# 11. Exam #3 Review

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## Review Topics for Midterm Exam #3

### Rules of Derivative

**Simplifying the given function before using the following rules!**

<table>
<tr><th>Additive Rule</th><th>Multiplicative</th><th colspan="2">Quotient Rule</th></tr>
<tr>
<td>$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$<br><br>**Example:** $f(x) = e^x - x^2 + \ln x$<br><br>$f'(x) = (e^x - x^2 + \ln x)'$<br>$= (e^x)' - (x^2)' + (\ln x)'$<br>$= e^x - 2x + 1/x$</td>
<td>$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$<br><br>**Example:** $f(x) = (x^2+1)e^x$<br><br>$f'(x) = [(x^2+1)e^x]'$<br>$= (x^2+1)'e^x + (x^2+1)(e^x)'$<br>$= 2xe^x + (x^2+1)e^x$<br>$= (x+1)^2 e^x$</td>
<td colspan="2">$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$<br><br>**Example:** $f(x) = x/(x^2+1)$<br><br>$f'(x) = \frac{(x)'(x^2+1) - x(x^2+1)'}{(x^2+1)^2}$<br>$= \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$<br>$= \frac{1-x^2}{(x^2+1)^2}$</td>
</tr>
<tr><th>Chain Rule</th><th colspan="3">Power Rule</th></tr>
<tr>
<td rowspan="3">$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$<br><br>**Example:** $f(x) = \ln(e^x - x^2 + \ln x)$<br><br>$f'(x) = \frac{(e^x - x^2 + \ln x)'}{e^x - x^2 + \ln x}$<br>$= \frac{e^x - 2x + 1/x}{e^x - x^2 + \ln x}$</td>
<td>$f(x) = \frac{x^b}{x^a} = x^{b-a}$</td>
<td colspan="2">$f(x) = \sqrt[a]{x^b} = x^{b/a}$</td>
</tr>
<tr><td colspan="3">$\frac{d}{dx}(x^n) = nx^{n-1}$</td></tr>
<tr>
<td>**Example:** $f(x) = \frac{e^x}{x^a}$<br><br>$f'(x) = \left(\frac{e^x}{x^a}\right)' = (x^{-a}e^x)'$<br>$= (x^{-a})'e^x + (x^{-a})(e^x)'$<br>$= -ax^{-a-1}e^x + x^{-a}e^x$</td>
<td colspan="2">**Example:** $f(x) = \sqrt[3]{5^{x+1}}$<br><br>$f'(x) = \left(\sqrt[3]{5^{x+1}}\right)' = \left(5^{\frac{x+1}{3}}\right)'$<br>$= 5^{\frac{x+1}{3}} \ln 5 \left(\frac{x+1}{3}\right)' = 5^{\frac{x+1}{3}} \frac{\ln 5}{3}$</td>
</tr>
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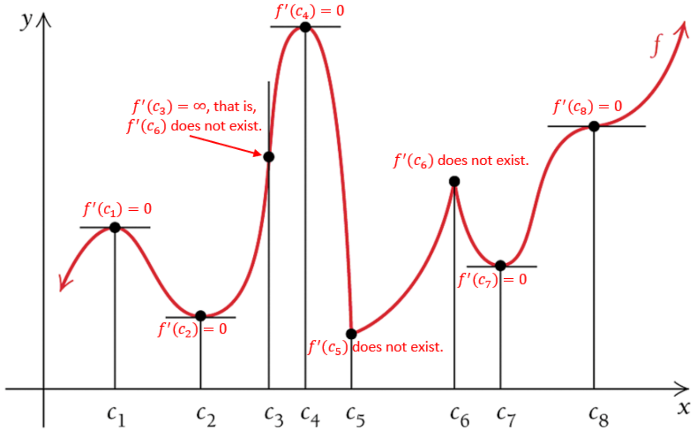
### Optimizations

- Critical Values of a function and first-order derivatives
  - Types of critical values: $f'(x) = 0$ or $f'(x)$ do not exist.

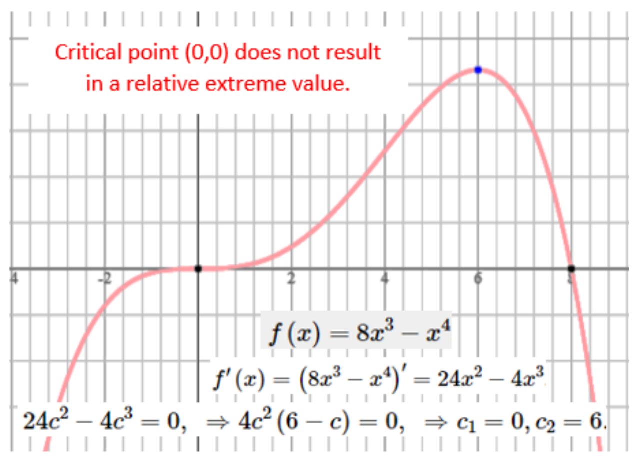



**Example 1**: Find the critical value(s) of $f(x) = x^3 - 3$.

**Solution** $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1) = 0$ yields $x = 1$ or $x = -1$. The two critical values are 1 and $-1$.

- Relative extrema
  - A critical value may not result in relative extrema

    **Example 2**: See the following figure



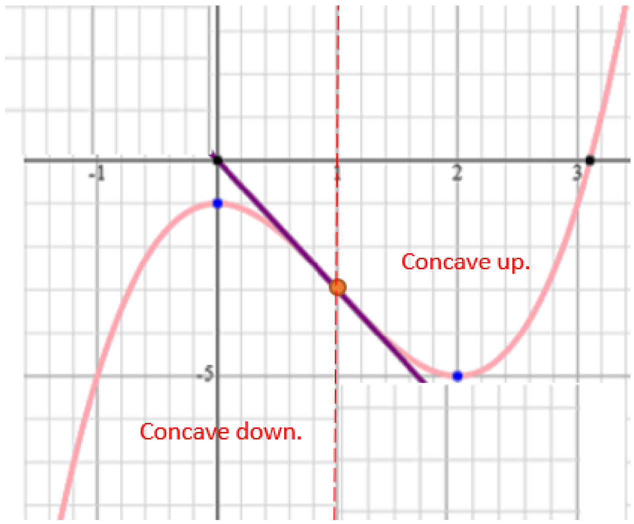

- Evaluate the function at critical points

**Example 3** From example 2, we found two critical values $c_1 = 0$ and $c_2 = 6$. Since $c_1 = 0$ is

- Concavity of a function and the second-order derivative
  - Definition of concave up/down of a function over intervals determined by the solutions to $f''(x) = 0$.

    **Example 4**: Determine the concavity intervals of $f(x) = x^3 - 3x^2 - 1$

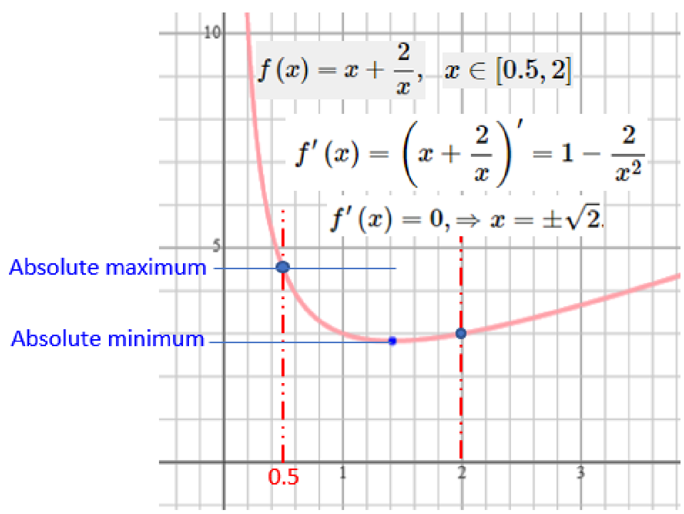
    **Solution**: $f'(X) = (x^3 - 3x^2 - 1)' = 3x^2 - 6x$ and $f''(x) = (3x^2 - 6x)' = 6x - 6$. Setting $f''(x) = 0$, we have $6x - 6 = 0$, which gives $x = 1$. Therefore, the inflection point is $(1, -3)$. The concave down interval is $(-\infty, 1)$ and concave up interval $(1, \infty)$.




- Finding absolute maximum and minimum:
  1. identify critical values by (1) solving $f'(x) = 0$; (2) finding $x$ such that $f'(x)$ does not exist.
  2. Evaluate $f(x)$ at all critical values and the end values of interval $[a, b]$
  3. The biggest finite value is the absolute maximum and the smallest finite value is the absolute minimum.

**Example 5**: Find the absolute maximum and minimum of the function $f(x) = x + 2/x$ on the given interval $[0.5, 2]$

**Solution**: *Step 1* - finding critical values. $f'(x) = 1 + 2 \times (-x^{-1-1}) = 1 - 2/x^{-2} = (x^2 - 2)/2$. $f'(x) = 0$ yields $x^2 - 2 = 0$, i.e., $x = \sqrt{2}$ is the only root in $[0.5, 2]$. *Step 2*. Evaluate $f(x)$ at $x = 0.5, \sqrt{2}, 2$: $f(0.5) = 0.5 + 2/0.5 = 4.5$, $f(\sqrt{2}) = \sqrt{2} + 2/\sqrt{2} = 2\sqrt{2} \approx 2.83$, and $f(2) = 2 + 2/2 = 3$. *Step 3*: The absolute maximum of $f(x)$ on $[0.5, 2]$ is attained at $x = 0.5$ and $f_{\text{maximum}}(0.5) = 4.5$ and the absolute minimum is attained at $x = \sqrt{2}$ and $f_{\text{maximum}}(\sqrt{2}) \approx 2.83$.




- Implicit differentiation
  - Implicit function:

**Example 6**: If $y = f(x)$ is defined implicitly by equation $F(x, y) = x^3 + y^3 - 3x^2y^5 = 0$, then function $y = f(x)$ is called implicit function.

– Derivative of implicit functions: Finding the derivative of an implicit function involves two basic steps- (1). Differentiate both sides of the equation with respect to $x$, assuming that $y$ is a differentiable function of $x$ and using the chain rule; (2). Solve the resulting equation for the derivative $y'$.

**Example 7** $y$ is implicitly defined in equation $x^3 + 2y^3 + yx^2 = 3$. Find $y'$.

**Solution**: $(x^3 + 2y^3 + yx^2)' = 3' \quad \Rightarrow \quad (x^3)' + 2(y^3)' + (yx^2)' = 3$. Note that the second and third terms on the left-hand side call for the chain rule: $3x^2 + 3y^2y' + [y'x^2 + y(x^2)'] = 0$, which is equivalent to $3x^2 + 3y^2y' + y'x^2 + 2xy = 0$. Solve for $y'$ from this equation, we have $y' = (3x^2 + 2yx)/(x^2 + 6y^2)$.

– Applications of the method of implicit differentiation

## Applications

- Cost, revenue, and profit functions
  - The relationship between the three functions $P(x) = R(x) - C(x)$
  - Average cost, revenue, and profit functions

$$\overline{C(x)} = \frac{C(x)}{x}, \quad \overline{R(x)} = \frac{R(x)}{x}, \quad \text{and} \quad \overline{P(x)} = \frac{P(x)}{x}$$

**Example 8**: The cost, in thousands of dollars, for producing $x$ thousand cellphone cases is given by $C(x) = 22 + x - 0.004x^2$. Find the average cost function.

**Solution**: By the definition, the average cost is defined to be $\overline{C(x)} = C(x)/x = (22 + x - 0.004x^2)/x = 22/x + 1 - 0.004x$.

- Definitions of average cost, revenue, and profit functions
- Definitions of marginal cost, revenue, and profit functions: the marginal functions are defined to be of the following form

$$\text{MC}(x) \approx C'(x), \quad \text{MR}(x) \approx R'(X), \quad \text{and} \quad \text{MP}(x) = P'(x)$$

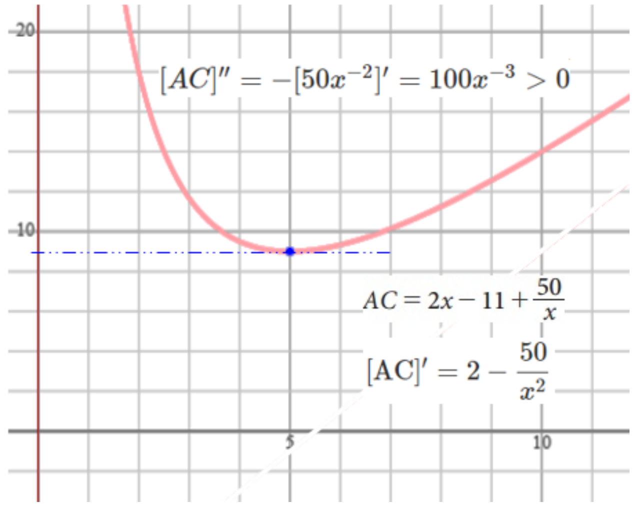
**Example 9**: We use the same cost function that was used in *example 8*.

**Solution**: The marginal cost function is given by $MC(x) = C'(x) = 1 - 0.08x$. For a specific value of $x$, say $x =$

- Maximization and minimization of cost, revenue, and profit functions

**Example 10**: The average cost for making $q$ items of certain product is given by $\text{AC} = 2x - 11 + 50/x$. Find the minimum average cost on ver interval $(0, \infty)$.

**Solution** Set $[\text{AC}]' = 2 - \frac{50}{x^2} = 0$, we have $2x^2 - 50 = 0$ which is equivalent to $(5x)(x + 5) = 0$. Since the average price is positive, the critical of the average cost over interval $(0, \infty)$ is $c = 5$. Note that $[AC]'' = -[50x^{-2}]' = 100x^{-3} > 0$. Therefore, the absolute minimum average cost is minimized at $x = 5$. The minimum cost is $\text{AC}(5) = 10 - 11 + 50/5 = 9$.




- Elastic Demand: **See the examples in the last lecture note**.
  - Percentage change = (new - old)/old

    **Example 11**: The price of a toy car is decreased from \$20 to \$15 after a discount. Find the percentage change in the amount.

    **Solution**: percentage change in the price = (15-20)/20 = -0.25.
  - Demand function: $q = D(p)$, $p$ is the unit price. For example, $D(p) = 280 - 7p$.
  - Elastic demand = - (percentage change in demand)/(percentage change in price)
  - Elastic demand $= -\frac{xD'(x)}{D(x)}$

  **Example 12**: The demand for organic chewing gum is given by $q = D(x) = 30 - 5x$. Find the elasticity of demand as a function of x.

  **Solution**: Note that $D'(x) = -5$. By the definition, $E(x) = -xD'(x)/D(x) = 5x/(30-5x) = x/(6-x)$.