Communication over a noisy channel.

(a) Let A be the event that a 0 is transmitted. Using the total probability theorem, the desired probability is

$$\mathbf{P}(A)(1-\epsilon_0)+(1-\mathbf{P}(A))(1-\epsilon_1)=p(1-\epsilon_0)+(1-p)(1-\epsilon_1).$$

(b) By independence, the probability that the string 1011 is received correctly is

$$(1-\epsilon_0)(1-\epsilon_1)^3.$$

(c) In order for a 0 to be decoded correctly, the received string must be 000, 001, 010, or 100. Given that the string transmitted was 000, the probability of receiving 000 is $(1-\epsilon_0)^3$, and the probability of each of the strings 001, 010, and 100 is $\epsilon_0(1-\epsilon_0)^2$. Thus, the probability of correct decoding is

$$3\epsilon_0(1-\epsilon_0)^2+(1-\epsilon_0)^3.$$

(d) When the symbol is 0, the probabilities of correct decoding with and without the scheme of part (c) are $3\epsilon_0(1-\epsilon_0)^2+(1-\epsilon_0)^3$ and $1-\epsilon_0$, respectively. Thus, the probability is improved with the scheme of part (c) if

$$3\epsilon_0(1-\epsilon_0)^2+(1-\epsilon_0)^3>(1-\epsilon_0),$$

or

$$(1-\epsilon_0)(1+2\epsilon_0)>1,$$

which is equivalent to $0<\epsilon_0<1/2$.

(e) Using Bayes' rule, we have

$$\mathbf{P}(0\mid 101)=\frac{\mathbf{P}(0)\mathbf{P}(101\mid 0)}{\mathbf{P}(0)\mathbf{P}(101\mid 0)+\mathbf{P}(1)\mathbf{P}(101\mid 1)}.$$

The probabilities needed in the above formula are

$$\mathbf{P}(0)=p,\quad \mathbf{P}(1)=1-p,\quad \mathbf{P}(101\mid 0)=\epsilon_0^2(1-\epsilon_0),\quad \mathbf{P}(101\mid 1)=\epsilon_1(1-\epsilon_1)^2.$$