3. Rules of Derivative

Cheng Peng

West Chester University

Contents

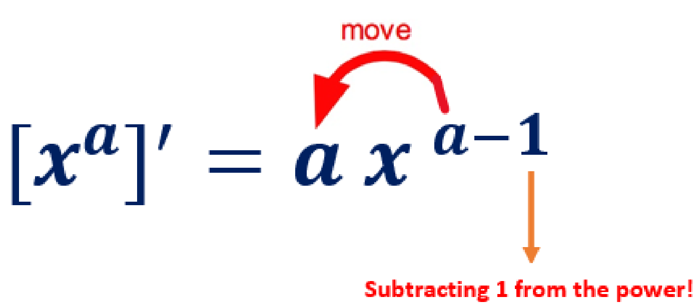
1 Review: Basic Rules of Derivative

We defined the derivaive of a function $y = f(x)$, denoted by $y' = f'(x)$, to be instantaneous rate of change

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The following basic rules were also introduced to calculate derivatives easier.

1. $f(x) = c$, then $f'(x) = (c)' = 0$.
2. $f(x) = x$, then $f'(x) = (x)' = 1$.
3. $f(x) = x^a$, then $f'(x) = (x^a)' = ax^{a-1}$. for any real number a.



Properties:

1. $[bf(x)]' = b[f(x)]'$
2. $[f(x) + g(x)]' = [f(x)]' + [g(x)]'$

We will continue to introduce more rules and properties of derivatives this week.

2 Leibniz Notation

In calculus, **Leibniz's notation** uses the symbols dx and dy to represent infinitely small (or infinitesimal) increments of x and y, respectively, just as Δx and Δy represent finite increments of x and y, respectively.

Consider y as a function of a variable x, or $y = f(x)$. If this is the case, then the derivative of y with respect to x, which later came to be viewed as the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

was, according to Leibniz, the quotient of an **infinitesimal increment** of y by an **infinitesimal increment** of x, or

$$\frac{dy}{dx} = f'(x),$$

The **infinitesimal increments** are called **differentials**. From now on, we will use the following notations interchangeably.

$$\frac{df(x)}{dx}, \ \frac{d}{dx}f(x), \ , \text{ and } f'(x)$$

3 Multiplicative Rule

If a function that has a complex form can be re-expressed into a product of two relatively simple functions, then we can use the **multiplicative rule** to find the derivative.

Let $f(x)$ and $g(x)$ be the two differentiable functions (i.e., the derivative of both functions exists everywhere in the domain).

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-product-rule-calculus-animation.gif

Example 1: Find the derivative of the following function.

(a). $y = \sqrt[3]{x^2}(2x - x^2)$

(b). $y = (6x^3 - x)(10 - 20x)$

Solution We use the multiplicative rule to calculate the above derivative.

(a). $y' = [x^{2/3}(2x - x^2)]' = (x^{2/3})'(2x - x^2) + x^{2/3}(2x - x^2)'$

$$= \frac{2}{3}x^{2/3-1}(2x - x^2) + x^{2/3}(2 - 2x) = \frac{2}{3}x^{-1/3}(2x - x^2) + x^{2/3}(2 - 2x)$$

$$= \frac{2}{3}x^{2/3}(2 - x) + 2x^{2/3}(1 - x) = 2x^{2/3}[(2 - x)/3 + 1 - x]$$

$$= 2x^{2/3}\frac{2 - x + 3(1 - x)}{3} = \frac{2x^{2/3}(5 - 4x)}{3}$$

(b). $y' = [(6x^3 - x)(10 - 20x)]' = (6x^3 - x)'(10 - 20x) + (6x^3 - x)(10 - 20x)'$

$= (18x^2 - x)(10 - 20x) + (6x^3 - x) \times (-20) = -480x^3 + 180x^2 + 40x - 10$

Example 2: (This example is taken from the textbook)

EXAMPLE 2 Let $F(x) = (2x + 1)(x^2 - 3)$. Find $F'(x)$ two ways: first by multiplying the two binomials and differentiating the product, then by using the Product Rule. Verify that both methods give the same result.

Solution Multiplying the binomials, we have

$$F(x) = (2x + 1)(x^2 - 3) = 2x^3 + x^2 - 6x - 3.$$

Now, using the Sum–Difference Rule, we have

$$\begin{aligned} F'(x) &= \frac{d}{dx}(2x^3) + \frac{d}{dx}(x^2) - \frac{d}{dx}(6x) - \frac{d}{dx}(3) \\ &= 6x^2 + 2x - 6. \end{aligned}$$

Using the Product Rule, where the two factors are $f(x) = 2x + 1$ and $g(x) = x^2$ we have

$$\begin{aligned} F'(x) &= \overbrace{(2x + 1)}^{f(x)}\overbrace{(2x)}^{g'(x)} + \overbrace{(x^2 - 3)}^{g(x)}\;\overbrace{(2)}^{f'(x)} \\ &= 4x^2 + 2x + 2x^2 - 6 \qquad \text{Using the distributive law} \\ &= 6x^2 + 2x - 6. \end{aligned}$$

4 Quotient Rule

The quotient rule can be used for differentiation when taking the derivative of a function divided by another function. For example, rational functions are this type of question.

Let $f(x)$ and $g(x)$ be two differentiable functions. The quotient rule of the derivative is given in the following.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

The following animated graph shows how to manipulate the terms algebraically.

https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-quotient-rule-formula-animation.gif

Example 3. Find the derivative of the following functions.

(a). $W(z) = (3z+9)/(2-z)$

(b). $h(x) = 4\sqrt{x}/(x^2-2)$

Solution: Using the quotient rule, we have

(a). $W'(z) = [(3z+9)'(2-z) - (3z+9)(2-z)']/(2-z)^2$

$$= \frac{3(2-z)-(3z+9)(-1)}{(2-z)^2} = \frac{15}{(2-z)^2}$$

(b). $y' = \left[(4\sqrt{x})'(x^2-2) - (4\sqrt{x})(x^2-2)'\right]/(x^2-2)^2$

$$= \frac{(2/\sqrt{x})(x^2-2) - 8x\sqrt{x}}{(x^2-2)^2} = -\frac{6x\sqrt{x}+4/\sqrt{x}}{(x^2-2)^2}.$$

Example 4 (This example is taken from the textbook)

EXAMPLE 3 Differentiate: $f(x) = \dfrac{1+x^2}{x^3+1}$.

Solution We let $f(x) = 1 + x^2$ and $g(x) = x^3 + 1$. We have

$$\frac{dy}{dx} = \frac{\overbrace{(x^3+1)}^{g(x)}\overbrace{(2x)}^{f'(x)} - \overbrace{(1+x^2)}^{f(x)}\overbrace{(3x^2)}^{g'(x)}}{\underbrace{(x^3+1)^2}_{(g(x))^2}} \qquad \text{Using the Quotient Rule}$$

$$= \frac{2x^4+2x-3x^2-3x^4}{(x^3+1)^2} \qquad \text{Using the distributive law}$$

$$= \frac{-x^4-3x^2+2x}{(x^3+1)^2}. \qquad \text{Simplifying}$$

Important Business Functions: Cost, Revenue, and Profit-related functions and applications.

DEFINITION

If $C(x)$ is the cost of producing x items, then the **average cost** of producing x items is $\dfrac{C(x)}{x}$.

If $R(x)$ is the revenue from the sale of x items, then the **average revenue** from selling x items is $\dfrac{R(x)}{x}$.

If $P(x)$ is the profit from the sale of x items, then the **average profit** from selling x items is $\dfrac{P(x)}{x}$.

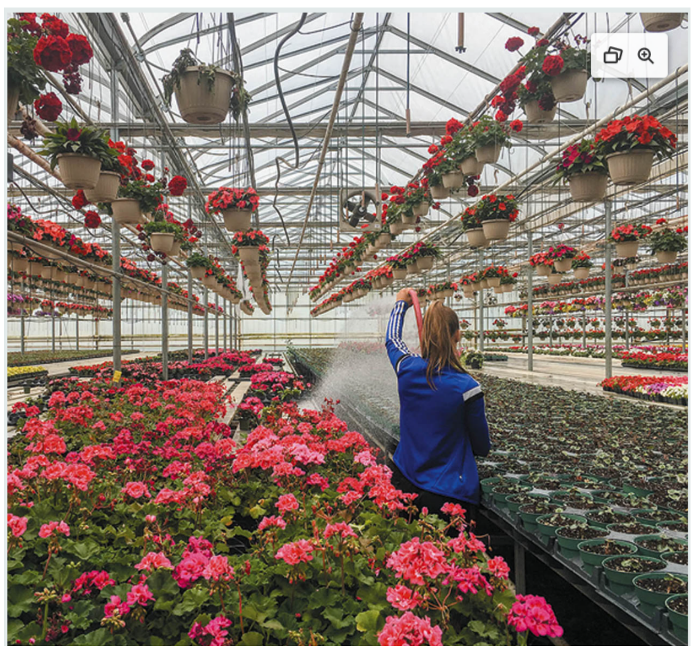
Example 5 (This example is also taken from the textbook)

Paulsen's Greenhouse finds that the cost, in dollars, of growing x hundred geraniums is modeled by

$$C(x) = 200 + 100\sqrt[4]{x}$$

If revenue from the sale of x hundred geraniums is modeled by

$$R(x) = 120 + 90\sqrt{x}$$



find each of the following.

(a) The average cost, average revenue, and average profit when 300 geraniums are grown and sold.

(b) The rate at which the average profit is changing when 300 geraniums are grown and sold.

a. We let $\overline{C}$, $\overline{R}$, and $\overline{P}$ represent average cost, average revenue, and average profit, respectively. Thus,

$$\overline{C}\left(x\right) = \frac{C(x)}{x} = \frac{200 + 100\sqrt[4]{x}}{x};$$
$$\overline{R}\left(x\right) = \frac{R(x)}{x} = \frac{120 + 90\sqrt{x}}{x};$$
$$\overline{P}\left(x\right) = \frac{R\left(x\right) - C(x)}{x} = \frac{-80 + 90\sqrt{x} - 100\sqrt[4]{x}}{x}.$$

To find the average cost, average revenue, and average profit when 300 geraniums are grown and sold, we evaluate each function at $x = 3$:

$$\overline{C}\left(3\right) = \frac{200 + 100\sqrt[4]{3}}{3} \approx \$110.54 \text{ per hundred geraniums;}$$
$$\overline{R}\left(3\right) = \frac{120 + 90\sqrt{3}}{3} \approx \$91.96 \text{ per hundred geraniums;}$$
$$\overline{P}\left(3\right) = \frac{-80 + 90\sqrt{3} - 100\sqrt[4]{3}}{3} \approx -\$18.57 \text{ per hundred geraniums.}$$

b. To find the rate at which average profit is changing when 300 geraniums are grown and sold, we calculate $\overline{P}'\left(3\right)$:

$$\overline{P}'\left(x\right) = \frac{d}{dx}\left[\frac{-80 + 90x^{1/2} - 100x^{1/4}}{x}\right]$$
$$= \frac{x\left(\frac{1}{2} \cdot 90x^{1/2-1} - \frac{1}{4} \cdot 100x^{1/4-1}\right) - \left(-80 + 90x^{1/2} - 100x^{1/4}\right) \cdot 1}{x^2}$$
$$= \frac{45x^{1/2} - 25x^{1/4} + 80 - 90x^{1/2} + 100x^{1/4}}{x^2} = \frac{75x^{1/4} - 45x^{1/2} + 80}{x^2};$$
$$\overline{P}'\left(3\right) = \frac{75\sqrt[4]{3} - 45\sqrt{3} + 80}{3^2} \approx 11.20.$$

When 300 geraniums have been grown and sold, the average profit is increasing by \$11.20 per hundred geraniums for each additional 100 geraniums sold.

5 Chain Rule

Before introducing the Chain Rule, we first review the definition of a composite function.

5.1 Composite Function

Suppose the two given functions are f and g, the composition of f and g, denoted by $f \circ g$, is given by

$$f \circ g(x) = f(g(x)) \quad \text{and} \quad g \circ f(x) = g(f(x))$$

$$f \circ g = f\left[\overset{\substack{\text{input}\\ \downarrow}}{g(x)}\right] \qquad g \circ f = g\left[\overset{\substack{\text{input}\\ \downarrow}}{f(x)}\right]$$

Example 6 Let $f(x) = 2x^2 - 3x - 1$ and $g(x) = -x + 5$. Find $f \circ g$.

$$\begin{aligned}
f(x) &= 2x^2 - 3x - 1 && \texttt{Write down the main function}\\
f[g(x)] &= 2[g(x)]^2 - 3[g(x)] - 1 && \texttt{Plug in the input function}\\
&= 2[-x+5]^2 - 3[-x+5] - 1 && \texttt{Square the binomial}\\
&= 2[x^2 - 10x + 25] - 3[-x+5] - 1 && \texttt{Apply distributive property}\\
&= 2x^2 - 20x + 50 + 3x - 15 - 1 && \texttt{Combine like terms}\\
f[g(x)] &= 2x^2 - 17x + 34
\end{aligned}$$

$$f \circ g = f[g(x)] = 2x^2 - 17x + 34 \longleftarrow \texttt{Final answer!}$$

Example 7 Let $f(x) = 1/(x+3)$ and $g(x) = (-3x-2)/x$. Find $f \circ g$.

$$f(x) = \frac{2}{x+3} \qquad \texttt{Write down the main function}$$

$$f[g(x)] = \frac{2}{g(x)+3} \qquad \texttt{Substitute the input function}$$

$$= \frac{2}{\frac{-3x-2}{x}+3}$$

$$= \frac{2}{\frac{-3x-2}{x}+3\left(\frac{x}{x}\right)} \qquad \texttt{Add the complex denominator using LCD}$$

$$= \frac{2}{\frac{-3x-2}{x}+\frac{3x}{x}}$$

$$= \frac{2}{\frac{-3x-2+3x}{x}} \qquad \texttt{Combine like terms on the numerator of the complex denominator}$$

$$= \frac{2}{\frac{-2}{x}} \qquad \texttt{Simplify}$$

$$= 2 \cdot \frac{x}{-2} \qquad \texttt{Divide by multiplying the numerator to the reciprocal of the denominator}$$

$$f[g(x)] = -x \qquad \texttt{Simplify by canceling common factors}$$

$$f \circ g = f[g(x)] = -x \longleftarrow \quad \texttt{Final answer!}$$

5.2 Chain Rule

The following Chain Rule is used to find the derivative of composite functions.

$$[f(g(x))]' = f'(g(x)) \times g'(x)$$

https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-Chain-Rule-1.gif

Example 8 Find the derivative of the following function

$$f(x) = \sqrt[4]{\frac{x+3}{x-2}}.$$

Here we use the Quotient Rule to differentiate the inside function:

$$\frac{d}{dx}\sqrt[4]{\frac{x+3}{x-2}} = \frac{d}{dx}\left(\frac{x+3}{x-2}\right)^{1/4} = \frac{1}{4}\left(\frac{x+3}{x-2}\right)^{1/4-1}\left[\frac{(x-2)1-1(x+3)}{(x-2)^2}\right]$$

$$= \frac{1}{4}\left(\frac{x+3}{x-2}\right)^{-3/4}\left[\frac{x-2-x-3}{(x-2)^2}\right]$$

$$= \frac{1}{4}\left(\frac{x+3}{x-2}\right)^{-3/4}\left[\frac{-5}{(x-2)^2}\right], \quad \text{or} \quad \frac{-5}{4(x+3)^{3/4}(x-2)^{5/4}}.$$

Example 9: Find the derivative of $f(x) = \sqrt[3]{x^5 + 6x}$.