Joint PMF drill #3.

1. We can find c knowing that the probability of the entire sample space must equal 1:

$$\begin{aligned} 1 &= \sum_{x=1}^{3}\sum_{y=1}^{3} p_{X,Y}(x,y) \\ &= c+c+2c+2c+4c+3c+c+6c \\ &= 20c, \end{aligned}$$

   and therefore, $c = \frac{1}{20}$.

2. $p_Y(2) = \sum_{x=1}^{3} p_{X,Y}(x,2) = 2c + 0 + 4c = 6c = \frac{3}{10}$.

3. Given $Z = YX^2$,

$$\begin{aligned} \mathbf{E}[Z \mid Y=2] &= \mathbf{E}[YX^2 \mid Y=2] \\ &= \mathbf{E}[2X^2 \mid Y=2] \\ &= 2\mathbf{E}[X^2 \mid Y=2]. \end{aligned}$$

   To calculate this conditional expectation, we first find the conditional PMF:

$$\begin{aligned} p_{X|Y}(x \mid 2) &= \frac{p_{X,Y}(x,2)}{p_Y(2)} \\ &= \begin{cases} \frac{1/10}{3/10} = \frac{1}{3}, & \text{if } x = 1, \\ \frac{1/5}{3/10} = \frac{2}{3}, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

   Therefore,

$$\begin{aligned} \mathbf{E}[Z \mid Y=2] &= 2\sum_{x=1}^{3} x^2 p_{X|Y}(x \mid 2) \\ &= 2\left((1^2)\cdot\frac{1}{3} + (3^2)\cdot\frac{2}{3}\right) \\ &= \frac{38}{3}. \end{aligned}$$

4. Yes. Given $X \neq 2$, the conditional distribution of X is the same no matter if $Y = 1$, 2, or 3: $\mathbf{P}(X = x \mid Y = y, X \neq 2) = \mathbf{P}(X = x \mid X \neq 2)$ for $x \in \{1,3\}$ and $y \in \{1,2,3\}$.

   For example,

$$\mathbf{P}(X = 1 \mid Y = 1, X \neq 2) = \mathbf{P}(X = 1 \mid Y = 3, X \neq 2) = \mathbf{P}(X = 1 \mid X \neq 2) = \frac{1}{3}.$$

   .

5. To calculate the conditional variance of Y given $X = 2$, we first find the conditional PMF:

$$p_{Y|X}(y \mid 2) = \frac{p_{X,Y}(2,y)}{p_X(2)}.$$

   The denominator can be calculated by summing over the y's:

$$p_X(2) = \sum_{y=1}^{3} p_{X,Y}(2,y) = c + 0 + c = 2c = \frac{1}{10}.$$

Therefore,

$$p_{Y|X}(y \mid 2) \quad = \quad \begin{cases} \frac{1/20}{1/10} = \frac{1}{2}, & \text{if } y = 1, \\ \frac{1/20}{1/10} = \frac{1}{2}, & \text{if } y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

With this conditional PMF, we can calculate the following two conditional expectations:

$$\mathbf{E}[Y^2 \mid X = 2] = \sum_{y=1}^{3} y^2 p_{Y|X}(y \mid 2) = (1^2) \cdot \frac{1}{2} + (3^2) \cdot \frac{1}{2} = 5,$$

$$\mathbf{E}[Y \mid X = 2] = \sum_{y=1}^{3} y p_{Y|X}(y \mid 2) = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2.$$

Finally, the conditional variance is calculated from these two conditional expectations as follows:

$$\mathrm{var}(Y \mid X = 2) = \mathbf{E}[Y^2 \mid X = 2] - \mathbf{E}[Y \mid X = 2]^2 = 5 - 2^2 = 1.$$