Midterm Exam #2 Review Sheet

MAT143 Brief Calculus

1. Basic Properties of Natural Log and Exponential Functions

	Natural-base Logarithm	Natural-base Exponential
Product Rule	ln(xy) = ln(x) + ln(y)	$e^x e^y = e^{x+y}$
Quotient Rule	ln(x/y) = ln(x) – ln(y)	$e^x/e^y = e^{x-y}$
Power Rule	$\ln(x^p) = p\ln(x)$	$(e^x)^p = e^{px}$
Equality Rule	ln(x) = ln(y) ➔ x = y	$e^x = e^y$ ➔ x = y

Example 1: Simplify following functions

(1). $f(x) = \ln[x/\sqrt{x+1}] = \ln(x) - \ln\sqrt{x+1} = \ln(x) - \ln(x+1)^{\frac{1}{2}} = \ln(x) - \frac{1}{2}\ln(x+1)$.

(2). $f(x) = \ln[x^3 e^{2x+1}/(x^3+1)] = \ln[x^3 e^{2x+1}] - \ln(x^3+1) = \ln x^3 + \ln e^{2x+1} - \ln(x^3+1) = 3\ln(x) + (2x+1) - \ln(x^3+1)$

(3). $f(x) = \ln[(x+1)e^{x+1}/(x\sqrt{e^x})] = \ln(x+1) + \ln e^{x+1} - \ln[xe^{x/2}] = \ln(x+1) + (x+1) - [\ln(x) + \ln e^{x/2}]$ $= \ln(x+1) - \ln(x) + (x+1) - x/2 = \ln(x+1) - \ln(x) + x/2 + 1$

2. Derivative of Natural Base Exponential Functions

Results: $[e^x]' = e^x$ and $[e^{f(x)}]' = f'(x)e^{f(x)}$

Example 2: Find the derivative of the following functions.

(1). $f(x) = \frac{x^2+1}{e^x}$

(2). $f(x) = \ln[e^{x^2+1}]$

Solution:

(1). $f'(x) = [\frac{x^2+1}{e^x}]' = \frac{(x^2+1)'e^x - (x^2+1)e^x}{[e^x]^2} = \frac{2xe^x - (x^2+1)e^x}{e^{2x}} = \frac{e^x(2x - x^2 - 1)}{e^{2x}} = \frac{2x - x^2 - 1}{e^x} = \frac{-(x-1)^2}{e^x}$.

(2). We first simplify the given function. $f(x) = \ln[(x^2+1)e^{x^2+1}] = \ln e^{x^2+1} = (x^2+1)$. Next, we take the derivative based on the simplified function in the following. $f'(x) = [(x^2+1)]' = [x^2+1]' = 2x$

Example 3: It is reasonable for a manufacturer to expect the daily output of a new worker to be low at first, increase over time, and then level off. A manufacturer of LED flashlights determines that after t workdays, the number of flashlights produced per day by the average worker can be modeled by

$$N(t) = 80 - 70e^{-0.13t}$$

Find $N'(t)$ and interpret this result as a rate of change.

Solution: $N'(t) = [80 - 70e^{-0.13t}]' = (80)' - [70e^{-0.13t}]' = -70e^{0.13t} \times (-0.13t)' = -70 \times (-0.13)e^{-0.13t} = 9.1e^{-0.13t}$

3. Derivatives of Natural Logarithmic Functions

Results: $[\ln(x)]' = 1/x$ and $[\ln f(x)]' = f'(x)/f(x)$.

Example 4: Find the derivative of $f(x) = x\ln(x^2 + 1)$

Solution: The given function cannot be simplified further. We simply take the derivative of the function $f'(x) = [x\ln(x^2 + 1)]' = [x]'\ln(x^2 + 1) + x[\ln(x^2 + 1)]' = \ln(x^2 + 1) + x\frac{(x^2+1)'}{x^2+1} = \ln(x^2 + 1) + \frac{2x}{x^2+1}$

Example 5: **Marginal profit**. The profit, in thousands of dollars, from the sale of x thousand candles can be estimated by

$$P(x) = 2x - 0.3x\ln(x)$$

(a). Find the marginal profit, $P'(x)$.

(b). Find $P'(150)$ and explain what this number represents.

Solution:

(a). $P'(x) = [2x - 0.3x\ln(x)]' = (2x)' - 0.3[x\ln(x)]' = 2 - 0.3[(x)'\ln(x) + x[\ln(x)]'] = 2 - 0.3(\ln(x) + x \times (1/x)) = 2 - 0.3(\ln x + 1) = 2 - 0.3\ln(x) - 0.3 = 1.7 - 0.3\ln(x)$

(b). $P'(150) = 1.7 - 0.3\ln(150) = 1.7 - 0.3 \times 5.01063529409626 = 0.196809411771123 \approx 0.197$. At the current production capacity (x = 150), producing one additional item will increase additional profit by 20 cents.

4. Derivative of General Exponential and Logarithmic Functions

- The following properties are useful in simplification.

Properties-Review

Logarithmic	Exponential
$\log_a[xy] = \log_a[x] + \log_a[y]$	$a^x a^y = a^{x+y}$
$\log_a[x/y] = \log_a[x] - \log_a[y]$	$a^x / a^y = a^{x-y}$
$\log_a[x]^p = p\log_a[x]$	$[a^x]^p = [a^{px}]$
Change Base Formula	
$\log_a[x] = \dfrac{\log_b x}{\log_b a}$	$a^x = b^{(\log_b a)x}$

- The key derivatives of general exponential and logarithmic functions.

$$[a^x]' = \ln(a)\, a^x \quad \text{and} \quad \left[a^{f(x)}\right]' = \ln(a)\, a^{f(x)} f'(x)$$

$$[\log_a x]' = \frac{1}{x\ln(a)} \quad \text{and} \quad [\log_a f(x)]' = \frac{f'(x)}{\ln(a) f(x)}$$

Example 6. Find the derivatives of the following functions

(1). $f(x) = 2^x$

(2). $f(x) = 5^{x^2+1}$

(4). $f(x) = \log_5(x)$

(5). $f(x) = log_2(x^2+1)$

Solution: (1). $f'(x) = [2^x]' = 2^x \times \ln(2)$

(2). $f'(x) = [5^{x^2+1}]' = \ln(5) \times 5^{x^2+1}(x^2+1)' = \ln(5)5^{x^2+1} \times 2x = 2\ln(5)x5^{x^2+1}$

(3). $f'(x) = [\log_5 x]' = \frac{1}{x\ln(5)}$

(4). $f'(x) = [\log_2(x^2+1)]' = \frac{(x^2+1)'}{\ln(2)\times(x^2+1)} = \frac{2x}{\ln(2)\times(x^2+1)}$

5. Business Applications - Exponential Models

Suppose P_0, in dollars, is invested in the Von Neumann Hi-Yield Fund, with interest compounded continuously at 7% per year. That is, at any point in time after t years, the balance P, in dollars per year, is growing at the rate

$$\frac{P(t)}{dt} = 0.07P(t)$$

1. Find the function that satisfies the equation. Write it in terms of P_0 and 0.07.
2. Suppose that \$10,000 is invested. What is the balance after 1 year?
3. If \$10,000 is invested, **how fast** is the balance growing at $t = 1$ year?

Solution: (1). According to the exponential model, the account balance is: $P(t) = P_0 e^{0.07t}$

(2). $P(1) = 10000 \times e^{0.07} \approx 10725.08$

(3). **how fast** means **rate**. We need to find the derivative and then evaluate it at $t = 1$.

$$P'(t) = 10000 \times e^{0.07t}(0.07t)' = 700e^{0.07t}$$

Therefore, $P'(1) = \$750.76$.