12. Areas and Definite Integral

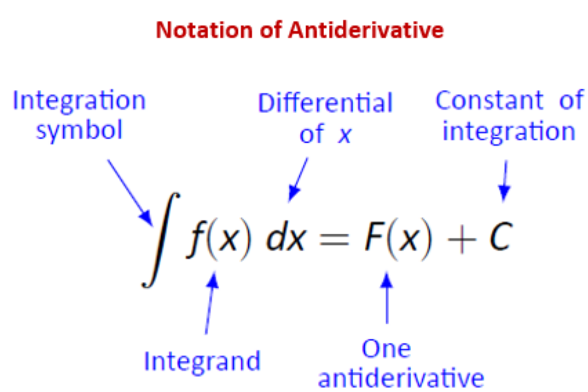
Cheng Peng

West Chester University

Contents

1 General Review and Exercises

- **Definition of Antiderivative**



- **Basic Rules of Derivatives and Integral**.

Rules of Derivative & Antiderivative of Basic Functions

$\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$
$\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}(C) = 0$	$\int 0\,dx = C$
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = \int dx = x + C$
$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}(a^x) = \ln a \cdot a^x$	$\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Example 1: Find the following integrals:

1. $\int \frac{1}{x^3} dx$.
2. $\int e^{4x} dx$
3. $\int (x^3 + 3e^x + 1/x + \sqrt[3]{x}) dx$

Solution:

1. Rewrite the fractional function as a power function and then use the power rule.

$$\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C.$$

2. Recall that $\int e^x dx = e^x + C$. We need to some algebraic manipulation before using the exponential rule.

$$\int e^{4x} dx = \int e^{4x} d\left(\frac{4x}{4}\right) = \frac{1}{4} \int e^{4x} d(4x)$$

$$\overset{y=4x}{=} \frac{1}{4} \int e^y dy = \frac{1}{4}(e^y + C) = \frac{e^{4x}}{4} + C.$$

3. This problem combines several rules of integration.

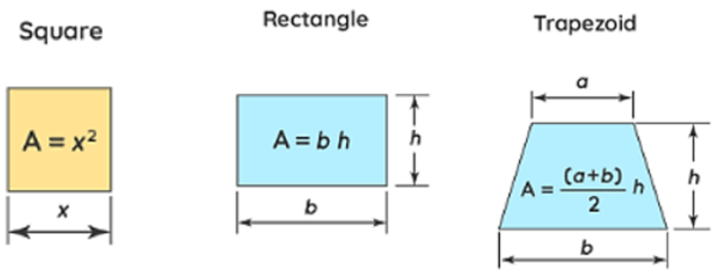
$$\int (x^3 + 3e^x + 1/x + \sqrt[3]{x}) dx = \int x^3 dx + 3 \int e^x dx + \int \frac{1}{x} dx + \int x^{1/3} dx$$

$$= \frac{x^{3+1}}{3+1} + 3e^x + \ln|x| + \frac{x^{1/3+1}}{1/3+1} + C = \frac{x^4}{4} + 3e^x + \ln|x| + \frac{x^{4/3}}{4/3}$$

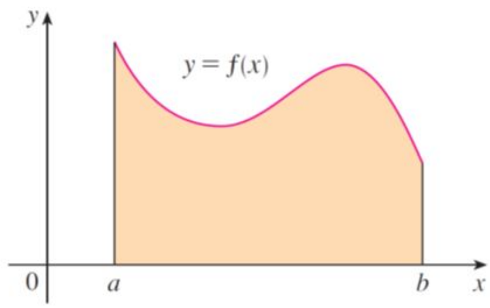
$$= \frac{x^4}{4} + 3e^x + \ln|x| + \frac{3x^{4/3}}{4}.$$

2 Areas Defined by Curves of Functions

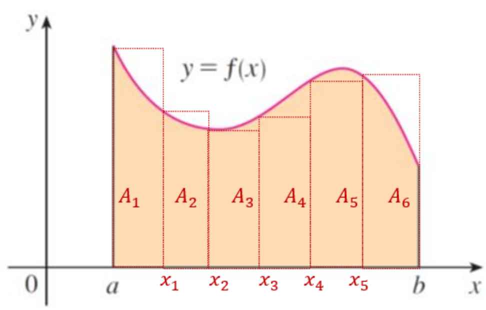
We have learned several area formulas in algebra. For example,



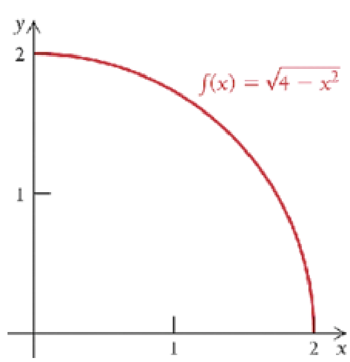
However, if have a region defined by curves and lines such as the one shown below. How find the area?



Although we don't have a formula to find the area, however, We can approximate the area by adding up the areas of rectangles with equal width as showing in the following figure.



Example 1: Consider the graph of $f(x) = \sqrt{4 - x^2}$ over the interval $[0, 2]$.



We demonstrate the accuracy of approximation using difference number of rectangles in the following. We first look at the case of 4 rectangles with the following calculation

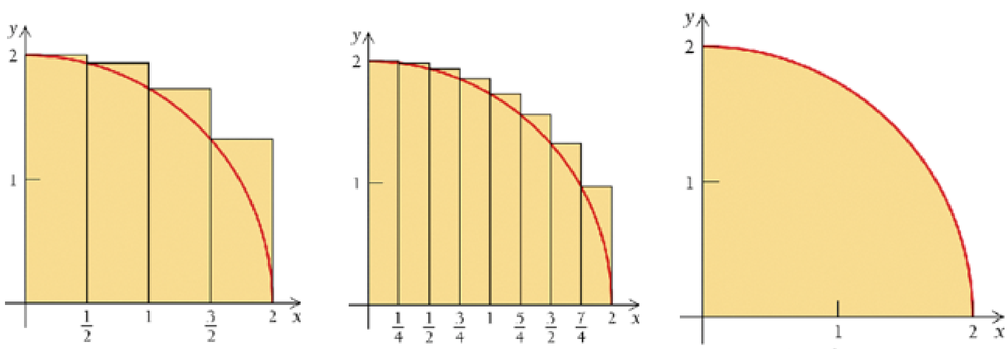
Dividing [0, 2] into 4 subintervals of equal width, we have

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}. \quad \text{This is the width of each rectangle.}$$

We then let x_i range from $x_1 = 0$ to $x_4 = \frac{3}{2}$ in increments of $\frac{1}{2}$. The area under the graph of f is then approximately

$$\begin{aligned}
\sum_{i=1}^{4} f(x_i)\,\Delta x &= f(0)\cdot\frac{1}{2} + f\left(\frac{1}{2}\right)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f\left(\frac{3}{2}\right)\cdot\frac{1}{2} \\
&= \frac{1}{2}\left(f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right) && \text{Factoring} \\
&\approx \frac{1}{2}\left(2 + 1.93649 + 1.73205 + 1.32288\right) && \text{Using a calculator} \\
&= \frac{1}{2}\left(6.99142\right) \\
&= 3.49571.
\end{aligned}$$

As we split the interval $[0, 2]$ with more subintervals with equal width, we will get more accurate approximations.

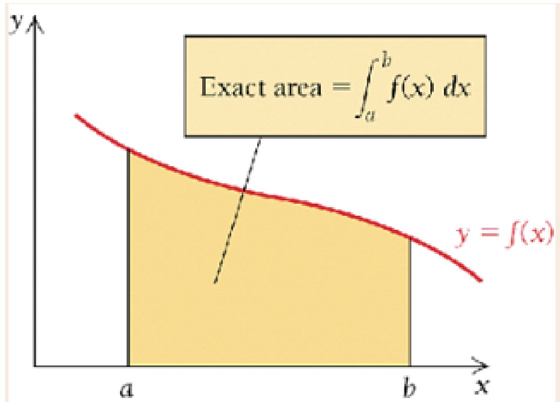


With 4 subintervals, area ≈ 3.49571. With 8 subintervals, area ≈ 3.33977. Actual area is π = 3.1415926....

If the number of the subintervals goes to infinity, we expect to obtain the actual area - this limiting process leads to the definition of definition integral over an interval.

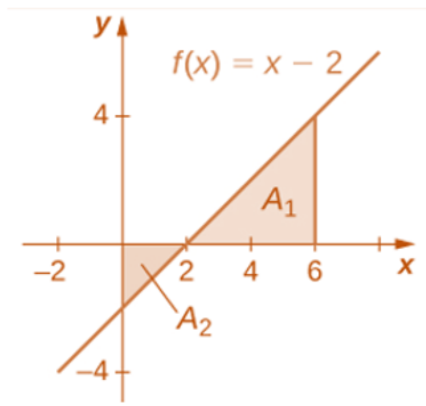
3 Definite Integral

Definition: Let $y = f(x)$ be continuous and non-negative over an interval $[a, b]$. A definite integral of $f(x)$ over interval $[a, b]$ is the limit as $n \to \infty$ (equivalently, the equal width $\Delta x \to 0$) of the **Riemann sum** of the areas of rectangles under the graph of $y = f(x)$ over $[a, b]$.



$$\text{exact area} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx.$$

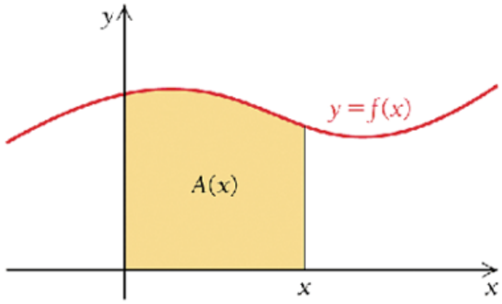
Caution: The above definition assumes that the function is positive. If the function is negative over $[a, b]$, the corresponding area is negative. If a function has both negative and positive components, then negative defines negative area and the positive part defined the positive area. For example, A_1 is positive and A_2 is negative in the following figure.



4 Fundmental Theorem of Calculus

To develop a technical formula to calculate the area under the curve of a positive function $f(x)$ over an interval, we define the following **following area function** $A(x)$, an antiderivative of $f(x)$, that is,

$$\frac{dA(x)}{dx} = f(x).$$

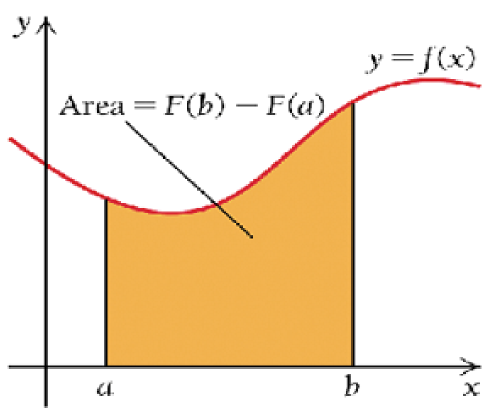


Theorem: Let $f(x)$ be a non-negative continuous function over $[0, b]$, and let $A'(x) = f(x)$ be the area between the graph of $f(x)$ and the x-axis over $[0, x]$, with $A(x)$. Then $0 < x < b$ is a differentiable function of x and $A'(x) = f(x)$.

Definition: Let $f(x)$ be any continuous function over $[a, b]$ and $F(x)$ be any antiderivative of $f(x)$.

$$\frac{dF(x)}{dx} = f(x)$$

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$$



Results from Fundamental Theorem of Calculus

$$1.\ \int_a^a f(x)\,dx = F(a) - F(a) = 0$$

$$2.\ \int_a^b f(x)\,dx = F(b) - F(a)$$

$$= -[F(a) - F(b)] = -\int_b^a f(x)\,dx$$

Example 2: Evaluate the following definite integrals

1. $\int_{-1}^{4}(x^2 - x)dx$
2. $\int_0^2 e^x dx$

Solution: Use the fundamental theorem of calculus.

1. $\int_{-1}^{4}(x^2 - x)dx = \int_{-1}^{4} x^2 dx - \int_{-1}^{4} x dx = \frac{x^3}{3}\Big|_{-1}^{4} - \frac{x^2}{2}\Big|_{-1}^{4} = \frac{4^3-(-1)^3}{3} - [\frac{4^2-(-1)^2}{2}] = \frac{65}{3} - \frac{15}{2} = \frac{85}{6}$.
2. $\int_0^2 e^x dx = e^x\Big|_0^2 = e^2 - 1$.

Example 3: *Business application* Melanie's Crafts estimates that its sales are growing continuously at a rate given by

$$S'(t) = 20e^t,$$

where $S'(t)$ is in dollars per day, on day t.

1. Find the accumulated sales for the first 5 days.
2. Find the accumulated sales from the beginning of the 2nd day through the 5th day.

Solution: Note that the accumulated sale is given by

$$f(t) = S'(t) = 20e^t.$$

1. The accumulated sales for the first 5 days the accumulated sales for the first 5 days is

$$\int_0^5 f(t)dt = \int_0^5 20e^t dt = 20\int_0^5 e^t dt = 20\left(e^t\Big|_0^5\right) = 20(e^5 - e^0) \approx 2948.26.$$

2. The accumulated sales from the beginning of the 2nd day through the 5th day is

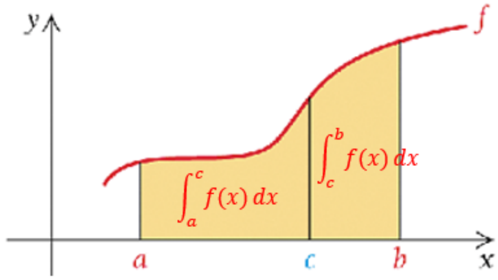
$$\int_2^5 20e^t dt = 20\int_2^5 e^t dt = 20\left(e^t\Big|_2^5\right) = 20(e^5 - e^2) \approx 2820.48.$$

5 Properties of Definite Integrals

The additive property of definite integrals is summarized in the following theorem.

Theorem 2: For any c in interval $[a, b]$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

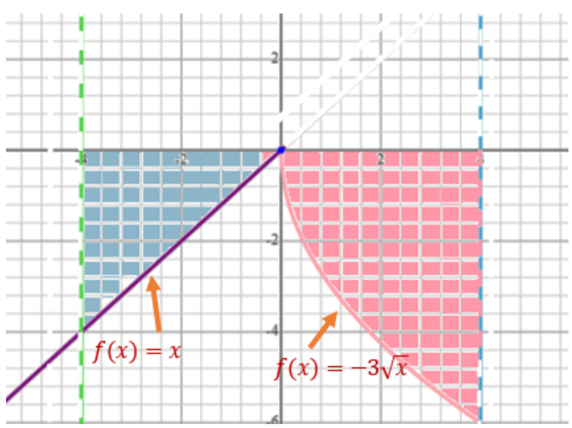


Example 3: Find the definite integral of $f(x)$ over interval $[-4, 4]$ where

$$f(x) = \begin{cases} -3\sqrt{x} & \text{for} \quad x > 0 \\ x & \text{for} \quad x \leq 0 \end{cases}.$$

Solution: Since the function on interval has two different expressions. So the additive property of integral should be used.

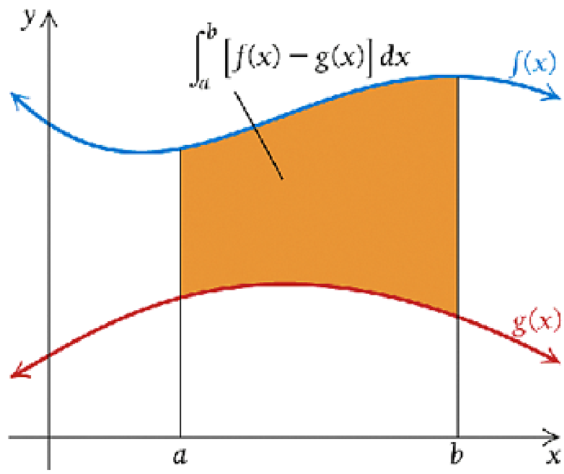
$$\begin{aligned}
\int_{-4}^{4} f(x)dx &= \int_{-4}^{0} f(x)dx + \int_0^4 f(x)dx = \int_0^4 -3\sqrt{x}dx + \int_{-4}^{0} xdx \\
&= -3\int_0^4 x^{1/2}dx + \int_{-4}^{0} xdx = -3\frac{x^{1/2+1}}{1/2+1}\Big|_0^4 + \frac{x^2}{2}\Big|_{-4}^{0} \\
&= -3\frac{x^{3/2}}{3/2}\Big|_0^4 + \left[0 - \frac{(-4)^2}{2}\right] = -2\sqrt{x^3}\Big|_0^4 - 8 = -16 - 8 = -24
\end{aligned}$$



The next property is related to the area of a region bounded by two graphs.

Theorem 3: Let $f(x)$ and $g(x)$ to be continuous functions with $f(x) \geq g(x)$ over $[a, b]$. Then the area of the region between the two curves, from $x = a$ to $x = b$, is

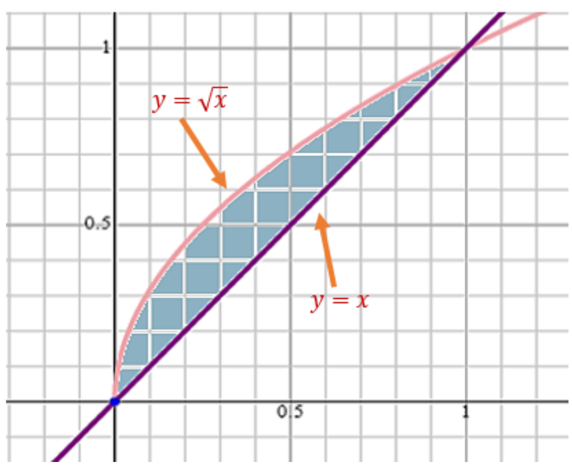
$$\int_a^b [f(x) - g(x)]dx$$



Clearly,

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

Example: Find the area of the region **enclosed** by $f(x) = \sqrt{x}$ and $g(x) = x$.



Solution: From the enclosed region is defined on interval $[0, 1]$. Therefore,

$$\text{eclosed area} = \int_0^1 [f(x) - g(x)]dx = \int_0^1 \sqrt{x}dx - \int_0^1 xdx$$

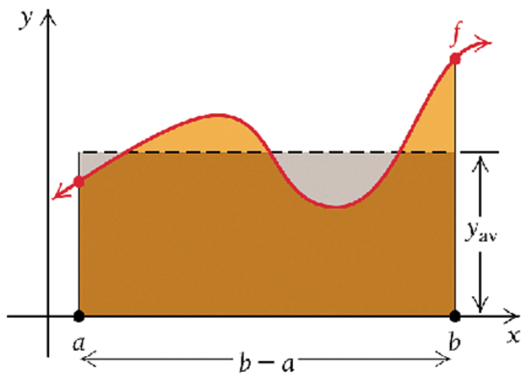
$$= \frac{x^{1/2+1}}{1/2+1}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 = \frac{1^{1/2+1}}{1/2+1} - \frac{1^2}{2} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

6 Average Value of A Continuous Function

From the definition of definite integral and its geometry we can see that integrate a function over an interval is a process of taking cumulative summation. It is meaningful to define the average value of a continuous function.

Definition For a continuous function $f(x)$ over $[a, b]$, the average of $f(x)$ over $[a, b]$, denoted by y_{av} or $\overline{f(x)}$, is defined to be

$$y_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)dx$$



Example 5 The population of the United States can be approximated by

$$P(t) = 310.65e^{0.00722t},$$

where $P(t)$ is in millions and **t is in the number of years since 2010**. (*Source: Population Division, U.S. Census Bureau*) Find the average size of the population from 2012 and 2019.

Solution: This problem is equivalent to finding the average of the population growth function over interval $[2012-2010, 2019-2010] = [2, 9]$.

$$\begin{aligned}
\text{average population} &= \frac{1}{9-2}\int_2^9 310.65e^{0.00722t}dt \\
&= \frac{1}{7}\times 310.65\int_2^9 e^{0.00722t}d\left(\frac{0.00722t}{0.00722}\right) \\
&= \frac{1}{7}\times\frac{310.65}{0.00722}\int_2^9 e^{0.00722t}d(0.00722t) \\
&= \frac{1}{7}\times\frac{310.65}{0.00722}\times e^{0.00722t}\Big|_2^9 \\
&\approx 6146.617\times(e^{0.00722\times 9} - e^{0.00722\times 2}) \approx 323.2685
\end{aligned}$$