Topic #2. Numerical Measure of Numerical Data Sets

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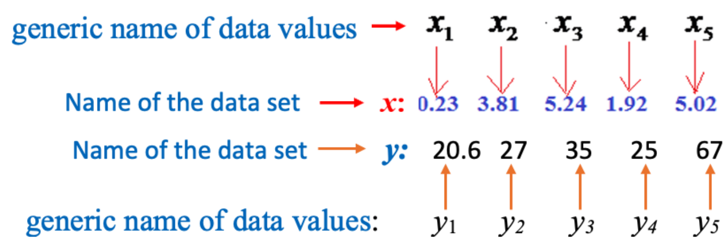
Contents

1 Introduction

This note focuses on using numerical measures to characterize numerical data sets. The numerical measures are used to describe the features such as mean, variance, and percentiles of a given numerical data set. These numeric measures are classified into three categories: central tendency, variation, and locations.

2 Notations Using Greek Letters for Parameters

Every data set has a name. For example, the heights of a group of WCU students is a data set that can be named `h` or `height`. We can give each data value has a "generic name" such as $h_1, h_2, \cdots, h_5$, etc. The following figure gives other examples of generic names of values in different data sets.



2.1 Big Sigma (Σ) Notation

These "generic names" were used to make compact formulas in some numeric measures. The sum of all data values in the data set with the name x (in the above figure) is given by the following **big sigma** notation.

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$= 0.23 + 3.81 + 5.24 + 1.92 + 5.02 = 16.22$$

Example 1: Consider the following two data sets with names **x** and **y**, we want to take the product of the corresponding values and sum up the product of the corresponding values. The following is the **big sigma** notation of the **sum of the cross-product**.

x	1.1	1.2	1.3	1.4	1.5	1.6
y	5	4	3	2	1	0

$$\sum_{i=1}^{4} x_i\, y_i = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$
$$= 1.1 \times 5 + 1.2 \times 4 + 1.3 \times 3 + 1.4 \times 2 = 17$$

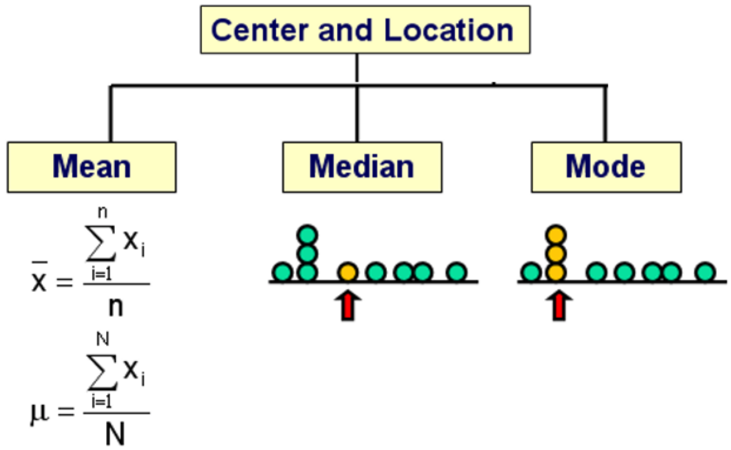
2.2 Notations for Parameters and Statistics

We use Greek letters to denote the population parameters of populations and English letters to denote statistics from the samples.

	Parameter (Always about population)	**Statistics** (Always about sample)
Definition	A number that describes the population characteristic.	A number that describes the sample characteristic.
Example	The average of PA population is 40 as of 2020. **Parameter value**: μ = 40	A random sample of 5000 Pennsylvanians was collected and found the average age to be 39.5 **Sample value**: $\bar{x}$ = 39.5

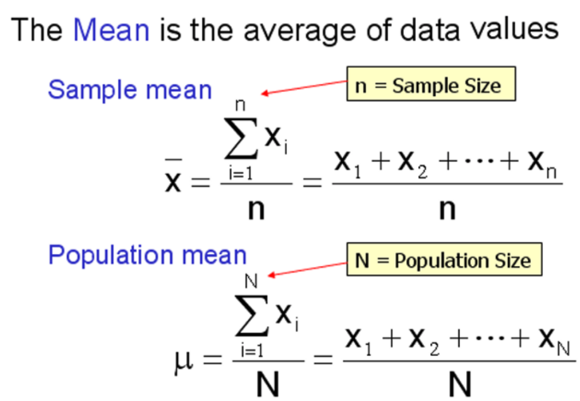
3 Measures of Center

Three measures are used as the center of a given numeric data set.



3.1 Mean

The mean of a given data set is defined as the average of all data values. The big sigma notations of sample and population means are given by



Remark: the mean can be affected significantly by outliers (extreme values). For example,

3.2 Median

The middle value of a sorted data set is called the median of the data set.

- If a data set has an **odd number** of data values, there is a unique “middle” value in the **sorted data** set.
- If a data set has an **even number** of data values, there will be two **middle** values in the **sorted data** set, in this case, the **average** of the two "middle** values is defined to be the median.

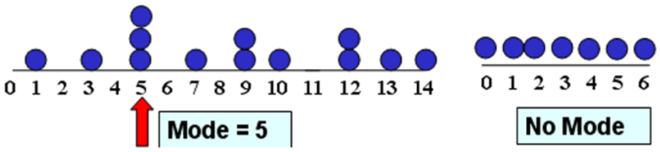
For example,

- {2, 6, 7} → median = 6
- {1, 2, 6, 7} → median = (2 + 6) / 2 = 4.

3.3 Mode

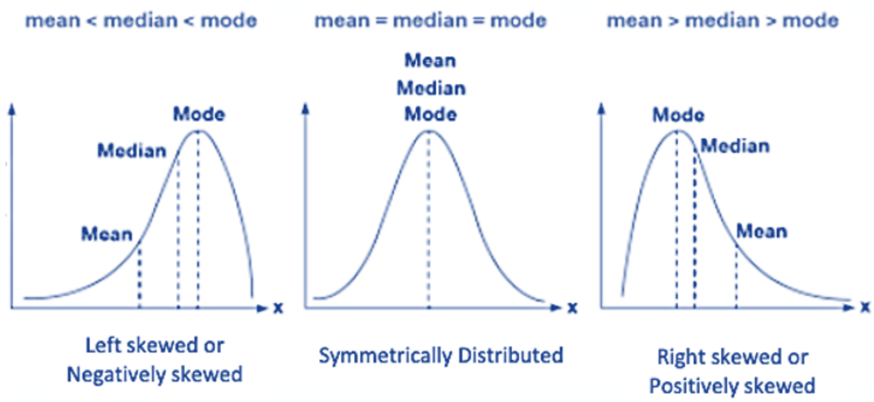
The mode(s) is (are) the data value(s) with highest frequency.

- If there is only one mode, the data set is unimodal.
- If there are two modes, the data set is bimodal.
- If there are more than two modes, the data is multi-modal.



3.4 Relationship between Mean, Median, and Mode

The relationship between mean, median, and mode is dependent on the shape of the distribution. The following figure illustrates this relationship.



4 Measures of Variation

Measures of variation are used to characterize the shape of the distribution. There are some different measures used in different situations. We only introduce the variance and the standard deviation in this course. We will also briefly introduce IQR in the applications of numerical measures.

4.1 Variance

Since the definitions of sample and population variances are different, we need to choose an appropriate formula based on whether the data set is a population or a sample. This information is provided to you before you select a formula to calculate the variances. The exact definitions using big sigma notation are given below

- Population Variance

$$\sigma^2 = \frac{\Sigma_{i=1}^{N}(x_i - \mu)^2}{N} = \frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}$$

- Sample Variance

$$s^2 = \frac{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}{n-1} = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1}$$

We can see the only difference is in the denominator of the two definitions.

4.2 Standard Deviation

Once the variance is calculated, we simply take the square root to obtain the standard deviation

- Population standard deviation

$$\sigma = \sqrt{\frac{\Sigma_{i=1}^{N}(x_i - \mu)^2}{N}} = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_N - \mu)^2}{N}}$$

- Sample Standard Deviation

$$s = \sqrt{\frac{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1}}$$

4.3 Steps for Calculating Variance

The following are steps for calculating the variance of a data set.

Step 1: Compute the sample mean $\bar{x}$.

Step 2: For each sample value x, compute the difference $x - \bar{x}$. This quantity is called a deviation.

Step 3: Square the deviations, to obtain quantities $(x - \bar{x})^2$.

Step 4: Sum the squared deviations, obtaining $\Sigma\,(x - \bar{x})^2$.

Step 5: Divide the sum obtained in Step 4 by $n - 1$ to obtain the sample variance s^2.

Example 2 The following table illustrates how to use the above steps to calculate the variance of a small **sample** toy data set: A = {1,4,7}.

X (dataset)	Deviation: $X - \bar{X}$	Squared Deviation: $(X - \bar{X})^2$
1	$1 - 4 = -3$	$(-3)^2 = 9$
4	$4 - 4 = 0$	$0^2 = 0$
7	$7 - 4 = 3$	$3^2 = 9$
$\bar{X} = 12/3$		$s^2 = \frac{18}{3-1} = 9$

Based on the above table, we can see that the standard deviation is $\sqrt{9} = 3$.

5 Measures of Location

Two important types of measures of location will be introduced in this course: z-score and percentiles.

5.1 z-score

A Z-score of a value of a **sample** data set is a standardized score that is defined by

$$z = \frac{x - \bar{x}}{s}.$$

We can easily adjust the above formula for a population as

$$z = \frac{x - \mu}{\sigma}.$$

Example 3: We still use the same sample toy data, A = {1,4,7}, used in **Example 2** to illustrate how to find z-scores of corresponding data values.

Solution: We know from **Example 2** that $\bar{x} = 4$ and $s = 3$. Therefore, the z-scores of the corresponding data values are calculated in the following.

$x_1 = 1 \rightarrow z_1 = \frac{1-4}{3} = -1$

$x_2 = 4 \rightarrow z_2 = \frac{4-4}{3} = 0$

$x_3 = 7 \rightarrow z_3 = \frac{7-4}{3} = 1$

That is, the standardized set of z-scores is $\{-1, 0, 1\}$. Note this is a sample z-scores. We can easily verify that the mean and standard deviation of the above three z-scores are 0 and 1 respectively.

5.2 Percentile

A **percentile** indicates the percentage of scores that fall below a particular value.

Example 4 Consider the following PSAT percentile table.

PSAT PERCENTILES

PSAT SECTION SCORE	EVIDENCE-BASED READING & WRITING PERCENTILE	MATH PERCENTILE
760	99+	99+
750	99+	99
740	99	98
730	99	97
720	98	97
710	97	96
700	96	95
690	95	94
680	94	93
670	93	92
660	91	91
650	90	90
640	88	89
630	86	87
620	84	85
610	[illegible]	[illegible]

If you took PSAT and scored 640 in MATH section, according to the above table, your MATH percentile is 89% meaning that 89% of all examinees scored below 640 in the PSAT. This also means that you did better than 89% of your peers on the PSAT.

Steps for Calculating Percentiles

Assume that we have a data set $\{x_1, x_2, \cdots, x_n\}$. we want to find $k - th$ percentile, denoted by P_k.

Step 1: Sort the data in ascending order: $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$.

Step 2: Calculate the `rough location` of the $k - th$ percentile

$$L = \frac{k}{100} \times n.$$

Step 3: The k^{th} percentiles is obtained depending on the form of L.

- if L is a whole number, then the k^{th} percentile is the average of the number in position L and the number in position $L+1$ in the sorted data set.
- if L is NOT a whole number, round it **up** to the next higher whole number. The k^{th} percentile is the number in the position corresponding to the **rounded-up** value.

Example 5: Consider the following data set

```
9, 13, 7, 7, 12, 15, 10, 10, 6, 19, 17, 10, 15, 9, 14, 12, 9, 13, 7,
7, 4, 8, 19, 5, 18, 20, 14, 1, 23, 10, 10, 7, 22, 9, 1
```

Find P_{40} and P_{55} percentiles respectively.

Solution we first sort the data in ascending order.

```
1, 1, 4, 5, 6, 7, 7, 7, 7, 7, 8, 9, 9, 9, 9, 10, 10, 10, 10, 10, 12, 12,
13, 13, 14, 14, 15, 15, 17, 18, 19, 19, 20, 22, 23
```

To find 40th percentile,

$$L = \frac{40}{100} \times 35 = 14.$$

Since $L = 14$ is an integer, the 20-th percentile is the average of 14th and 15th data values in the sorted data set. That is, $(9 + 9)/2 = 9$.

To find 55th percentile,

$$L = \frac{55}{100} \times 35 = 19.25.$$

Since $L = 19.25$ is NOT an integer, we round up L to get **20**. The 55th percentile is the 20th data value in the sorted data set which is 10.

5.3 Applications of Numeric Measures

Three concepts based on the numeric measures will be introduced in the following.

5.3.1 Five Number Summary

The five number summary consists of the minimum, 25th, 50th, 75th percentiles and the maximum. The 25th, 50th, 75th percentiles are also called the first (Q_1), second (Q_2) and third quartiles (Q_3), respectively.

Example 6: We use the **length of CD** data to show the five-number-summary. The unit of data values is minute. The following is the sorted data.

```
20.5, 29, 32, 32, 32, 33, 36, 37, 38, 39, 39, 43, 47, 48, 49, 49, 49,
50, 50, 51, 51, 52, 52, 52, 53, 54, 54, 54, 56, 56, 57, 58, 60, 61, 62,
62, 69, 73, 74, 74.5
```

Solution, The minimum and maximum are 20.5 and 74.5 minutes. The quartiles are calculated in the following.

$Q_1 : L = (25/100) \times 40 = 10$, Q_1 is the average of the 10th and 11th data values $= (39 + 39)/2 = 39$.

$Q_2 : L = (50/100) \times 40 = 20$, Q_1 is the average of the 20th and 21st data values $= (51 + 51)/2 = 51$.

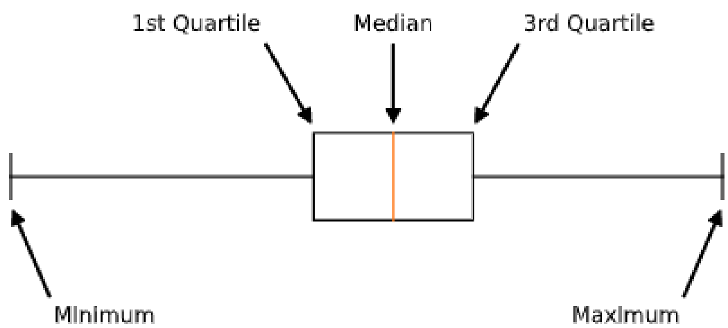
$Q_3 : L = (75/100) \times 40 = 30$, Q_1 is the average of the 30th and 31st data values $= (56 + 57)/2 = 56.5$.

Therefore, the five-number-summary is given by

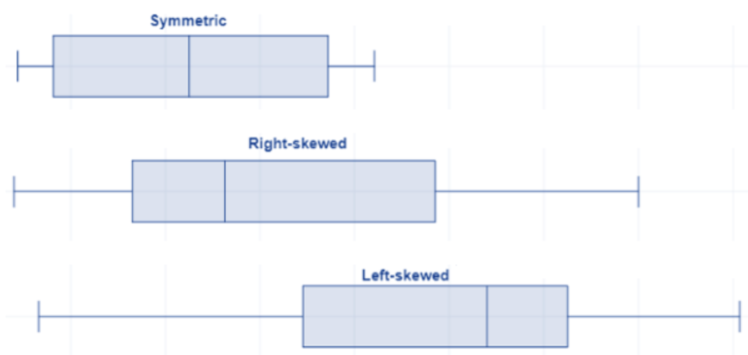
```
  Min      Q1      Q2      Q3      Max
 20.5      39      51    56.5     74.5
```

5.3.2 Box-plot

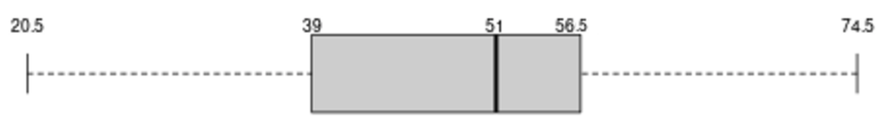
The box-plot is a geometric representation of the five-number-summary that given in the following figure



Box-plots are used to describe the distribution of data. The following three box-plots represent three different types of distributions:

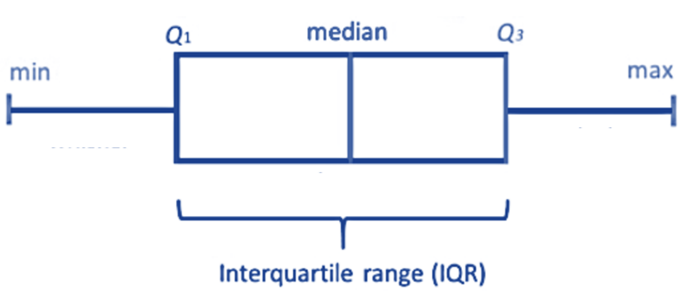


Example 7: (Length of CS Continued) The box-plot is given by



5.3.3 Inter-quartile Range (IQR)

The inter-quartile range of data is defined by $IQR = Q3 - Q1$. IQR is used to measure the variation of the data set. It is NOT sensitive to extremely large and small values since IQR is defined only based on the "middle 50% of data values".



6 Use of Technology

The following video shows you how to use **IntroStatsApps** to find numeric measures and their applications.