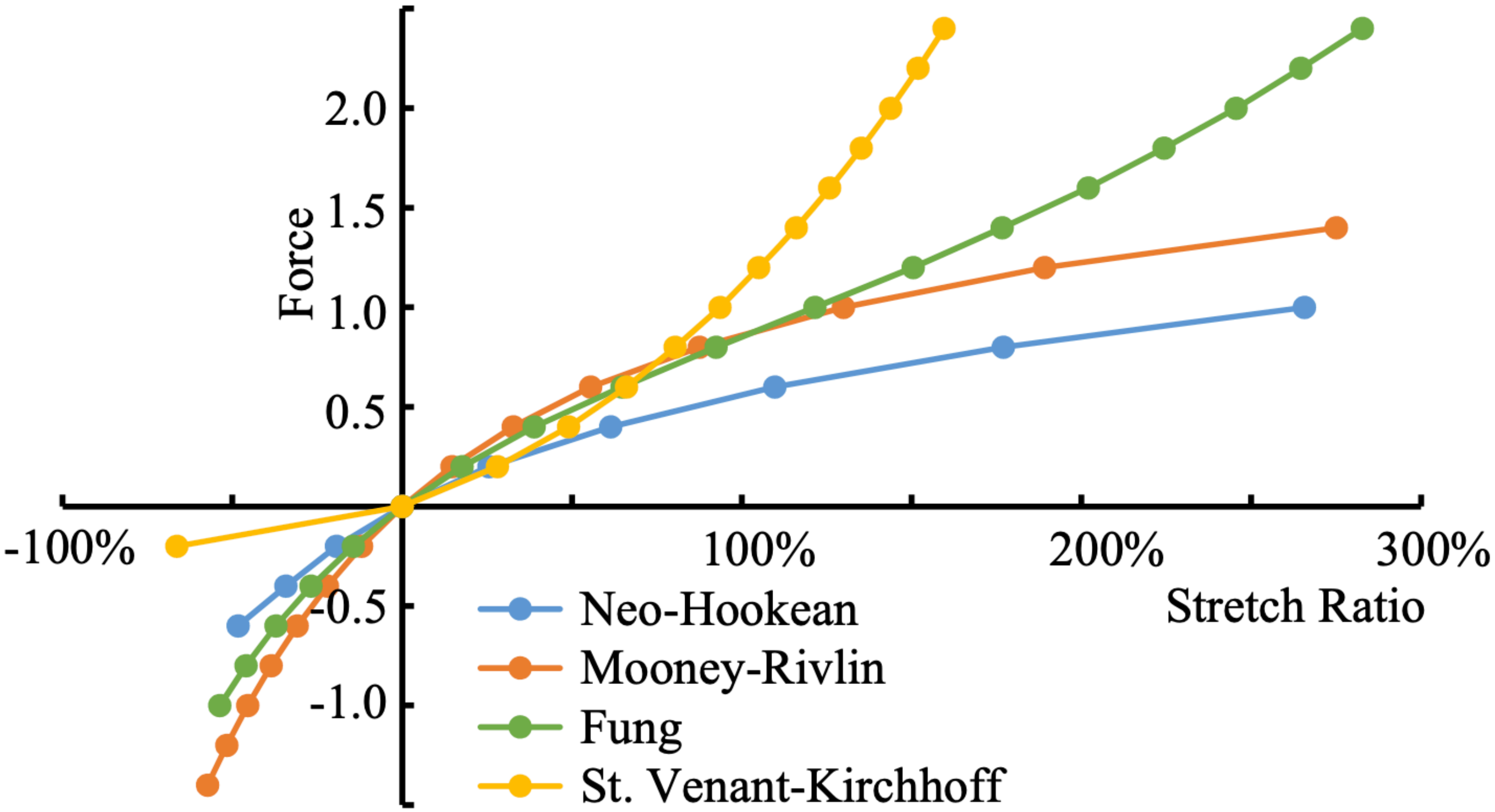


Figure 11: *The force-displacement curves generated by the box example. These curves are consistent with the stress-strain relationships of the underlying hyperelastic models.*

Isotropic Materials

It can then be shown [Irving et al. 2004] that:

$$\mathbf{P}(\lambda_0, \lambda_1, \lambda_2) = \begin{bmatrix} \frac{\partial W}{\partial \lambda_0} & & \\ & \frac{\partial W}{\partial \lambda_1} & \\ & & \frac{\partial W}{\partial \lambda_2} \end{bmatrix}$$

And we compute the first Piola-Kirchhoff stress as:

$$\mathbf{P} = \mathbf{U}\mathbf{P}(\lambda_0, \lambda_1, \lambda_2)\mathbf{V}^{\mathrm{T}}$$

A Quick Summary (cont.)

A FEM/FVM Framework

$$\mathbf{D}_\mathrm{m} = [\mathbf{X}_{10} \quad \mathbf{X}_{20} \quad \mathbf{X}_{30}]$$

Deformation gradient → $$\mathbf{F} = [\mathbf{x}_{10} \quad \mathbf{x}_{20} \quad \mathbf{x}_{30}]\mathbf{D}_\mathrm{m}^{-1}$$

Principal stretches $\mathbf{\Lambda}$ → $$[\mathbf{U} \quad \mathbf{\Lambda} \quad \mathbf{V}^\mathrm{T}] = \mathrm{svd}(\mathbf{F})$$

Second PK Stress → $$\mathbf{P} = \mathbf{U}\mathrm{diag}\left(\frac{\partial W}{\partial \lambda_0}, \frac{\partial W}{\partial \lambda_1}, \frac{\partial W}{\partial \lambda_2}\right)\mathbf{V}^\mathrm{T}$$

Forces → $$[\mathbf{f}_1 \quad \mathbf{f}_2 \quad \mathbf{f}_3] = -\frac{1}{6\det(\mathbf{D}_\mathrm{m}^{-1})}\mathbf{P}\mathbf{D}_\mathrm{m}^{-\mathrm{T}}$$

$$\mathbf{f}_0 = -\mathbf{f}_1 - \mathbf{f}_2 - \mathbf{f}_3$$

A Summary For the Day

- FEM uses the derivates of the strain energy function to obtain the force.
- FVM uses the integral of the interface traction to obtain the force.
- The two approaches lead to the identical outcome, in two different formulations. FVM is more implementation friendly.
- Hyperelastic models define the strain energy function by principal stretches, i.e., the singular values of the deformation gradient.
- For isotropic materials, we can simply the stress through diagonalization.