Hw11

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1 Find the loglikelihood function

We have $x_1, x_2, \cdots, x_{100}$ data. The likelihood function is

$$L = \delta P(x_1)\Gamma P(x_2)\cdots\Gamma P(x_{100})I \tag{1.1}$$

Here $\delta = [0,1,0], I = [1,1,1]^T$

$$\Gamma = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P(x) = \begin{bmatrix} p_1(x) & 0 & 0 \\ 0 & p_2(x) & 0 \\ 0 & 0 & p_3(x) \end{bmatrix}$$

$p_1(x) = \frac{1}{\sqrt{2*\pi}*0.5}e^{-\frac{(x-1)^2}{2*(0.5)^2}}, p_2(x) = \frac{1}{\sqrt{2*\pi}}e^{-\frac{(x-6)^2}{2}}, p_3(x) = \frac{1}{\sqrt{2*\pi}*0.5}e^{-\frac{(x-3)^2}{2*(0.5)^2}}$. Then the

$$loglikelihood = log(L) = -172.66 \tag{1.2}$$

```
> da=read.table("~/Downloads/time series analysis/data_for_HW11.txt",sep=',',header=F)
> xt=da$V2
> gamma=matrix(c(1/2,1/2,0,1/3,1/3,1/3,0,1/2,1/2),nrow=3,byrow=T)
> delta=t(c(0,1,0))
> p=matrix(0,nrow=3,ncol=3)
> mu=c(1,6,3)
> sigma=c(0.5,1,0.5)
> one=rep(1,3)
> l=matrix(0,nrow=1,ncol=3)
> alp=matrix(0,nrow=1,ncol=3)
> beta=matrix(0,nrow=3,ncol=1)
> bet=matrix(0,nrow=3,ncol=3)
> state=matrix(0,nrow=1,ncol=3)
> for (i in c(1:100))
+ {
+   for(j in c(1:3))
+   {p[j,j]=dnorm(xt[i],mean=mu[j],sd=sigma[j])}
+   if(i==1){l=delta%*%p}
+   else{l=l%*%gamma%*%p}
+   if(i==50){alpha=l}
```

```
+   if(i>50){ if(i==51){bet=gamma%*%p}else{bet=bet%*%gamma%*%p} }
+
+ }
> L=log(l%*%one)[1,1]
> L

[1] -172.6599
```

2 Find the forward probability $\alpha_{50}(j)$ for $j = 1, 2, 3$

We know the formula for α is simple.

$$\alpha_1 = \delta P(x_1), \alpha_t = \alpha_{t-1} * \Gamma * P \tag{2.1}$$

So we can get the result $\alpha_{50} = [2.5e - 68, 1.87e - 37, 5.68e - 50]$

```
> alpha

             [,1]         [,2]         [,3]
[1,] 2.542636e-68 1.187351e-37 5.683856e-50
```

3 Find the backward probability $\beta_{50}(j)$ for $j = 1, 2, 3$

The definition for β is

$$\beta_t(j) := pr(X_{t+1:T} = x_{t+1:T} | C_t = j) \tag{3.1}$$

And we have a formula for it

$$\beta_t = \prod_{s=t+1}^{T} (\Gamma * P(x_s)) * I \tag{3.2}$$

Then we get the result. $\beta = [4.17e - 41, 8.71e - 39, 1.31e - 38]^T$

```
> beta=bet%*%one
> beta

             [,1]
[1,] 4.166726e-41
[2,] 8.713305e-39
[3,] 1.306969e-38
```

4 Predict the state of the Markov chain at time T+h,for h=5

The definition of state is

$$Pr(C_{T+h=i} | X_{1:T} = x_{1:T}) = \frac{pr(C_{T+h=i}, X_{1:T} = x_{1:T})}{pr(X_{1:T} = x_{1:T})} \tag{4.1}$$

The state formula is

$$C_t = \frac{\delta P(x_1)\Gamma P(x_2)\cdots\Gamma P(x_{100})\Gamma^h}{\delta P(x_1)\Gamma P(x_2)\cdots\Gamma P(x_{100})I} \tag{4.2}$$

So the result is $C_t = [0.286, 0.428, 0.286]$

```
> d=l%*%one
> state=l%*%gamma%*%gamma%*%gamma%*%gamma%*%gamma/(d[1,1])
> state
```

```
          [,1]      [,2]      [,3]
[1,] 0.2857486 0.428498 0.2857534
```

5 Predict the emission distribution at time T+h,for h=5

$$pr(X_{T+h} = x|X_{1:T} = x_{1:T}) = \frac{\delta P(x_1)\Gamma P(x_2)\cdots\Gamma P(x_{100})\Gamma^h * P(x)I}{\delta P(x_1)\Gamma P(x_2)\cdots\Gamma P(x_{100})I} \tag{5.1}$$

So the distribution function is

$$f(x) = 0.286 * p_1(x) + 0.428 * p_2(x) + 0.286 * p_3(x) \tag{5.2}$$

$$= \frac{0.286}{\sqrt{2*\pi}*0.5}e^{-\frac{(x-1)^2}{2*(0.5)^2}} + \frac{0.428}{\sqrt{2*\pi}}e^{-\frac{(x-6)^2}{2}} + \frac{0.286}{\sqrt{2*\pi}*0.5}e^{-\frac{(x-3)^2}{2*(0.5)^2}} \tag{5.3}$$

Here is the graph

```
> tt=seq(-5,15,0.1)
> yy=rep(0,201)
> yy=state[1]*dnorm(tt,mean=mu[1],sd=sigma[1])+state[2]*dnorm(tt,mean=mu[2],sd=sigma[2])+state[3]*dnorm(tt
> plot(tt,yy,type="l")
```

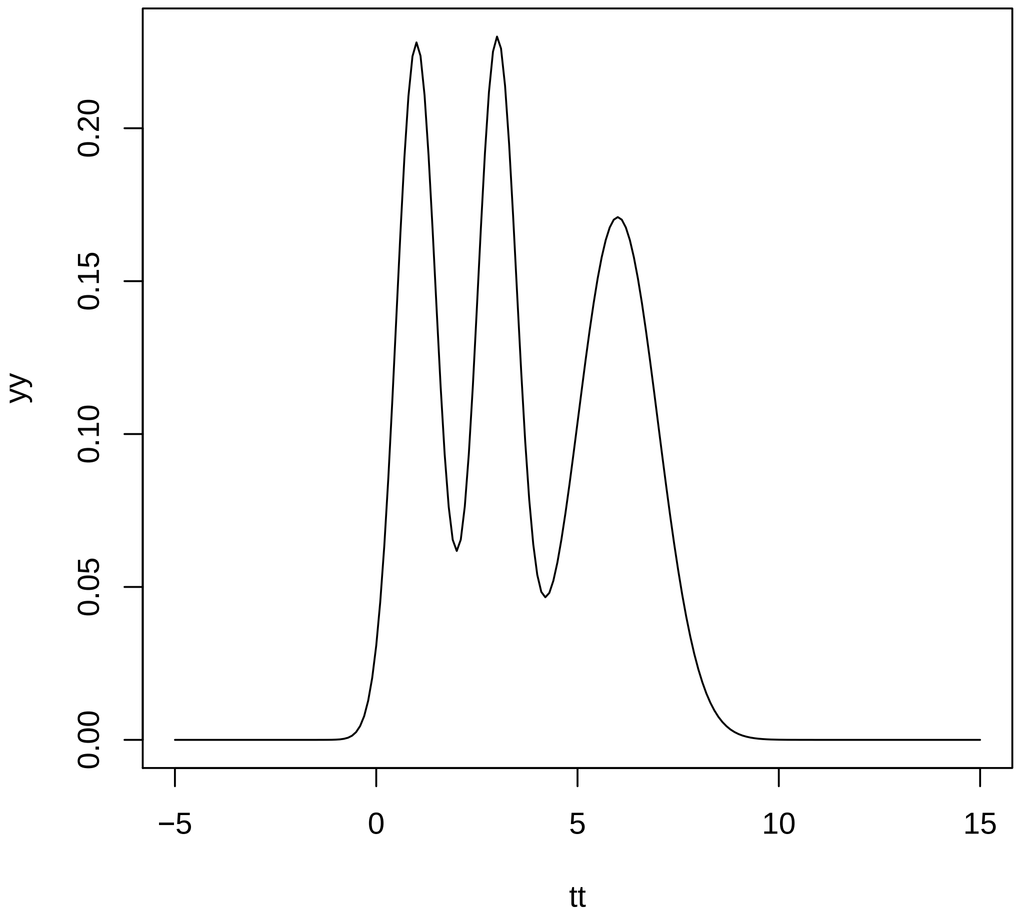

Figure 1: distribution function