7. Critical Values

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Contents

1 Review Topics

We first review the Five Rules of Derivative

These rules of the derivative will be used frequently throughout the semester.

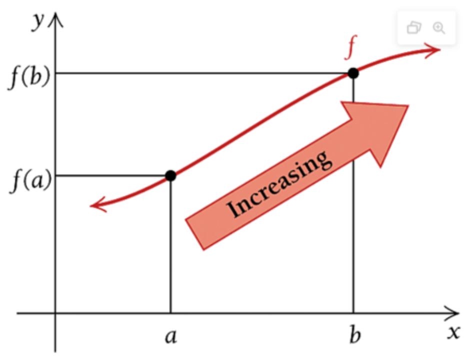
Rules of Derivative-Review

Power Rule	$(x^a)' = ax^{a-1}$
Additive Rule	$[f(x)+g(x)]' = f'(x)+g'(x)$
Multiplicative Rule	$[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x)$
Quotient Rule	$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)+f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\{f[g(x)]\}' = f[g(x)] \times g'(x)$

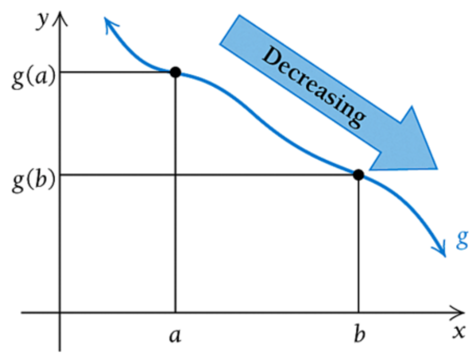
We will apply derivatives to solve real-world optimization problems maximizing profit or revenue and minimizing the cost, etc. This note focuses on the concepts of critical values and using derivation to find critical values.

2 Increasing and decreasing functions

We reviewed monotonic function in the first week. The following figures demonstrate increasing and decreasing functions.



If the input a is less than the input b, then the output for a is less than the output for b.



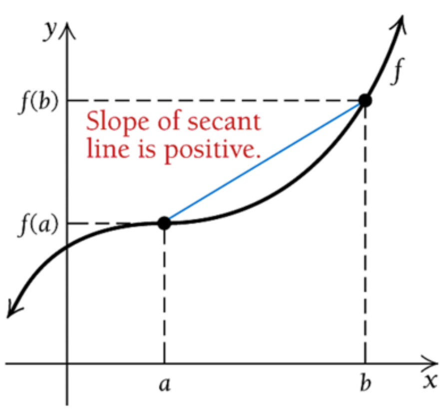
If the input a is less than the input b, then the output for a is greater than the output for b.

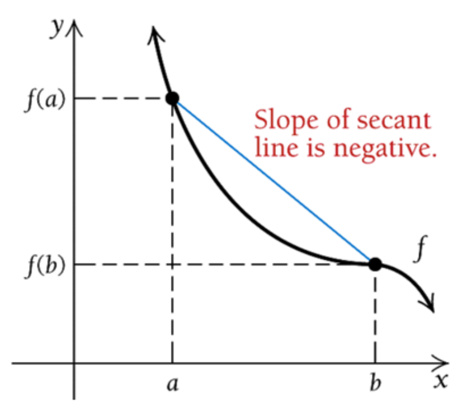
Using the definition of the derivative, we can see that

- If $f(x)$ is increasing over $[a, b]$, then $f'(x) > 0$ over $[a, b]$;
- If $f(x)$ is decreasing over $[a, b]$, then $f'(x) < 0$ over $[a, b]$.

Graphically, we have the following figures explain the above observation

$$\text{Increasing} : \frac{f(b) - f(a)}{b - a} > 0. \qquad \text{Decreasing} : \frac{f(b) - f(a)}{b - a} < 0.$$





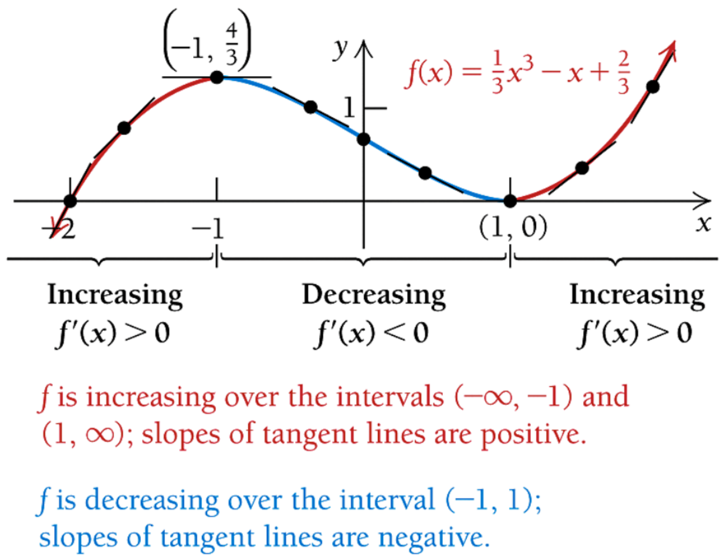
We next formalize the above observation and present the following theorem.

Theorem 1: Let $f(x)$ be differential over interval $I = [a, b]$

1. If $f'(x) > 0$ for all x in $I = [a, b]$, then $f(x)$ is increasing over $I = [a, b]$
2. If $f'(x) < 0$ for all x in $I = [a, b]$, then $f(x)$ is deceasing over $I = [a, b]$.

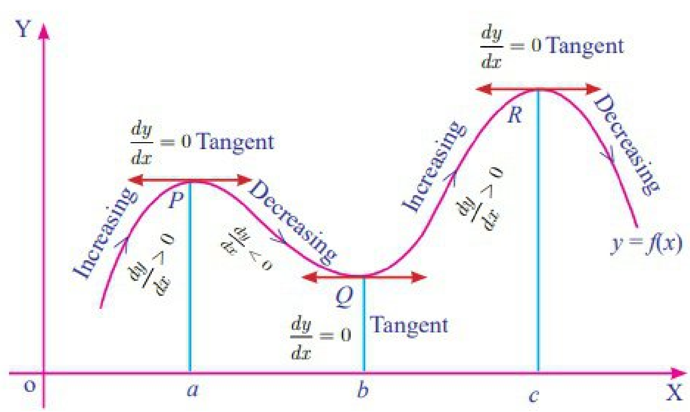
Example 1: Using the above Theorem 1 to find the intervals on which the function $f(x) = x^3/3 - x + 2/3$.

Solution: The following figure answers the above question.



3 Critival Value

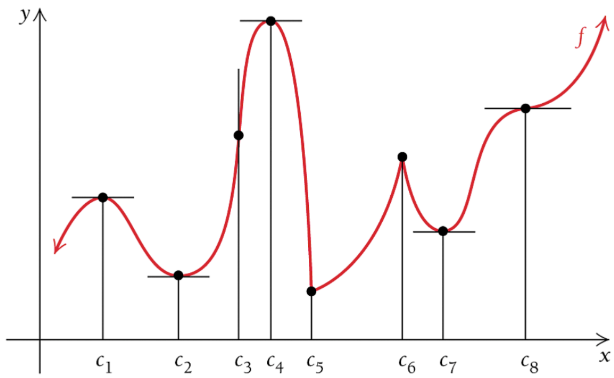
In the above curve, two points A and B are special because the monotonicity of the function changed at these points (from increasing to decreasing at A and from decreasing to increasing at B). If the derivative of a function exists over an interval $[a, b]$, if the function changes its monotonicity, its sign of derivative also changes. That means, there exists a value, say c, in $[a, b]$ that satisfies $f'(c) = 0$.



Definition: A **critical value** of a function $f(x)$ is any number c in the **domain** of $f(x)$ for which the tangent line at $(c, f(c))$ is horizontal or for which the derivative does **not** eixst. That is, c is a **critical** value if $f(c)$ exists and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}$$

If c is a critical value of $f(x)$, then $(c, f(c))$ is called a **critical point**.



All labeled points on the above figure are critical points since the corresponding derivative is either 0 or does not exist.

1. $f'(x) = 0$ for $x = c_1, c_2, c_4, c_7$ and c_8. That is, the tangent line to the graph is horizontal at these values.
2. $f'(x)$ does not exist for $x = c_3, c_5$ and c_6. The tangent line is vertical at c_3 and there are corners at both c_5 and c_6.

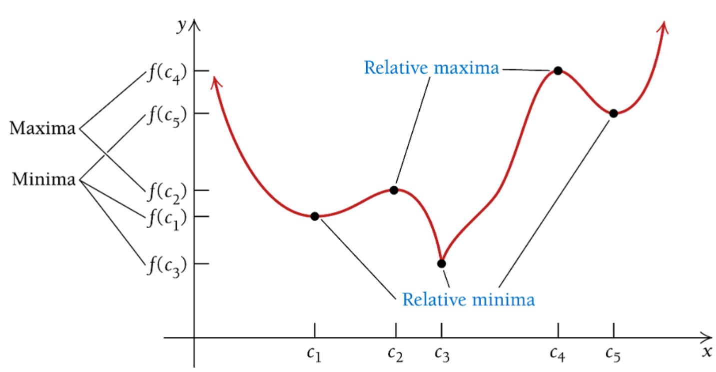
Example 2 Find critical values of the function $f(x) = x^3/3 - x + 2/3$.

Solution: By the definition, we need to the solution to $f'(x) = 0$ and those values on which the derivative of $f(x)$ does not exist.

Note that $f'(x) = x^2 - 1$. Therefore, equation $x^2 - 1 = 0$ has solutions $x = \pm 1$. These two critical values are the same as shown in example 1.

4 Relative (Local) Maximum and Minimum Values

Graphically, the local maximum and minimum are the second coordinates of the points that are labeled in the following figure.



Here, $f(c_2)$ and $f(c_4)$ are each an example of a **relative**, or **local**, **maximum (plural: maxima)**, and $f(c_1), f(c_3)$ and $f(c_5)$ are each an example of a relative, or local, **minimum (plural: minima)**. Collectively, maximum and minimum values are called **extrema (singular: extremum)**. Note that a relative minimum can be greater than a relative maximum; for example, $f(c_5) > f(c_2)$ in the graph at the bottom of the preceding page.

Note that x-values at which a continuous function has relative extrema are those values for which the derivative is 0 or for which the derivative does not exist—the critical values.

Theorem: If a function $f(x)$ has a relative extreme value $f(c)$ on an **open interval**, then c is a critical value, and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \quad \text{does not exist.}$$

The next theorem gives a test for relative extrema: **The First-Derivative Test for Relative Extrema**

Theorem: For any continuous function $f(x)$ that has exactly one critical value c over an **open interval** (a, b):

1. $f(x)$ has a relative minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b). That is, $f(x)$ is decreasing to the left of c and increasing to the right of c.
2. 1. $f(x)$ has a relative maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b). That is, $f(x)$ is increasing to the left of c and decreasing to the right of c.
3. $f(x)$ has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on both sides of c.

Example 3: Consider the relative maximum and relative minimum of function $f(x) = 4x^3 - 9x^2 - 30x + 25$.

Solution The derivative $f'(x) = 12x^2 - 18x - 30 = 6(2x^2 - 9x - 5) = 6(ax - 5)(x + 1)$. Set $f'(x) = 0$, we have $6(ax - 5)(x + 1) = 0$, therefore, $x = 2.5$ or $x = -1$.