

3. Rules of Derivative

Cheng Peng

West Chester University

Contents

1	Review: Basic Rules of Derivative	1
2	Leibniz Notation	2
3	Multiplicative Rule	2
4	Quotient Rule	4
5	Chain Rule	7
5.1	Composite Function (Reviewed in the note of Topic #1)	7
5.2	Chain Rule	9

1 Review: Basic Rules of Derivative

We defined the derivaive of a function $y = f(x)$, denoted by $y' = f'(x)$, to be instantaneous rate of change

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The following basic rules were also introduced to calculate derivatives easier.

1. $f(x) = c$, then $f'(x) = (c)' = 0$.
2. $f(x) = x$, then $f'(x) = (x)' = 1$.
3. $f(x) = x^a$, then $f'(x) = (x^a)' = ax^{a-1}$. for any real number a .

$$[x^a]' = a x^{a-1}$$

Subtracting 1 from the power!

Properties:

1. $[bf(x)]' = b[f(x)]'$
2. $[f(x) + g(x)]' = [f(x)]' + [g(x)]'$

We will continue to introduce more rules and properties of derivatives this week.

2 Leibniz Notation

In calculus, **Leibniz's notation** uses the symbols dx and dy to represent infinitely small (or infinitesimal) increments of x and y , respectively, just as Δx and Δy represent finite increments of x and y , respectively.

Consider y as a function of a variable x , or $y = f(x)$. If this is the case, then the derivative of y with respect to x , which later came to be viewed as the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

was, according to Leibniz, the quotient of an **infinitesimal increment** of y by an **infinitesimal increment** of x , or

$$\frac{dy}{dx} = f'(x),$$

The **infinitesimal increments** are called **differentials**. From now on, we will use the following notations interchangeably.

$$\frac{df(x)}{dx}, \frac{d}{dx}f(x), \text{ , and } f'(x)$$

3 Multiplicative Rule

If a function that has a complex form can be re-expressed into a product of two relatively simple functions, then we can use the **multiplicative rule** to find the derivative.

Let $f(x)$ and $g(x)$ be the two differentiable functions (i.e., the derivative of both functions exists everywhere in the domain).

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

<https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-product-rule-calculus-animation.gif>

Caution:

$$[f(x) \times g(x)]' \neq f'(x) \times g'(x) !$$

Example 1: Find the derivative of the following function.

(a). $y = \sqrt[3]{x^2}(2x - x^2)$

(b). $y = (6x^3 - x)(10 - 20x)$

Solution We use the multiplicative rule to calculate the above derivative.

$$\begin{aligned} \text{(a). } y' &= [x^{2/3}(2x - x^2)]' = (x^{2/3})'(2x - x^2) + x^{2/3}(2x - x^2)' \\ &= \frac{2}{3}x^{2/3-1}(2x - x^2) + x^{2/3}(2 - 2x) = \frac{2}{3}x^{-1/3}(2x - x^2) + x^{2/3}(2 - 2x) \\ &= \frac{2}{3}x^{2/3}(2 - x) + 2x^{2/3}(1 - x) = 2x^{2/3}[(2 - x)/3 + 1 - x] \\ &= 2x^{2/3} \frac{2 - x + 3(1 - x)}{3} = \frac{2x^{2/3}(5 - 4x)}{3} \end{aligned}$$

$$\begin{aligned} \text{(b). } y' &= [(6x^3 - x)(10 - 20x)]' = (6x^3 - x)'(10 - 20x) + (6x^3 - x)(10 - 20x)' \\ &= (18x^2 - 1)(10 - 20x) + (6x^3 - x) \times (-20) = -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Example 2: (This example is taken from the textbook)

EXAMPLE 2 Let $F(x) = (2x + 1)(x^2 - 3)$. Find $F'(x)$ two ways: first by multiplying the two binomials and differentiating the product, then by using the Product Rule. Verify that both methods give the same result.

Solution Multiplying the binomials, we have

$$F(x) = (2x + 1)(x^2 - 3) = 2x^3 + x^2 - 6x - 3.$$

Now, using the Sum-Difference Rule, we have

$$\begin{aligned} F'(x) &= \frac{d}{dx}(2x^3) + \frac{d}{dx}(x^2) - \frac{d}{dx}(6x) - \frac{d}{dx}(3) \\ &= 6x^2 + 2x - 6. \end{aligned}$$

Using the Product Rule, where the two factors are $f(x) = 2x + 1$ and $g(x) = x^2 - 3$ we have

$$\begin{aligned} F'(x) &= \overbrace{(2x + 1)}^{f(x)} \overbrace{(2x)}^{g'(x)} + \overbrace{(x^2 - 3)}^{g(x)} \overbrace{(2)}^{f'(x)} \\ &= 4x^2 + 2x + 2x^2 - 6 \quad \text{Using the distributive law} \\ &= 6x^2 + 2x - 6. \end{aligned}$$

4 Quotient Rule

The quotient rule can be used for differentiation when taking the derivative of a function divided by another function. For example, rational functions are this type of question.

Let $f(x)$ and $g(x)$ be two differentiable functions. The quotient rule of the derivative is given in the following.

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

The following animated graph shows how to manipulate the terms algebraically.

<https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-quotient-rule-formula-animation.gif>

Caution:

$$\left[\frac{f(x)}{g(x)} \right]' \neq \frac{f'(x)}{g'(x)}!$$

Example 3. Find the derivative of the following functions.

(a). $W(z) = (3z + 9)/(2 - z)$

(b). $h(x) = 4\sqrt{x}/(x^2 - 2)$

Solution: Using the quotient rule, we have

$$\begin{aligned} \text{(a). } W'(z) &= [(3z + 9)'(2 - z) - (3z + 9)(2 - z)'] / (2 - z)^2 \\ &= \frac{3(2 - z) - (3z + 9)(-1)}{(2 - z)^2} = \frac{15}{(2 - z)^2} \end{aligned}$$

$$\begin{aligned} \text{(b). } y' &= [(4\sqrt{x})'(x^2 - 2) - (4\sqrt{x})(x^2 - 2)'] / (x^2 - 2)^2 \\ &= \frac{(2/\sqrt{x})(x^2 - 2) - 8x\sqrt{x}}{(x^2 - 2)^2} = -\frac{6x\sqrt{x} + 4/\sqrt{x}}{(x^2 - 2)^2}. \end{aligned}$$

Example 4 (This example is taken from the textbook)

EXAMPLE 3 Differentiate: $f(x) = \frac{1 + x^2}{x^3 + 1}$.

Solution We let $f(x) = 1 + x^2$ and $g(x) = x^3 + 1$. We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\overbrace{(x^3 + 1)}^{g(x)} \overbrace{(2x)}^{f'(x)} - \overbrace{(1 + x^2)}^{f(x)} \overbrace{(3x^2)}^{g'(x)}}{\underbrace{(x^3 + 1)^2}_{(g(x))^2}} && \text{Using the Quotient Rule} \\ &= \frac{2x^4 + 2x - 3x^2 - 3x^4}{(x^3 + 1)^2} && \text{Using the distributive law} \\ &= \frac{-x^4 - 3x^2 + 2x}{(x^3 + 1)^2}. && \text{Simplifying} \end{aligned}$$

Important Business Functions: Cost, Revenue, and Profit-related functions and applications.

DEFINITION

If $C(x)$ is the cost of producing x items, then the **average cost** of producing x items is $\frac{C(x)}{x}$.

If $R(x)$ is the revenue from the sale of x items, then the **average revenue** from selling x items is $\frac{R(x)}{x}$.

If $P(x)$ is the profit from the sale of x items, then the **average profit** from selling x items is $\frac{P(x)}{x}$.

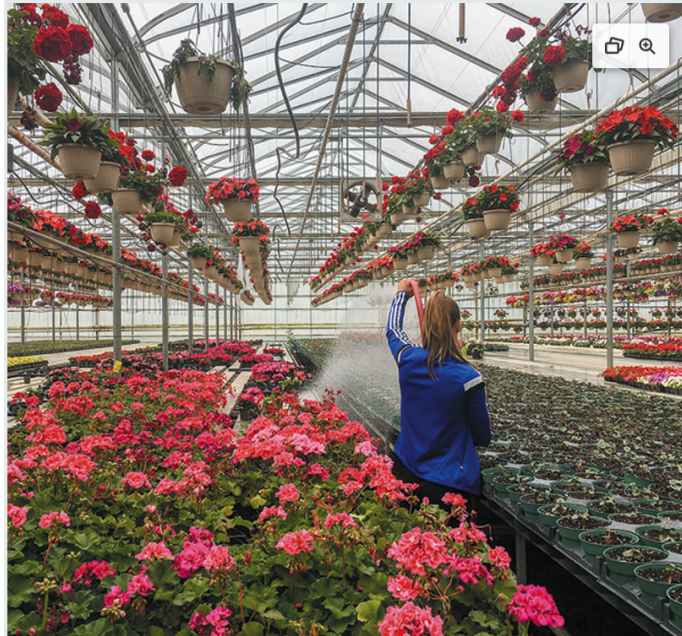
Example 5 (This example is also taken from the textbook)

Paulsen's Greenhouse finds that the cost, in dollars, of growing x hundred geraniums is modeled by

$$C(x) = 200 + 100\sqrt[4]{x}$$

If revenue from the sale of x hundred geraniums is modeled by

$$R(x) = 120 + 90\sqrt{x}$$



find each of the following.

- (a) The average cost, average revenue, and average profit when 300 geraniums are grown and sold.
- (b) The rate at which the average profit is changing when 300 geraniums are grown and sold.

- a. We let \bar{C} , \bar{R} , and \bar{P} represent average cost, average revenue, and average profit, respectively. Thus,

$$\begin{aligned}\bar{C}(x) &= \frac{C(x)}{x} = \frac{200 + 100\sqrt[4]{x}}{x}; \\ \bar{R}(x) &= \frac{R(x)}{x} = \frac{120 + 90\sqrt{x}}{x}; \\ \bar{P}(x) &= \frac{R(x) - C(x)}{x} = \frac{-80 + 90\sqrt{x} - 100\sqrt[4]{x}}{x}.\end{aligned}$$

To find the average cost, average revenue, and average profit when 300 geraniums are grown and sold, we evaluate each function at $x = 3$:

$$\begin{aligned}\bar{C}(3) &= \frac{200 + 100\sqrt[4]{3}}{3} \approx \$110.54 \text{ per hundred geraniums}; \\ \bar{R}(3) &= \frac{120 + 90\sqrt{3}}{3} \approx \$91.96 \text{ per hundred geraniums}; \\ \bar{P}(3) &= \frac{-80 + 90\sqrt{3} - 100\sqrt[4]{3}}{3} \approx -\$18.57 \text{ per hundred geraniums}.\end{aligned}$$

- b. To find the rate at which average profit is changing when 300 geraniums are grown and sold, we calculate $\bar{P}'(3)$:

$$\begin{aligned}\bar{P}'(x) &= \frac{d}{dx} \left[\frac{-80 + 90x^{1/2} - 100x^{1/4}}{x} \right] \\ &= \frac{x \left(\frac{1}{2} \cdot 90x^{1/2-1} - \frac{1}{4} \cdot 100x^{1/4-1} \right) - (-80 + 90x^{1/2} - 100x^{1/4}) \cdot 1}{x^2} \\ &= \frac{45x^{1/2} - 25x^{1/4} + 80 - 90x^{1/2} + 100x^{1/4}}{x^2} = \frac{75x^{1/4} - 45x^{1/2} + 80}{x^2}; \\ \bar{P}'(3) &= \frac{75\sqrt[4]{3} - 45\sqrt{3} + 80}{3^2} \approx 11.20.\end{aligned}$$

When 300 geraniums have been grown and sold, the average profit is increasing by \$11.20 per hundred geraniums for each additional 100 geraniums sold.

5 Chain Rule

Before introducing the Chain Rule, we first review the definition of a composite function.

5.1 Composite Function (Reviewed in the note of Topic #1)

We have reviewed this concept in topic #1. If you still remember the definition and how to find the composite function, you skip this subsection. Otherwise, read this subsection and again and **do** the examples by yourself.

Suppose the two given functions are f and g , the composition of f and g , denoted by $f \circ g$, is given by

$$f \circ g(x) = f(g(x)) \quad \text{and} \quad g \circ f(x) = g(f(x))$$

$$\begin{array}{cc}
 \text{input} & \text{input} \\
 \downarrow & \downarrow \\
 f \circ g = f[g(x)] & g \circ f = g[f(x)]
 \end{array}$$

Example 6 Let $f(x) = 2x^2 - 3x - 1$ and $g(x) = -x + 5$. Find $f \circ g$.

$$\begin{array}{ll}
 f(x) = 2x^2 - 3x - 1 & \text{Write down the main function} \\
 f[g(x)] = 2[g(x)]^2 - 3[g(x)] - 1 & \text{Plug in the input function} \\
 = 2[-x + 5]^2 - 3[-x + 5] - 1 & \text{Square the binomial} \\
 = 2[x^2 - 10x + 25] - 3[-x + 5] - 1 & \text{Apply distributive property} \\
 = 2x^2 - 20x + 50 + 3x - 15 - 1 & \text{Combine like terms} \\
 f[g(x)] = 2x^2 - 17x + 34 & \\
 f \circ g = f[g(x)] = 2x^2 - 17x + 34 & \leftarrow \text{Final answer!}
 \end{array}$$

Example 7 Let $f(x) = 1/(x + 3)$ and $g(x) = (-3x - 2)/x$. Find $f \circ g$.

$f(x) = \frac{2}{x+3}$	Write down the main function
$f[g(x)] = \frac{2}{g(x)+3}$	Substitute the input function
$= \frac{2}{\frac{-3x-2}{x}+3}$	
$= \frac{2}{\frac{-3x-2}{x}+3\left(\frac{x}{x}\right)}$	Add the complex denominator using LCD
$= \frac{2}{\frac{-3x-2}{x}+\frac{3x}{x}}$	
$= \frac{2}{\frac{-3x-2+3x}{x}}$	Combine like terms on the numerator of the complex denominator
$= \frac{2}{\frac{-2}{x}}$	Simplify
$= 2 \cdot \frac{x}{-2}$	Divide by multiplying the numerator to the reciprocal of the denominator
$f[g(x)] = -x$	Simplify by canceling common factors

$$f \circ g = f[g(x)] = -x \longleftarrow \text{Final answer!}$$

5.2 Chain Rule

The following Chain Rule is used to find the derivative of composite functions.

$$[f(g(x))]' = f'(g(x)) \times g'(x)$$

<https://github.com/pengdsci/MAT143/raw/main/w03/img/w03-Chain-Rule-1.gif>

Example 8 Find the derivative of the following function

$$f(x) = \sqrt[4]{\frac{x+3}{x-2}}.$$

Here we use the Quotient Rule to differentiate the inside function:

$$\begin{aligned}\frac{d}{dx} \sqrt[4]{\frac{x+3}{x-2}} &= \frac{d}{dx} \left(\frac{x+3}{x-2} \right)^{1/4} = \frac{1}{4} \left(\frac{x+3}{x-2} \right)^{1/4-1} \left[\frac{(x-2)1 - 1(x+3)}{(x-2)^2} \right] \\ &= \frac{1}{4} \left(\frac{x+3}{x-2} \right)^{-3/4} \left[\frac{x-2-x-3}{(x-2)^2} \right] \\ &= \frac{1}{4} \left(\frac{x+3}{x-2} \right)^{-3/4} \left[\frac{-5}{(x-2)^2} \right], \quad \text{or} \quad \frac{-5}{4(x+3)^{3/4}(x-2)^{5/4}}.\end{aligned}$$

Example 9: Find the derivative of $f(x) = \sqrt[3]{x^5 + 6x}$.