



中国科学技术大学
University of Science and Technology of China

GAMES 301（专题课程）

曲面参数化

刘利刚 陈仁杰 傅孝明 方清
中国科学技术大学

Welcome

欢迎来到GAMES 301 !

創寰宇學府
育天下英才
嚴濟慈題
一九八八年五月



中国科学技术大学
University of Science and Technology of China

GAMES 301：第1讲

曲面参数化介绍

刘利刚
中国科学技术大学

提纲



1. GAMES在线课程介绍
2. GAMES 301课程信息
3. 曲面参数化及其应用
4. 曲面参数化的表达及度量
5. 曲面参数化的主要方法
6. 广义参数化及数据降维

1

GAMES在线课程介绍

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GAMES简介



GAMES: Graphics And Mixed Environment Symposium
计算机图形学与混合现实在线平台

- 主页: <http://games-cn.org>
- 宗旨: 图形学及相关领域交流的华人在线社区
- 隶属: 中国计算机学会计算机辅助设计与图形学专委会
- 每周在线学术报告: 7大方向, 250+期
- 高质量在线课程 (稍后详细介绍)
- 其他活动: 年度线下会议、学术沙龙、名企大咖面对面...
- 微信群: 23个群 (11000+人)

加入微信群的方法: 在微信中搜索微信号, 加gameswebinar为好友;
然后回复“GAMES”即可获取群聊邀请。

丰富的学习资料！！
总播放量: 300+万



GAMES 组织架构



GAMES 指导委员会委员

- 鲍虎军 (浙江大学)
- 陈宝权 (北京大学)
- 陈 为 (浙江大学)
- 冯结青 (浙江大学)
- 胡事民 (清华大学)
- 刘利刚 (中国科学技术大学)
- 马利庄 (上海交通大学)
- 童 欣 (微软亚洲研究院)
- 汪国平 (北京大学)

GAMES 执行委员会

- 2017年6月1日, GAMES 线上平台创建并启动
 - 主席: 刘利刚 (中国科学技术大学), 线上活动运营负责人
- 2019年1月1日, GAMES 执委会:
 - 主席: 刘利刚 (中国科学技术大学)
 - 秘书长: 周晓巍 (浙江大学), 线上活动运营负责人
 - 常务委员会: 15人
- 2021年10月1日, GAMES 执委会:
 - 主席: 周晓巍 (浙江大学)
 - 秘书长: 胡瑞珍 (深圳大学), 线上活动运营负责人
 - 常务委员会: 14人
 - 执行委员会: 39人

GAMES 组织架构（第三届执委会）



几何	徐凯	曹明伟	视觉	周晓巍	崔兆鹏	混合现实	杨旭波	龚江涛
		陈仁杰			李瑞辉			郭诗辉
		高林			李镇			黄程韦
	胡瑞珍	郭建伟		韩晓光	廖依伊		王莉莉	潘烨
		连宙辉			林迪			汪淼
		马锐			刘昊			邢冠宇
		孟敏		朱俊彦	晏轶超	渲染	王锐	过洁
		唐可可			易冉			彭袆帆
		徐豪			夏佳志			孙启霖
	模拟与动画	朱晨阳		陈思明	陆旻	CAD/CAE/CAM	徐岗	曹娟
		许威威			马昱欣			陈中贵
		刘利斌			王叙萌		贾晓红	王伟明
		王笑琨			曾伟			邹强
		杨垠						

GAMES 在线课程



基础课程 (1**)	GAMES101	现代计算机图形学入门	闫令琪	高级课程 (2**)	GAMES201	高级物理引擎实战指南	胡渊鸣			
	GAMES102	几何建模与处理	刘利刚		GAMES202	高质量实时渲染	闫令琪			
	GAMES103	基于物理的计算机动画入门	王华民		GAMES203	三维重建和理解	黄其兴			
	GAMES104	现代游戏引擎	王希		GAMES204 计算成像		孙启霖			
	GAMES105	计算机角色动画	刘利斌				彭祎帆			
专题课程 (3**)	New		GAMES 301 曲面参数化, GAMES 302、303 (规划中)							
开发课程 (4**)	New		GAMES 401、402 (规划中)							



GAMES 301课程信息

創寰宇學府
育天下英才
嚴濟慈題
一九八八年五月

GAMES 301 专题课程：曲面参数化



GAMES 301



**刘利刚 & 陈仁杰
傅孝明 & 方 清**

中国科学技术大学

2022年10月8日起 | 北京时间每周六、周日上午10:00-11:30 | WEBINAR.GAMES-CN.ORG

▲ GAMES301：曲面参数化 [↗](#)



课程首页

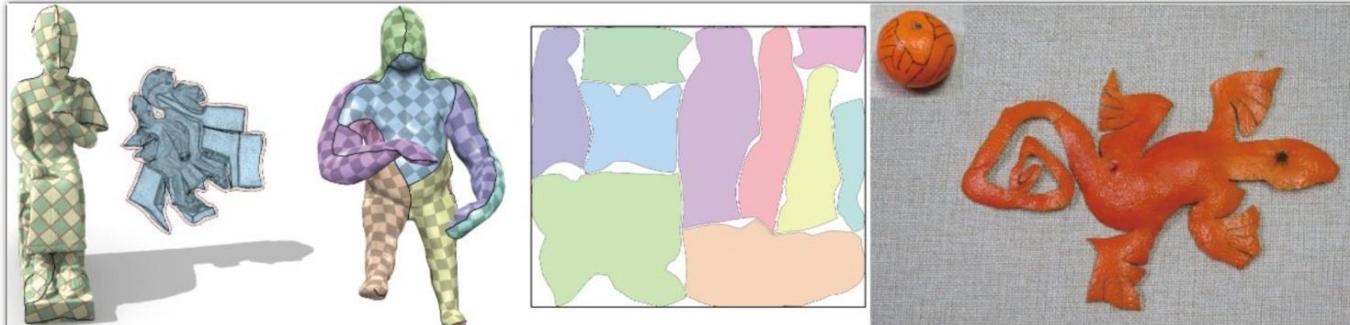
课程安排

授课教师

课程作业

其它学习资料

GAMES 301 : 曲面参数化 Surface Parameterization



授课教师团队



刘利刚



陈仁杰



傅孝明



方清

中国科学技术大学图形与几何计算实验室

<http://gcl.ustc.edu.cn>

课程安排



时间		授课老师	课程题目
第一周	第1讲	10月7日, 10:00-11:30	刘利刚
	第2讲	10月9日, 10:00-11:30	傅孝明
第二周	第3讲	10月15日, 10:00-11:30	傅孝明
	第4讲	10月16日, 10:00-11:30	傅孝明
第三周	第5讲	10月22日, 10:00-11:30	傅孝明
	第6讲	10月23日, 10:00-11:30	傅孝明
第四周	第7讲	10月29日, 10:00-11:30	傅孝明
	第8讲	10月30日, 10:00-11:30	陈仁杰
第五周	第9讲	11月5日, 10:00-11:30	陈仁杰
	第10讲	11月6日, 10:00-11:30	傅孝明
第六周	第11讲	11月12日, 10:00-11:30	方清
	第12讲	11月13日, 10:00-11:30	方清
第七周	第13讲	11月19日, 10:00-11:30	方清
	第14讲	11月20日, 10:00-11:30	刘利刚

课程信息



- 课程目标：曲面参数化的基础及进阶知识和方法
- 授课时间：每周六、周日上午10:00-11:30（14次）
- 在线直播：<http://webinar.games-cn.org>（GAMES直播号）
- 课程作业
 - 课程作业：4个编程作业（C++或Matlab为主）
 - 提交系统：<http://cn.ces-alpha.org/course/register/GAMES301>
- 课程交流
 - 课程QQ群：549032092
 - 课程BBS：<http://games-cn.org/forums/forum/games301-forum>
- 课程助教：柳士博, 梁仲轩
- Q&A:
 - 直播间（上课时）、301 QQ群、GAMES微信群
 - 使用腾讯文档维护平时的问题及解答

基础知识及参考资料



- 数学
 - 线性代数、微积分、数值方法
 - 最优化
 - 参考: GAMES 102课程的第12讲 (00:48-)
<https://www.bilibili.com/video/BV1NA411E7Yr>
- 计算机图形学
 - 基础图形学知识, 比如GAMES 101
 - 基础数字几何处理知识, 比如GAMES 102, 或中科大《计算机图形学》(2020)
- 编程能力
 - C/C++/Matlab
- 其他参考资料
 - 中国科学技术大学《计算机图形学前沿》暑期课程
<http://staff.ustc.edu.cn/~lgliu/Courses/SummerSchool/USTC-summer-school.html>
 - 傅孝明, 《数字几何处理》, <https://www.bilibili.com/video/BV1B54y1B7Uc>

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曲面参数化及其应用

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GAMES 102 《几何建模与处理》



- 课程主页：http://staff.ustc.edu.cn/~lgliu/Courses/GAMES102_2020/default.html
- B站录屏：<https://www.bilibili.com/video/BV1NA411E7Yr>

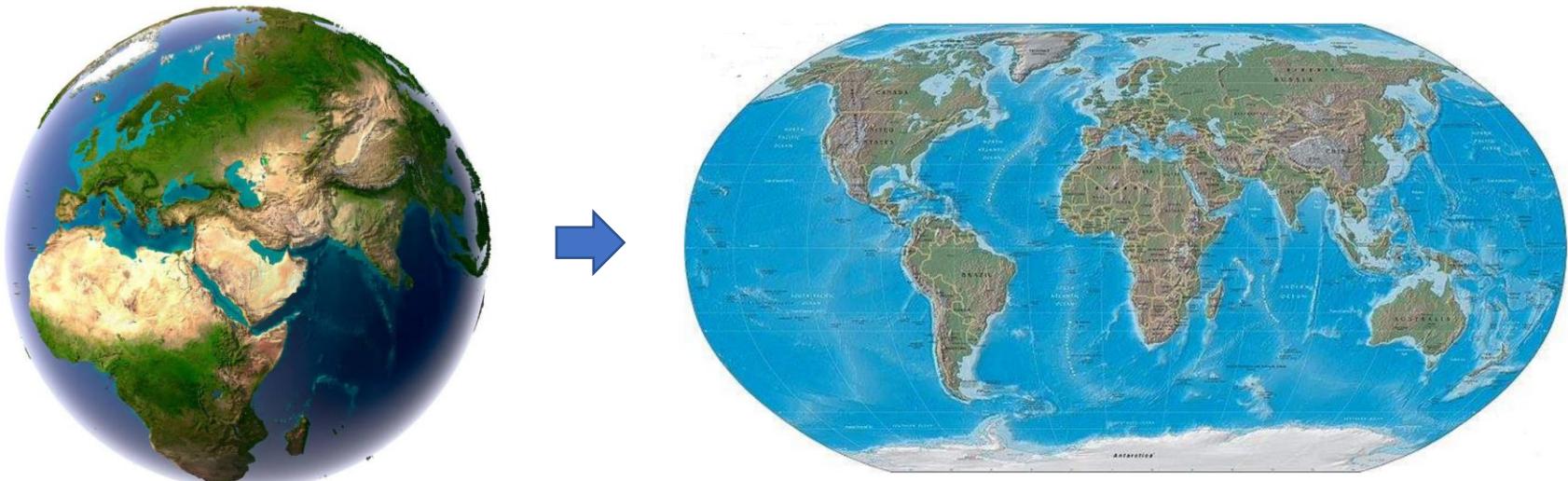


第11讲



第12讲

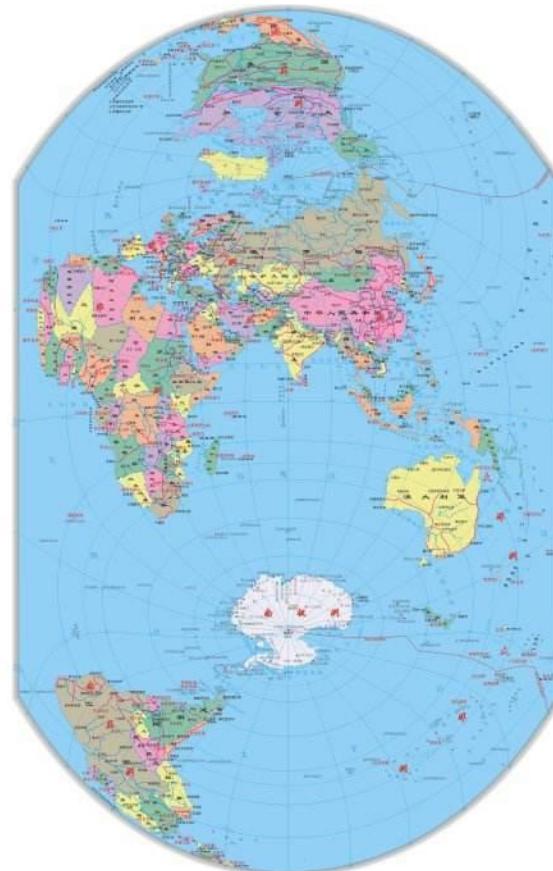
例子1：地图--地球表面展开



例子1：地图--地球表面展开



例子1：地图--地球表面展开



例子2：橘子剥皮



例子2：橘子剥皮艺术



www.okadas.com

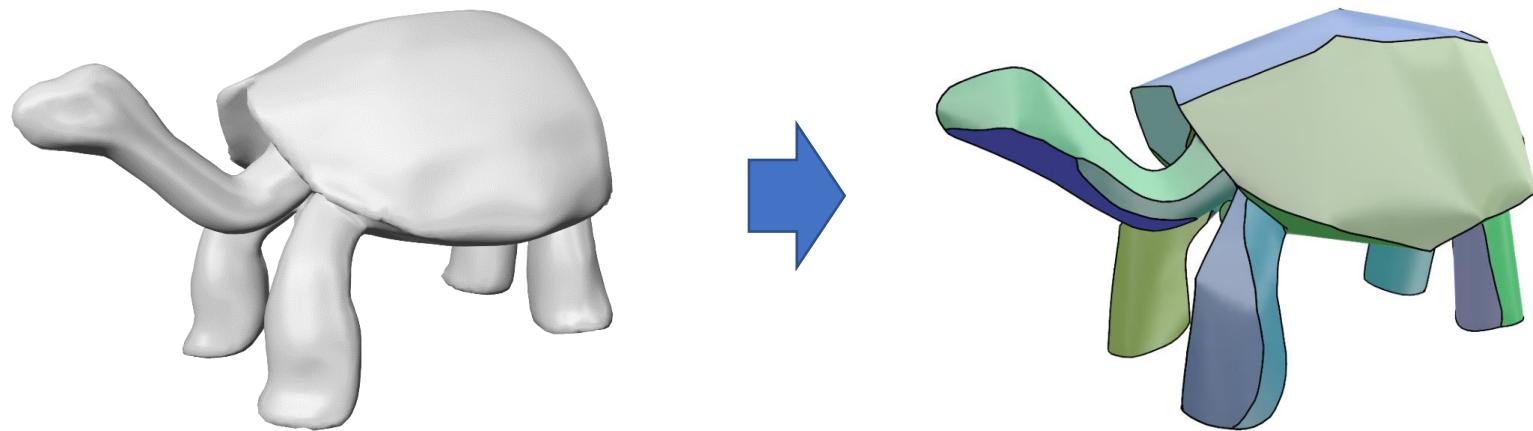
<https://www.youtube.com/watch?v=JlOUHAKQdc4>



例子2：橘子剥皮艺术



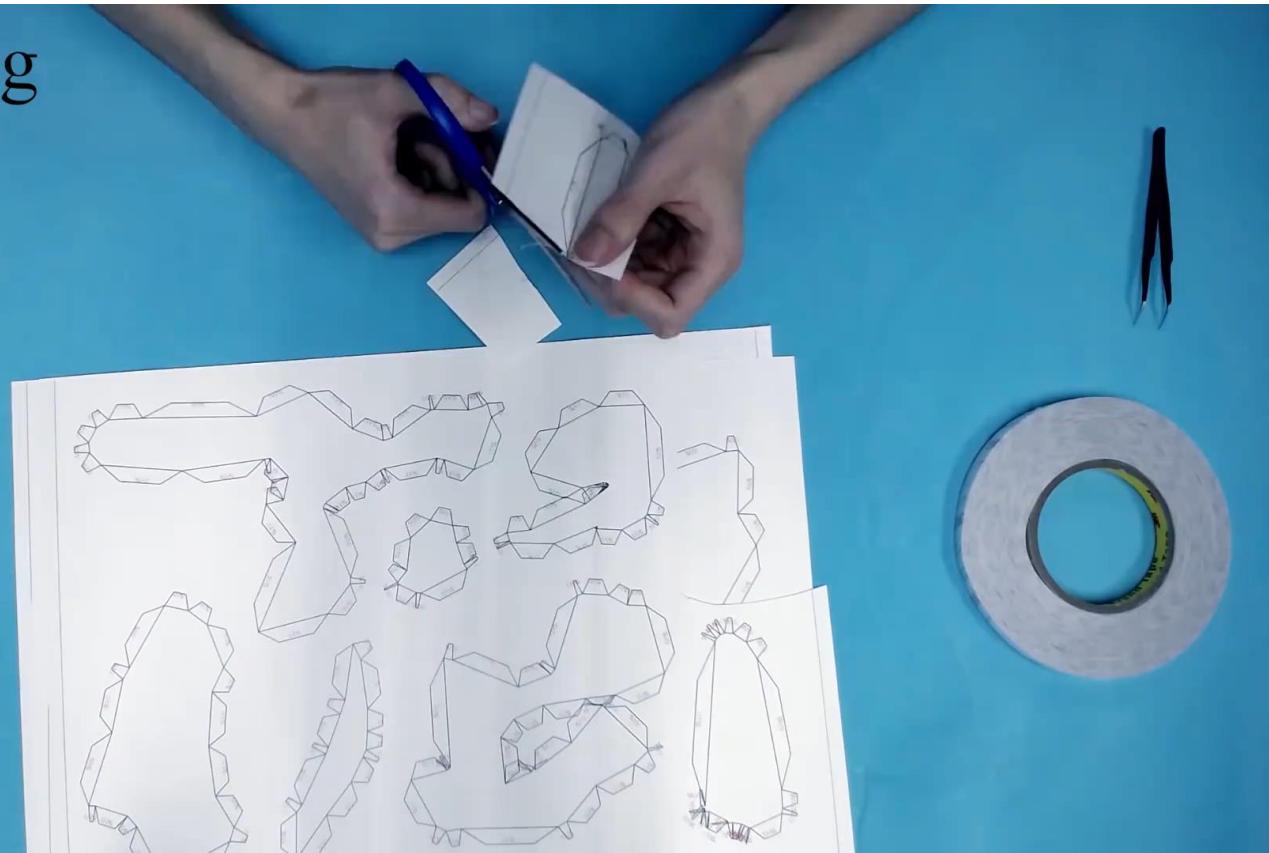
例子3：多片展开





例子2：多片折纸

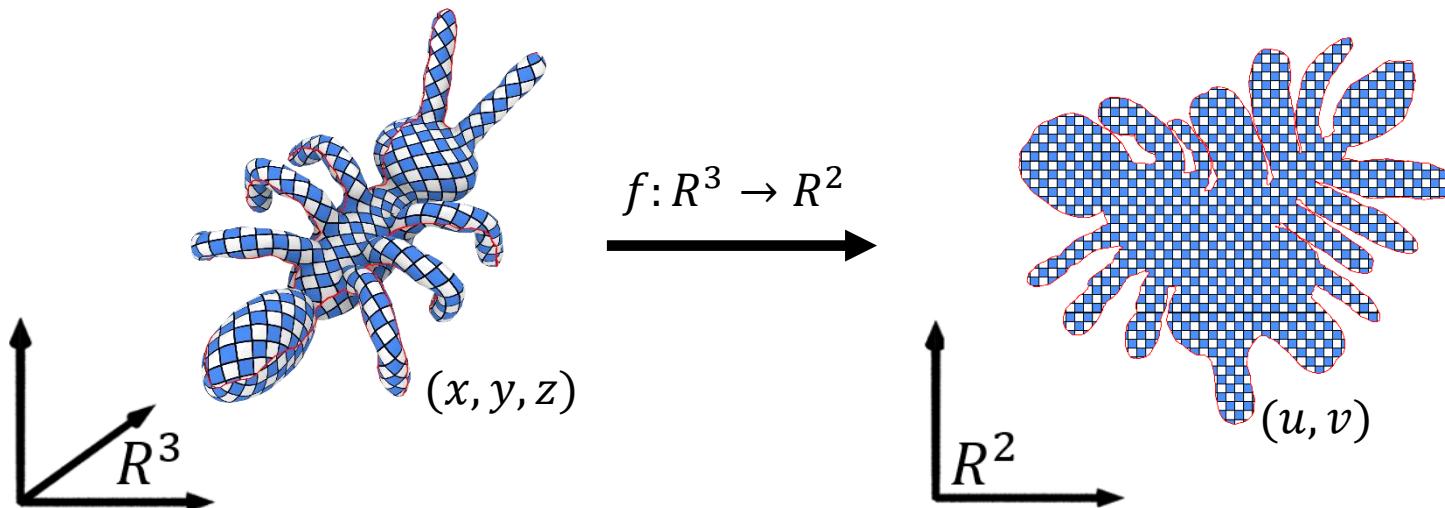
Cutting



曲面参数化 (Surface Parameterization)



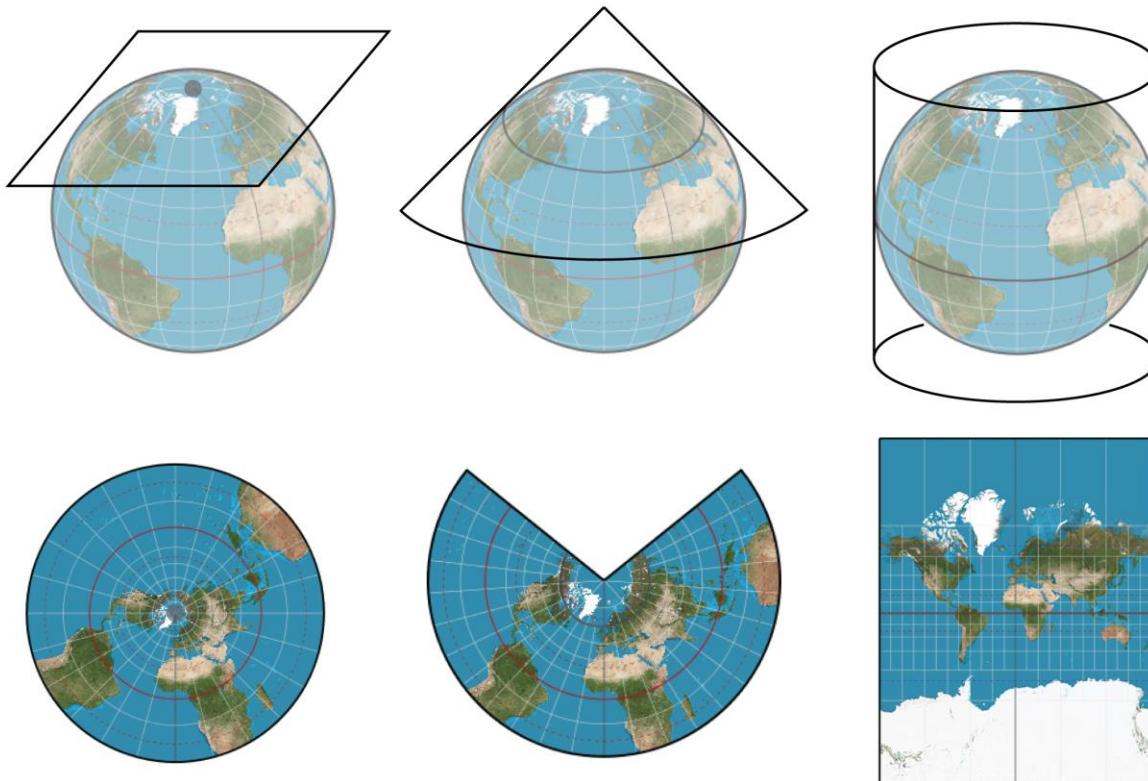
- 几何直观：3D曲面展开成2D平面
- 数学本质：3D曲面在2D平面的嵌入 (embedding) / 映射 (mapping)
 - 构建曲面到平面区域的一一对应
 - 3D曲面本质是2D的：2D流形 (manifold)



曲面参数化的应用-1



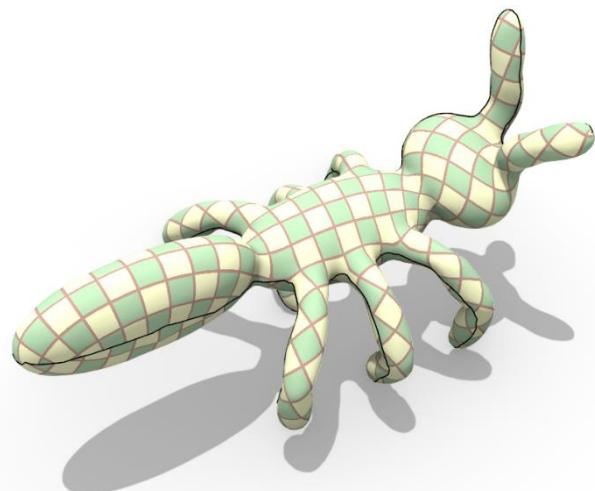
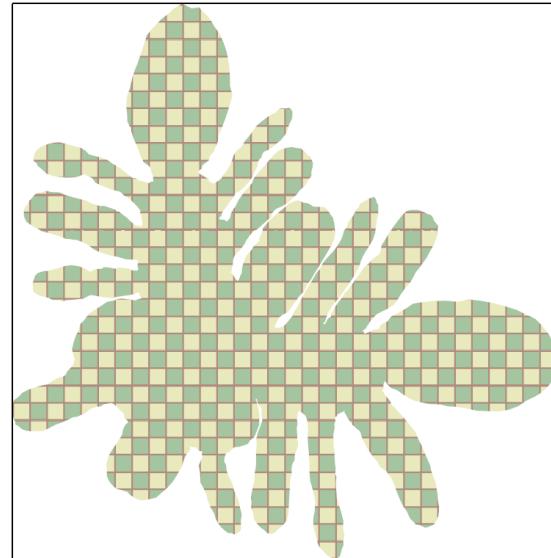
- 地图制作



曲面参数化的应用-2



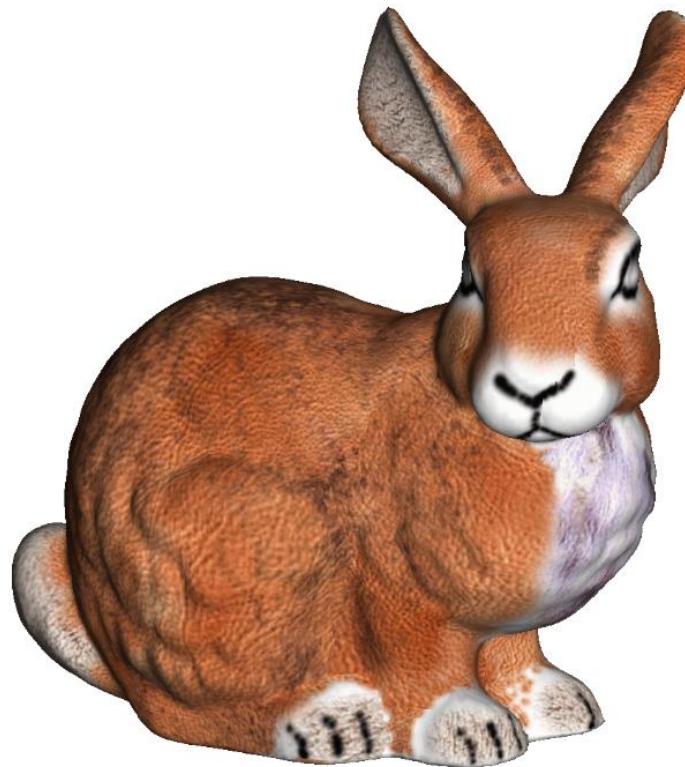
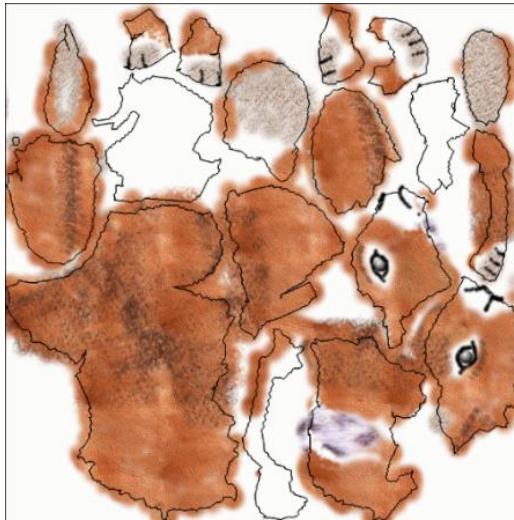
- 曲面贴图：存储和表达曲面上的各种信息
 - 纹理(texture)贴图、法向(normal)贴图、偏置(displacement)贴图、颜色(albedo)、材质(material/BRDF)...



曲面参数化的应用-3



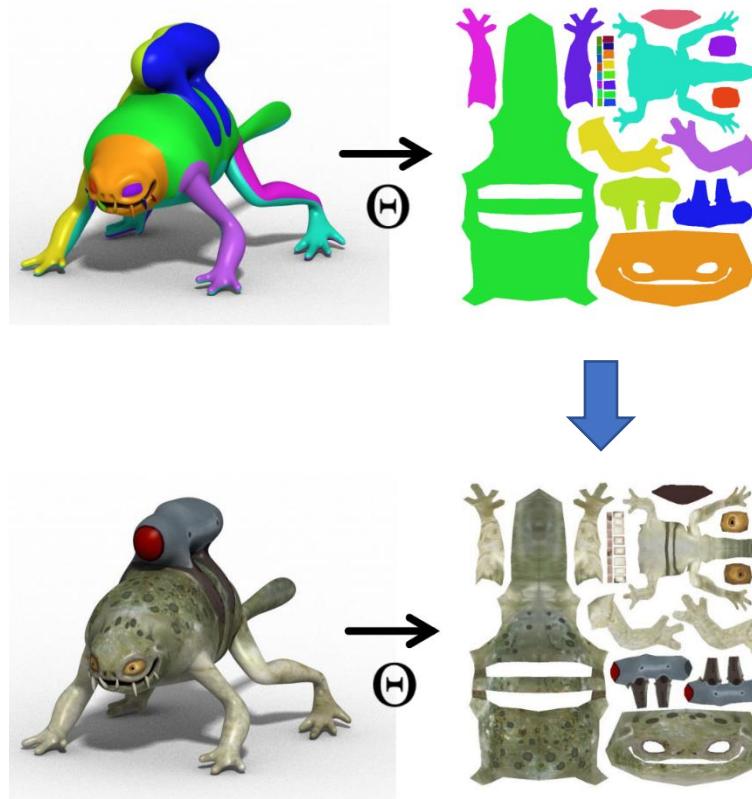
- 纹理地图 (texture atlas)
 - Segmentation and packing



曲面参数化的应用-3



- 纹理地图
 - 曲面绘画

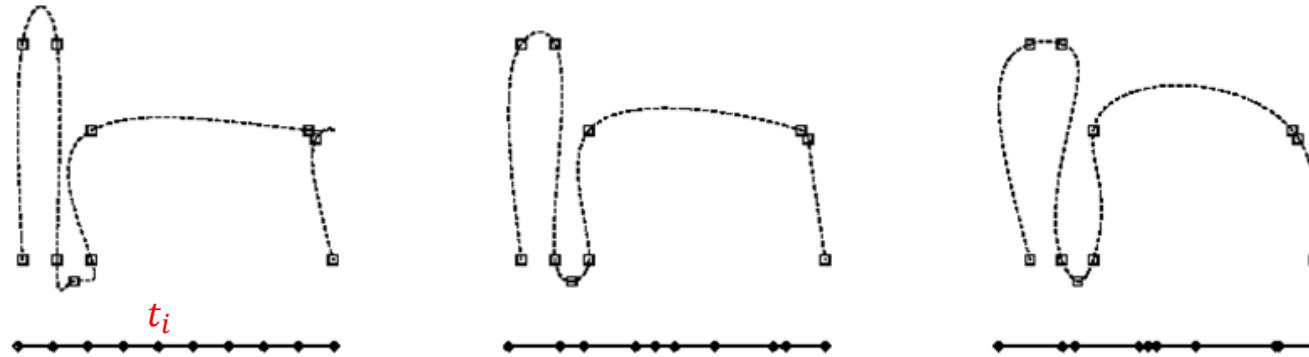


曲面参数化的应用-4



- 曲线拟合

2D点列的B样条曲线拟合（不同的参数化对拟合曲线结果的影响）



$$f: R^1 \rightarrow R^2 \quad \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in [0,1]$$

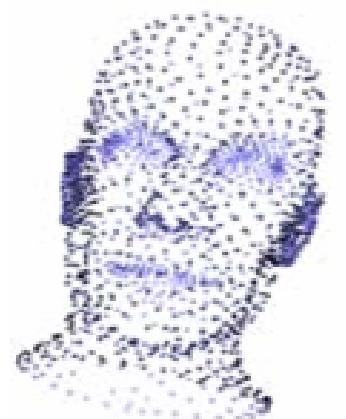
$$\min E = \sum_{i=1}^n \| \mathbf{p}(\mathbf{t}_i) - \mathbf{p}_i \|^2$$

曲面参数化的应用-4

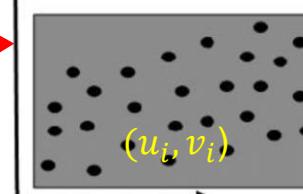


- 曲面拟合

3D点云的B样条(NURBS/T样条)曲面拟合



参数化

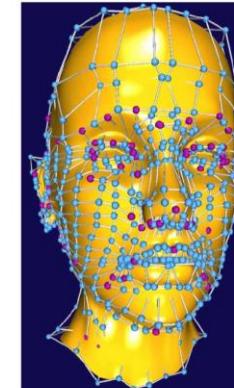


$$f: R^2 \rightarrow R^3$$

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \quad (u, v) \in [0, 1] \times [0, 1]$$



NURBS
4712 control points

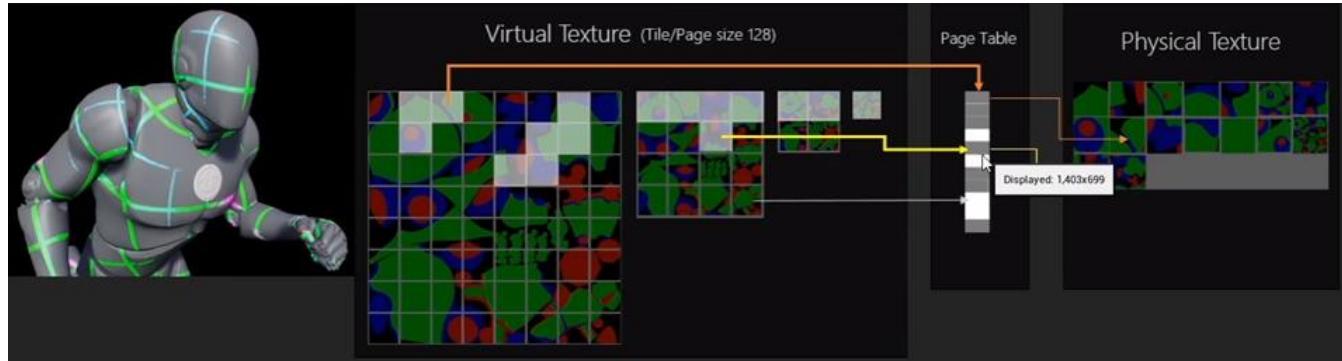


T-Splines:
1109 control points

曲面参数化的应用-5



- 曲面渲染
 - 虚拟纹理
 - 虚拟几何
 - Mipmap
 - LOD

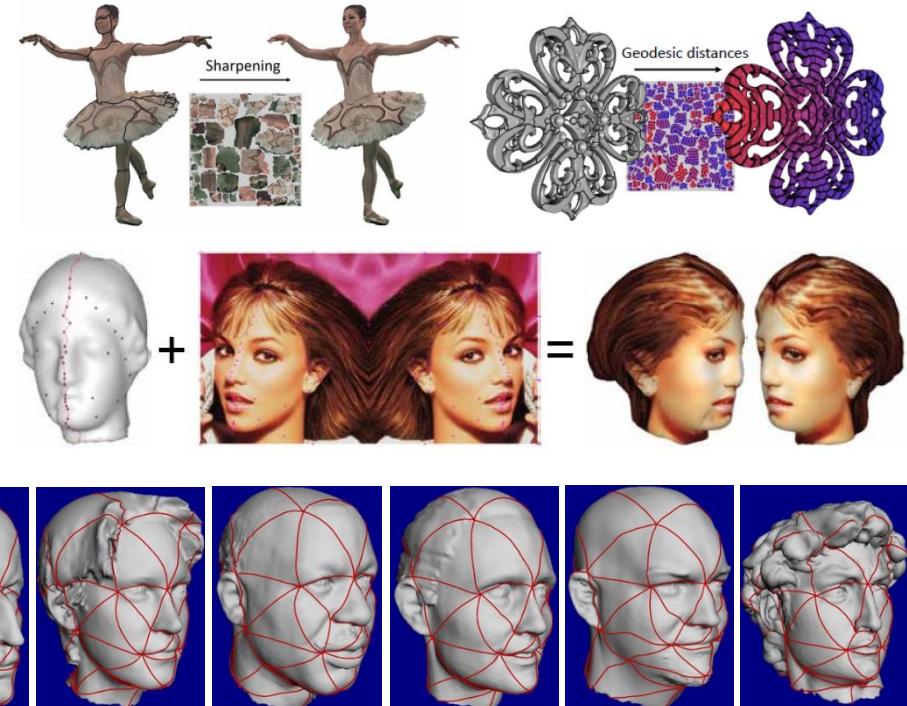


曲面参数化的应用-6



- 大部分几何处理的基础（基本问题）

- Visualization
- Compression
- Transmission
- Simplification
- Matching
- Remeshing
- Reconstruction
- Repairing
- Texture synthesis
- Rendering
- Animation
- Morphing
- ...

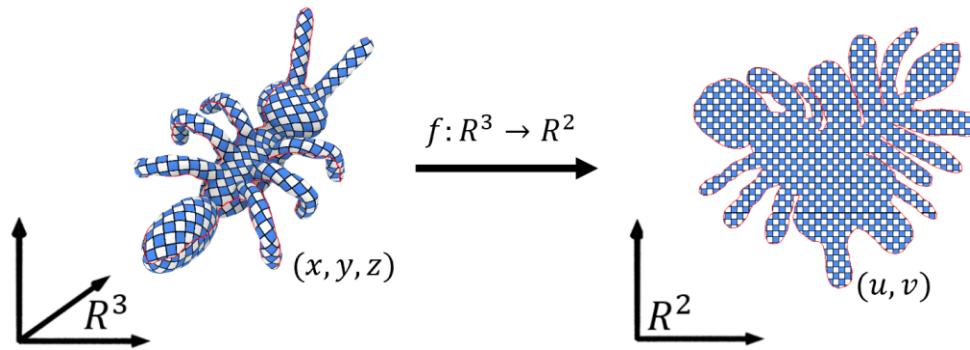




曲面参数化的表达及度量

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嚴濟慈題
一九八八年五月

曲面参数化的表达



- 参数化映射的表达：高维 (3D) 函数的表达
- 基本方法：化繁为简
 - **函数**的简化：线性函数空间 -- 基函数的线性组合
 - **定义域**的简化：将定义域分解为简单区域（比如三角形）的拼接，然后在三角形上定义简单函数（有限元方法）
 - 线性函数：分片线性函数，光滑性低 (C^0)
 - 高阶多项式函数：分片多项式函数，光滑性较高 (C^k)

曲面参数化的表达-1：连续映射方法



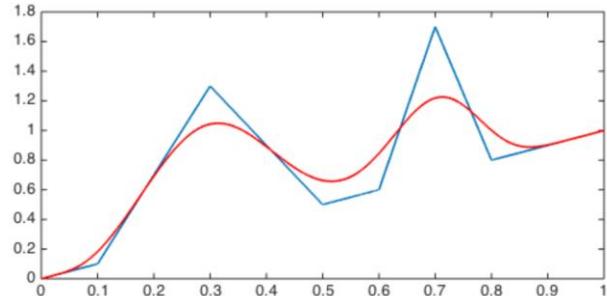
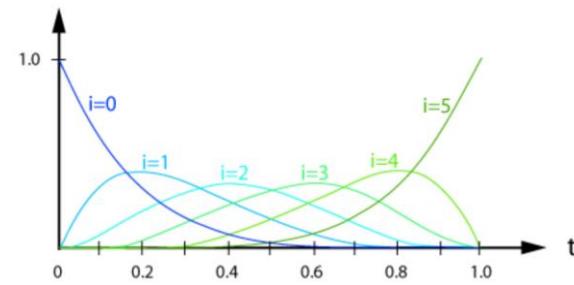
- 基函数 (basis functions)

- Bernstein基, B样条基, Fourier基, 小波基, ...
- 广义重心坐标, 调和映射, RBF, ...

$$f_1, f_2, f_3, \dots, f_n$$

- 线性函数空间: 基函数的线性组合

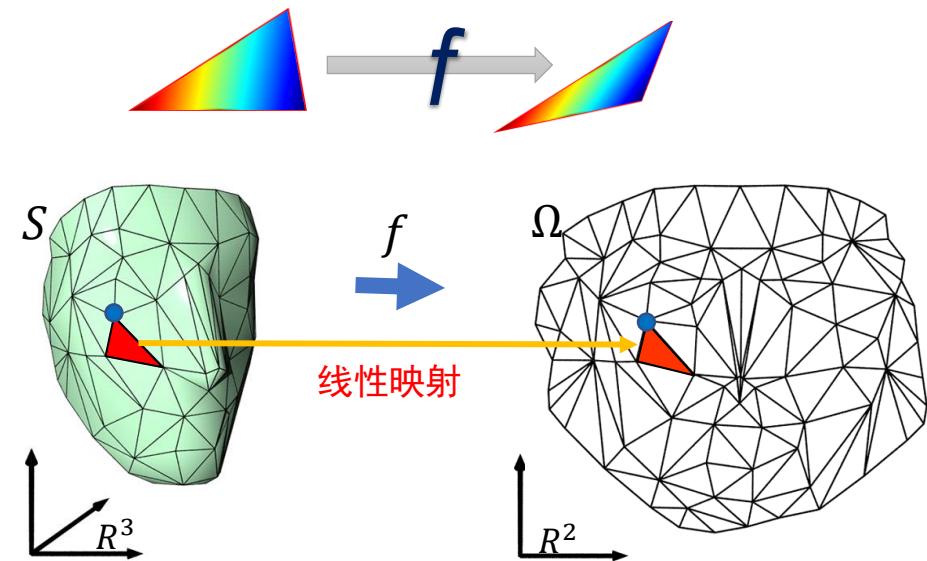
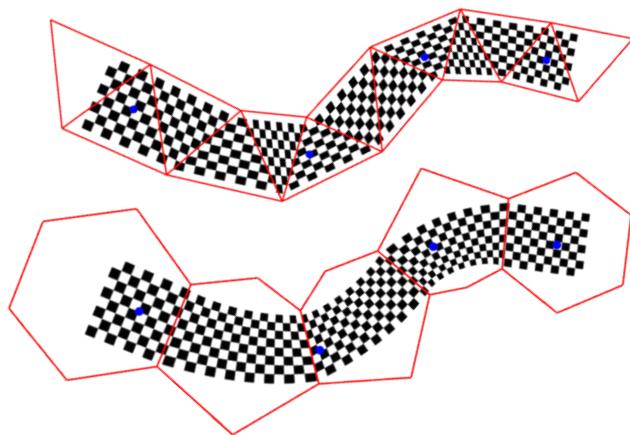
$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \sum a_i f_i(\mathbf{x}) \\ \sum b_i f_i(\mathbf{x}) \end{pmatrix}$$



曲面参数化的表达-2：分片映射方法



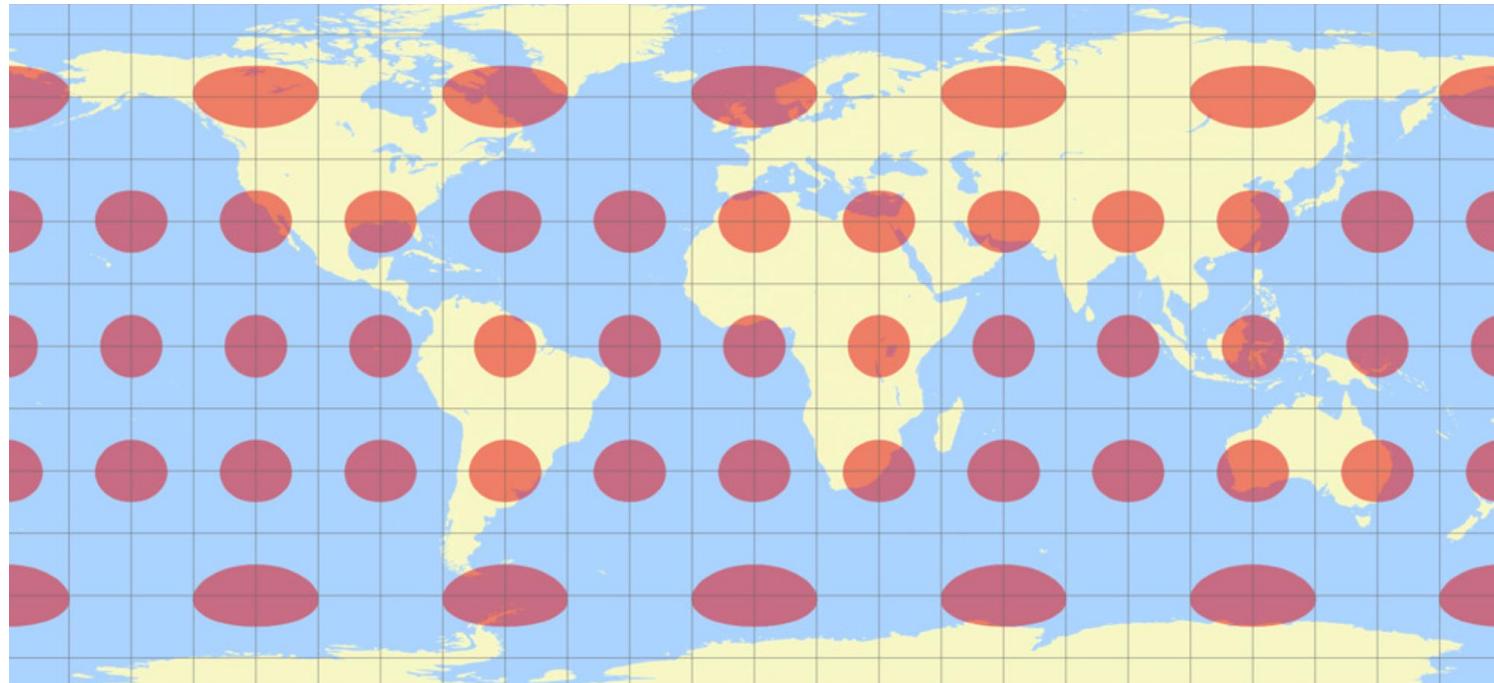
- 曲面表达为三角网格
 - 曲面的离散采样（点、线、面），即分片线性逼近表达
- 参数化映射表达为对应三角片之间的线性映射的拼接
 - 分片线性元：相邻面片映射的导数不连续
 - 分片高阶元
 - ...



参数化的度量：什么是“最优”参数化？



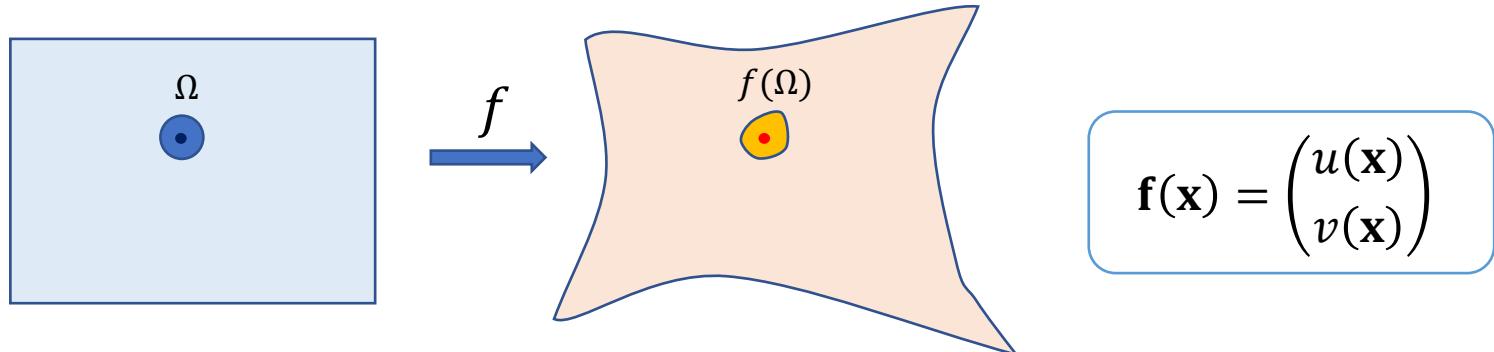
- 不同的最优准则：度量曲面的几何性质（几何量）的变化
 - 角度、面积、距离...



雅可比 (Jacobian) 的几何意义



- 函数 f 在某点 x 的 Jacobian $Jf(x)$ 度量了其局部的形变量
 - $Jf(x)$ 的行列式值度量了局部有向面积的伸缩比



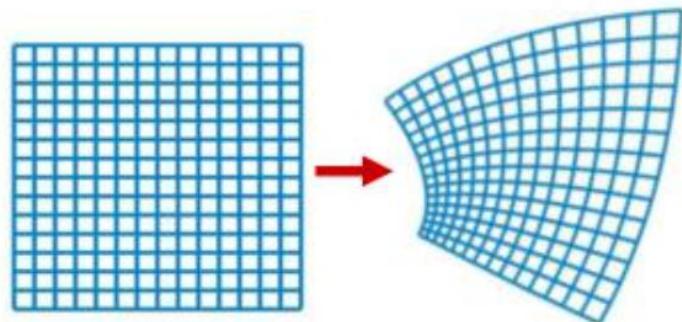
The Jacobian:

$$J\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \partial_x u(\mathbf{x}) & \partial_y u(\mathbf{x}) \\ \partial_x v(\mathbf{x}) & \partial_y v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \nabla u(\mathbf{x}) \\ \nabla v(\mathbf{x}) \end{pmatrix}$$

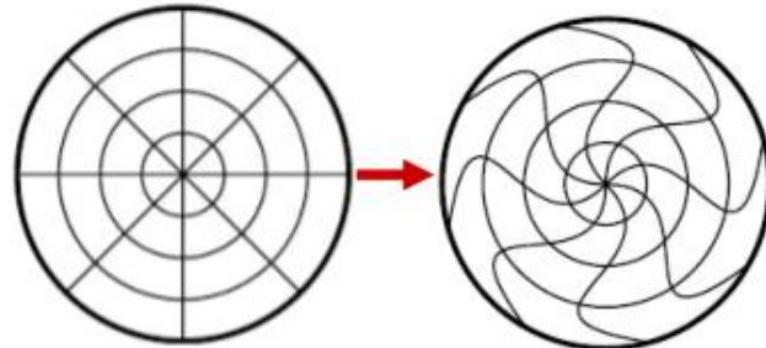
保持不同几何性质的映射



- 保角映射(angle-preserving): conformal (共形)
- 保面积映射(area-preserving): authalic (等积)
- 等距映射(isometric): conformal + authalic



Conformal mapping

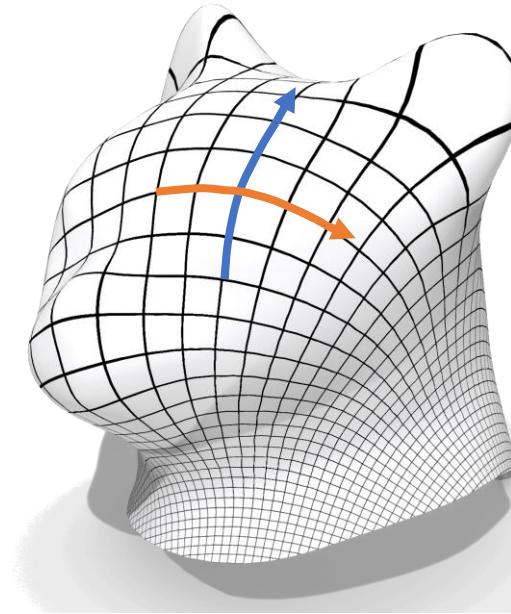
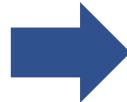
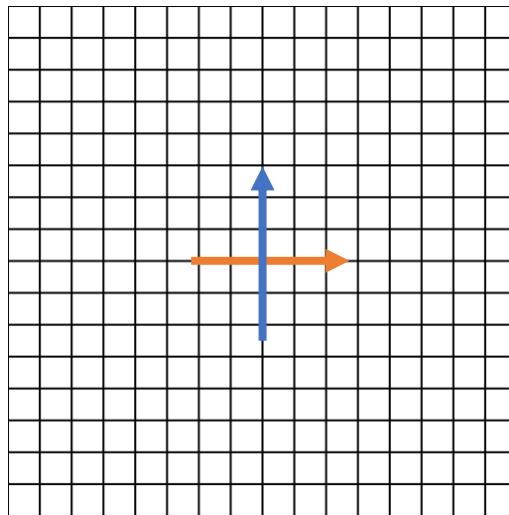


Authalic mapping

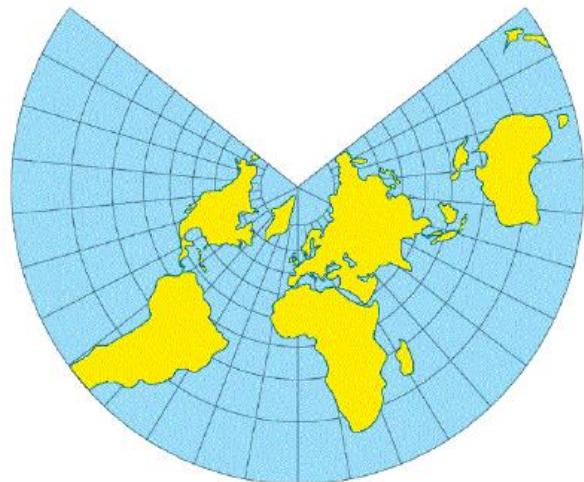
Conformal Mapping



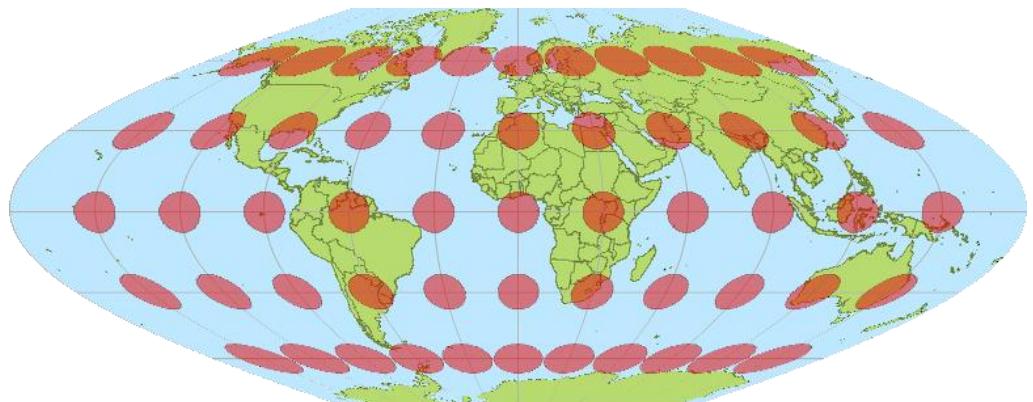
- 空间曲线在交点处的夹角：切线之间的角度



共形映射及等积映射的地图

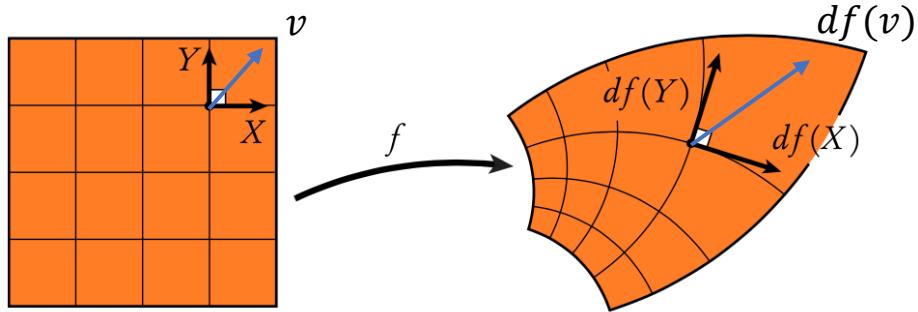


Conformal mapping



Authalic mapping

共形映射：对数共形因子 (Conformal factor)



$$df(v) = \lambda R dv$$



$$\begin{aligned}\|df(v)\|^2 &= \lambda^2 \|dv\|^2 \\ &= e^{2\Psi} \|dv\|^2\end{aligned}$$

对数共形因子：

$$\Psi = \log \lambda : \mathcal{M} \rightarrow \mathbb{R}$$

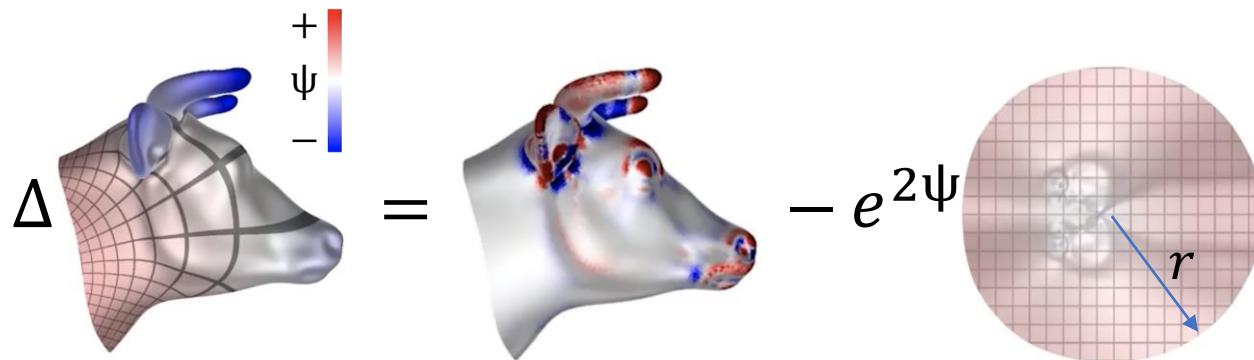
共形映射：Yamabe 方程



曲面度量 \Rightarrow 内蕴高斯曲率, 共形映射下:

$$\Delta\psi = K^{ori} - e^{2\psi}K$$

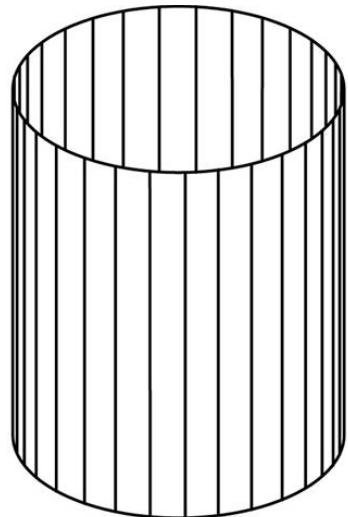
↑ ↑ ↑
Laplace-Beltrami 原始 变化后
 高斯曲率 高斯曲率



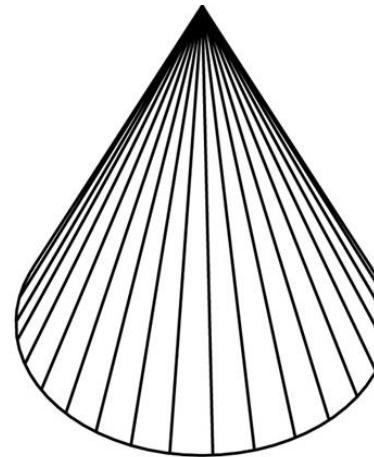
等距映射：保持任意两点的测地距离



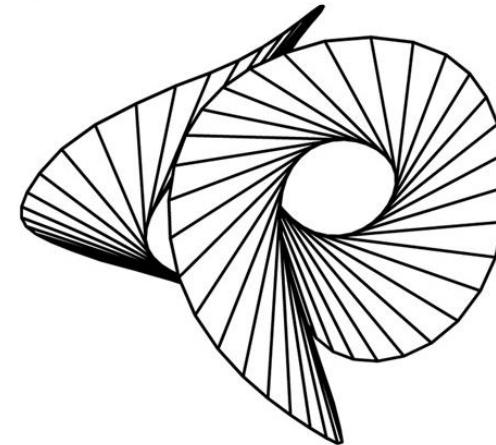
- 可展曲面 (developable surfaces): 高斯曲率处处为0



柱面



锥面

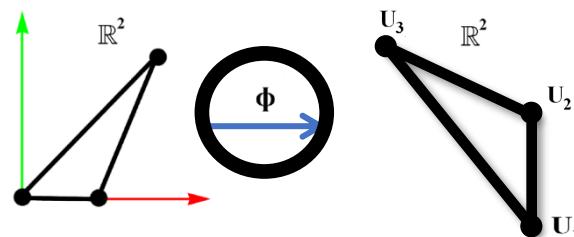
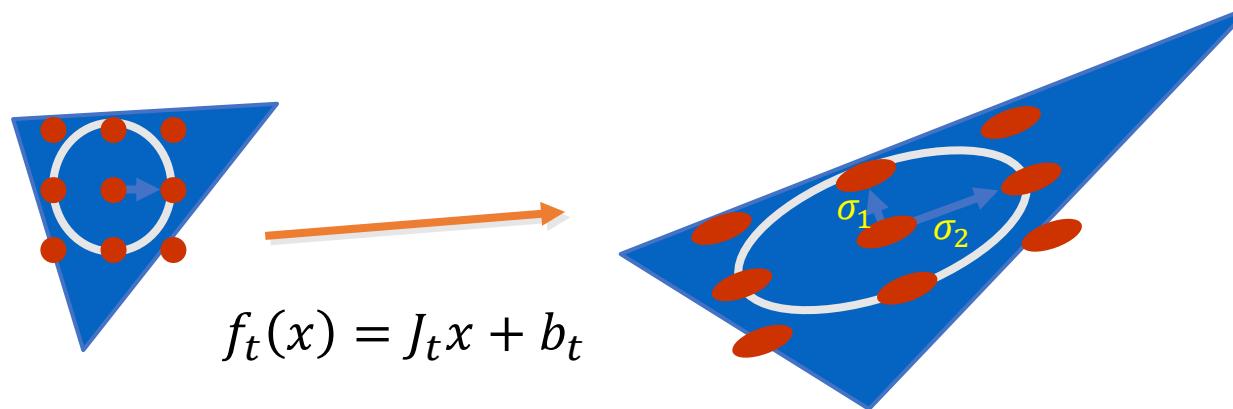


切线面

几何形变/扭曲 (distortion)



- 非可展曲面的参数化结果必然有几何形变



$$L = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} V^*$$
$$\sigma_2 \geq \sigma_1$$

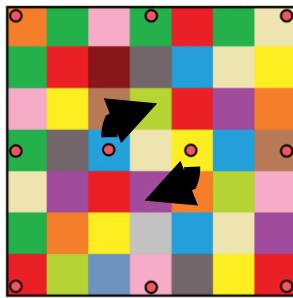
- 保角: $\sigma_1 = \sigma_2$
- 等积: $\sigma_1 \sigma_2 = 1$
- 等距: $\sigma_1 = \sigma_2 = 1$

What are ‘good’ parameterizations?



Local

Bijectivity



Not
Bijective

Low distortion

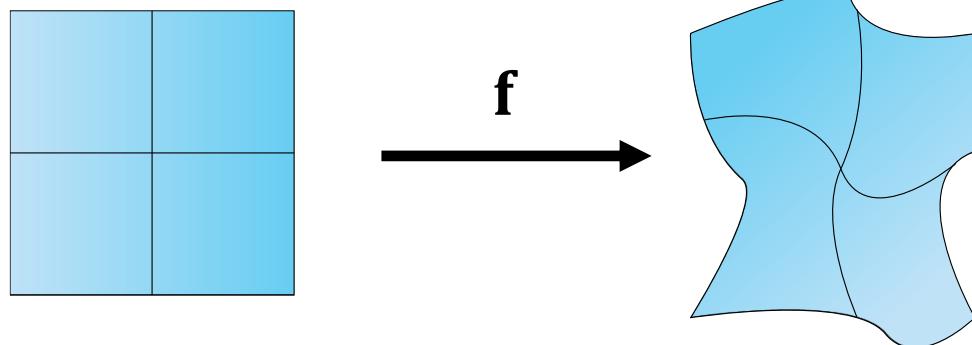
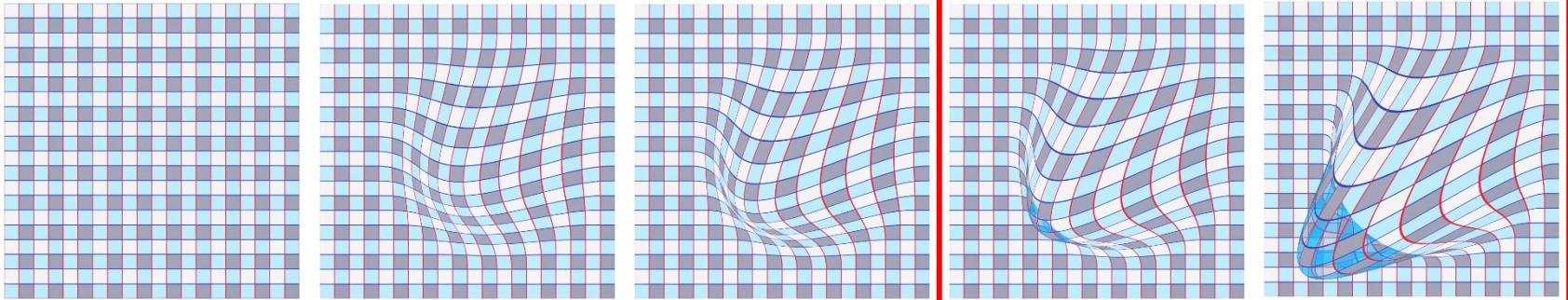


Bijective



Lower
distortion

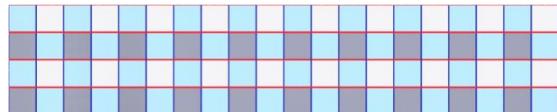
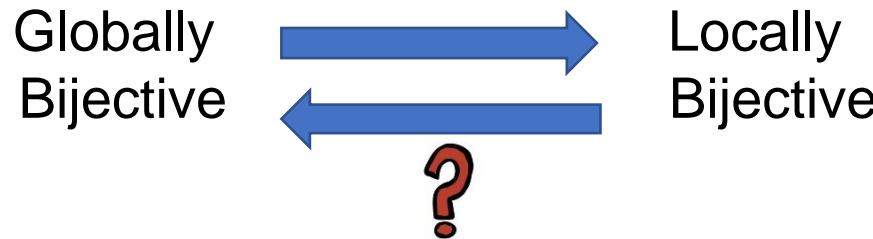
Locally Bijective = Injective



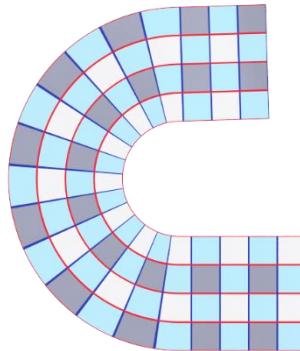
The sufficient condition:

$$\det \mathcal{J}\mathbf{f}(\mathbf{x}) > 0, \forall \mathbf{x}$$

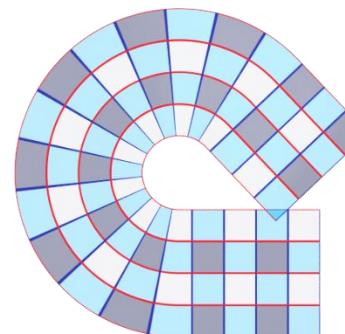
Globally Bijective VS. Locally Bijective



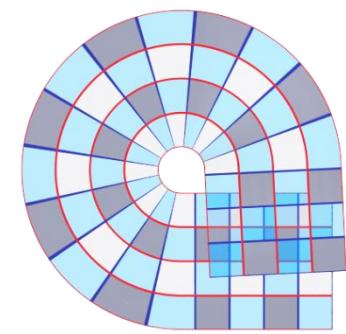
Input



Injective



Injective, but not bijective



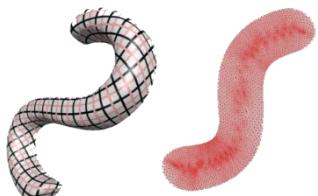
Distortion metric



- Conformal

[Degener et al. 2003]

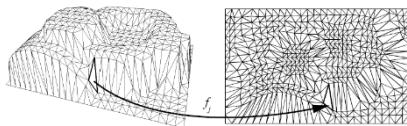
$$\frac{\sigma_2}{\sigma_1}$$



- MIPS

[Hormann and Greiner 2000]

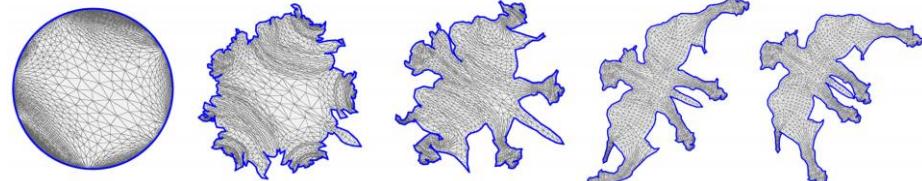
$$\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$



- Symmetric Dirichlet energy

[Smith and Schaefer 2015]

$$\sigma_1^2 + \frac{1}{\sigma_1^2} + \sigma_2^2 + \frac{1}{\sigma_2^2}$$



- Maximal Isometric Distortion

[Sorkine et al. 2002]

$$\max(\sigma_2, \frac{1}{\sigma_1})$$



- Isometric

[Aigermann et al. 2014]

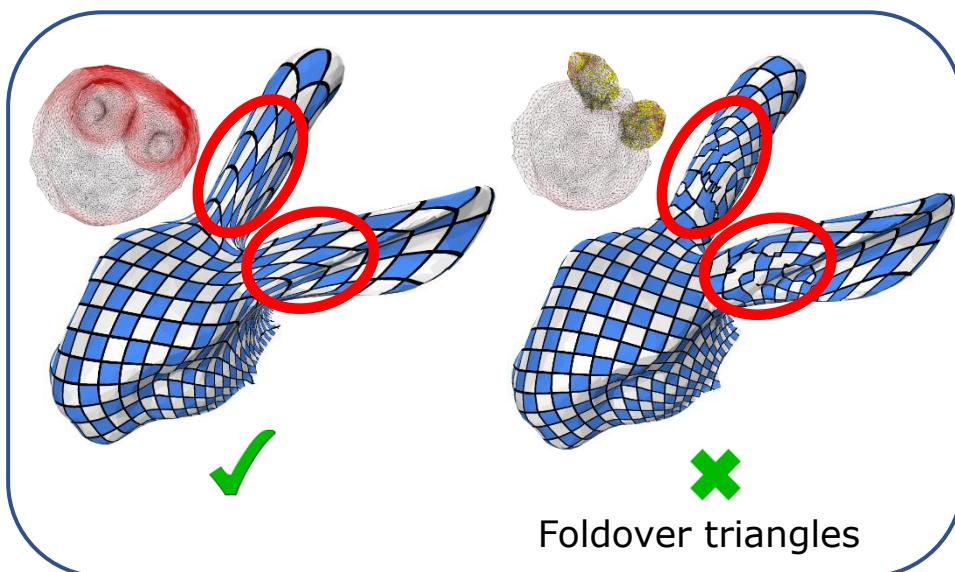
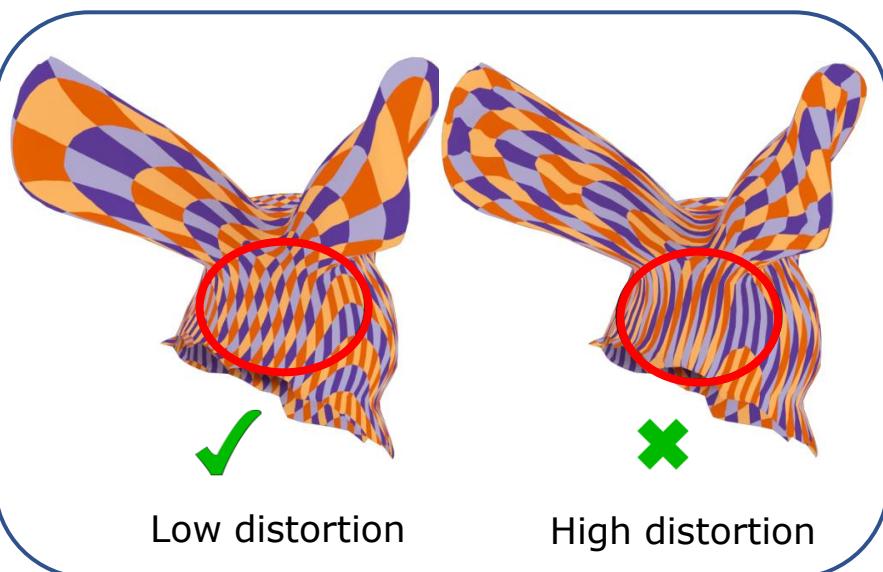
$$\sqrt{\sigma_2^2 + \frac{1}{\sigma_1^2}}$$



Desired Properties of Good Parameterization



- Low distortion
- Validity (injective, or even bijective): foldover / flip free (无翻转)



5

曲面参数化的主要方法

創寰宇學府
育天下英才
嚴濟慈題
一九八八年五月

曲面参数化的主要方法



- Tutte's method and its variants
 - Tutte's method [Tutte 1963; Floater 1997, 2003]
 - Variants [Weber and Zorin 2014; Aigerman and Lipman 2015, 2016; Aigerman et al. 2017; Bright et al. 2017;]
- Geometry-based optimization methods
 - Representation based methods [Sheffer and Sturler 2001; Sheffer et al. 2005; Chien et al. 2016b; Fu and Liu 2016]
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 - Bounded distortion methods [Lipman 2012; Aigerman et al. 2014; Kovalsky et al. 2015]
- Foldover free guaranteed optimization methods
 - [Smith and Schaefer 2015; Kovalsky et al. 2016; Jiang et al. 2017; Claici et al. 2017; Rabinovich et al. 2017; Shtengel et al. 2017; Zhu et al. 2018]

曲面参数化的主要方法

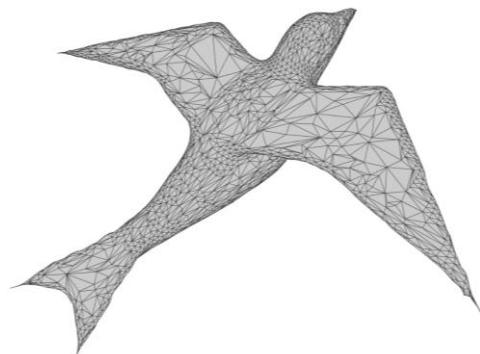


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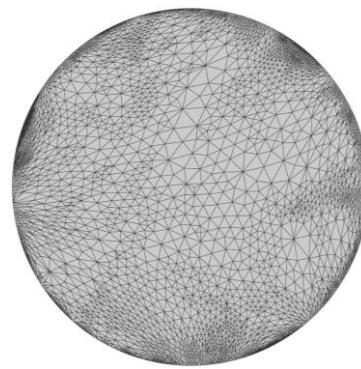
Tutte's embedding method (线性方法)



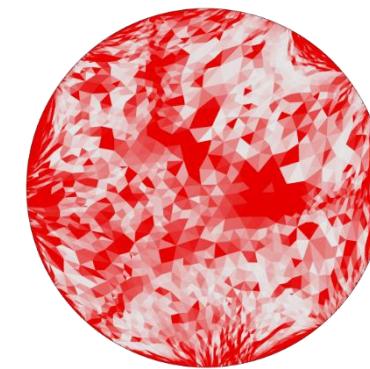
- Map the triangulation within a **convex** boundary by solving a linear system
- Foldover-free result with a **theoretical guarantee**
- Usually **high** distortion



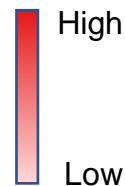
[Tutte 1963; Floater 1997, 2003]



Convex boundary



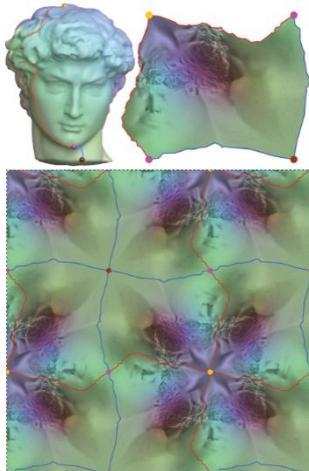
High distortion



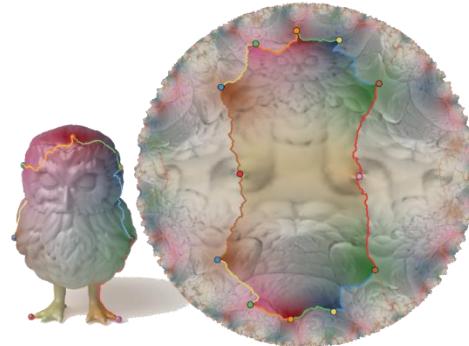
Variants of Tutte's embedding methods



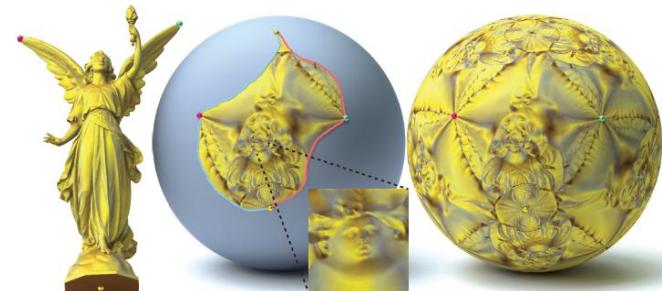
- Foldover-free result with theoretical guarantees



Euclidean-orbifold
[Aigerman et al. 2015]



Hyperbolic-orbifold
[Aigerman et al. 2016]



Spherical-orbifold
[Aigerman et al. 2017]

曲面参数化的主要方法



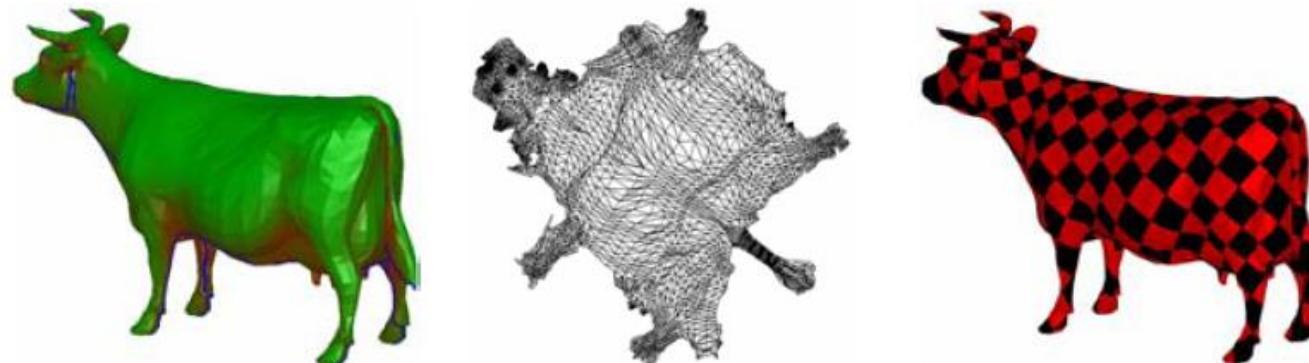
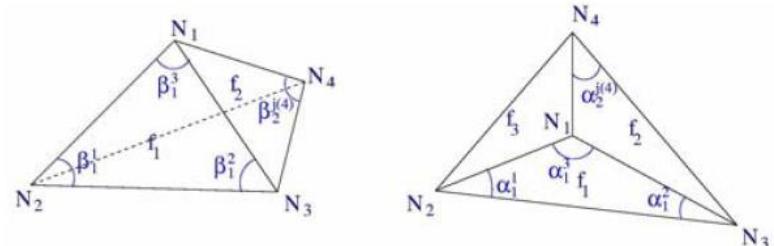
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ABF/ABF++

[Sheffer and Sturler 2001; Sheffer et al. 2005]



- Triangular 2D mesh is defined by its angles
 - Define problem in angle space
- Angle based formulation
 - Distortion as function of angles
 - Validity - set of angle constraints



As-rigid-as-possible (ARAP)

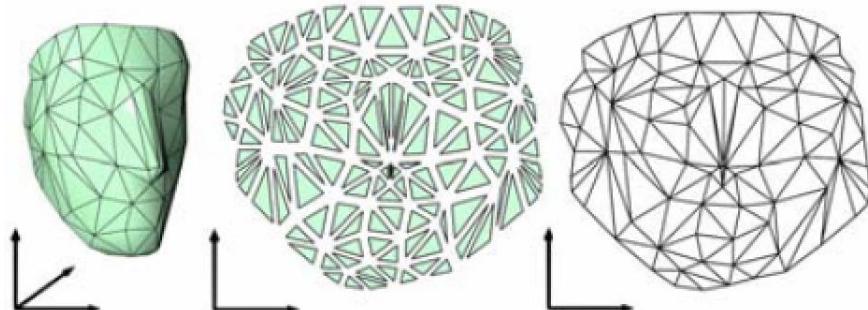
[Liu et al. 2008]



- Measuring the approximation between the linear mapping and the rigidity (the optimal rotation)
- Local/global optimization
 - Local: compute singular values
 - Global: solve linear systems
- Not guaranteed to avoid foldovers

被授权集成于商业软件 RapidForm 中
(全球四大逆向工程软件之一)

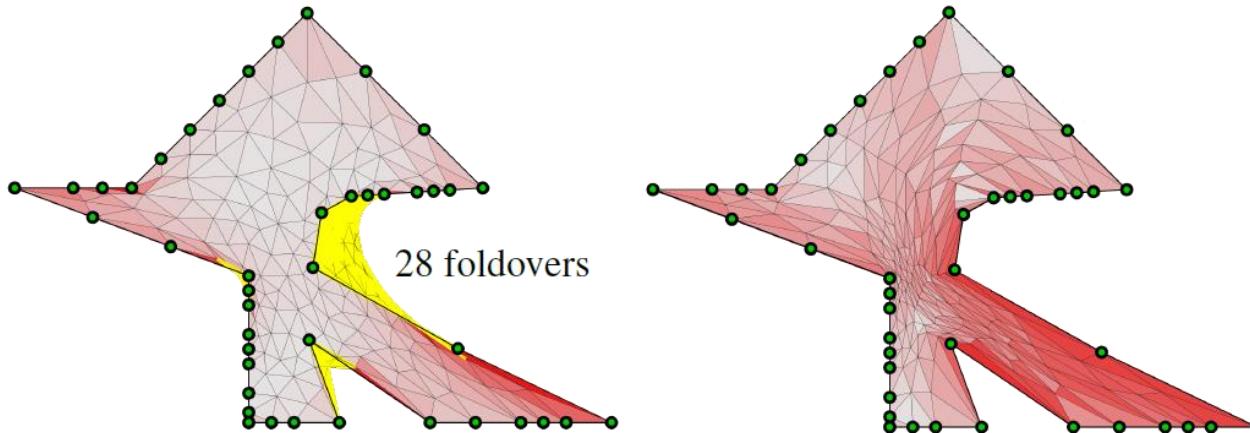
集成于著名的计算几何开源软件 **CGAL** (国际上应用最广泛的计算几何算法库) 中的基础算法之一



Simplex Assembly [Fu and Liu 2016]



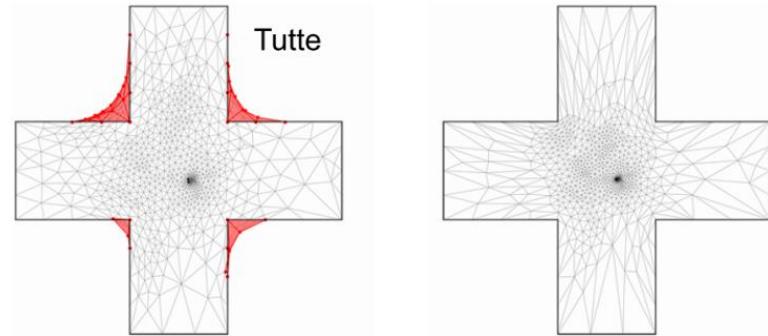
- Instead of vertex positions, treat the affine transformation as variables
- Use a barrier function to prevent the inversion
- No theoretically guaranteed to avoid foldovers



消除翻转的优化方法



- 这些参数化方法可能会产生翻转三角形
- 需要靠后续方法来尽量消除翻转
- 常用方法
 - Penalty functions
 - Bounding the conformal distortion
 - Iterative projection
 - Area-based methods



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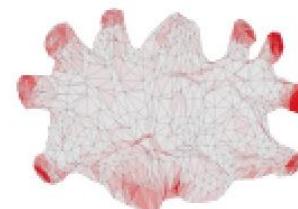
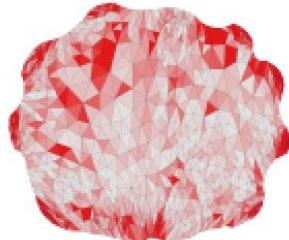
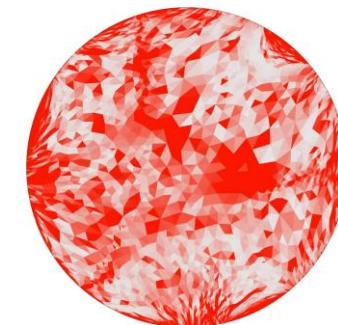
保证无翻转参数化方法的主要思想



Valid initialization

Low distortion cost

Optimization



Valid Initialization

Optimization

Reducing the distortion cost while
guaranteeing the validity

Optimized result

High
Low

保证无翻转参数化的优化方法



Valid initialization



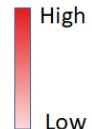
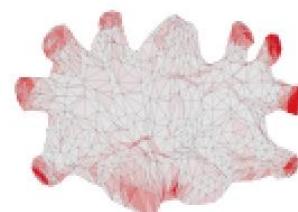
Validity guaranteed methods (e.g. Tutte's)

Low distortion cost



Preventing foldover objective function

Optimization



Valid Initialization

Optimization

Reducing the distortion cost while
guaranteeing the validity

Optimized result

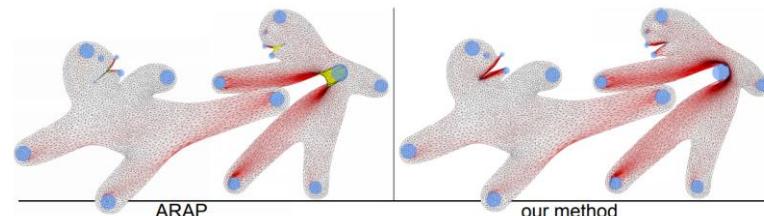
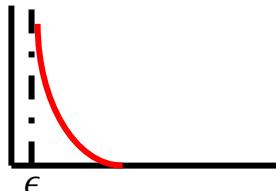
防止翻转的目标函数的设计策略-1



- 在目标函数中增加一个防止翻转的能量项：当越来越接近翻转的时候，该能量项的值很大，且趋向于正无穷（翻转发生时）
- 策略1：增加一个度量几何碰撞的障碍函数 [Schüller et al. 2013]

$$B(x) = \begin{cases} \frac{1}{x - \epsilon}, & x > \epsilon \\ +\infty, & \text{other} \end{cases}$$

x : area/volume



防止翻转的目标函数的设计策略-2



- 在目标函数中增加一个防止翻转的能量项：当越来越接近翻转的时候，该能量项的值很大，且趋向于正无穷（翻转发生时）
- 策略2：增加一个度量几何形变的障碍 (barrier) 函数
 - 包含 $\frac{1}{\sigma_1}$ 项 (因为当三角形趋向退化时， σ_1 的值趋向于 0)

- Conformal

[Degener et al. 2003]

$$\frac{\sigma_2}{\sigma_1}$$

- Maximal Isometric Distortion

[Sorkine et al. 2002]

$$\max(\sigma_2, \frac{1}{\sigma_1})$$

- MIPS

[Hormann and Greiner 2000]

$$\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$

- Isometric

[Aigermann et al. 2014]

$$\sqrt{\sigma_2^2 + \frac{1}{\sigma_1^2}}$$

- Symmetric Dirichlet energy

[Smith and Schaefer 2015]

$$\sigma_1^2 + \frac{1}{\sigma_1^2} + \sigma_2^2 + \frac{1}{\sigma_2^2}$$

优化问题的形式



$$\begin{aligned} \min_V E(V) &= \sum_{t \in T} \left(\sigma_1^2 + \frac{1}{\sigma_1^2} + \sigma_2^2 + \frac{1}{\sigma_2^2} \right) \\ \text{s.t. } \sigma_1 \sigma_2 > 0, \quad \forall t \end{aligned}$$

- The cost function is highly nonlinear and nonconvex
- The constraints are nonlinear
- The Hessian matrix is highly non-definite

Computationally expensive for large scale meshes!

保证无翻转参数化的优化方法



Valid initialization



Validity guaranteed methods (e.g. Tutte's)

Low distortion cost

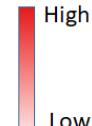
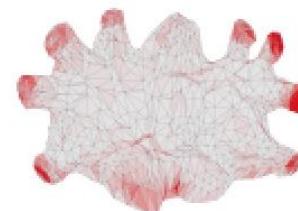
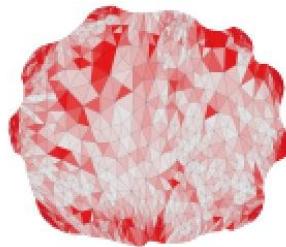


Preventing foldover objective function

Optimization



Efficient solvers



Valid Initialization

Optimization

Reducing the distortion cost while
guaranteeing the validity

Optimized result

优化问题的求解



Optimization formulation: $\min_x E(x) = \sum_t E_t(x)$

Input: a valid parameterization initialization x_0

Repeat

$$p = -H^{-1} \nabla E(x)$$

How to find a good decent direction?

$\alpha_{max} \leftarrow$ injective maximal search step

$\alpha \leftarrow$ line search by backtracking from α_{max}

$$x \leftarrow x + \alpha p$$

Until converged

Output: a locally injective parameterization

Methodologies



- First-order method
 - Pros: only pre-factorization once, cost and storage remain tractable for large-scale model
 - Cons: slow convergence, easily trapped at a high distortion point
- Quasi-Newton methods
 - Iteratively update H_i to approximate second derivatives
 - Using just differences in gradients and variables
- Second-order (Newton type) method
 - Pros: rapid convergence
 - Cons: costly assemble, factorize and backsolve a new linear equation every iteration, huge storage requirement for large-scale model

Methodologies

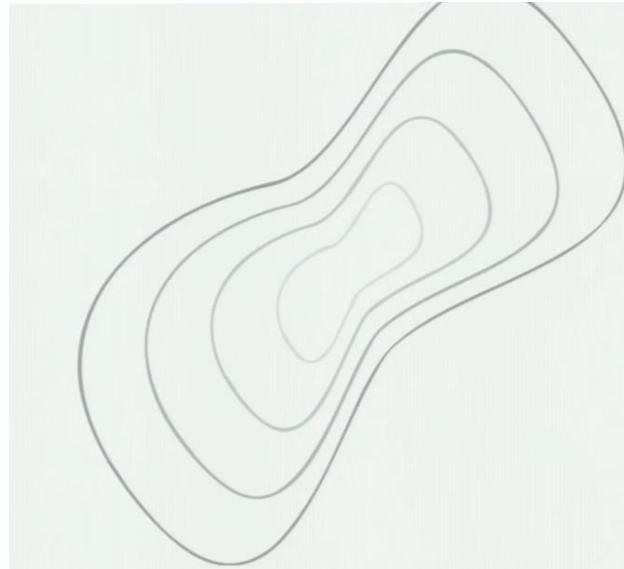
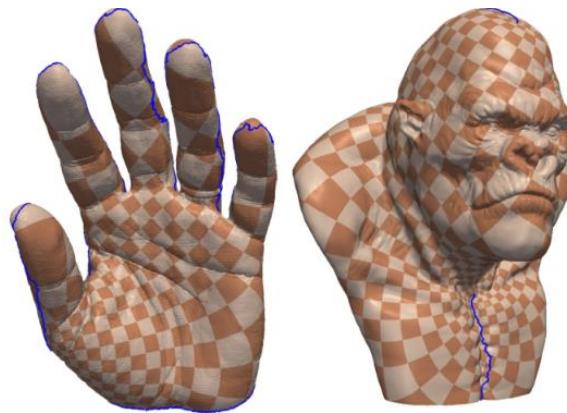


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Accelerated Quadratic Proxy (AQP)



- H = "discrete Laplacian L"
- Acceleration
- First-order method



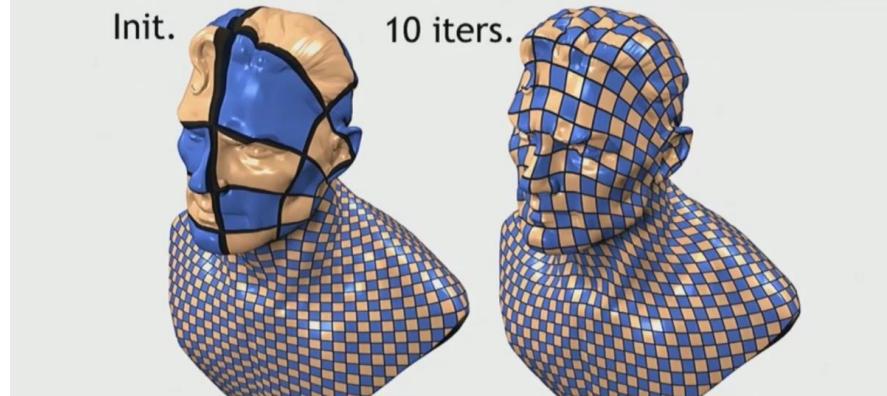
Scalable Locally Injective Mappings (SLIM)



- ✓ H = reweighted Laplacian L
- ✓ Compute the weight matrix W_J by the matching gradients condition

$$\nabla_J E_W^R = \nabla_J E_{RI}$$

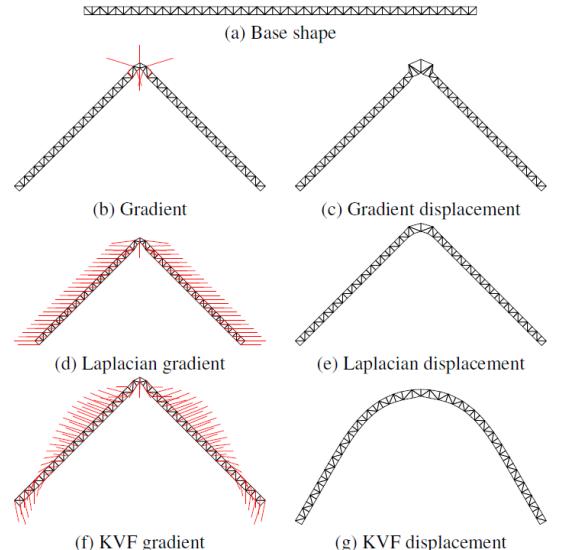
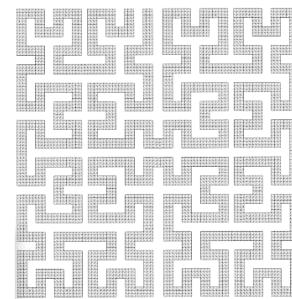
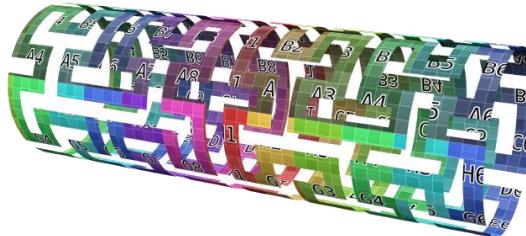
- ✓ Quickly recovers from a bad initialization, but slowly converge to a local minimum
- ✓ First-order method



Isometry-Aware Preconditioning (AKVF)



- ✓ H = approximate killing vector field operator $K(x)$
- ✓ $K(x)$ converts the local distortion gradient into a global near-rigid decent direction
- ✓ First-order method



Methodologies



- First-order method
 - Pros: only pre-factorization once, cost and storage remain tractable for large-scale model
 - Cons: slow convergence, easily trapped at a high distortion point
- Quasi-Newton methods
 - Iteratively update H_i to approximate second derivatives
 - Using just differences in gradients and variables
- Second-order (Newton type) method
 - Pros: rapid convergence
 - Cons: costly assemble, factorize and backsolve a new linear equation every iteration, huge storage requirement for large-scale model

Maximal Search Step



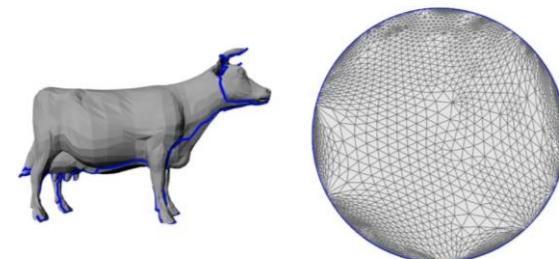
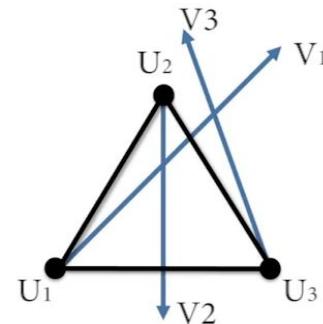
- Explicitly limit the maximal line search step to prevent foldover

$$\det \begin{pmatrix} (U_2 + V_2 t) - (U_1 + V_1 t) \\ (U_3 + V_3 t) - (U_1 + V_1 t) \end{pmatrix} = 0$$

- Use a locally supported barrier function to prevent the boundary collision

$$\max\left(0, \frac{\epsilon}{dist(U_i, \overline{U_1 U_2})} - 1\right)^2$$

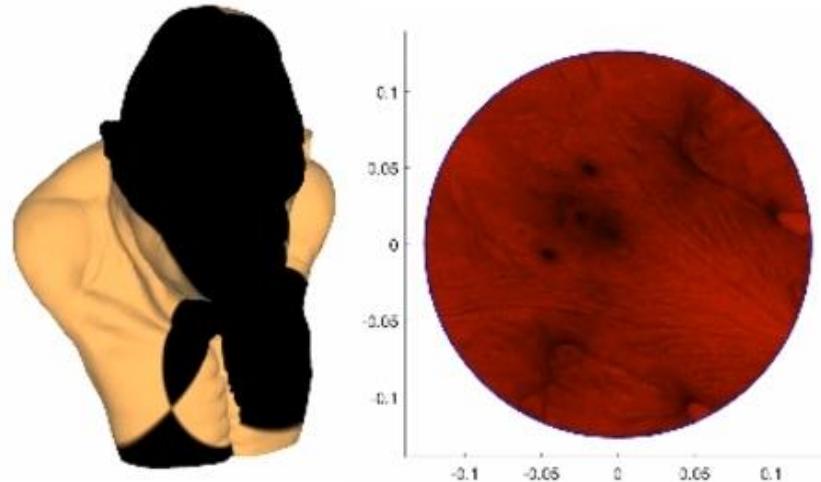
- Solver: L-BFGS (Quasi-Newton)



Blended Cured Quasi-Newton (BCQN)



- ✓ Blended quasi-Newton method
- ✓ Barrier-aware line search filtering
- ✓ Quasi-Newton method



Methodologies



- First-order method
 - Pros: only pre-factorization once, cost and storage remain tractable for large-scale model
 - Cons: slow convergence, easily trapped at a high distortion point
- Quasi-Newton methods
 - Iteratively update H_i to approximate second derivatives
 - Using just differences in gradients and variables
- Second-order (Newton type) method
 - Pros: rapid convergence
 - Cons: costly assemble, factorize and backsolve a new linear equation every iteration, huge storage requirement for large-scale model

Composite Majorization (CM)

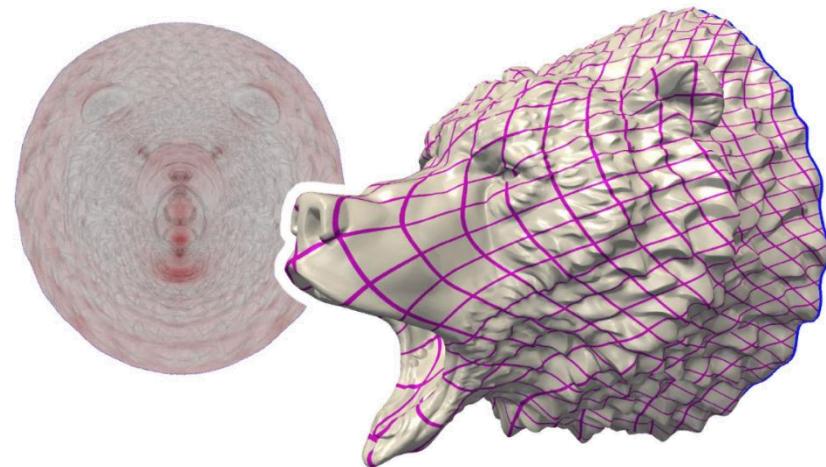


- ✓ Use a tight convex proxy to approximate the objective function by convex-concave decomposition

$$E = E^+ + E^-$$

✓ $H = \nabla^2 E^+$

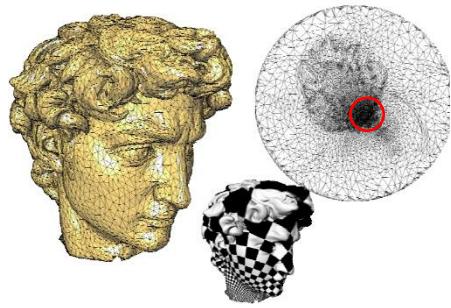
- ✓ Second-order method



Progressive Parameterization

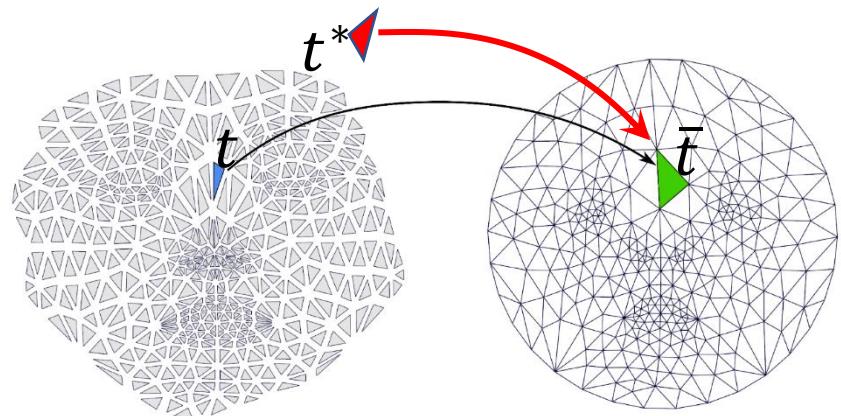


- Observation: even one extremely large distortion term can restrict the line search step size!



- Key Idea: If we kill extremely large distortion terms, we may obtain larger line search step size and thus faster convergence!

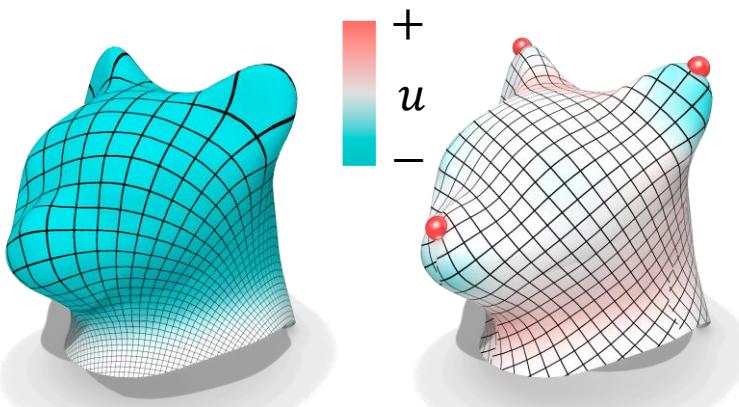
- Hybrid
 - First perform SLIM solver
 - Then use the CM solver



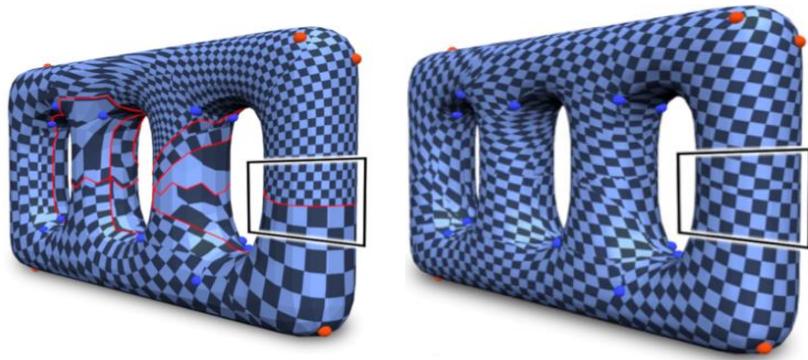
Ideal Reference triangles
(Isometric to their counterparts in S)

Intermediate
Parameterization

其他



Cone parameterization



Seamless parameterization

6

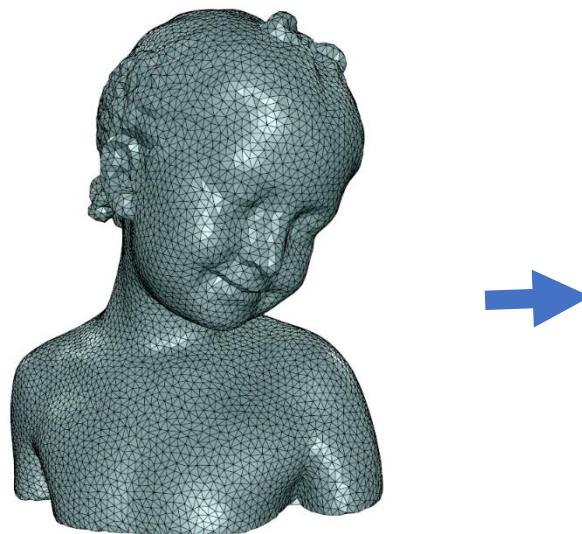
广义参数化及数据降维

創寰宇學府
育天下英才
嚴濟慈題
一九八八年五月

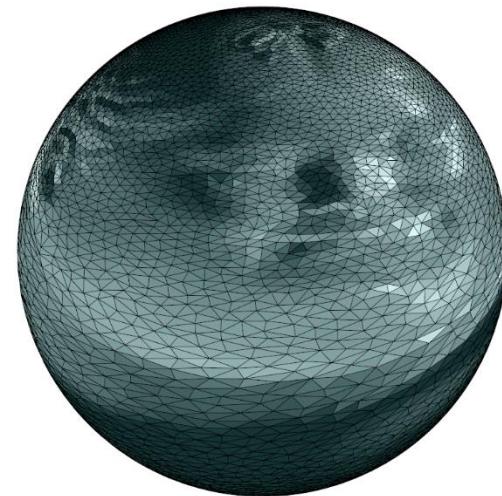
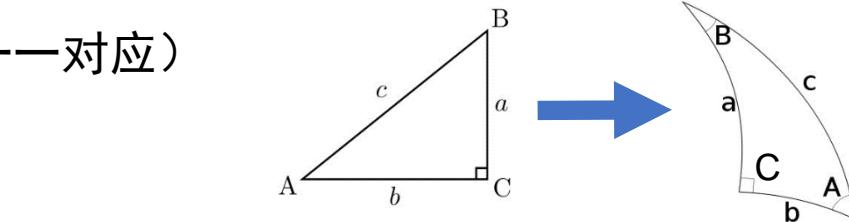
球面参数化 (Spherical parameterization)



- 构建封闭曲面与球面的映射（一一对应）



Genus-0 closed surface



Sphere

球面参数化



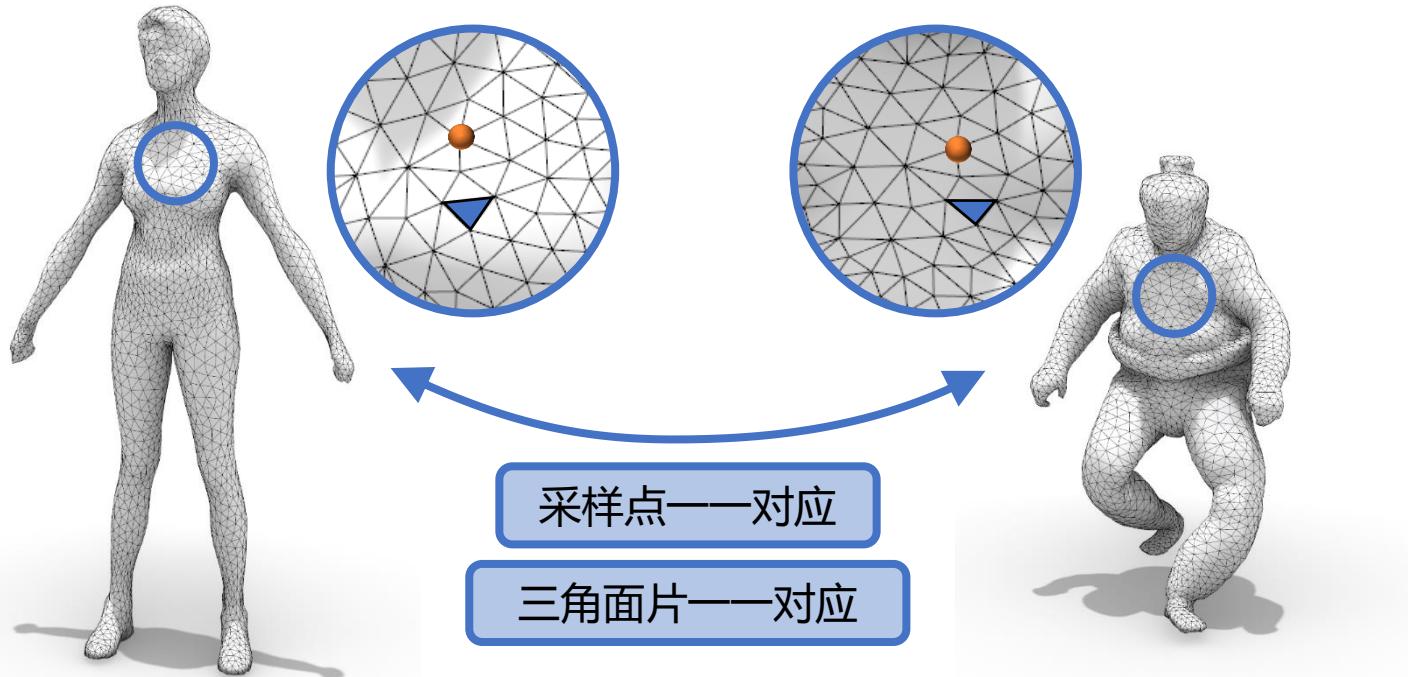
- 球面上的几何形变度量



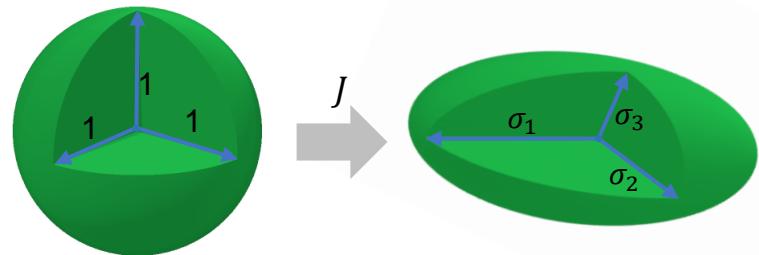
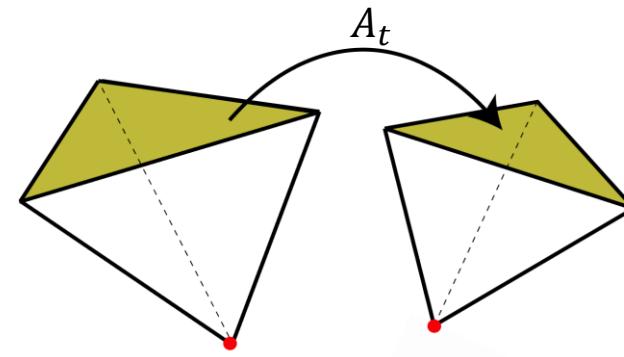
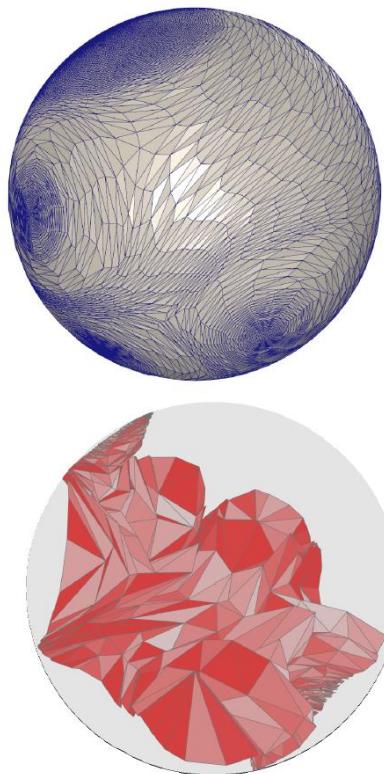
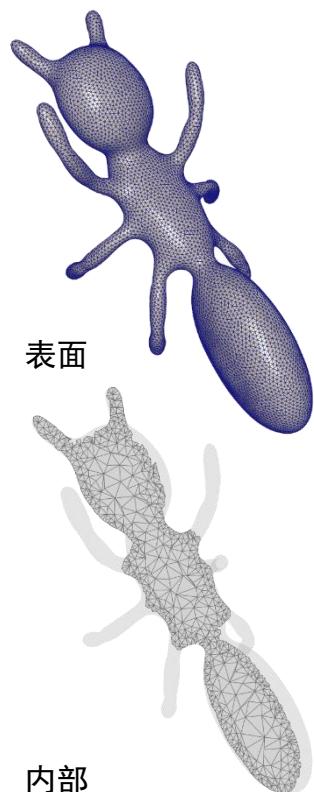
相容性网格 (Compatible mesh)



- 构建两个曲面之间的映射（一一对应）



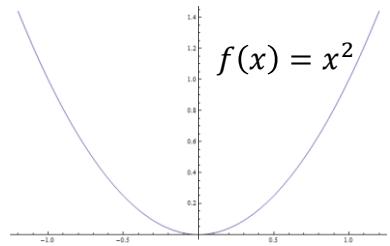
体参数化 (Volumetric parameterization)



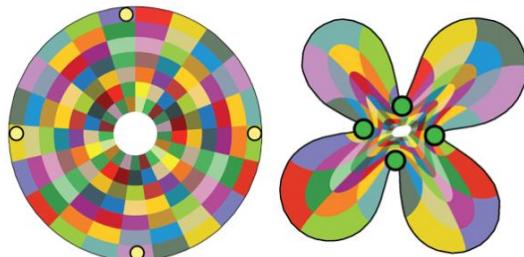
$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

Locally injective: $\det(J) > 0 \Leftrightarrow \sigma_3 > 0$

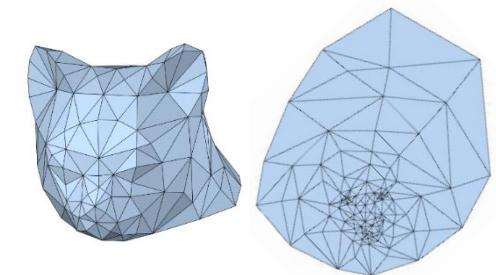
几何映射 (Geometric mapping)



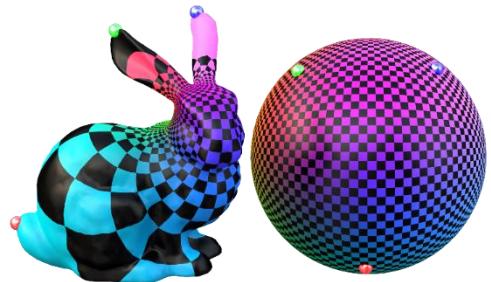
$$f: \mathbb{R} \rightarrow \mathbb{R}$$



$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



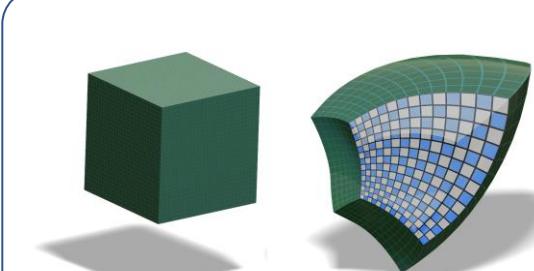
$$f: \mathbb{M} \rightarrow \mathbb{R}^2$$



$$f: \mathbb{M} \rightarrow \mathbb{S}^2$$

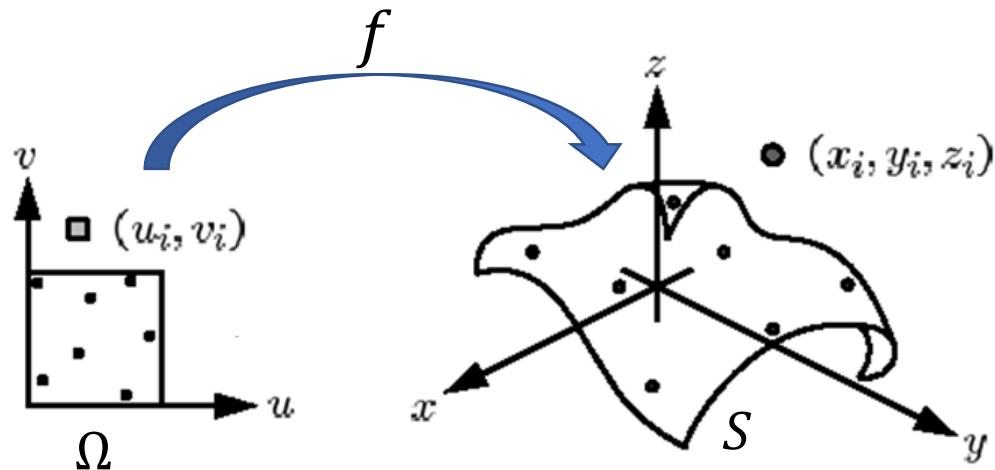


$$f: \mathbb{M} \rightarrow \mathbb{M}'$$



$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

Recap: 参数曲面 (Parametric surfaces)

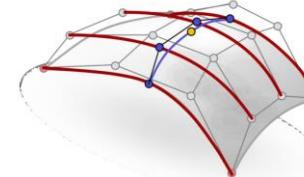


$$f: \Omega \rightarrow S$$

$$(u, v) \mapsto \begin{cases} x = x(u, v), \\ y = y(u, v), \\ z = z(u, v), \end{cases}$$

Tensor Product Surfaces

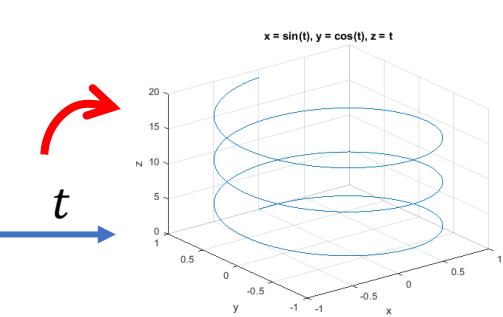
$$f(u, v) = \sum_{i=1}^n \sum_{j=1}^n b_i(u) b_j(v) \mathbf{p}_{i,j}$$



参数化表达（什么是参数？）

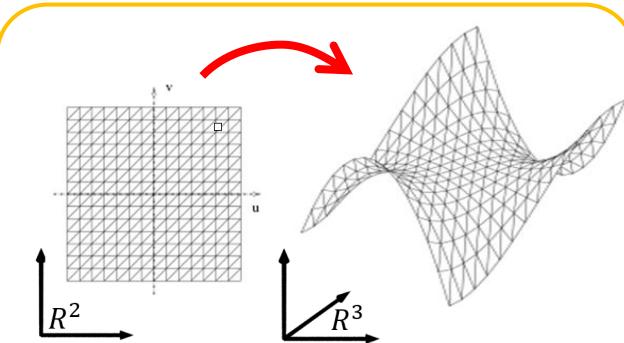


- 本征维度 (Intrinsic dimension): 相互独立的变量的个数



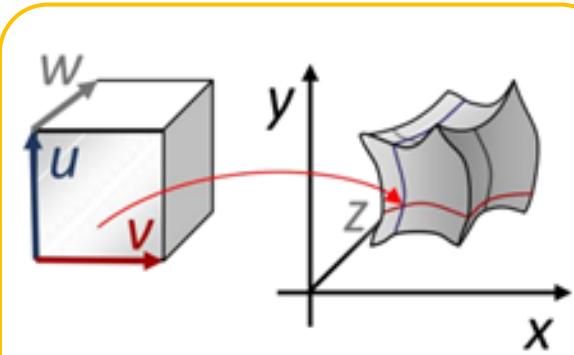
$$f: R^1 \rightarrow R^3$$

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$



$$f: R^2 \rightarrow R^3$$

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$



$$f: R^3 \rightarrow R^3$$

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$

参数化表达（什么是参数？）

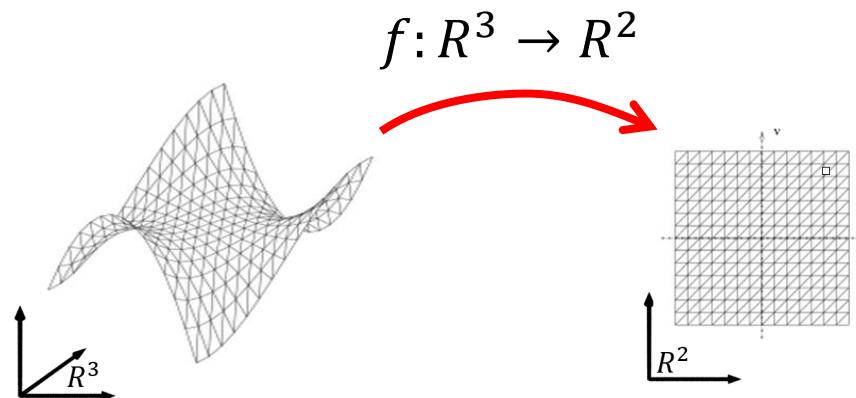


	Output: 1D	Output: 2D	Output: 3D
Input: 1D	 Function graph	 Plane curve	 Space curve
Input: 2D		 Plane warp	 Surface
Input: 3D			 Space warp

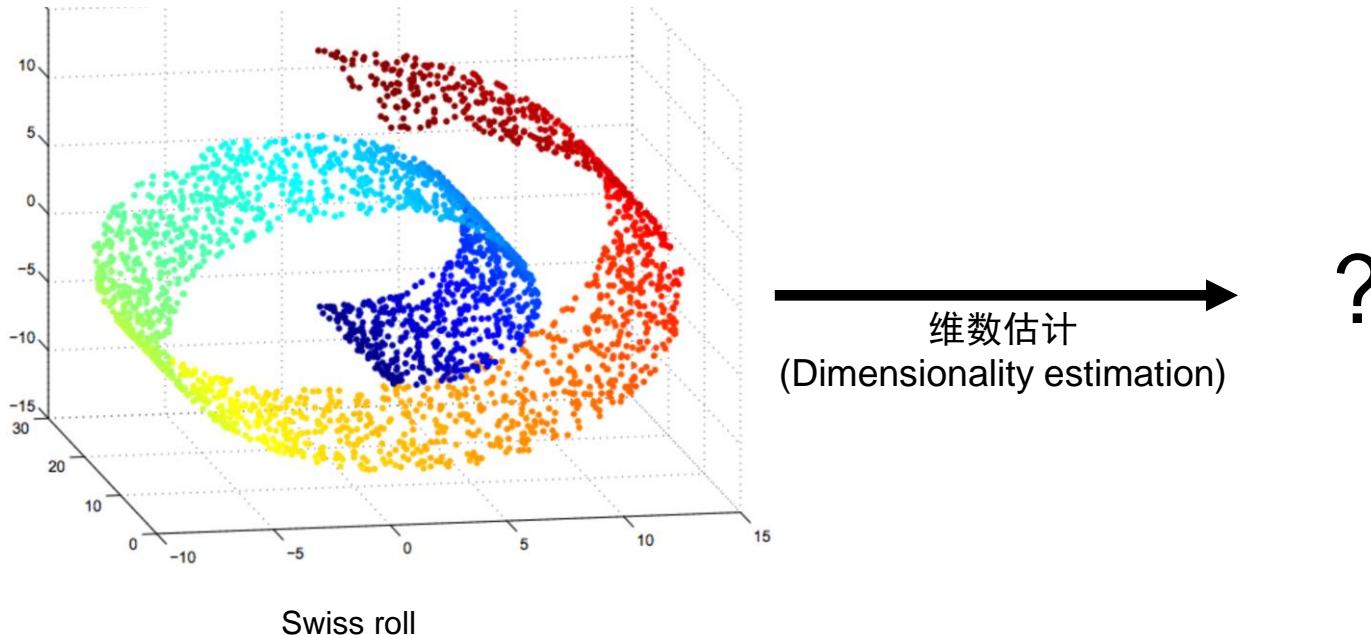
参数化：低维嵌入



- 降维映射一般有信息丢失
 - 丢失的信息大部分情况下不可逆（如低于本征维度），即无法恢复
- 寻求数据的本征维度仍然是个未解难题！



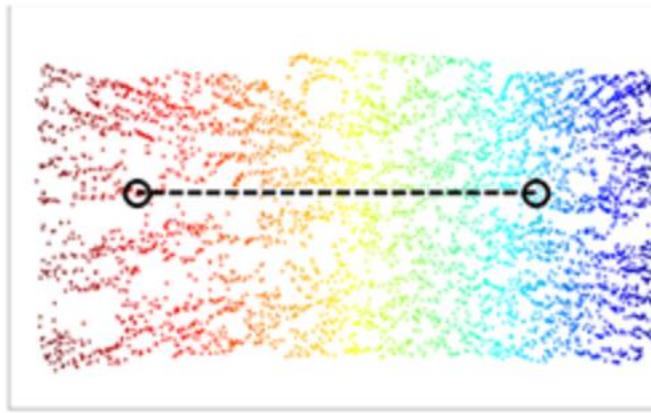
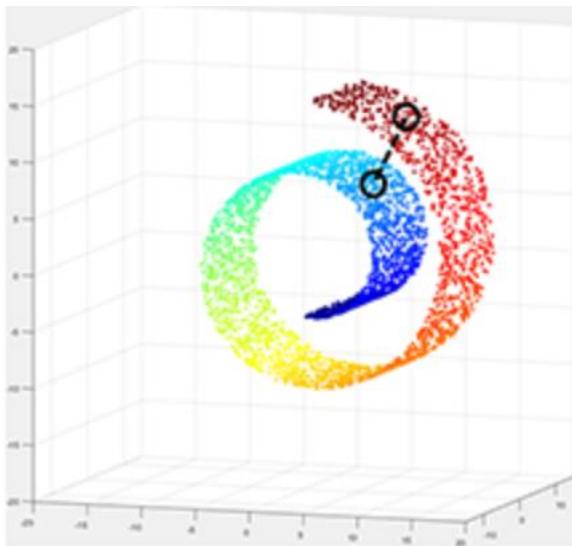
高维数据可能存在低维结构



高维数据可能存在低维结构



Dim = 2



一般映射

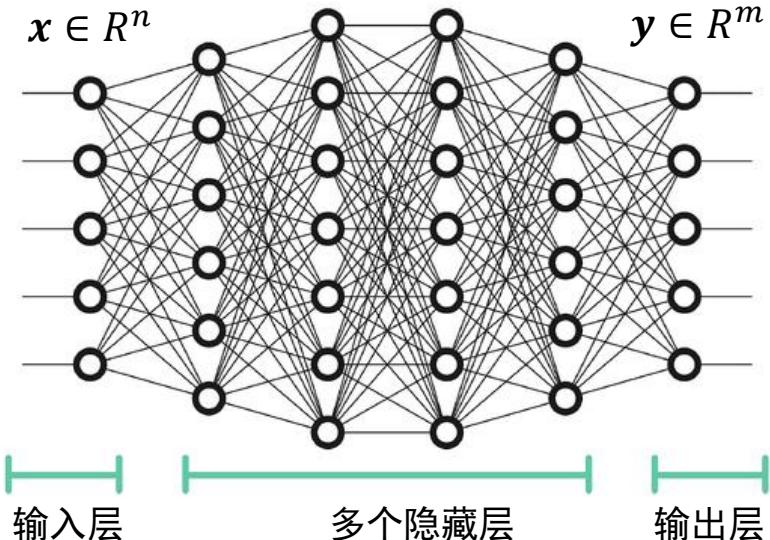


$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$y = f(x)$$

- 如果 $n < m$, 为低维到高维的映射 (高维的超曲面, n 维流形曲面)
 - x 为 y 的参数
 - 如果 $n > m$, 为降维映射
 - 一般信息有损失
 - 如果 R^n 中的点集刚好位于一个 m 维 (或小于 m) 的流形上, 则映射可能是无损的, 即可以被恢复的

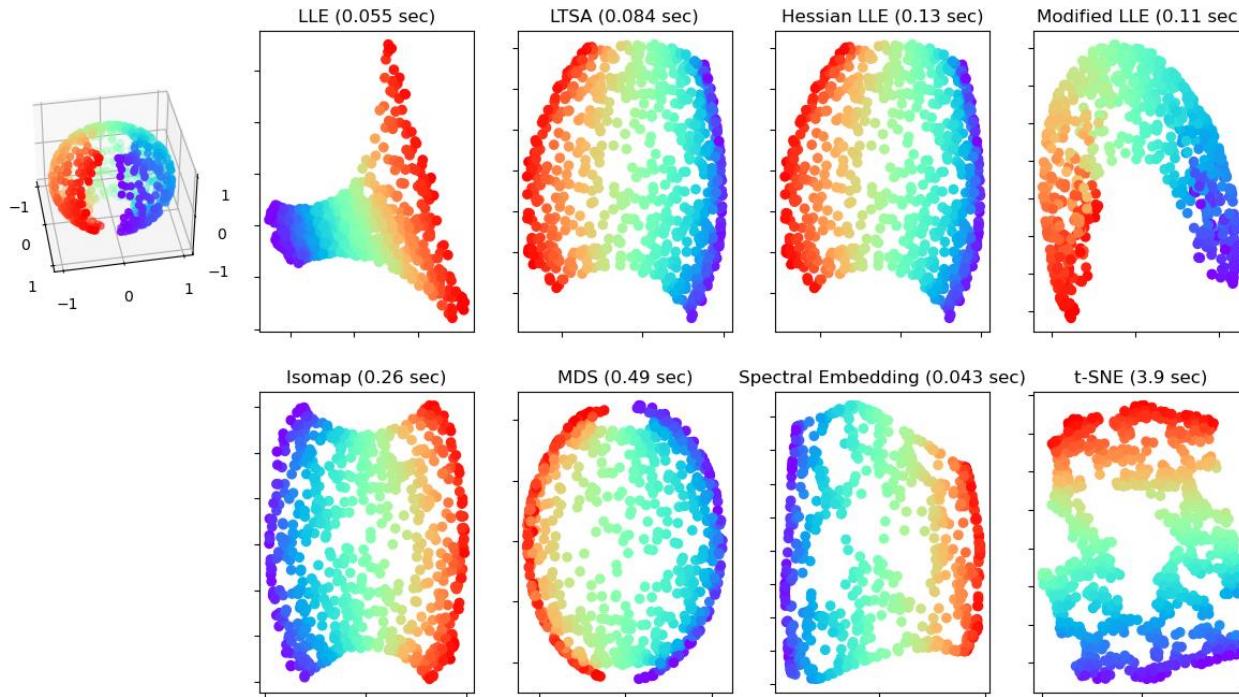
神经网络函数 f



高维数据的参数化



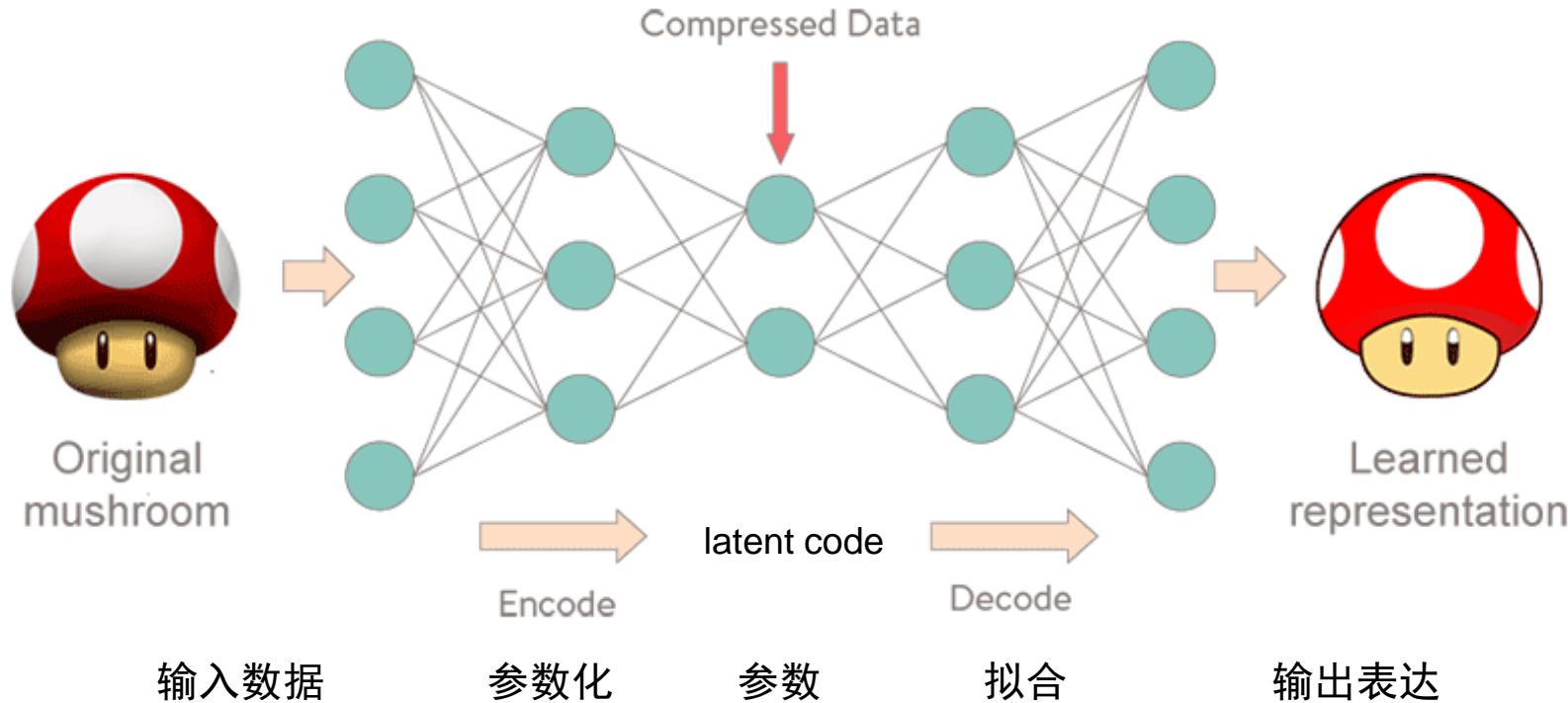
- 流形学习 (Manifold learning) / 降维 (Dimension reduction)



高维数据的参数化



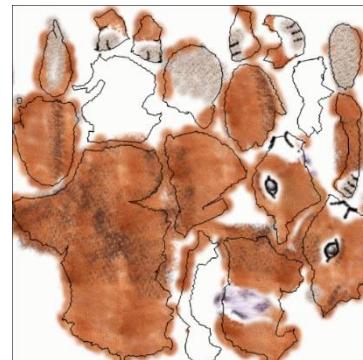
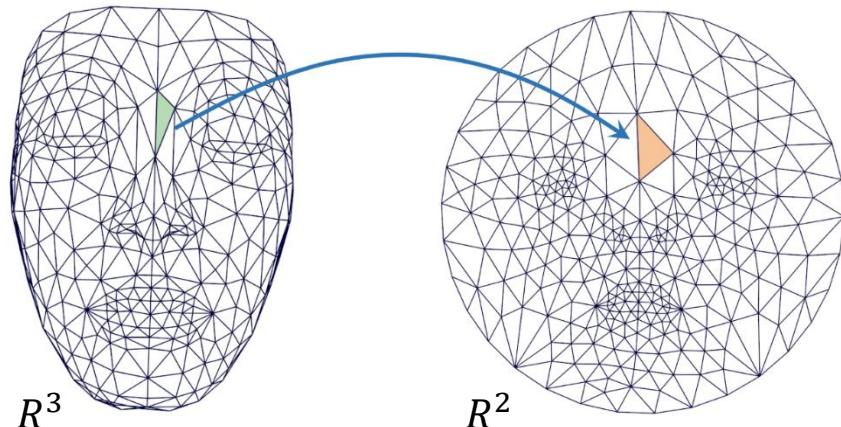
自编码神经网络 (AutoEncoder)



本节课小结



- 曲面参数化
 - 几何处理的基本问题
 - 有着非常广泛的应用
- 高质量参数化
 - 低扭曲、无翻转
 - 无自交
- 几何映射
- 高维数据处理
 - 数据降维/流形学习
 - 深度学习
- 广义参数化
 - 学习模型
 - 学习参数
 - 下游任务





中国科学技术大学
University of Science and Technology of China

谢 谢 !

