

# BA Jacobian矩阵推导

计算空间中第 $i$ 个三维点在第 $j$ 个相机投影的Jacobian矩阵。为书写方便，以下省略掉空间三维点的下标 $i$ 和相机的下标 $j$ 。假设空间三维点 $P$ ，坐标为 $\mathbf{X} = [X, Y, Z]^T$ ，在相机 $C$ 中的投影点为 $p$ ，坐标为 $\mathbf{x} = [u, v]^T$ ，其中与相机有关的参数是焦距 $f$ ，径向畸变系数 $k_0, k_1$ ，旋转矩阵(角轴法表示) $\mathbf{w} = [w_0, w_1, w_2]^T$ ，平移向量 $\mathbf{t} = [t_0, t_1, t_2]^T$ 。 $p$ 相对于相机参数以及三维点的Jacobian矩阵可以表示为

$$\begin{bmatrix} \frac{\partial u(\mathbf{C}, \mathbf{X})}{\partial \mathbf{C}} & \frac{\partial u(\mathbf{C}, \mathbf{X})}{\partial \mathbf{X}} \\ \frac{\partial v(\mathbf{C}, \mathbf{X})}{\partial \mathbf{C}} & \frac{\partial v(\mathbf{C}, \mathbf{X})}{\partial \mathbf{X}} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial f}, \frac{\partial u}{\partial k_0}, \frac{\partial u}{\partial k_1}, \frac{\partial u}{\partial w_0}, \frac{\partial u}{\partial w_1}, \frac{\partial u}{\partial w_2}, \frac{\partial u}{\partial t_0}, \frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}, \frac{\partial u}{\partial X}, \frac{\partial u}{\partial Y}, \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial f}, \frac{\partial v}{\partial k_0}, \frac{\partial v}{\partial k_1}, \frac{\partial v}{\partial w_0}, \frac{\partial v}{\partial w_1}, \frac{\partial v}{\partial w_2}, \frac{\partial v}{\partial t_0}, \frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_2}, \frac{\partial v}{\partial X}, \frac{\partial v}{\partial Y}, \frac{\partial v}{\partial Z} \end{bmatrix}$$

## 1. 相机投影过程

为了更为方便的求解每一项的偏导数，我们首先给出三维点到二维点的投影过程。

### 1.1 世界坐标系到相机坐标系的变换

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \mathbf{R}\mathbf{X} + \mathbf{t} = \mathbf{R} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix}$$

### 1.2 归一化像平面

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x_c}{z_c} \\ \frac{y_c}{z_c} \end{bmatrix}$$

### 1.3 相机成像平面

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} fd(k_0, k_1, r^2)x \\ fd(k_0, k_1, r^2)y \end{bmatrix},$$

$$\text{其中 } d(k_0, k_1, r^2) = 1 + (k_0 + k_1 r^2)r^2,$$

$$r^2 = x^2 + y^2$$

## 2. 链式法则推到 Jacobian 矩阵

### 2.1 计算关于焦距的偏导数 $\frac{\partial u}{\partial f}, \frac{\partial v}{\partial f}$

$$\frac{\partial u}{\partial f} = d(k_0, k_1, r^2)x$$

$$\frac{\partial v}{\partial f} = d(k_0, k_1, r^2)y$$

2.2 计算关于径向畸变系数 $\frac{\partial u}{\partial k_0}, \frac{\partial u}{\partial k_1}, \frac{\partial v}{\partial k_0}, \frac{\partial v}{\partial k_1}$

引入中间变量

$$\frac{\partial u}{\partial d(k_0, k_1, r^2)} = fx$$

$$\frac{\partial v}{\partial d(k_0, k_1, r^2)} = fy$$

$$\frac{\partial d(k_0, k_1, r^2)}{\partial k_0} = r^2$$

$$\frac{\partial d(k_0, k_1, r^2)}{\partial k_1} = r^4$$

根据链式法则，我们可以得到

$$\frac{\partial u}{\partial k_0} = \frac{\partial u}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial k_0} = fxr^2$$

$$\frac{\partial u}{\partial k_1} = \frac{\partial u}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial k_1} = fxr^4$$

$$\frac{\partial v}{\partial k_0} = \frac{\partial v}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial k_0} = fyr^2$$

$$\frac{\partial v}{\partial k_1} = \frac{\partial v}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial k_1} = fyr^4$$

计算关于平移向量和旋转向量的偏导数略微复杂，为了方便采用链式法则进行求导，引入一些中间量

$$\frac{\partial x}{\partial x_c} = \frac{1}{z_c}, \frac{\partial x}{\partial y_c} = 0, \frac{\partial x}{\partial z_c} = -\frac{x_c}{z_c^2} = -\frac{x}{z_c}$$

$$\frac{\partial y}{\partial x_c} = 0, \frac{\partial y}{\partial y_c} = \frac{1}{z_c}, \frac{\partial y}{\partial z_c} = -\frac{y_c}{z_c^2} = -\frac{y}{z_c}$$

$$\frac{\partial u}{\partial x} = fd(k_0, k_1, r^2)$$

$$\frac{\partial v}{\partial y} = fd(k_0, k_1, r^2)$$

$$\frac{\partial d(k_0, k_1, r^2)}{\partial r^2} = k_0 + 2k_1 r^2$$

$$\frac{\partial r^2}{\partial x_c} = 2x \frac{\partial x}{\partial x_c} = \frac{2x}{z_c}$$

$$\frac{\partial r^2}{\partial y_c} = 2y \frac{\partial y}{\partial y_c} = \frac{2y}{z_c}$$

$$\begin{aligned} \frac{\partial r^2}{\partial z_c} &= 2x \frac{\partial x}{\partial z_c} + 2y \frac{\partial y}{\partial z_c} = -2x \frac{x}{z_c} - 2y * \frac{y}{z_c} \\ &= -2 \frac{x^2 + y^2}{z_c} = -2 \frac{r^2}{z_c} \end{aligned}$$

$$\frac{\partial d(k_0, k_1, r^2)}{\partial x_c} = \frac{\partial d(k_0, k_1, r^2)}{\partial r^2} \frac{\partial r^2}{\partial x_c} = (k_0 + 2k_1 r^2) \frac{2x}{z_c}$$

$$\frac{\partial d(k_0, k_1, r^2)}{\partial y_c} = \frac{\partial d(k_0, k_1, r^2)}{\partial r^2} \frac{\partial r^2}{\partial y_c} = (k_0 + 2k_1 r^2) \frac{2y}{z_c}$$

$$\frac{\partial d(k_0, k_1, r^2)}{\partial z_c} = \frac{\partial d(k_0, k_1, r^2)}{\partial r^2} \frac{\partial r^2}{\partial z_c} = -(k_0 + 2k_1 r^2) \frac{2r^2}{z_c}$$

$$\frac{\partial u}{\partial x_c} = \frac{\partial u}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial x_c} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial x_c}$$

$$\frac{\partial u}{\partial y_c} = \frac{\partial u}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial y_c} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial y_c}$$

$$\frac{\partial u}{\partial z_c} = \frac{\partial u}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial z_c} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial z_c}$$

$$\frac{\partial v}{\partial x_c} = \frac{\partial v}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial x_c} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial x_c}$$

$$\frac{\partial v}{\partial y_c} = \frac{\partial v}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial y_c} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial y_c}$$

$$\frac{\partial v}{\partial z_c} = \frac{\partial v}{\partial d(k_0, k_1, r^2)} \frac{\partial d(k_0, k_1, r^2)}{\partial z_c} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial z_c}$$

2.3 计算关于平移向量的偏导数  $\frac{\partial u}{\partial t_0}, \frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}, \frac{\partial u}{\partial t_0}, \frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}$

引入中间变量

$$\frac{\partial x_c}{\partial t_0} = 1, \frac{\partial x_c}{\partial t_1} = 0, \frac{\partial x_c}{\partial t_2} = 0$$

$$\frac{\partial y_c}{\partial t_0} = 0, \frac{\partial y_c}{\partial t_1} = 1, \frac{\partial y_c}{\partial t_2} = 0$$

$$\frac{\partial z_c}{\partial t_0} = 0, \frac{\partial z_c}{\partial t_1} = 0, \frac{\partial z_c}{\partial t_2} = 1$$

根据链式法则求取关于平移向量的偏导数

$$\frac{\partial u}{\partial t_0} = \frac{\partial u}{\partial x_c} \frac{\partial x_c}{\partial t_0} = \frac{\partial u}{\partial x_c}$$

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial y_c} \frac{\partial y_c}{\partial t_1} = \frac{\partial u}{\partial y_c}$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial z_c} \frac{\partial z_c}{\partial t_2} = \frac{\partial u}{\partial z_c}$$

$$\frac{\partial v}{\partial t_0} = \frac{\partial v}{\partial x_c} \frac{\partial x_c}{\partial t_0} = \frac{\partial v}{\partial x_c}$$

$$\frac{\partial v}{\partial t_1} = \frac{\partial v}{\partial y_c} \frac{\partial y_c}{\partial t_1} = \frac{\partial v}{\partial y_c}$$

$$\frac{\partial v}{\partial t_2} = \frac{\partial v}{\partial z_c} \frac{\partial z_c}{\partial t_2} = \frac{\partial v}{\partial z_c}$$

2.4 计算关于旋转(角轴向量)的偏导数  $\frac{\partial u}{\partial w_0}, \frac{\partial u}{\partial w_1}, \frac{\partial u}{\partial w_2}, \frac{\partial v}{\partial w_0}, \frac{\partial v}{\partial w_1}, \frac{\partial v}{\partial w_2}$

根据 Rodrigues 公式

$$\mathbf{R} = \cos\|\mathbf{w}\| \mathbf{I}_{3 \times 3} + \frac{[\mathbf{w}]_{\times}}{\|\mathbf{w}\|} \sin(\|\mathbf{w}\|) + \frac{[\mathbf{w}]_{\times}^2}{\|\mathbf{w}\|^2} (1 - \cos(\|\mathbf{w}\|))$$

对于微小变化,  $\cos(\|\mathbf{w}\|) \approx 1, \sin(\|\mathbf{w}\|) \approx \|\mathbf{w}\|$

$$\delta \mathbf{R} \approx \mathbf{I}_{3 \times 3} + [\mathbf{w}]_{\times}$$

其中

$$[\mathbf{w}]_{\times} = \begin{bmatrix} 0, -w_2, w_1 \\ w_2, 0, -w_0 \\ -w_1, w_0, 0 \end{bmatrix}$$

在世界坐标系向相机坐标系转换过程中添加左扰动

$$\begin{aligned} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} &= \delta \mathbf{R} \mathbf{R} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \mathbf{T} = (\mathbf{I}_{3 \times 3} + [\mathbf{w}]_{\times}) \mathbf{R} \mathbf{X} + \mathbf{T} \\ &= \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} + \begin{bmatrix} 0, -w_2, w_1 \\ w_2, 0, -w_0 \\ -w_1, w_0, 0 \end{bmatrix} \begin{bmatrix} r_0 \mathbf{X} \\ r_1 \mathbf{X} \\ r_2 \mathbf{X} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} + \begin{bmatrix} -w_2 \mathbf{r}_1 \mathbf{X} + w_1 \mathbf{r}_2 \mathbf{X} \\ w_2 \mathbf{r}_0 \mathbf{X} - w_0 \mathbf{r}_2 \mathbf{X} \\ -w_1 \mathbf{r}_0 \mathbf{X} + w_0 \mathbf{r}_1 \mathbf{X} \end{bmatrix}$$

其中 $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ 是旋转矩阵 $\mathbf{R}$ 的第 1-3 行。

引入中间变量

$$\frac{\partial x_c}{\partial w_0} = 0, \frac{\partial x_c}{\partial w_1} = \mathbf{r}_2 \mathbf{X}, \frac{\partial x_c}{\partial w_2} = -\mathbf{r}_1 \mathbf{X}$$

$$\frac{\partial y_c}{\partial w_0} = -\mathbf{r}_2 \mathbf{X}, \frac{\partial y_c}{\partial w_1} = 0, \frac{\partial y_c}{\partial w_2} = \mathbf{r}_0 \mathbf{X}$$

$$\frac{\partial z_c}{\partial w_0} = \mathbf{r}_1 \mathbf{X}, \frac{\partial z_c}{\partial w_1} = -\mathbf{r}_0 \mathbf{X}, \frac{\partial z_c}{\partial w_2} = 0$$

根据链式法则求取对于旋转向量的偏导数

$$\frac{\partial u}{\partial w_0} = \frac{\partial u}{\partial y_c} \frac{\partial y_c}{\partial w_0} + \frac{\partial u}{\partial z_c} \frac{\partial z_c}{\partial w_0}$$

$$\frac{\partial u}{\partial w_1} = \frac{\partial u}{\partial x_c} \frac{\partial x_c}{\partial w_1} + \frac{\partial u}{\partial z_c} \frac{\partial z_c}{\partial w_1}$$

$$\frac{\partial u}{\partial w_2} = \frac{\partial u}{\partial x_c} \frac{\partial x_c}{\partial w_2} + \frac{\partial u}{\partial y_c} \frac{\partial y_c}{\partial w_2}$$

$$\frac{\partial v}{\partial w_0} = \frac{\partial v}{\partial y_c} \frac{\partial y_c}{\partial w_0} + \frac{\partial v}{\partial z_c} \frac{\partial z_c}{\partial w_0}$$

$$\frac{\partial v}{\partial w_1} = \frac{\partial v}{\partial x_c} \frac{\partial x_c}{\partial w_1} + \frac{\partial v}{\partial z_c} \frac{\partial z_c}{\partial w_1}$$

$$\frac{\partial v}{\partial w_2} = \frac{\partial v}{\partial x_c} \frac{\partial x_c}{\partial w_2} + \frac{\partial v}{\partial y_c} \frac{\partial y_c}{\partial w_2}$$

## 2.5 求关于三维点的偏导数

将世界坐标系到相机坐标系的变换写成如下形式

$$\begin{aligned} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} &= \mathbf{R} \mathbf{X} + \mathbf{t} = \mathbf{R} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix} \\ &= \begin{bmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix} \\ &= \begin{bmatrix} R_{00}X + R_{01}Y + R_{02}Z + t_0 \\ R_{10}X + R_{11}Y + R_{12}Z + t_1 \\ R_{20}X + R_{21}Y + R_{22}Z + t_2 \end{bmatrix} \end{aligned}$$

我们可以得到

$$\frac{\partial x_c}{\partial X} = R_{00}, \frac{\partial x_c}{\partial Y} = R_{01}, \frac{\partial x_c}{\partial Z} = R_{02}$$

$$\frac{\partial y_c}{\partial X} = R_{10}, \frac{\partial y_c}{\partial Y} = R_{11}, \frac{\partial y_c}{\partial Z} = R_{12}$$

$$\frac{\partial z_c}{\partial X} = R_{20}, \frac{\partial z_c}{\partial Y} = R_{21}, \frac{\partial z_c}{\partial Z} = R_{22}$$

根据链式法则，可以得到

$$\frac{\partial u}{\partial X} = \frac{\partial u}{\partial x_c} \frac{\partial x_c}{\partial X} + \frac{\partial u}{\partial y_c} \frac{\partial y_c}{\partial X} + \frac{\partial u}{\partial z_c} \frac{\partial z_c}{\partial X}$$

$$\frac{\partial u}{\partial Y} = \frac{\partial u}{\partial x_c} \frac{\partial x_c}{\partial Y} + \frac{\partial u}{\partial y_c} \frac{\partial y_c}{\partial Y} + \frac{\partial u}{\partial z_c} \frac{\partial z_c}{\partial Y}$$

$$\frac{\partial u}{\partial Z} = \frac{\partial u}{\partial x_c} \frac{\partial x_c}{\partial Z} + \frac{\partial u}{\partial y_c} \frac{\partial y_c}{\partial Z} + \frac{\partial u}{\partial z_c} \frac{\partial z_c}{\partial Z}$$

$$\frac{\partial v}{\partial X} = \frac{\partial v}{\partial x_c} \frac{\partial x_c}{\partial X} + \frac{\partial v}{\partial y_c} \frac{\partial y_c}{\partial X} + \frac{\partial v}{\partial z_c} \frac{\partial z_c}{\partial X}$$

$$\frac{\partial v}{\partial Y} = \frac{\partial v}{\partial x_c} \frac{\partial x_c}{\partial Y} + \frac{\partial v}{\partial y_c} \frac{\partial y_c}{\partial Y} + \frac{\partial v}{\partial z_c} \frac{\partial z_c}{\partial Y}$$

$$\frac{\partial v}{\partial Z} = \frac{\partial v}{\partial x_c} \frac{\partial x_c}{\partial Z} + \frac{\partial v}{\partial y_c} \frac{\partial y_c}{\partial Z} + \frac{\partial v}{\partial z_c} \frac{\partial z_c}{\partial Z}$$