14. Applications of Mutivariable Functions

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Contents

1 General Review and Exercises

- **Basic Rules and Properties of Integrals**.

A1. Constant Rule:

$$\int k\,dx = kx + C.$$

A2. Power Rule (where $n \neq -1$):

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

A3. Natural Logarithm Rule:

$$\int \frac{1}{x}\,dx = \ln\left|x\right| + C,$$

A4. Exponential Rule (base e):

$$\int e^x\,dx = e^x + C$$

P1. A constant multiplier can be factored to the front of the indefinite integral:

$$\int [c \cdot f(x)]\,dx = c \cdot \int f(x)\,dx.$$

P2. The antiderivative of a sum or difference is the sum or difference of the antiderivatives:

$$\int \left[f(x) \pm g(x)\right]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Let f be any continuous function over $[a, b]$ and F be any antiderivative of f. Then the definition integral of f from a to b is

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- **Notations of Differentials**

$$f'(x)\,dx = d f(x)$$

Examples: 1. $2x\,dx = dx^2$

2. $x^k dx = d\left(\frac{x^{k+1}}{k+1}\right)$

3. $e^x dx = de^x$

4. $\left(\frac{1}{x}\right) dx = d\ln x$

The last example assumes that $x > 0$.

- ***Integration by Substitution*

$$\int g[f(x)]f'(x)dx = \int g[f(x)]\,df(x)$$

$$\xrightarrow{u=f(x)} \int g[u]\,du$$

2 More on Partial Derivatives

We first work on two examples of finding the partial derivatives of functions with multiple variables to review the steps and the relevant rules. Then present a pictorial interpretation of partial derivative.

2.1 Examples

Recall that when taking the derivative of a multivariable function with respect to one variable, all other variables are treated as constant scalars.

Example 1: Find all first order partial derivatives of function $f(x, y, z) = x^2y^3z^4$

Solution: The derivative rule of scalar multiplication will be used repeatedly in the following.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 \overbrace{y^3 z^4}^{\text{Constant scalar}}\right) = \overbrace{y^3 z^4}^{\text{Constant scalar}} \frac{\partial}{\partial x}\left(x^2\right) = 2xy^3 z^4$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\overbrace{x^2 y^3 z^4}^{\text{Constant scalar}}\right) = \overbrace{x^2 z^4}^{\text{Constant scalar}} \frac{\partial}{\partial y}\left(y^3\right) = 3x^2 y^2 z^4$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\overbrace{x^2 y^3}^{\text{Constant scalar}} z^4\right) = \overbrace{x^2 y^3}^{\text{Constant scalar}} \frac{\partial}{\partial z}\left(z^4\right) = 4x^2 y^3 z^3$$

Example 2: Let $g(x, y) = \ln(x^2 + xy^4)$. Find partial derivative $g_x(x, y)$ and $g_y(x, y)$.

Solution: We do the first part. g_y is used for practice.

$$g_x = \frac{\partial}{\partial x} g(x,y) = \frac{\partial}{\partial x} \ln\left(\overbrace{x^2 + xy^4}^{\text{Inner function}}\right)$$

$$= \frac{1}{x^2+xy^4} \frac{\partial}{\partial x}\left(x^2 + x\overset{\text{Constant scalar}}{y^4}\right)$$

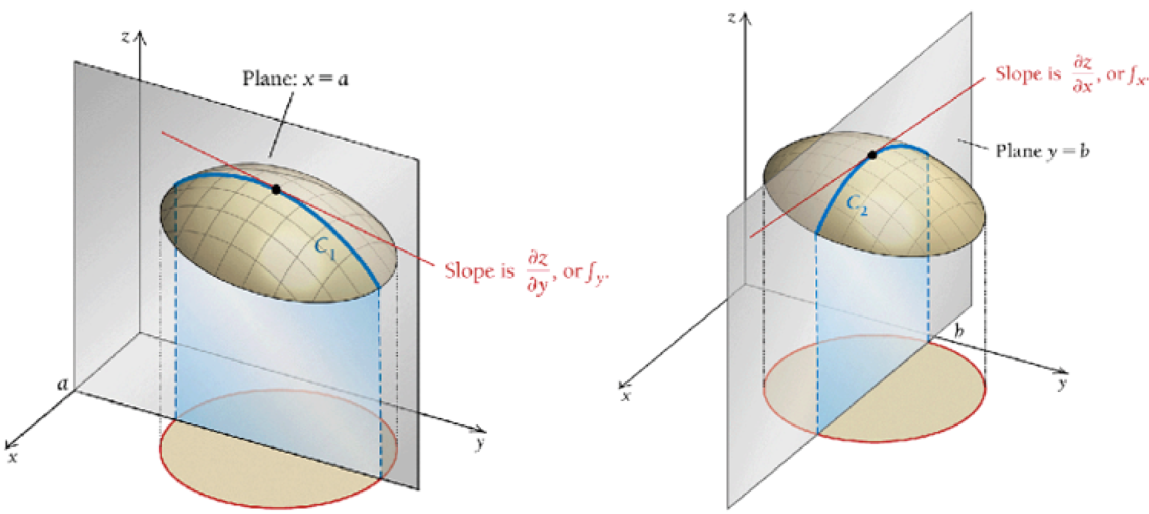
$$= \frac{1}{x^2+xy^4}\left[\frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial x}\left(x\overset{\text{Constant scalar}}{y^4}\right)\right]$$

$$= \frac{1}{x^2+xy^4}\left[2x + y^4 \frac{\partial}{\partial x}(x)\right]$$

$$= \frac{1}{x^2+xy^4}\left[2x + y^4\right] = \frac{2x+y^4}{x^2+xy^4}$$

2.2 Geometry of Partial Derivatives

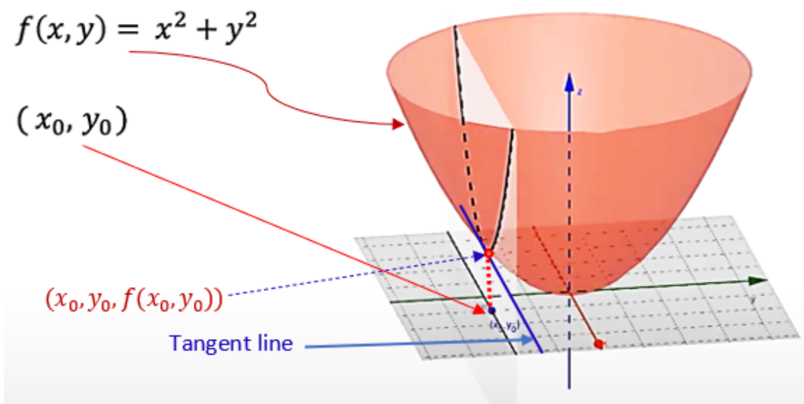
The derivative of a single variable function is the slope of the tangent line at x. Similarly, a partial derivative also represents the slope of a curve on the surface of the underlying function. The following figure shows the geometry of partial derivatives.



Example 3: Consider the partial derivative $\partial f/\partial x$ of function $f(x,y) = x^2 + y^2$ at (x_0, y_0).

Solution. By definition, the partial derivative of $f(x, y)$ with respect to x is given by

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) \overset{\text{y is a constant}}{=} 2x + 0 = 2x.$$



3 Business Applications - Marginal Productivity

One model of production that is frequently in business and economics is the Cobb-Douglass production function

$$p(x, y) = Ax^{\alpha}y^{1-\alpha} \quad \text{for} \quad A > 0 \quad \text{and} \quad 0 < \alpha < 1.$$

where $p(x, y)$ is the number of units produced with x units of labor and y units of capital (Capital is the cost of machinery, buildings, tools, and supplies). The partial derivatives

$$\frac{\partial p(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial p(x,y)}{\partial y}$$

are called, respectively, the **marginal productivity of labor** and the **marginal productivity of capital**.

Remark: α *is a system parameter determined by the production process based on historical production data. It could also be controlled by the management.*

Example 4. MyTell Cellular has the following production function for a smartphone:

$$p(x, y) = 50x^{2/3}y^{1/3}$$

where $p(x,y) is the number of units produced with x units of labor and y units of capital.

1). Find the number of units produced with 125 units of labor and 64 units of capital.

2). Find the **marginal productivity**.

3). Evaluate the **marginal productivity** at $x = 125$ and $y = 64$.

Solution (1). We simply evaluate the function with the given information about labor and capital.

$$p(125, 64) = 125^{2/3} \times 64^{1/3} = (5^3)^{2/3} \times (4^3)^{1/3} = 5^2 \times 4 = 100.$$

(2). We first find the marginal productivity functions of labor and capital respectively in the following.

$$\frac{\partial p(x,y)}{\partial x} = \frac{\partial}{\partial x}(x^{2/3}y^{1/3}) \overset{\text{y is constant}}{=} y^{1/3}\frac{\partial}{\partial x}(x^{2/3}) = y^{1/3}[(2/3)x^{2/3-1}] = \frac{2}{3}x^{-1/3}y^{1/3}$$

$$\frac{\partial p(x,y)}{\partial y} = \frac{\partial}{\partial y}(x^{2/3}y^{1/3}) \overset{\text{x is constant}}{=} x^{2/3}\frac{\partial}{\partial y}(y^{1/3}) = x^{2/3}[(1/3)y^{1/3-1}] = \frac{1}{3}x^{2/3}y^{-2/3}$$

(3). We evaluate the above two partial derivatives respectively in the following

$$\text{Marginal productivity of labor} = \frac{2}{3}125^{2/3} \times 64^{-1/3} = \frac{2}{3} \times \frac{25}{4} = \frac{25}{6}.$$

and

$$\text{Marginal productivity of capital} = \frac{1}{3}125^{1/3} \times 64^{-2/3} = \frac{1}{3} \times \frac{5}{16} = \frac{5}{48}.$$