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The n-fold convolution of a finite mixture of densities

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ABSTRACT

In this paper, the probability density function of the n-fold convolution of a finite mixture of densities is obtained. The new density is again a finite mixture of densities. In this way, the formula recently given by Ma [N.-Y. Ma, A comment on "On the distribution of Ma and King, Applied Mathematics and Computation" 218 (2011) 202–203] for a two-fold convolution is interpreted and extended and a correct expression for the formula provided by Nadarajah [S. Nadarajah, On the distribution of Ma and King, Applied Mathematics and Computation 189 (2007) 732–733] is given. Several relevant examples are provided, including the convolution for the generalized exponential-sum distribution.

1. Introduction

Let X be a random variable with probability density function (PDF) given by,

$$f_X(t) = \sum_{i=1}^{m} \pi_i f_i(t), \tag{1}$$

where $\pi_i \geqslant 0,\ \sum_{i=1}^{m} \pi_i = 1$ and $f_i(t),\ i = 1, 2, \ldots, m$ are genuine PDF, that is, $f_i(t)$ are non-negative and $\int_{-\infty}^{\infty} f_i(t)dt = 1$.

Model (1) corresponds to a finite mixture of densities. This model presents many applications in probability, statistics and applied mathematics (see [1,2]).

The objective of this paper is to obtain the PDF of the n-fold convolution of a finite mixture of densities given by (1). The contents of this paper are the following. The main result is presented in Section 2. Several specific models are presented in Section 3, including normal, Cauchy, gamma and stable distributions. Finally, some conclusions are given in Section 4.

2. Main result

We introduce some notation and previous results. The convolution of two densities f_i and f_j is represented by $f_i \otimes f_j$ and is defined as

$$f_i \otimes f_j(t) = \int_{-\infty}^{\infty} f_i(x) f_j(t - x)dx.$$

The n-fold convolution of f_i is represented by $f_i^{(n)}$, where $f_i^{(n)} = f_i^{(n-1)} \otimes f_i$, with $f_i^{(1)} = f_i$ and $f_i \otimes f_j^{(0)} = f_i$.

The characteristic function of a random variable Y is defined as (see [3])

$$\varphi_Y(t) = E[\exp(itY)] = E[\cos(tY)] + iE[\sin(tY)].$$

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For a random variable having a PDF $f_Y(y)$ the characteristic function takes the form,

$$\varphi_Y(t) = \int_{-\infty}^{\infty} e^{ity} f_Y(y) dy.$$

It is well known that $\varphi_Y(t)$ uniquely determines the distribution of the corresponding random variable Y. Additionally, if we have random variables $Y_1, \cdots, Y_n$ which are independent with characteristic functions $\varphi_{Y_1}(t), \cdots, \varphi_{Y_n}(t)$, respectively, the characteristic function of the sum $Y = Y_1 + \cdots + Y_n$ is given by $\varphi_Y(t) = \prod_{k=1}^n \varphi_{Y_k}(t)$. If $Y_1, \cdots, Y_n$ are independent and identically distributed, the characteristic function of the convolution Y is $\varphi_Y(t) = [\varphi_{Y_1}(t)]^n$.

Let $X_1, X_2, \ldots, X_n$ be independent random variables with common density (1). In the following theorem we obtain the pdf of the n-fold convolution $X = X_1 + \cdots + X_n$.

Theorem 1. *The probability density function of the n-fold convolution of a finite mixture of densities of the form (1) is given by*

$$f_X^{(n)}(t) = \sum \frac{n!}{n_1! \ldots n_m!} \pi_1^{n_1} \ldots \pi_m^{n_m} \cdot f_1^{(n_1)} \otimes \cdots \otimes f_m^{(n_m)}(t), \tag{2}$$

where the sum is taken over all combinations of non-negative integers $n_1, \ldots, n_m$ so such that $\sum_{i=1}^m n_i = n$ and $f_1^{(n_1)} \otimes \cdots \otimes f_m^{(n_m)}$ represents the pdf of the m convolutions of the n_i-fold convolution of f_i, $i = 1, 2, \ldots, m$.

Proof. The proof of this Theorem is based on characteristic functions. The characteristic function of (1) is $\varphi_X(t) = \sum_{i=1}^m \pi_i \varphi_i(t)$, where φ_i is the characteristic function of f_i, $i = 1, 2, \ldots, m$. Now, the characteristic function of the n-fold convolution of X is $[\varphi_X(t)]^n$, and using the multinomial formula we get

$$\left[\sum_{i=1}^m \pi_i \varphi_i(t)\right]^n = \sum \frac{n!}{n_1! \ldots n_m!} \pi_1^{n_1} \cdots \pi_m^{n_m} \cdot \varphi_1(t)^{n_1} \varphi_2(t)^{n_2} \cdots \varphi_m(t)^{n_m}.$$

Now, we need to identify the PDF of the expression

$$\varphi_1(t)^{n_1} \varphi_2(t)^{n_2} \cdots \varphi_m(t)^{n_m},$$

which corresponds to the m convolutions of the n_i-fold convolution of f_i, $i = 1, 2, \ldots, m$. Finally, using the inversion formula of the characteristic function we obtain the result. □

Remark 1. Because $f_1^{(n_1)} \otimes \cdots \otimes f_m^{(n_m)}(t)$ is a PDF and $\sum \frac{n!}{n_1! \cdots n_p!} \pi_1^{n_1} \cdots \pi_p^{n_p} = 1$, formula (2) is again a finite mixture of densities.

Remark 2. In general, formula (2) is difficult to compute because we need to know the convolution $f_1^{(n_1)} \otimes \cdots \otimes f_m^{(n_m)}(t)$.

Remark 3. Formula (2) in Theorem 1 corrects formula (3) in [4]. See also [5] for the case $m = 2$ and $n = 2$-fold convolution.

2.1. Some special cases

For illustrating previous formula we provide some special cases. If we have $m = 2$ components and $n = 2$, formula (2) becomes,

$$f_X^{(2)}(t) = \pi_1^2 f_1^{(2)}(t) + \pi_2^2 f_2^{(2)}(t) + 2\pi_1\pi_2 f_1 \otimes f_2(t), \tag{3}$$

where $0 < \pi_1 < 1$, $\pi_2 = 1 - \pi_1$. Formula (3) corresponds to the [5] expression (Section 3) for $f^{(2)}(t)$ because (assuming f_1 and f_2 are positive),

$$\int_0^t [f_1(t-x) f_2(x) + f_2(t-x) f_1(x)] dx = f_1 \otimes f_2(t) + f_2 \otimes f_1(t) = 2 f_1 \otimes f_2(t).$$

Formula (3) is a finite mixture of the densities $f_1^{(2)}$, $f_2^{(2)}$ and $f_1 \otimes f_2$ with weights π_1^2, π_2^2 and $2\pi_1\pi_2$, respectively.

If $m = 2$ components and $n = 3$, formula (2) becomes,

$$f_X^{(3)}(t) = \sum_{i=1}^2 \pi_i^3 f_i^{(3)}(t) + 3\pi_1^2 \pi_2 f_1^{(2)} \otimes f_2(t) + 3\pi_1 \pi_2^2 f_1 \otimes f_2^{(2)}(t).$$

If we have m components and $n = 2$, we obtain

$$f_X^{(2)}(t) = \sum_{i=1}^m \pi_i^2 f_i^{(2)}(t) + 2\sum_{i<j} \pi_i \pi_j f_i \otimes f_j(t).$$

3. Models

In this section we present some specific models of the n-fold convolution of a finite mixture of densities.

3.1. Finite mixture of normal distributions

Our first model corresponds to the normal case. Assume that the components of the mixture (1) are distributed as normal distributions with PDF

$$f_i(t) = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{t-\mu_i}{\sigma_i}\right)^2\right\}, \quad -\infty < t < \infty,$$

where $\mu_i \in \mathbb{R},\ \sigma_i > 0,\ i = 1, 2, \ldots, p$. We will represent $f_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$. It is well-known that

$$f_i \otimes f_j \sim \mathcal{N}(\mu_i + \mu_j, \sigma_i^2 + \sigma_j^2),$$

in consequence, the n-fold convolution can be written as,

$$f_X^{(n)}(t) = \sum \frac{n!}{n_1! \cdots n_m!} \pi_1^{n_1} \cdots \pi_m^{n_m} \cdot \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{t-\tilde{\mu}}{\tilde{\sigma}}\right)^2\right\}, \tag{4}$$

where $\tilde{\mu} = \sum_{i=1}^m n_i\mu_i$ and $\tilde{\sigma}^2 = \sum_{i=1}^m n_i\sigma_i^2$. The cumulative distribution function (CDF) of (4) can be written easily in closed form.

3.2. Finite mixture of Cauchy distributions

Let X_i be a Cauchy distribution (see [3], Chapter 12) with parameters λ_i and θ_i and PDF,

$$f_i(t) = \frac{\lambda_i}{\pi} \frac{1}{\lambda_i^2 + (t-\theta_i)^2}, \quad -\infty < t < \infty, \tag{5}$$

with $\lambda_i > 0$ and $\theta_i \in \mathbb{R}$. A random variable with PDF (5) will be denoted $X_i \sim \mathcal{C}(\lambda_i, \theta_i)$. The characteristic function of X_k is $\exp\{i\theta_k t - \lambda_k|t|\}$, in consequence,

$$f_i \otimes f_j \sim \mathcal{C}(\lambda_i + \lambda_j, \theta_i + \theta_j),$$

and then,

$$f_X^{(n)}(t) = \sum \frac{n!}{n_1! \cdots n_m!} \pi_1^{n_1} \cdots \pi_m^{n_m} \cdot \frac{\tilde{\lambda}}{\pi} \frac{1}{\tilde{\lambda}^2 + (t-\tilde{\theta})^2},$$

where $\tilde{\lambda} = \sum_{i=1}^m n_i\lambda_i$ and $\tilde{\theta} = \sum_{i=1}^m n_i\theta_i$.

3.3. Finite mixture of gamma distributions

In this section we will study the n-fold distribution of a mixture of gamma distributions. Assume that f_i in (1) are distributed according to gamma distributions with PDF

$$f_i(t) = \frac{t^{\alpha_i - 1}\exp(-t/\lambda_i)}{\lambda_i^{\alpha_i}\Gamma(\alpha_i)}, \quad t > 0;\ i = 1, 2, \ldots, m, \tag{6}$$

with $\alpha_i, \lambda_i > 0$ and $\Gamma(\cdot)$ representing the gamma function. A PDF of type (6) will be represented as $f_k \sim \mathcal{G}(\alpha_k, \lambda_k)$. The characteristic function of (6) is $\varphi_k(t) = (1 - \lambda_k it)^{-\alpha_k}$. To set our results we distinguish two cases.

3.3.1. The case with the same scale parameters

In this case we assume that all the scale parameters are the same, that is, $\lambda_1 = \cdots = \lambda_m = \lambda$. Then, if $Z_i \sim \mathcal{G}(\alpha_i, \lambda),\ i = 1, 2, \ldots, m$ are independent gamma variables, the convolution $Z = Z_1 + \cdots + Z_m$ is distributed according to a gamma distribution $\mathcal{G}(\alpha_1 + \cdots + \alpha_m, \lambda)$, in consequence

$$f_1^{(n_1)} \otimes \cdots \otimes f_m^{(n_m)} \sim \mathcal{G}(\tilde{\alpha}, \lambda),$$

where $\tilde{\alpha} = \sum_{i=1}^m n_i\alpha_i$. Using (2), the PDF of the n-fold convolution of a mixture of gamma distributions is,

$$f_X^{(n)}(t) = \sum \frac{n!}{n_1! \cdots n_m!} \pi_1^{n_1} \cdots \pi_m^{n_m} \cdot \frac{t^{\tilde{\alpha}-1}\exp(-t/\lambda)}{\lambda^{\tilde{\alpha}}\Gamma(\tilde{\alpha})}, \quad t > 0. \tag{7}$$

The CDF of (7) can be obtained in closed form in terms of the incomplete beta function.

3.3.2. The general case

We consider now the general case where all the parameters α_i and λ_i are different. The case with different scale parameters λ_i and where the shape parameters α_i are integers (generalized exponential-sum distribution), has been studied by Ma and King [6,7].

The distribution of $Z = Z_1 + \cdots + Z_m$ with $Z_i \sim \mathcal{G}(\alpha_i, \lambda_i)$ has received special attention in recent statistical and probabilistic literature. There are several representations for the PDF of Z in terms of infinite series [8,9], infinite integrals [10] and special function (see [11], Chapter 8; [12]). In this work we will pay attention to two of these representations. The first representation is provided by Moschopoulos [8] and is given by

$$g_Z(t;\alpha_1,\ldots,\alpha_m;\lambda_1,\ldots,\lambda_m) = \prod_{i=1}^{m}\left(\frac{\lambda_1}{\lambda_i}\right)^{\alpha_i}\sum_{k=0}^{\infty}\frac{\delta_k t^{\alpha+k-1}e^{-t/\lambda_1}}{\Gamma(\alpha+k)\lambda_1^{\alpha+k}}, \quad t>0 \tag{8}$$

and 0 elsewhere, where $\alpha = \sum_{i=1}^{m}\alpha_i,\ \lambda_1 = \min\{\lambda_i\}$ and δ_k is defined as

$$\delta_{k+1} = \frac{1}{k+1}\sum_{i=1}^{k+1} i\gamma_i\delta_{k+i-1}, \quad k=0,1,2,\ldots,$$

with $\delta_0 = 1$ and

$$\gamma_k = \sum_{i=1}^{k}\alpha_i\left(1-\frac{\lambda_1}{\lambda_i}\right)^k\frac{1}{k}, \quad k=1,2,\ldots$$

This derivation is based on inversion of the moment generating function. Mathai [13] has studied the cases when all the shape parameters α_i's are equal and when all α_i's are integers.

The second representation [10] is given by,

$$g_Z(t;\alpha_1,\ldots,\alpha_m;\lambda_1,\ldots,\lambda_m) = \frac{\lambda_1^{\alpha_1}\cdots\lambda_m^{\alpha_m}t^{\alpha-1}}{\Gamma(\alpha)}\Phi_2^{(m)}\left(\alpha_1,\ldots,\alpha_m;\alpha;-\frac{t}{\lambda_1},\ldots,-\frac{t}{\lambda_m}\right), \tag{9}$$

where $\Phi_2^m(\cdot;\cdot;\cdot)$ denotes the confluent Lauricella multivariate hypergeometric function, and $\alpha = \sum_{i=1}^{m}\alpha_i$.

Now, as $f_i^{n_i} \sim \mathcal{G}(n_i\alpha_i, \lambda_i),\ i=1,2,\ldots,m$, using (8) or (9) we have,

$$f_X^{(n)}(t) = \sum\frac{n!}{n_1!\cdots n_m!}\pi_1^{n_1}\cdots\pi_m^{n_m}g_T(t;\ n_1\alpha_1,\ldots,n_m\alpha_m;\ \lambda_1,\ldots,\lambda_m),$$

where $g_T(t;\ \alpha_1,\ldots,\alpha_m;\ \lambda_1,\ldots,\lambda_m)$ is defined in (8) or (9).

3.4. Mixture of stable distributions

Models in Sections 3.1 and 3.2 are special cases of the stable distributions (see [3], Chapter 24). A random variable X is distributed according to a stable distribution if its characteristic function is given by

$$\varphi_X(t) = \exp\left(i\theta t - \lambda|t|^\alpha\right), \tag{10}$$

where $\theta \in \mathbb{R},\ \lambda > 0$ and $0 < \alpha \leqslant 2$. If we set $\alpha = 1$ in (10) we obtain the characteristic function of the Cauchy distribution and for $\alpha = 2$ we obtain the characteristic function of the normal distribution.

Assume that the components of the mixture (1) are distributed according to stable distributions $f_i \sim \mathcal{S}(\lambda_i, \theta_i, \alpha)$, $i=1,2,\ldots,m$. We have,

$$f_i \otimes f_j \sim \mathcal{S}(\lambda_i+\lambda_j, \theta_i+\theta_j, \alpha),$$

and then,

$$f_1^{(n_1)}\otimes\cdots\otimes f_m^{(n_m)} \sim \mathcal{S}\left(\sum_{i=1}^{m}n_i\lambda_i, \sum_{i=1}^{m}n_i\theta_i, \alpha\right).$$

In consequence, the n-convolution of stable distributions is a mixture of stable distributions with weights $\frac{n!}{n_1\cdots n_m}\pi_1^{n_1}\cdots\pi_m^{n_m}$.

4. Conclusions

In this work, the probability density function of the n-fold convolution of a finite mixture of densities is obtained. The main result provides an interpretation and extension to the work of Ma [5] for a two-fold convolution. In addition, the general formula developed by Nadarajah [4] is corrected. Several relevant models are provided, including normal, Cauchy, gamma and stable distributions.

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