Reliability parameter estimation of the Inverse Flexible Weibull Distribution for Type-I censored samples

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3 authors:

A. Nasr
Ecole Supérieure des Sciences et Techniques de Tunis

Soufiane Gasmi
Université de Tunis

Fayçal Ben Hmida
Université de Tunis-Ecole Nationale Supérieure d'Ingénieurs de Tunis ENSIT (ex E…

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Parameter estimation of the flexible Weibull distribution for type I censored samples

Arwa Nasr[a], Soufiane Gasmi[b] and Fayçal Ben Hmida[a]

[a]Department of Electrical engineering, University of Tunis, The National Higher Engineering School of Tunis, Tunisia; [b]Department of Mathematics, University of Tunis, The National Higher Engineering School of Tunis, Tunisia

ABSTRACT

Often in practice one is interested in the situation where the lifetime data are censored. Censorship is a common phenomenon frequently encountered when analyzing lifetime data due to time constraints. In this paper, the flexible Weibull distribution proposed in Bebbington *et al.* [*A flexible Weibull extension*, Reliab. Eng. Syst. Safety 92 (2007), pp. 719–726] is studied using maximum likelihood technics based on three different algorithms: Newton Raphson, Levenberg Marquardt and Trust Region reflective. The proposed parameter estimation method is introduced and proved to work from theoretical and practical point of view. On one hand, we apply a maximum likelihood estimation method using complete simulated and real data. On the other hand, we study for the first time the model using simulated and real data for type I censored samples. The estimation results are approved by a statistical test.

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Reliability; flexible Weibull distribution; parameter estimation; statistical test; censoreddata

1. Introduction

The Weibull distribution was proposed by Waloddi Weibull [15], this distribution was widely used to resolve many reliability engineering problems and to model lifetime data. Since 1970, this law was developed and modified to have several forms and extensions. We can quote some of them: the four-parameter Weibull distribution proposed by Kies [7], the exponentiated Weibull distribution studied by Mudholkar and Srivastave [11], then Lai *et al.* [8] suggested the modified Weibull distribution. Famoye *et al.* [4] proposed the beta-Weibull distribution by coupling the beta density and the Weibull distribution function. Nikulin and Haghighi introduced the generalized power Weibull family [13].

Often, during a study for an industrial system the results are incomplete. In this case, we obtain incomplete data called: censored data. These data come from the fact that we have no access to all information. We will focus our work on the study of models based on censored data which are not widely dealt by researchers. Notice that censorship is the most frequently encountered phenomenon when collecting survival data. In the last few

CONTACT Arwa Nasr arwa.nasr@gmail.com Department of Electrical engineering, University of Tunis, The National Higher Engineering School of Tunis, 5, Avenue Taha Hussein, B.P.56, Bab Menara 1008, Tunis, Tunisia

decades, the progressive censoring schemes have become commonly used in the literature because of its flexibility in removing failed units from the test.

We cite some of these works: to estimate the model parameters, Lee *et al.* [9] extract the derivatives of the log-likelihood function to obtain the maximum likelihood estimates (MLEs) of the parameters in the case of censored simulated data. Also, Wang *et al.* [14] used the MLE via the expectation maximization algorithm to estimate the BUR XII parameters using censored data. The MLEs of the inverse Weibull distribution parameters with censored data were studied and obtained by Gusmao *et al.* [6]. The progressive type II censoring schemes have been applied to exponential distributions in [10], also it is used by Zhang *et al.* [17] for several popular lifetime models.

The beta-Weibull model was applied by Lee *et al.* [9] for two censored data sets of bus-motor failures rate. In Nasr *et al.* [12], the reliability and the maintainability parameters of the Weibull distribution were estimated based on type I censored data and by using the virtual age model of Kijima type I and type II. Gasmi *et al.* [5] estimated the modified Weibull distribution parameters proposed by Zaindin and Sarhan [16] in the case of type I censored data. Bebbington *et al.* [2] introduced a two-parameter aging distribution, the flexible Weibull distribution (FWD), which represents a generalization of the Weibull distribution.

The research objective of this paper is to develop an MLE method to obtain more accurately estimated parameters based on complete and censored data and using the FWD which is quite simple and yet very flexible to model reliability data.

Graphical resolution techniques have been frequently used in the literature but they are not accurate enough. In this work, we present an analytical parameter estimation technique – the maximum likelihood method – which is widely used by researches in different fields. This technique is complicated for kinds of distributions because of the complexity of the likelihood function but leads to less estimation errors.

The rest of the paper is organized as follows. Section 2 is devoted to the study of the flexible Weibull model using simulated and real data sets in the case of complete samples. We mention that our study is based on the maximum likelihood method of estimation using three different algorithms – Newton Raphson (NR), Levenberg Marquardt (LM) and Trust Region reflective (TR). After that, we performed the estimation of model parameters which belongs to the family of Weibull. Our results were compared to those found by Bebbington [2]. The K–S statistical test was established to check the compatibility between the model and the real data sets.

Section 3 is dedicated to the parameter estimation of the FWD in the case of type I censored samples. In this section, simulated and real censored data are used. The flexible Weibull model has not been studied by researchers in the case of censored data hence we obtained new estimation results. The estimation results were validated by the K–S statistical test. The conclusion is presented in Section 4.

2. Estimation of the FWD in the case of complete data

We treat in this section the parameters estimation of the FWD in the maximum likelihood sense, which is characterized by its asymptotic normality and consistency, and based on complete simulated and real data.

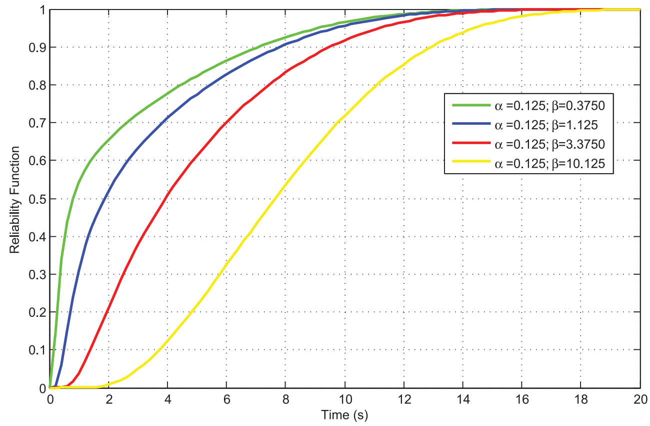

Figure 1. Reliability function.

2.1. Flexible Weibull distribution

The cumulative distribution function (CDF) and the survival function (SF) of the FWD are given, respectively, by the following expression:

$$F(t,\theta) = 1 - \mathrm{e}^{(-\mathrm{e}^{at-b/t})}, \quad t > 0, \tag{1}$$

$$S(t,\theta) = \mathrm{e}^{(-\mathrm{e}^{at-b/t})}, \quad t > 0, \tag{2}$$

where $\theta = (a, b)$ and a,$b > 0$.

The probability density function (PDF) and the hazard rate function (HRF) corresponding to Equation (1) are, respectively, as follows:

$$f(t,\theta) = \left(a + \frac{b}{t^2}\right) \mathrm{e}^{(at-b/t)}\, \mathrm{e}^{(-\mathrm{e}^{at-b/t})}, \tag{3}$$

$$h(t,\theta) = \left(a + \frac{b}{t^2}\right) \mathrm{e}^{(at-b/t)}. \tag{4}$$

In contrast to many other generalized Weibull distribution, we remark that the hazard rate expression of this distribution is not complicated.

Figures 1, 2, 3 and 4 represent, respectively, the CDF, the reliability function R, the PDF and the HRF for different selected values of a and b. We mention that when we decrease b, the HRF becomes more bathtub shape and when a increases, the bathtub becomes less deep. For different values of a and b, the HRF is considered as increasing failure rate (IFR), increasing failure rate average (IFRA) and modified bathtub (MBT) shaped which explains the flexibility of the studied model. Bebbington *et al.* [2] studied the behavior of the failure rate function, so we can conclude that:

- If $ab \geq \frac{27}{64}$, the distribution is IFR.
- If and only if $ab \geq \frac{1}{4}$, the distribution is IFRA which includes the IFR case, as needed.
- If $\frac{1}{4} < ab < \frac{27}{64}$, the distribution is IFRA and not IFR.
- If $ab < \frac{27}{64}$, the hazard rate has an MBT shaped.

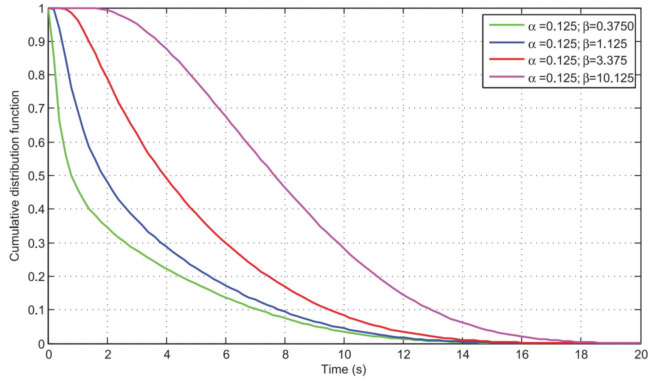

Figure 2. Cumulative distribution function.

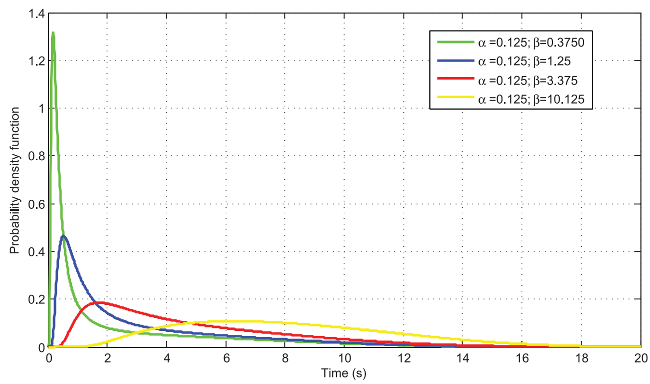

Figure 3. Probability density function.

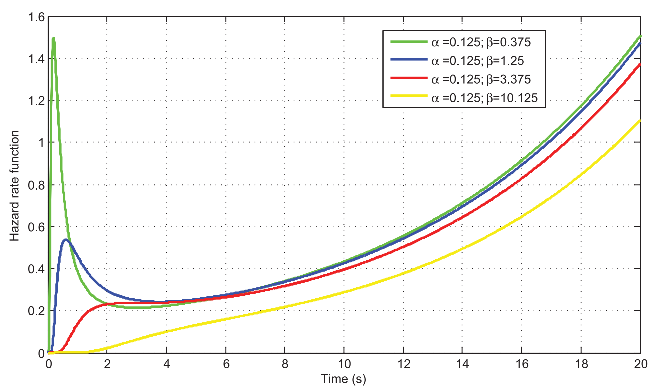

Figure 4. Hazard rate function.

We remark that when b decreases, the failure rate function becomes more bathtub-like, in consideration when a increases, the bathtub becomes shallower. So we can conclude that the model has a great deal of flexibility.

Table 1. Simulated data.

0.0372	0.0451	0.0479	0.0581	0.0595	0.0601	0.0638	0.0653	0.0669	0.0688	0.0714	0.0783	0.0783	0.0792
0.0805	0.0816	0.0818	0.0867	0.0870	0.0872	0.0909	0.0927	0.0959	0.0977	0.0992	0.1004	0.1012	0.1029
0.1040	0.1054	0.1067	0.1069	0.1085	0.1098	0.1120	0.1140	0.1142	0.1155	0.1170	0.1179	0.1227	0.1234
0.1236	0.1249	0.1259	0.1294	0.1427	0.1454	0.1484	0.1511	0.1511	0.1521	0.1530	0.1572	0.1590	0.1594
0.1608	0.1622	0.1640	0.1644	0.1718	0.1723	0.1733	0.1779	0.1847	0.1916	0.1950	0.1993	0.2147	0.2151
0.2173	0.2254	0.2262	0.2269	0.2282	0.2422	0.2429	0.2540	0.2552	0.2638	0.2665	0.2749	0.2773	0.2797
0.2961	0.3020	0.3151	0.3215	0.3257	0.3316	0.3361	0.3379	0.3439	0.3445	0.3764	0.3837	0.4077	0.4107
0.4159	0.4170	0.4194	0.4362	0.4543	0.4606	0.4664	0.4728	0.5152	0.5220	0.5455	0.5739	0.5744	0.5751
0.5839	0.5843	0.6531	0.6572	0.6845	0.7046	0.7305	0.7305	0.8158	0.8967	0.9339	0.9568	1.0016	1.0863
1.1011	1.1118	1.1222	1.1604	1.1684	1.1714	1.1734	1.3669	1.3756	1.4401	1.4629	1.5162	1.5574	1.6267
1.6604	1.8987	2.0276	2.0457	2.0859	2.1226	2.1518	2.1632	2.1934	2.1999	2.2949	2.4126	2.4228	2.4671
2.4807	2.4810	2.6383	2.6662	2.6736	2.6785	2.7448	2.7477	2.9442	2.9876	2.9923	3.0443	3.0776	3.1126
3.1879	3.3403	3.4742	3.5379	3.8037	3.8141	4.2147	4.3987	4.4541	4.5193	4.6251	4.6852	4.6860	4.7397
4.7734	5.0678	5.0710	5.1921	5.1925	5.4180	5.4995	5.6697	5.8701	5.9070	5.9193	5.9849	5.9924	6.0019
6.0845	6.2048	6.4564	8.6892										

Another advantage of this distribution is the fact that: $\lim_{t\to 0} h(t) = 0$ and $\lim_{t\to+\infty} h(t) = +\infty$. We can conclude that the HRF is always increasing.

2.2. MLEs using simulated data

This subsection is devoted to the lifetime data simulation and the parameter estimation in the case of simulated data. We mention that the maximum likelihood technique is a common statistical method used to infer the probability distribution parameters for a given sample. The parameter estimation of the model by the likelihood method requires two steps: Step 1: get the likelihood expression function from the model formulation and assumptions. Step 2: find the parameter values maximizing this function. This method consists in constructing of the likelihood function (constructed from the PDF of the period of system operation) and maximizes its logarithm with respect to the unknown parameters.

The simulation has been made by writing some computer programs using Matlab. To achieve this purpose, we need to set the FWD parameters as follows: $a = 0.2$ and $b = 0.2$, and generate samples from the FWD population by solving the following equation:

$$e^{at_i - b/t_i} + \log(1 - U) = 0, \tag{5}$$

where $i = 1, \ldots, n$, n indicates the sample size and U is a random variable uniformly distributed on the interval (0,1). Table 1 represents 200 simulated data generated from the FWD.

Let $x = (x_1, \ldots, x_n)$ be a random sample of the FWD with unknown parameter vector $\theta = (a, b)$.

The likelihood function $L(t, \theta)$ is given by

$$\begin{aligned} L(t,\theta) &= \prod_{i=1}^{n} f(t_i,\theta) \\ &= \prod_{i=1}^{n} \left(a + \frac{b}{t_i^2}\right) e^{(at - b/t_i)}\, e^{(-e^{at_i - b/t_i})}. \end{aligned} \tag{6}$$

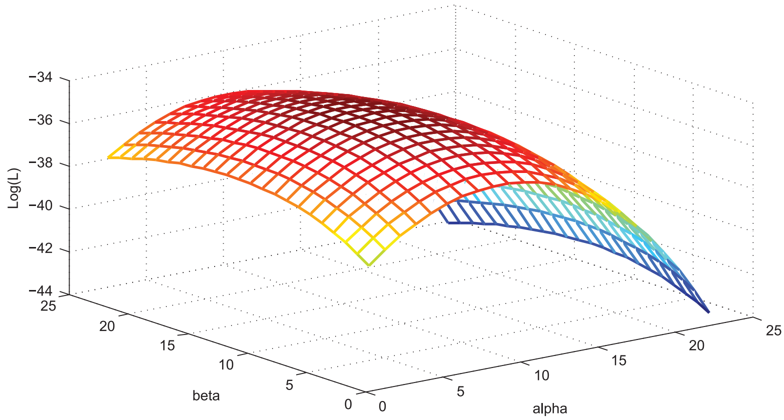

Figure 5. Log-likelihood function.

The corresponding log-likelihood function is given by

$$\log(L) = \sum_{i=1}^{n} \log\left(a + \frac{b}{t_i^2}\right) + \left(a\sum_{i=1}^{n} t_i - \sum_{i=1}^{n} \frac{b}{t_i}\right) - \sum_{i=1}^{n} \mathrm{e}^{(at_i - b/t_i)}. \tag{7}$$

Therefore, we obtained a nonlinear system consisting of two equations in two parameters as in [2]. To obtain the FWD estimates, we have to solve the first partial derivatives equations by equating it to zero. Since it is not possible to solve analytically the system, we have chosen the numerical resolution using iterative techniques such as TR and NR [3]. To estimate the FWD parameters, we use the following three algorithms: NR, LM and TR algorithms. To find the optimal values that maximizes the log-likelihood function, we solve the obtained nonlinear system using the simulated data mentioned in Table 1. The initial values introduced at the beginning of the program should belong to the domain convergence of the distribution. The estimation results using the MLE technique for complete simulated data are $\hat{a} = 0.2095$ and $\hat{b} = 0.1995$.

Figure 5 illustrates the empirical CDF, the CDF and the 95% lower and upper confidence bounds, noted, respectively, LCD and UCB, for the CDF of the 200 simulated data by setting $a = 0.2$ and $b = 0.2$. We remark that the CDF follows properly the empirical CDF.

2.3. Application

To demonstrate the performance of the FWD in practice, we consider two real data sets which represent the data of primary and secondary pumps installed inthe RSG gas reactor [1] mentioned, respectively, in Tables 2 and 3. The primary and secondary pumps are the mechanical components in the RSG-GAS reactor cooling system which are used for routine operation.

The data are taken from the operation log-book of the RSG-GAS reactor during 10 years of reactor operation since 1987.

For the two data sets, we compare the results of the FWD fits and its sub-model: the exponential distribution ED.

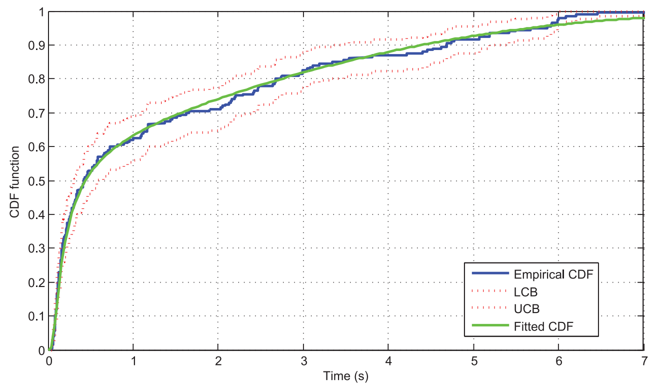

Figure 6. CDF and empirical CDF of the FWD(a,b) for complete simulated data.

Table 2. Time between failures (thousands of hours) of primary reactor pumps.

Pumps	TBF
JEOI-AP01	2.767 0.114 2.319 9.026 0.003 0.258
JEOI-AP02	2.805 0.008 1.132 10.136
JEOI-AP03	0.054 0.595 0.018 0.009

Table 3. Time between failures (thousands of hours) of secondary reactor pumps.

Pumps	TBF
PAOI-APO1	2.160 0.746 0.402 0.954 0.491 6.560 4.992
PAO2-APO2	3.474 0.150 0.358 0.101 1.359 3.465 1.060 0.614 1.921 4.082 0.199
PAO3-APO3	0.605 0.273 0.070 0.062 5.320

The SF of the Weibull distribution:

$$S(t) = \mathrm{e}^{(-t/\eta)^{\beta}}, \quad t \geqslant 0, \tag{8}$$

with $\beta, \eta > 0$.

$$S(t) = \mathrm{e}^{-\alpha t}, \quad t \geqslant 0, \tag{9}$$

with $\alpha > 0$.

Tables 2 and 3 represent the time between failure (TBF) of the primary pump and the secondary pump. We mention that JEOI-AP01, JEOI-AP02 and JEOI-AP03 are the three primary pumps and PAOI-APO1, PA02-AP02 and PA03-AP03 of the three secondary pumps of the RSG-GAS reactor, they are identically in type and duty [1].

We start modeling of the FWD in the case of complete data. For this purpose, we use the time between failure data set of the primary pumps mentioned in Table 2.

In this case, contrary to the authors of [2], we have to use X_{ij} in the likelihood function to represent the time between successive failures of pumps instead of t_i which materially represents the time to failure (TTF) and it is why authors have not found a good estimation results.

Table 4. Parameter estimation using TBF of the primary pumps data.

	$\hat{a}$	$\hat{b}$	*p*-Values	K–S	L
FWD	0.1342	0.0125	.7009	0.2476	−13.7660
ED	0.4787	–	.1846	0.3956	−24.3126

Table 5. Parameter estimation using TBF of the secondary pumps data.

	$\hat{a}$	$\hat{b}$	*p*-Values	K–S	L
FWD	0.2084	0.2519	.9150	0.1558	−32.7878
ED	0.5835	–	.7639	0.1866	−35.3908

So the log-likelihood function becomes

$$\mathrm{Log}(L) = \sum_{i=1}^{n}\sum_{j=1}^{k} \log\left(a + \frac{b}{(X_{ij})^2}\right) + \left(aX_{ij} - \frac{b}{X_{ij}}\right) - \mathrm{e}^{(aX_{ij}-b/X_{ij})}. \tag{10}$$

The first partial derivatives of the log-likelihood function with respect to the two parameters of the FWD are

$$\frac{\partial \mathrm{Log}(L)}{\partial a} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\left(\frac{1}{a + \frac{b}{X_{ij}{}^2}}\right) + X_{ij} - X_{ij}\,\mathrm{e}^{(aX_{ij}-b/X_{ij})}\right), \tag{11}$$

$$\frac{\partial \mathrm{Log}(L)}{\partial b} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\frac{1}{a(X_{ij})^2 + b} - \frac{1}{X_{ij}} + \frac{1}{X_{ij}}\mathrm{e}^{(aX_{ij}-b/X_{ij})}\right), \tag{12}$$

with n being the number of pumps and k representing the TBFs data for each pumps.

We obtain the MLEs $\hat{\theta} = (\hat{a}, \hat{b})$ by maximizing the log-likelihood function. The estimation results of the FWD and the ED using the TBF of the primary pumps in the case of complete data are summarized in Table 4.

Second, we proceed to the FWD reliability parameter estimation using the secondary pumps data sets. So we try to solve the system consisting the derivatives with respect to flexible Weibull parameters and we set them equal to zero.

The estimation results of the FWD and ED using the TBF for the secondary pumps data are summarized in Table 5.

From Table 5, we remark that the estimates of the FWD using TBF of the secondary pumps in the case of complete data are equal to (0.2084, 0.2519). We rectify the estimation results using the maximum likelihood estimator presented by Bebbington in [2] as $\hat{\theta} = (0.0207,\ 2.5875)$. To confirm our results, we check graphically the values of $\hat{a}$ and $\hat{b}$ that maximize the log-likelihood function. We mention that the parameter estimation results, in both cases, are equal using NR, LM and TR algorithms.

Figure 6 illustrates the log-likelihood function in 3D, it varies depending on the shape and the scale parameters. The optimum values of a and b that maximize the log-likelihood function are checked by curves in Figures 7 and 8. We remark that the log-likelihood function have a maximum for $\hat{a} = 0.19$ and $\hat{b} = 0.26$, which proves the results obtained using maximum likelihood method, we notice that these values verifies the numerical results found previously.

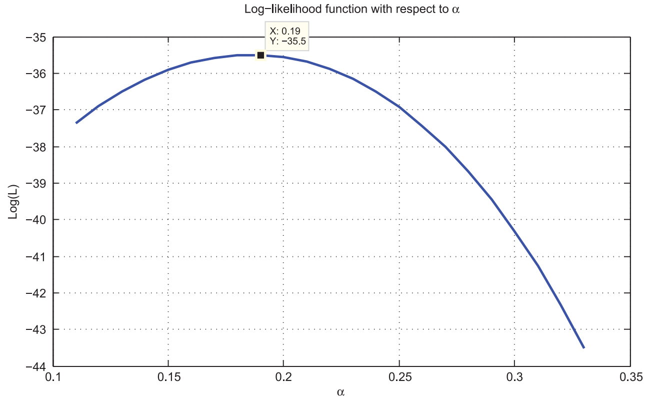

Figure 7. Log-likelihood function with respect to *a*.

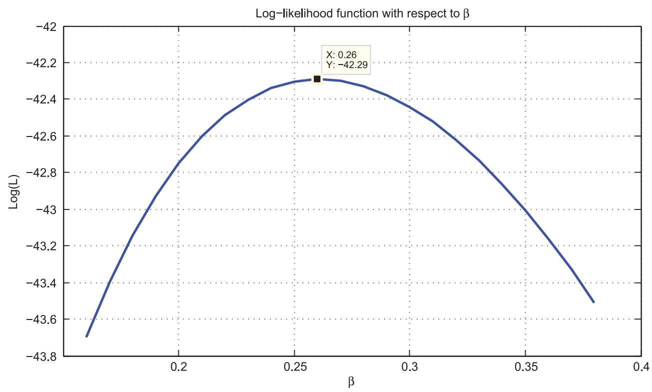

Figure 8. Log-likelihood function with respect to *b*.

As a second step, we perform the following null hypothesis test; H_0: $b=0$ and we set $a = \log(\lambda)$, the data follow the exponential distribution (ED). In favor of the alternative hypothesis H_a: the data follow the FWD.

We use a nonparametric test statistics, the K–S test with a level of significance equal to 0.05, to test the null hypothesis mentioned below against H_a. We accept H_0 with the p-value under the condition $p-$value > 0.05. In Tables 4 and 5, the K–S test values of the studied models are summarized also the p-values and the log-likelihood function of each models. We can conclude first from Table 4 that none of H_0 is rejected at level $\vartheta \leq 0.19$ and second from Table 5 that none of H_0 is rejected for the studied models at level $\vartheta \leq 0.77$. We deduce that the FWD represents the best model to fit the current data sets because it has the biggest p-value and the lowest K–S value.

Figures 9 and 10 illustrate the plots of the empirical and fitted scaled TTT-transforms, the empirical and parametric cumulative density functions, the empirical and fitted hazard and PDFs for the flexible Weibull model and its correspondent sub-models for the primary and the secondary pumps data.

We remark from Figures 9 and 10 that the FWD fits the data sets better than all other distributions used here, because its fitted curve is closer to the empirical curve.

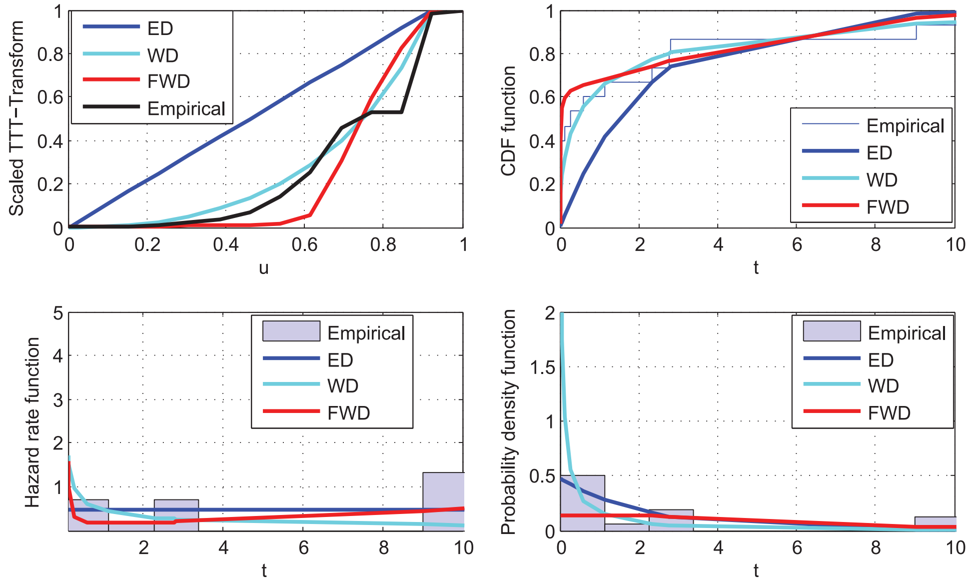

Figure 9. (a) The empirical and estimated scaled TTT-transform plots of the ED, WD and FWD models; (b) the empirical and estimated cumulative density function of the ED, WD and FWD models; (c) empirical and estimated HRFs of the ED, WD and FWD models; (d) empirical and estimated PDF of the ED, WD and FWD models, for TBF primary pumps data.

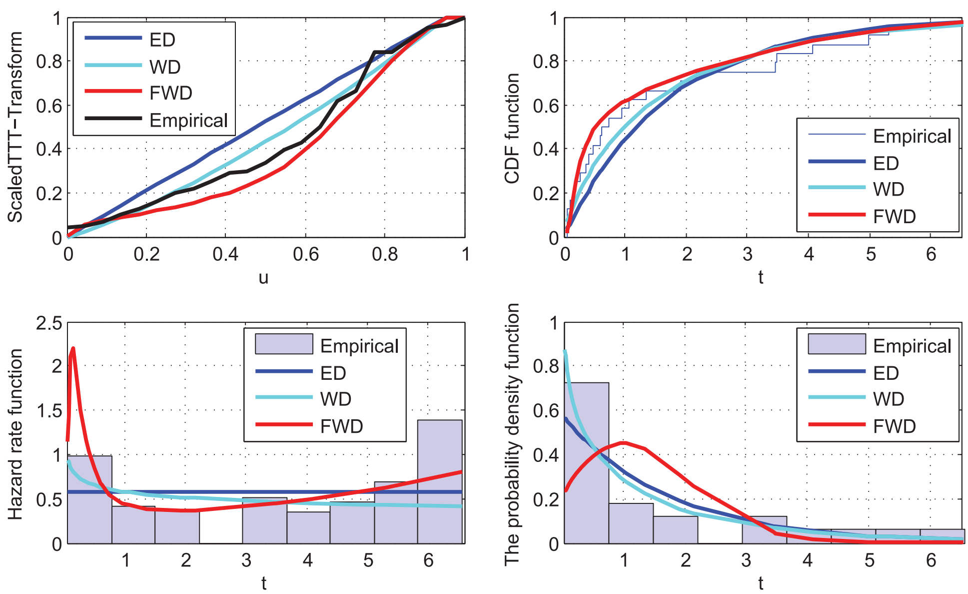

Figure 10. (a) The empirical and estimated scaled TTT-transform plots of the ED, WD and FWD models; (b) the empirical and estimated cumulative density function of the ED, WD and FWD models; (c) empirical and estimated HRFs of the ED, WD and FWD models; (d) empirical and estimated PDF of the ED, WD and FWD models, for TBF secondary pumps data.

3. Parameter estimation in the case of censored data

The phenomenon of censoring is encountered in the reliability data analysis which makes estimating parameters more difficult. The modeling problem of the FWD in the presence of censorship phenomenon was not treated in the literature. In this section, we discuss the reliability parameter estimation using the maximum likelihood method in the case of type I censored data. We have to use the reliability function in place of the PDF.

Consider n independent and identical components put on test. $X = (X_1, \dots, X_n)$ represent the random sample of the FWD distribution with unknown parameters vector $\theta = (a, b)$. Here, we consider maximum likelihood estimation for censored data without replacement. N items are independently observed and the observation of the ith item $(i = 1, \dots, N)$ is censored at time T_i.

The likelihood function becomes

$$L(t,\theta) = \prod_{i=1}^{n} f(t_i,\theta) \prod_{i=1}^{n} R(T_i,\theta), \tag{13}$$

$$L_i(m_i,\delta_i) = (f_X(t_i,\theta))^{\delta_i}(R_X(T_i,\theta))^{(1-\delta_i)},$$

$$m_i = \min(t_i, T_i) \quad \text{and}$$

$$\delta_i = \begin{cases} 1 & \text{if a failure was observed,} \\ 0 & \text{if the observation was censored at time } T_{\mathrm{i}}. \end{cases}$$

If all T_i are equal: $T_i = T, i = 1, \dots, N$, the likelihood function has the following form:

$$L_i(m_i,\delta_i) = \left\{\prod_{i=1}^{n} f_X(t_i,\theta)\right\}(R_X(T,\theta))^{(N-n)} \tag{14}$$

$n = n(T)$. In this case, we are using real data which represent the TBF of the primary pump and the secondary pump in the place of the TTF as in the previous section, so we have to introduce $X_{i,j}$ in the likelihood function instead of t_i.

Using Equations (2) and (3), we obtain the following likelihood function:

$$L(X_{ij},\theta) = \left\{\prod_{i=1}^{n}\prod_{j=1}^{k} X_{ij}\left(a + \frac{b}{{X_{ij}}^2}\right) \mathrm{e}^{(aX_{ij}-b/X_{ij})}\,\mathrm{e}^{(-\mathrm{e}^{aX_{ij}-b/X_{ij}})}\right\}(\mathrm{e}^{(-\mathrm{e}^{aT-b/T})})^{(N-n)}.$$

The log-likelihood function takes the following form:

$$\mathrm{Log}(L) = \sum_{i=1}^{n}\sum_{j=1}^{k} \log\left(a + \frac{b}{{X_{ij}}^2}\right) + \left(aX_{ij} - \frac{b}{X_{ij}}\right) - \mathrm{e}^{(aX_{ij}-b/X_{ij})} - (N-n)\,\mathrm{e}^{(aT-b/T)}.$$

Now, we must extract the log-likelihood function derivatives with respect to the FWD parameters and equating it to zero.

Table 6. Parameter estimation using censored simulated data.

Level of censoring	$\hat{a}$	$\hat{b}$
5% of censoring	0.2268	0.1982
10% of censoring	0.2343	0.1976
15% of censoring	0.2646	0.1954

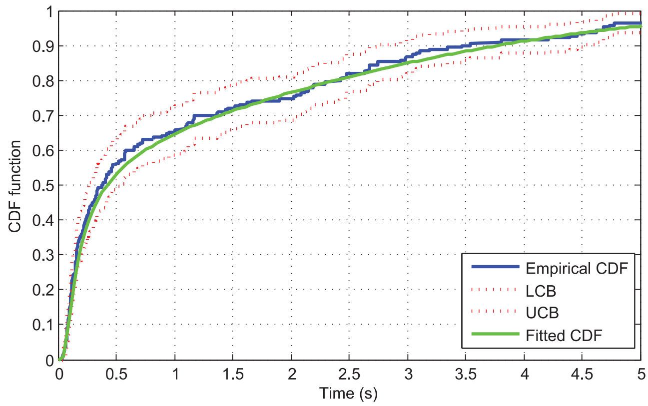

Figure 11. CDF and empiricalCDF of the FWD(a,b) for censored simulated data.

The first partial derivative of the log-likelihood function with respect to a and b is given by the following expressions:

$$\frac{\partial \mathrm{Log}(L)}{\partial a} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\frac{1}{a+\frac{b}{X_{ij}^2}} + X_{ij} - X_{ij}\,\mathrm{e}^{(aX_{ij}-b/X_{ij})}\right) - (N-n)T\,\mathrm{e}^{(aT-b/T)}, \quad (15)$$

$$\frac{\partial \mathrm{Log}(L)}{\partial b} = \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\left(\frac{1}{aX_{ij}^2+b}\right) - \frac{1}{X_{ij}} + \frac{1}{X_{ij}}\mathrm{e}^{(aX_{ij}-b/X_{ij})}\right) + (N-n)\frac{1}{T}\mathrm{e}^{(aT-b/T)}. \quad (16)$$

After obtaining the log-likelihood derivatives with respect to a and b, we solve the system of nonlinear equations numerically in **a** and **b** to obtain MLEs for censored data.

We perform three type I censored samples from the simulated data mentioned in Table 1 and we look for the MLE's of the FWD. We apply to our system the same steps to achieve optimum values that maximize the log-likelihood function. To find the FWD estimates, we need to introduce a_0 and b_0 at the beginning of the program. The choice of the initial values is arbitrary, it just requires that they belong to the domain of convergence.

In the first place, we consider the parameter estimation of the FWD based on the three performed censored samples. The system converges to the global maximum and gives the results mentioned in Table 6.

Figure 11 illustrates the empirical CDF, the CDF and the 95% LCB and UCB for the CDF of the censored simulated data. We remark that the CDF follows properly the empirical CDF using censored simulated data.

As a first step, we consider parameter estimation of the FWD using censored real data.

Table 7. Parameter estimation using TBF of the primary pumps censored data.

	Model	$\hat{a}$	$\hat{b}$	K–S	p-Value
Level 1	FWD	0.0320	0.0141	0.2857	.5635
	ED	0.1548	–	0.6154	.0930
Level 2	FWD	0.0319	0.0143	0.3333	.4595
	ED	0.1115	–	0.7500	.0120

Table 8. Parameters estimation using TBF of the secondary pumps censored data.

	Model	$\hat{a}$	$\hat{b}$	K–S	p-Value
Level 1	FWD	0.0325	0.3276	0.3478	.1021
	ED	0.1415	–	0.5119	.0320
Level 2	FWD	0.0020	0.3884	0.4091	.0389
	ED	0.0884	–	0.6364	.0140

We perform two artificial levels of censoring for each real data used in Section 2. In all the cases, the parameters estimation was carried out using the NR, LM and TR algorithms. We observe that the MLEs in the three cases turned out to be equals. Finally, our system converges to the global maximum and we get the results mentioned in Table 7 and 8.

We used the nonparametric statistical test, K–S, to test the null hypothesis against H_a as in Section 2. The FWD and ED estimates, the K–S statistic with p-value are cited in Tables 7 and 8 for two different levels of censoring. We can conclude first from Table 7 that none of H_0 is rejected at level $\vartheta \leq 0.013$ and second from Table 8 that none of H_0 is rejected for the studied models at level $\vartheta \leq 0.015$. The FWD is the best model compared to ED, to fit the censored real data sets because it has the biggest p-value and the lowest K–S value for all levels of censorship and for both used real data.

4. Conclusions

In this paper, we show the performance of the FWD. The maximum likelihood estimations of the FWD are discussed based on NR, LM and TR algorithms. The parameter estimation was established using first complete real and simulated data and second censored real and simulated data. Two real data sets are analyzed using the FWD and it is compared with sub-models. The results of the comparisons showed that the FWD provides a better fit for the two data sets using complete and censored data. Thus, in the presence of the phenomenon of censoring we recommend the use of MLE via NR, LM and TR algorithms to estimate the FWD parameters.

Disclosure statement

No potential conflict of interest was reported by the authors.

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