# 3. Rules of Derivative

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## Contents



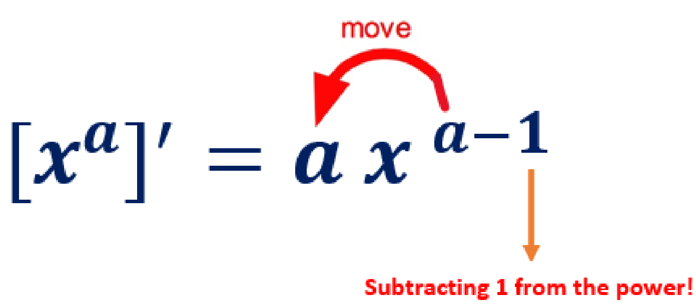
## 1 Review: Basic Rules of Derivative

We defined the derivaive of a function $y = f(x)$, denoted by $y' = f'(x)$, to be instantaneous rate of change

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The following basic rules were also introduced to calculate derivatives easier.

1. $f(x) = c$, then $f'(x) = (c)' = 0$.
2. $f(x) = x$, then $f'(x) = (x)' = 1$.
3. $f(x) = x^a$, then $f'(x) = (x^a)' = ax^{a-1}$. for any real number $a$.




Properties:

1. $[bf(x)]' = b[f(x)]'$
2. $[f(x) + g(x)]' = [f(x)]' + [g(x)]'$

We will continue to introduce more rules and properties of derivatives this week.

# 2 Leibniz Notation

In calculus, **Leibniz's notation** uses the symbols $dx$ and $dy$ to represent infinitely small (or infinitesimal) increments of $x$ and $y$, respectively, just as $\Delta x$ and $\Delta y$ represent finite increments of $x$ and $y$, respectively.

Consider $y$ as a function of a variable $x$, or $y = f(x)$. If this is the case, then the derivative of $y$ with respect to $x$, which later came to be viewed as the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

was, according to Leibniz, the quotient of an **infinitesimal increment** of $y$ by an **infinitesimal increment** of $x$, or

$$\frac{dy}{dx} = f'(x),$$

The **infinitesimal increments** are called **differentials**. From now on, we will use the following notations interchangeably.

$$\frac{df(x)}{dx}, \ \frac{d}{dx} f(x), \ , \text{ and } f'(x)$$

# 3 Multiplicative Rule

If a function that has a complex form can be re-expressed into a product of two relatively simple functions, then we can use the **multiplicative rule** to find the derivative.

Let $f(x)$ and $g(x)$ be the two differentiable functions (i.e., the derivative of both functions exists everywhere in the domain).

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-product-rule-calculus-animation.gif

**Example 1**: Find the derivative of the following function.

(a). $y = \sqrt[3]{x^2}(2x - x^2)$

(b). $y = (6x^3 - x)(10 - 20x)$

**Solution** We use the multiplicative rule to calculate the above derivative.

(a). $y' = [x^{2/3}(2x - x^2)]' = (x^{2/3})'(2x - x^2) + x^{2/3}(2x - x^2)'$

$$= \frac{2}{3}x^{2/3-1}(2x - x^2) + x^{2/3}(2 - 2x) = \frac{2}{3}x^{-1/3}(2x - x^2) + x^{2/3}(2 - 2x)$$

$$= \frac{2}{3}x^{2/3}(2 - x) + 2x^{2/3}(1 - x) = 2x^{2/3}[(2 - x)/3 + 1 - x]$$

$$= 2x^{2/3}\frac{2 - x + 3(1 - x)}{3} = \frac{2x^{2/3}(5 - 4x)}{3}$$

(b). $y' = [(6x^3 - x)(10 - 20x)]' = (6x^3 - x)'(10 - 20x) + (6x^3 - x)(10 - 20x)'$

$= (18x^2 - x)(10 - 20x) + (6x^3 - x) \times (-20) = -480x^3 + 180x^2 + 40x - 10$

**Example 2**: (This example is taken from the textbook)

**EXAMPLE 2** Let $F(x) = (2x + 1)(x^2 - 3)$. Find $F'(x)$ two ways: first by multiplying the two binomials and differentiating the product, then by using the Product Rule. Verify that both methods give the same result.

***Solution*** Multiplying the binomials, we have

$$F(x) = (2x + 1)(x^2 - 3) = 2x^3 + x^2 - 6x - 3.$$

Now, using the Sum–Difference Rule, we have

$$\begin{aligned} F'(x) &= \frac{d}{dx}(2x^3) + \frac{d}{dx}(x^2) - \frac{d}{dx}(6x) - \frac{d}{dx}(3) \\ &= 6x^2 + 2x - 6. \end{aligned}$$

Using the Product Rule, where the two factors are $f(x) = 2x + 1$ and $g(x) = x^2$ we have

$$\begin{aligned} F'(x) &= \overbrace{(2x + 1)}^{f(x)}\overbrace{(2x)}^{g'(x)} + \overbrace{(x^2 - 3)}^{g(x)}\overbrace{(2)}^{f'(x)} \\ &= 4x^2 + 2x + 2x^2 - 6 \qquad \text{Using the distributive law} \\ &= 6x^2 + 2x - 6. \end{aligned}$$

# 4 Quotient Rule

The quotient rule can be used for differentiation when taking the derivative of a function divided by another function. For example, rational functions are this type of question.

Let $f(x)$ and $g(x)$ be two differentiable functions. The quotient rule of the derivative is given in the following.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

The following animated graph shows how to manipulate the terms algebraically.

https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-quotient-rule-formula-animation.gif

**Example 3**. Find the derivative of the following functions.

(a). $W(z) = (3z+9)/(2-z)$

(b). $h(x) = 4\sqrt{x}/(x^2-2)$

**Solution**: Using the quotient rule, we have

(a). $W'(z) = [(3z+9)'(2-z) - (3z+9)(2-z)']/(2-z)^2$

$$= \frac{3(2-z) - (3z+9)(-1)}{(2-z)^2} = \frac{15}{(2-z)^2}$$

(b). $y' = \left[(4\sqrt{x})'(x^2-2) - (4\sqrt{x})(x^2-2)'\right]/(x^2-2)^2$

$$= \frac{(2/\sqrt{x})(x^2-2) - 8x\sqrt{x}}{(x^2-2)^2} = -\frac{6x\sqrt{x} + 4/\sqrt{x}}{(x^2-2)^2}.$$

**Example 4** (This example is taken from the textbook)

**EXAMPLE 3** Differentiate: $f(x) = \dfrac{1+x^2}{x^3+1}$.

***Solution*** We let $f(x) = 1 + x^2$ and $g(x) = x^3 + 1$. We have

$$\frac{dy}{dx} = \frac{\overbrace{(x^3+1)}^{g(x)}\overbrace{(2x)}^{f'(x)} - \overbrace{(1+x^2)}^{f(x)}\overbrace{(3x^2)}^{g'(x)}}{\underbrace{(x^3+1)^2}_{(g(x))^2}} \qquad \text{Using the Quotient Rule}$$

$$= \frac{2x^4 + 2x - 3x^2 - 3x^4}{(x^3+1)^2} \qquad \text{Using the distributive law}$$

$$= \frac{-x^4 - 3x^2 + 2x}{(x^3+1)^2}. \qquad \text{Simplifying}$$

**Important Business Functions**: Cost, Revenue, and Profit-related functions and applications.

**DEFINITION**

If $C(x)$ is the cost of producing $x$ items, then the **average cost** of producing $x$ items is $\frac{C(x)}{x}$.

If $R(x)$ is the revenue from the sale of $x$ items, then the **average revenue** from selling $x$ items is $\frac{R(x)}{x}$.

If $P(x)$ is the profit from the sale of $x$ items, then the **average profit** from selling $x$ items is $\frac{P(x)}{x}$.

**Example 5** (This example is also taken from the textbook)

**EXAMPLE 5** **Business.** Paulsen's Greenhouse finds that the cost, in dollars, of growing $x$ hundred geraniums is modeled by

$$C(x) = 200 + 100\sqrt[4]{x}.$$

If revenue from the sale of $x$ hundred geraniums is modeled by

$$R(x) = 120 + 90\sqrt{x},$$

find each of the following.

**a)** The average cost, average revenue, and average profit when 300 geraniums are grown and sold

**b)** The rate at which the average profit is changing when 300 geraniums are grown and sold

***Solution***

**a)** We let $\overline{C}$, $\overline{R}$, and $\overline{P}$ represent average cost, average revenue, and average profit, respectively. Thus,

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{200 + 100\sqrt[4]{x}}{x};$$

$$\overline{R}(x) = \frac{R(x)}{x} = \frac{120 + 90\sqrt{x}}{x};$$

$$\overline{P}(x) = \frac{R(x) - C(x)}{x} = \frac{-80 + 90\sqrt{x} - 100\sqrt[4]{x}}{x}.$$

To find the average cost, average revenue, and average profit when 300 geraniums are grown and sold, we evaluate each function at $x = 3$:

$$\overline{C}(3) = \frac{200 + 100\sqrt[4]{3}}{3} \approx \$110.54 \text{ per hundred geraniums;}$$

$$\overline{R}(3) = \frac{120 + 90\sqrt{3}}{3} \approx \$91.96 \text{ per hundred geraniums;}$$

$$\overline{P}(3) = \frac{-80 + 90\sqrt{3} - 100\sqrt[4]{3}}{3} \approx -\$18.57 \text{ per hundred geraniums.}$$

# 5 Chain Rule

Before introducing the Chain Rule, we first review the definition of a composite function.

## 5.1 Composite Function

Suppose the two given functions are $f$ and $g$, the composition of $f$ and $g$, denoted by $f \circ g$, is given by

$$f \circ g(x) = f(g(x)) \quad \text{and} \quad g \circ f(x) = g(f(x))$$

$$f \circ g = f\left[\overset{\substack{\text{input} \\ \downarrow}}{g(x)}\right] \qquad\qquad g \circ f = g\left[\overset{\substack{\text{input} \\ \downarrow}}{f(x)}\right]$$

**Example 6** Let $f(x) = 2x^2 - 3x - 1$ and $g(x) = -x + 5$. Find $f \circ g$.

$$\begin{aligned}
f(x) &= 2x^2 - 3x - 1 && \texttt{Write down the main function} \\
f[g(x)] &= 2[g(x)]^2 - 3[g(x)] - 1 && \texttt{Plug in the input function} \\
&= 2[-x+5]^2 - 3[-x+5] - 1 && \texttt{Square the binomial} \\
&= 2[x^2 - 10x + 25] - 3[-x+5] - 1 && \texttt{Apply distributive property} \\
&= 2x^2 - 20x + 50 + 3x - 15 - 1 && \texttt{Combine like terms} \\
f[g(x)] &= 2x^2 - 17x + 34
\end{aligned}$$

$$f \circ g = f[g(x)] = 2x^2 - 17x + 34 \longleftarrow \texttt{Final answer!}$$

**Example 7** Let $f(x) = 1/(x+3)$ and $g(x) = (-3x-2)/x$. Find $f \circ g$.

$$f(x) = \frac{2}{x+3}$$ Write down the main function

$$f[g(x)] = \frac{2}{g(x)+3}$$ Substitute the input function

$$= \frac{2}{\frac{-3x-2}{x}+3}$$

$$= \frac{2}{\frac{-3x-2}{x}+3\left(\frac{x}{x}\right)}$$ Add the complex denominator using LCD

$$= \frac{2}{\frac{-3x-2}{x}+\frac{3x}{x}}$$

$$= \frac{2}{\frac{-3x-2+3x}{x}}$$ Combine like terms on the numerator of the complex denominator

$$= \frac{2}{\frac{-2}{x}}$$ Simplify

$$= 2 \cdot \frac{x}{-2}$$ Divide by multiplying the numerator to the reciprocal of the denominator

$$f[g(x)] = -x$$ Simplify by canceling common factors

$$f \circ g = f[g(x)] = -x \longleftarrow$$ Final answer!

## 5.2 Chain Rule

The following Chain Rule is used to find the derivative of composite functions.

$$[f(g(x))]' = f'(g(x)) \times g'(x)$$

https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-Chain-Rule-1.gif

**Example 8** Find the derivative of the following function

$$f(x) = \sqrt[4]{\frac{x+3}{x-2}}.$$

Here we use the Quotient Rule to differentiate the inside function:

$$\frac{d}{dx}\sqrt[4]{\frac{x+3}{x-2}} = \frac{d}{dx}\left(\frac{x+3}{x-2}\right)^{1/4} = \frac{1}{4}\left(\frac{x+3}{x-2}\right)^{1/4-1}\left[\frac{(x-2)1-1(x+3)}{(x-2)^2}\right]$$

$$= \frac{1}{4}\left(\frac{x+3}{x-2}\right)^{-3/4}\left[\frac{x-2-x-3}{(x-2)^2}\right]$$

$$= \frac{1}{4}\left(\frac{x+3}{x-2}\right)^{-3/4}\left[\frac{-5}{(x-2)^2}\right], \quad \text{or} \quad \frac{-5}{4(x+3)^{3/4}(x-2)^{5/4}}.$$

**Example 9**: Find the derivative of $f(x) = \sqrt[3]{x^5 + 6x}$.