根据FVM有

$$egin{aligned} \mathbf{P} &= \mathbf{F} rac{\partial W}{\partial \mathbf{G}} \ &= \mathbf{F} \mathbf{S} \ &= \mathbf{U} \mathbf{D} \mathbf{V}^T \mathbf{S} \ &= \mathbf{U} \mathbf{D} \mathbf{V}^T (2 \mu \mathbf{G} + \lambda \mathrm{tr}(\mathbf{G}) \mathbf{I}) \end{aligned}$$

其中

$$\mathbf{G} = rac{1}{2}(\mathbf{V}\mathbf{D}^2\mathbf{V}^T - \mathbf{I}) \ = rac{1}{2}\mathbf{V}(\mathbf{D}^2 - \mathbf{I})\mathbf{V}^T$$

把G带入P有

$$\begin{split} \mathbf{P} &= \mathbf{U}\mathbf{D}\mathbf{V}^T(2\mu\mathbf{G} + \lambda\mathrm{tr}(\mathbf{G})\mathbf{I}) \\ &= 2\mu\mathbf{U}\mathbf{D}\mathbf{V}^T\mathbf{G} + \lambda\mathrm{tr}(\mathbf{G})\mathbf{U}\mathbf{D}\mathbf{V}^T \\ &= \mu\mathbf{U}\mathbf{D}(\mathbf{D}^2 - \mathbf{I})\mathbf{V}^T + \frac{\lambda}{2}\mathrm{tr}(\mathbf{D}^2 - \mathbf{I})\mathbf{U}\mathbf{D}\mathbf{V}^T \\ &= \mathbf{U}\bigg(\mu\mathbf{D}(\mathbf{D}^2 - \mathbf{I}) + \frac{\lambda}{2}\mathrm{tr}(\mathbf{D}^2 - \mathbf{I})\mathbf{D}\bigg)\mathbf{V}^T \end{split}$$

另一方面,根据SVD有

$$\mathbf{P} = \mathbf{U} \operatorname{diag}\!\left(rac{\partial W}{\partial \lambda_i}
ight) \mathbf{V}^T$$

因此只需证明

$$\mu \mathbf{D}(\mathbf{D}^2 - \mathbf{I}) + rac{\lambda}{2} \mathrm{tr}(\mathbf{D}^2 - \mathbf{I}) \mathbf{D} = \mathrm{diag}igg(rac{\partial W}{\partial \lambda_i}igg)$$

以对角矩阵第一项 λ_0 为例,等式左边有

$$\mu\lambda_0(\lambda_0^2-1)+rac{\lambda}{2}\lambda_0(\lambda_0^2+\lambda_1^2+\lambda_2^2-3)$$

而右边有

$$rac{\partial W}{\partial \lambda_0} = \left[egin{array}{ccc} rac{\partial I}{\partial \lambda_0} & rac{\partial II}{\partial \lambda_0} & rac{\partial III}{\partial \lambda_0} \end{array}
ight] \left[egin{array}{c} rac{\partial W}{\partial I} \ rac{\partial W}{\partial II} \ rac{\partial W}{\partial III} \end{array}
ight]$$

对于StVK模型有

$$egin{bmatrix} rac{\partial I}{\partial \lambda_0} & rac{\partial II}{\partial \lambda_0} & rac{\partial III}{\partial \lambda_0} \end{bmatrix} = egin{bmatrix} 2\lambda_0 & 4\lambda_0^3 & * \end{bmatrix} \ egin{bmatrix} rac{\partial W}{\partial I} \ rac{\partial W}{\partial II} \ rac{\partial W}{\partial II} \end{bmatrix} = egin{bmatrix} s_0(I-3) - rac{s_1}{2} \ rac{s_1}{4} \ 0 \end{bmatrix}$$

因此

$$egin{aligned} rac{\partial W}{\partial \lambda_0} &= 2\lambda_0[s_0(I-3) - rac{s_1}{2}] + s_1\lambda_0^3 \ &= 2s_0\lambda_0(I-3) + s_1\lambda_0(\lambda_0^2 - 1) \ &= 2s_0\lambda_0(\lambda_0^2 + \lambda_1^2 + \lambda_2^2 - 3) + s_1\lambda_0(\lambda_0^2 - 1) \end{aligned}$$

因此只需令拉梅参数 $\lambda=4s_0$, $\mu=s_1$ 即可使左右两边相等。对于对角矩阵的另外两项可以类似地证明。