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Collaboration Statement:

Total hours spent: 2 hrs

I consulted the following resources:

- The recorded course videos
- Internet
- ...

Links: [HW0 instructions] [collab. policy]

Contents

Problem 1

A desk drawer contains 3 sheets of paper, identical in all respects (size etc) except:

- sheet 1: both sides are colored orange
- sheet 2: both sides are colored blue
- sheet 3: one blue side and one orange side

The sheets are shaken up together in the drawer. One sheet is drawn uniformly at random and put face down on the ground. Only one side (the ”face up” side) is visible.

You observe one random draw from this process: the chosen sheet’s face-up side shows a blue color.

Naturally, you are still uncertain about the color of the **face-down** side of the chosen sheet.

1a: Problem Statement

Formalize this problem by defining all relevant random variables. For each variable, clearly define the possible values.

1a: Solution

We could define two random variables and their sample space as follows:

S (i.e. the sheet we choose):= { sheet1, sheet2, sheet3 } (i.e. possible values)

FU (i.e. the face-up color) := { orange, blue } (i.e. possible colors)

1b: Problem Statement

Compute the joint probability table for all possible configurations of your two random variables.

Write your answers as a 3x2 table (rows should match to sheets, columns should match to face-up colors).

1b: Solution

We could use "product rule" to calculate the joint probability.

e.g. if we choose sheet1 and face-up color is orange, then $P(\mathbf{S} = sheet1, \mathbf{FU} = orange) = \frac{1}{3} \times 1 = \frac{1}{3}$ ... Then we could get the table as follows (rows match to sheets, columns match to face-up colors):

	orange	*blue*
sheet1	$\frac{1}{3}$	0
sheet2	0	$\frac{1}{3}$
sheet3	$\frac{1}{6}$	$\frac{1}{6}$

1c: Prompt

What is the probability that the face-down side of the chosen sheet is orange?

1c: Solution

Let **FD** be a r.v. denoting the face-down color, then the required probability is:

$\mathbf{P}(\mathbf{FD} = orange|\mathbf{FU} = blue)$

$= \frac{\mathbf{P}(\mathbf{FD}=orange,\mathbf{FU}=blue)}{\mathbf{P}(\mathbf{FU}=blue)}$ /* using definition of the conditional probability */

From the joint probability table we get in 1b), we could have: $\mathbf{P}(\mathbf{FU} = blue) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

The event "face-down has color orange, face-up has color blue" is actually equivalent to "choosing sheet3 and face-up color is blue", so $\mathbf{P}(\mathbf{FD} = orange, \mathbf{FU} = blue) = \mathbf{P}(\mathbf{S} = sheet3, \mathbf{FU} = blue) = \frac{1}{6}$ /* by looking up the joint probability table in 1b) */

Thus, $\mathbf{P}(\mathbf{FD} = orange|\mathbf{FU} = blue) = \frac{\mathbf{P}(\mathbf{FD}=orange,\mathbf{FU}=blue)}{\mathbf{P}(\mathbf{FU}=blue)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

Problem 2

Consider a joint model of 3 random variables:

- H : a discrete r.v. with K possible values
- E_1: a discrete r.v. with observed value e_1
- E_2: a discrete r.v. with observed value e_2

Imagine three possible sets of PMFs we could know completely:

- (i) $p(H)$, $p(e_1, e_2)$, $p(e_1|H), p(e_2|H)$
- (ii) $p(H)$, $p(e_1, e_2)$, $p(e_1, e_2|H)$
- (iii) $p(H)$, $p(e_1|H), p(e_2|H)$

2a: Problem Statement

Consider any joint distribution over H, E_1, E_2.

For each set of PMFs above (i) - (iii), do we have enough information to calculate $p(H|e_1, e_2)$? Provide 1-2 sentences of justification.

2a: Solution

Based on the definition of conditional probablity, we could rewrite $\mathbf{P}(H|e_1, e_2)$ as:

$\mathbf{P}(H|e_1, e_2) = \frac{\mathbf{P}(H, e_1, e_2)}{\mathbf{P}(e_1, e_2)} = \frac{\mathbf{P}(e_1, e_2|H)\mathbf{P}(H)}{\mathbf{P}(e_1, e_2)}$

Then,

for (i), it has $\mathbf{P}(e_1, e_2)$ and $\mathbf{P}(H)$, but no $\mathbf{P}(e_1, e_2|H)$ (and $\mathbf{P}(e_1, e_2|H)$ is not definitely equal to $\mathbf{P}(e_2|H) \times \mathbf{P}(e_1|H)$ unless E_1 and E_2 are conditional independent on H). So we don't have enough information.

for (ii), we have all the needed pieces.

for (iii), since it's the subset of (i), and we don't have enough information for (i). So does (iii).

2b: Problem Statement

Now suppose E_1 and E_2 are **conditionally independent** given H.

For each set of PMFs above (i) - (iii), does this additional assumption allow us to calculate $p(H|e_1, e_2)$? Provide 1-2 sentences of justification.

2b: Solution

Now if we have this conditional independence info, then

for (i), we have enough information, since now $\mathbf{P}(e_1, e_2|H) = \mathbf{P}(e_2|H) \times \mathbf{P}(e_1|H)$ because E1 and E2 are conditionally independent on H.

for (ii), since it has enough info even without conditional independence requirement, it still provides enough info now.

for (iii), it still doesn't have enough information, since the vital piece $\mathbf{P}(e_1, e_2)$ is still missed.