5. Fixed Point Method

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Lecture Note for MAT325 Numerical Analysis

Contents

1 Introduction

The number p is a fixed point for a given function g if $g(p) = p$. In other words, if function $g(x)$ has a fixed point p, then p is a root of equation $g(x) - x = 0$. Conversely, we could convert root finding problem $f(x) = 0$ to fixed-point problem $g(x) = af(x) - x = x$ for any real number $a \neq 0$. This section discusses how to approximate the root p using the **Fixed-point Method**.

2 Some Theories of the Fixed Point Method

Theorem 1 (Existence of fixed point)

Suppose that $f(x)$ is a continuous function that maps its domain, D, onto a subset of itself, $S = f(D)$, i.e., $f(x) \in C[a, b]$, such that

$$f(x) : [a, b] \in S \subset [a, b]$$

Then $f(x)$ has a fixed point in $[a, b]$.

Proof: If $f(a) = a$ or $f(b) = b$, we are done.

Assume $f(a) \neq a$ and $f(b) \neq b$. Since $f([a, b]) \subset [a, b]$, we have $f(a) > a$ and $f(b) < b$. Let $h(x) = f(x) - x$. Since $f(x) \in C[a, b]$, so is $h(x) \in C[a, b]$. Observe that $h(a) = f(a) - a > 0$, $h(b) = f(b) - b < 0$. By intermediate value theorem, there exists at least one value r in $[a, b]$ such that $h(r) = 0$. That implies that $f(r) = r$. This completes the proof.

Corollary: Every continuous bounded function on the real numbers has a fixed point.

Theorem 2 (Uniqueness of fixed point, also called Contraction Mapping Theorem)

Suppose that $f(x)$ is a continuous function that maps its domain, D, onto a subset of itself, $S = f(D)$, i.e., $f(x) \in C[a, b]$, such that

$$f(x) : [a, b] \in S \subset [a, b]$$

Suppose further that there exists some positive constant $0 < K < 1$ such that $|f'(x)| < K$ for all x in $[a, b]$. Then $f(x)$ has a unique fixed point p in $[a, b]$.

Proof: Assume there at least two fixed points, say p_1 and p_2 in $[a, b]$, such that $f(p_1) = p_1$ and $f(p_2) = p_2$ with $p_1 < p_2$. That is, $f(p_2) - f(p_1) = p_2 - p_1$.

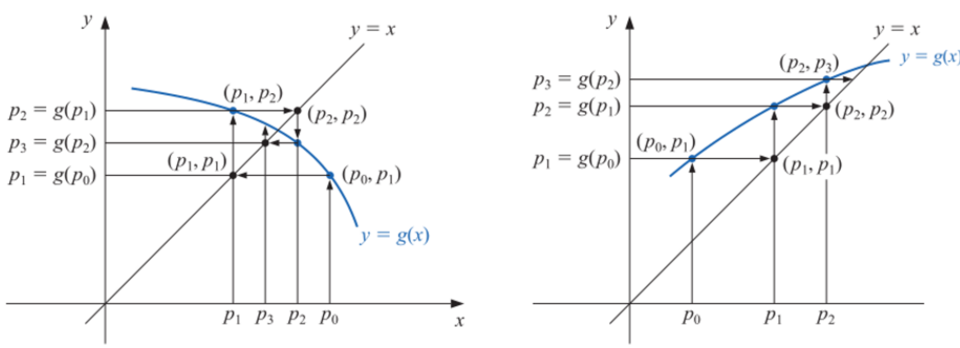
On the other hand, the mean value theorem indicated that $f(p_2) - f(p_1) = f'(c)(p_2 - p_1)$ for some c in $[p_1, p_2]$. Since , we have $f(p_2) - f(p_1) < p_2 \breve{} p_1$. This **contradicts** with $f(p_2) - f(p_1) = p_2 - p_1$. Therefore, we have proved the uniqueness of the fixed point. The proof is completed.

3 Fixed Point Iteration

With the theory developed previously, we focus on the following recursive equation for finding the fixed point numerically.

Choose an initial approximation p_0, generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$. If the sequence converges to p and $g(x)$ is continuous at p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(p)$$



The following animated graph demonstrates the process of searching the fixed point.

https://github.com/pengdsci/MAT325/raw/main/w03/img/Fixed_point_anime.gif

Fixed-point Iteration Algorithm

```
INPUT    f(x),
         initial p0,
         TOL
         N                        (max iterations)
STEP 1.  ERR = infinity           (initial error - must be a big number)
         x                        (initial value)
         n = 0                    (initial iterator)
STEP 2.  WHILE ERR > TOL DO:
         STEP 3. n = n + 1
         STEP 4. new.x = f(x)
                 ERR = new.x - x
                 IF ERR < TOL DO:
                    OUTPUT (results)
```

```
   STOP
 ENDIF
IF ERR > TOL DO:
  OUTPUT (intermediate results)
  x = new.x      (update x value)
ENDIF
IF n == N DO:
  OUTPUT ()
  STOP
ENDIF
```

4 Error Analysis

By the Lipschitz condition in the definition of **Contraction Mapping Theorem**, the approximated error is easily obtained as $|p_{n+1} - p_n| \leq K^n |p_1 \breve{} p_0|$. Next, we present a result that gives the bound the true error.

Theorem 3 (Error Estimation)

If fixed point iteration is terminated after $n > 1$ steps then the error is limited by

$$|p_n - p| \leq \frac{K^n |p_1 - p_0|}{1 - K}$$

where $0 < K < 1$ is the Lipschitz constant.

Proof We use mathematical induction to prove this theorem. For $n = 1$, we need to show that

$$|p_1 - p| \leq \frac{K}{K-1} |p_1 - p_0|.$$

To this end, we use the Mean Value Theorem, there exists a constant $c \in [\min\{p, p_0\}, \max\{p, p_0\}]$ such that

$$|g'(c)| = \left| \frac{f(p_0) - f(p)}{p_0 - p} \right| = \left| \frac{p_1 - p}{p_0 - p} \right| \leq K.$$

That is, $|p_1 - p| \leq K|p_0 - p|$. Using the triangular inequality, we have

$$|p_1 - p| \leq K|p_0 - p| = K|p_0 - p_1 + p_1 - p| \leq K|p_1 - p_0| + K|p_1 - p|$$

Therefore

$$|p_1 - p| \leq \frac{K}{K-1} |p_1 - p_0|.$$

We assume that original inequality is true when $n = m$, i.e.,

$$|p_m - p| \leq \frac{K^m |p_1 - p_0|}{1 - K}.$$

We want to show that

$$|p_{m+1} - p| \leq \frac{K^{m+1} |p_1 - p_0|}{1 - K}.$$

We still use the MVT on interval $[\min\{p, p_m\}, \max\{p, p_m\}]$ to get

$$|g'(c)| = \left|\frac{f(p_m) - f(p)}{p_m - p}\right| = \left|\frac{p_{m+1} - p}{p_m - p}\right| \leq K.$$

where $c \in [\min\{p, p_m\}, \max\{p, p_m\}]$. Therefore,

$$|p_{n+1} - p| \leq K|p_n - p|$$

and

$$|p_{m+1} - p| \leq K|p_m - p_0| = \frac{K^{m+1}}{1-L}|p_1 - p_0|.$$

This completes the proof.

Corollary (Order of Convergence)

If the fixed-point method converges, it has a linear convergence order.

The proof has already been given in the proof of the above theorem.

5 Numerical Example

We know that there is a solution for the equation $x^3 - 7x + 2 = 0$ in $[0, 1]$. We rewrite the equation in the form $x = (x^3 + 2)/7$ and denote the process $x_{n+1} = (x_n^3 + 2)/7$. We see from the following figures that if $0 \leq x_0 \leq 1$ then (x_n) converges to a root of the above equation. We also note that if we start with (for example) $x_0 = 2.5$ then the recursive process does not converge.

```
## input values
TOL = 10^(-6)
  M = 200
####
gfun = function(x) (x^3 + 2)/7
####
 a = 0
 b = 2
 x = 2        # initial value of x
 n = 0       # initializing iterator
 ERR = Inf  # initial error
##
 while (ERR > TOL){
  n = n + 1
  new.x = gfun(x)
  ERR = abs(new.x - x)
  if(ERR < TOL){
    cat("\n\nThe algorithm converges!")
    cat("\nThe approximate root is:", new.x,".")
    cat("\nThe absolute error is:", ERR, ".")
    cat("\nThe number of iterations is:", n, ".")
    break
  } else{
    if(ERR > 10^7){
        cat("\n\nThe algorithm diverges!")
        break
    } else{
```

```r
        cat("\nIteration:",n,". Estimated root:", new.x, ". Absolute error:", ERR,".")
        x = new.x         # update x value!!!
    }
  }
  if(n == M){
    cat("\n\nThe maximum number of iterations is achieved!")
    break
  }
 }
```

```
##
## Iteration: 1 . Estimated root: 1.428571 . Absolute error: 0.5714286 .
## Iteration: 2 . Estimated root: 0.7022074 . Absolute error: 0.726364 .
## Iteration: 3 . Estimated root: 0.3351793 . Absolute error: 0.3670281 .
## Iteration: 4 . Estimated root: 0.2910937 . Absolute error: 0.04408562 .
## Iteration: 5 . Estimated root: 0.289238 . Absolute error: 0.001855685 .
## Iteration: 6 . Estimated root: 0.289171 . Absolute error: 6.696095e-05 .
## Iteration: 7 . Estimated root: 0.2891686 . Absolute error: 2.400242e-06 .
##
## The algorithm converges!
## The approximate root is: 0.2891685 .
## The absolute error is: 8.601697e-08 .
## The number of iterations is: 8 .
```

6 Chapter 2 Homework Part II

Section 2.2:

Problem 1. Page 18.

Problem 7. Page 18 (first part): Use Theorem 2.3 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$.

Problem 12(a). Page 19.