Recitation Note: A Testing Example with the Uniform Distribution

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This note will discuss some concepts in hypothesis testing with the $\mathsf{Unif}([0,\theta)])$ distribution.

Introduction

Recall that a hypothesis test can be thought of as a function

$$\psi(T(X_1,\dots,X_n)) = \mathbb{1}(T(X_1,\dots,X_n) \in R), \tag{1}$$

where the function outputs a 1 if we reject the null hypothesis and a 0 if we fail to reject our null hypothesis. T is the statistic we calculate from our data $X_1,\dots,X_n$.

We consider testing a null hypothesis H_0 against an alternative hypothesis H_1. A type I error occurs when we reject H_0 when it is true. A type II error occurs when we fail to reject H_0 when H_1 is true. The type I error rate is the probability of a type I error, and is also called the significance level of our test. The probability of a type II error is one minus the power of our test (this is how we define power).

Question

Suppose that we observe $X_1,\dots,X_n \overset{i.i.d.}{\sim} \mathsf{Unif}([0,\theta])$. The ultmate goal is to test $H_0 : \theta = 1$ versus $H_1 : \theta < 1$.

1. Give a test with significance level $\alpha = 0$.
2. Significance level of a test that has rejection region $R = [1-c, 1]$ $(c < 1)$
3. Significance level and max power of a test that has rejection region $R = [0, c]$ $(c < 1)$
4. What is the likelihood ratio test?
5. Suppose we collect 10 observations that are all less than 0.9. Run a test at significance level $\alpha = 0.05$ for the above hypotheses.

Solution

In order to do any sort of test, we need a statistic and it's distribution (and sometimes, the asymptotic distribution of this statistic suffices).

Let's find the maximum likelihood estimator. The likelihood function is

$$\begin{aligned} L(X_1, \ldots, X_n; \theta) &= \prod_{i=1}^{n} \frac{\mathbb{1}(X_i \in [0, \theta])}{\theta} \\ &= \frac{\mathbb{1}(0 \leq X_{(1)} \leq X_{(n)} \leq \theta)}{\theta^n}, \end{aligned} \tag{2}$$

where $X_{(1)}$ is the minimum X_i and $X_{(n)}$ is the maximum X_i. Notice that θ^{-n} is a decreasing function of theta, and $\mathbb{1}(0 \leq X_{(1)} \leq X_{(n)} \leq \theta)$ is only non-zero when $\theta \geq X_{(n)}$. Therefore, the maximum likelihood estimator of θ is $\hat{\theta}_{MLE} = X_{(n)}$.

We can try to find the distribution of $X_{(n)}$ by finding the cdf. This is

$$\begin{aligned} F_{X_{(n)}}(x) &= P(X_{(n)} < x) \\ &= (P(X_1 < x))^n \\ &= \left(\frac{x}{\theta}\right)^n. \end{aligned} \tag{3}$$

Therefore, any test we could do might involve the test statistic $T(X_1, \ldots, X_n) = X_{(n)}$. In this case, our test is

$$\psi(X_{(n)}) = \mathbb{1}(X_{(n)} \in R), \tag{4}$$

1. Let's first write down the definition of signficance level, which is also the type I error rate. For a test of level α, this is

$$\begin{aligned} \alpha &= P(\text{rejecting } H_0 | H_0 \text{ is true}) \\ &= P(\psi(X_{(n)}) = 1 | \theta = 1) \\ &= P(X_{(n)} \in R | \theta = 1) \end{aligned} \tag{5}$$

To find a test with significance level 0, we need $P(X_{(n)} \in R | \theta = 1) = 0$. Suppose that our region takes the form $[0, c]$. Then,

$$P(X_{(n)} \in [0, c] | \theta = 1) = \left(\frac{c}{1}\right)^n = 0 \tag{6}$$

implies that we must take $c = 0$. Thus, the test with significance level 0 only rejects the null hypothesis when $X_{(n)} = 0$ (which essentially means it never rejects the null hypothesis).

2. Next, let's look at the test that rejects when $X_{(n)} \in [1 - c, 1]$. This is a weird test that doesn't make any sense to do in practice. However, our definitions still apply and so we can still calculate things like power and significance levels. The significance level is

$$P(X_{(n)} \in [1 - c, 1] | \theta = 1) = 1 - \left(\frac{1 - c}{1}\right)^n = 1 - (1 - c)^n. \tag{7}$$

On the other hand, the probability of a type II error is

$$P(X_{(n)} \in [0, 1-c] | \theta < 1) = \left(\frac{1-c}{\theta}\right)^n. \tag{8}$$

Notice that if $\theta = 1 - c$, then the probability of a type II error is 1. and the power is

$$\inf_{\theta<1} 1 - P(X_{(n)} \in [0, 1-c]) = 1 - \sup_{\theta<1} \left(\frac{1-c}{\theta}\right)^n = 0. \tag{9}$$

Thus, we see that this test has a pretty poor significance level and a power of 0. This aligns with our intuition that this is a pretty bad test.

3. Now we go back to the test outlined in 1, which makes more intuitive sense. As calculated, in general, the significance level is c^n. Thus, we must take $c = \alpha^{1/n}$ for a test with level α. The power of this test is then

$$\inf_{\theta<1} 1 - P(X_{(n)} \in [\alpha^{1/n}, 1]) = \inf_{\theta<1} P(X_{(n)} \in [0, \alpha^{1/n}]) = \inf_{\theta<1} \left(\frac{\alpha^{1/n}}{\theta}\right)^n = \alpha. \tag{10}$$

Therefore, the power of this test (or minimum power at least) is α, when we take θ close to 1 under the alternative hypothesis.

4. To find the likelihood ratio test, we first must calculate the likelihood ratio statistic. This is given by

$$\Lambda = \frac{L(X_1, \ldots, X_n | \theta = 1)}{\max_\theta L(X_1, \ldots, X_n | \theta < 1)} = \frac{1}{1/X_{(n)}^n} = X_{(n)}^n. \tag{11}$$

The likelihood ratio test rejects when this ratio is small. As we can see, this is equivalent to the test we just derived in 3.

5. For a level $\alpha = 0.05$ test and 10 observations, our test in 3. gives a rejection region of $[0, 0.05^{.1}] \approx [0, 0.74]$. As we can see, if the maximum observation is 0.9, we fail to reject the null hypothesis with the given data (since the estimator $X_{(n)}$ would not fall into the rejection region).