Approach 1: Mathematical: Digital Root

Formula for the Digital Root

There is a known formula to compute a digital root in a decimal numeral system

$$dr_{10}(n) = 0, \qquad \text{if } n = 0$$

$$dr_{10}(n) = 9, \qquad \text{if } n = 9k$$

$$dr_{10}(n) = n \mod 9, \qquad \text{if } n = 9k$$

How to derive it? Probably, you already know the following proof from school, where it was used for a divisibility by 9: "The original number is divisible by 9 if and only if the sum of its digits is divisible by 9". Let's revise it briefly.

The input number could be presented in a standard way, where $d_0, d_1, ..d_k$ are digits of n:

$$n = d_0 + d_1 \cdot 10^1 + d_2 \cdot 10^2 + ... + d_k \cdot 10^k$$

One could expand each power of ten, using the following:

$$10 = 9 \cdot 1 + 1$$
$$100 = 99 + 1 = 9 \cdot 11 + 1$$
$$1000 = 999 + 1 = 9 \cdot 111 + 1$$
$$...$$
$$10^k = 1\underbrace{00..0}_{\text{k times}} = \underbrace{99..9}_{\text{k times}} + 1 = 9 \cdot \underbrace{11..1}_{\text{k times}} + 1$$

That results in

$$n = d_0 + d_1 \cdot (9 \cdot 1 + 1) + d_2 \cdot (9 \cdot 11 + 1) + ... + d_k \cdot (9 \cdot \underbrace{11..1}_{\text{k times}} + 1)$$

and could be simplified as

$$n = (d_0 + d_1 + d_2 + ... + d_k) + 9 \cdot (d_1 \cdot 1 + d_2 \cdot 11 + ... + d_k \cdot \underbrace{11..1}_{\text{k times}})$$

The last step is to take the modulo from both sides:

$$n \mod 9 = (d_0 + d_1 + d_2 + ... + d_k) \mod 9$$

and to consider separately three cases: the sum of digits is 0, the sum of digits is divisible by 9, and the sum of digits is *not* divisible by nine:

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