7. Secant Method

Cheng Peng

Lecture Note for MAT325 Numerical Analysis

Contents

1 Introduction

Recall that Newton's method uses Taylor expansion to derive the functional recursive relationship between adjacent approximated roots

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n = 0, 1, \cdots.$$

where $f'(x_n)$ is the slope of the tangent line passing through $x = x_n$. If we use the slope of a secant line that passes through points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$, we can use the x-coordinates of the intersection between the secant line and x-axis to approximate the root of $f(x) = 0$.

2 Secant Method

Assume we have two distinct initial values $x = x_0$ and $x = x_1$. Then slope of the secant line passing through A$(x_0, f(x_0))$ and B$(x_1, f(x_1))$ is

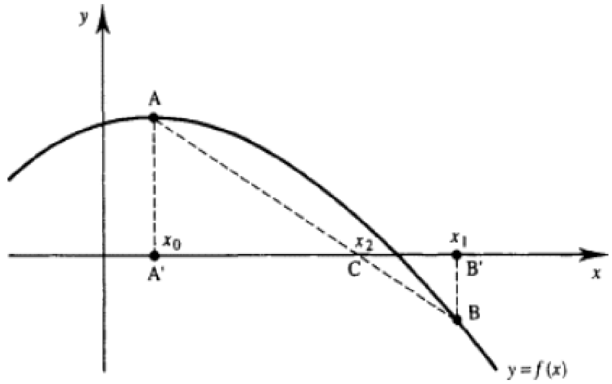
$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \approx f'(x_1) \text{ when } |x_1 - x_0| \text{ is small.}$$

The secant method uses the x-coordinate of the intersection of the secant line

$$f(x) = f(x_1) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

and $f(x) = 0$ (equation of the x-axis). Solving for x, we have

$$x = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) \equiv x_2.$$



In general, the recursive relationship between approximated roots of the **secant method** is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \text{ for } n = 0, 1, \cdots$$

https://github.com/pengdsci/MAT325/raw/main/w05/img/w05-betterSecantAnimation.gif

- **Secant Algorithm**

We develop the following pseudo-code of the secant method.

```
INPUT:  f(x)            (satisfying f(x) = 0)
        x0              (initial value 1)
        x1              (initial value 2)

STEP 1: x0
        x1              (f(x0)*f(x1) must be negative)
        M = 200
        TOL = 10^(-6)
        n = 0
        ERR = |x1 - x0|
STEP 2: WHILE ERR > TOL DO
           n = n + 1
           new.x = x1 - ((x1-x0)/(f(x1)-f(x0)))*f(x1)
           ERR = |new.x - x1|
           IF ERR < TOL DO:
              OUTPUT        (results and optional relevant info)
              STOP
           ENDIF
           IF ERR >= TOL DO:
              OUTPUT        (message or intermediate outputs)
              x1 = new.x    (update)
              x0 = x1
           ENDIF
           IF n == M DO:
             OUTPUT         (warning messages)
             STOP
           ENDIF
        ENDWHILE
```

- **Implementation with R**

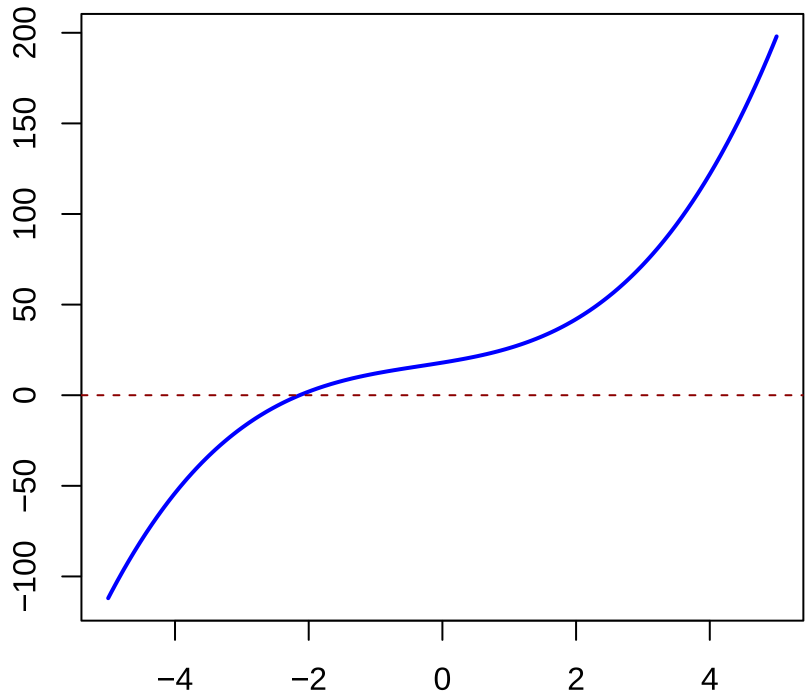
We next write an R function to implement the secant method.

```r
########################################
##     Root Finding: Secant Method
#########################################
Secant.Method = function(fn,            # input function
                  TOL,                  # error tolerance
                  max.iter,             # max allowed iterations
                  x1,                   # initial value #1
                  x2                    # initial value #2
                  ){
  ctr = 0                  # counter of iteration
  ERR = abs(x2 - x1)       # initial error - width of initial interval
  # Define a data frame (data table) to store the output of each iteration
  ERR.table =  data.frame(Iteration = 1:max.iter,
                           Est.root = rep(NA, max.iter),
                           Abs.error = rep(NA, max.iter))
  while(ERR > TOL){
      ctr = ctr + 1
      new.x = x2 - fn(x2) * (x2 - x1) / (fn(x2) - fn(x1))
      ERR = abs(new.x - x2)
      if(ERR < TOL){
         ERR.table[ctr,] = c(ctr, new.x, ERR)
         break
         } else{
            ERR.table[ctr,] = c(ctr, new.x, ERR)
            # updating the two values. CAUTION: order matters
            x1 = x2
            x2 = new.x
         }
        if(ctr == max.iter){
         cat("\n\nThe maximum number of iterations attained!")
         break
       }
   }                       # close the while-loop
  na.omit(ERR.table)       # delete rows with NAs (missing values)
}                          # close the function environment
```

Example 1: Find a root of equation $x^3 + x^2 + 6x + 18 = 0$.

Solution: we use the above R function of the Newton method to find the approximated root of the equation.

```r
# define the function f(x) that satisfies f(x) = 0
example01.func = function(x){x^3+x^2+6*x+18 }
###
xx = seq(-5,5, length=500)    # 500 evenly x-values evenly spread on [-5, 5]
yy = example01.func(xx)       # the corresponding y values
plot(xx, yy, type = "l", xlab ="", ylab="", main="", lwd = 2, col = "blue")
abline(h=0, col = "darkred", lty = 2)
```



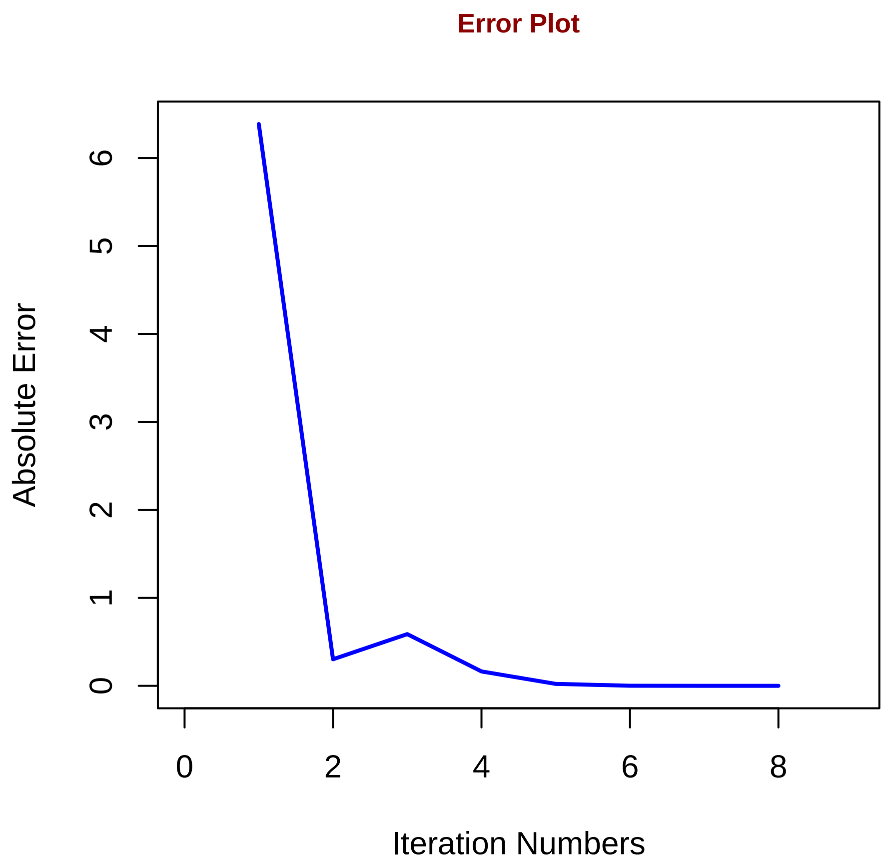
Based on the above graph, we search a root over $[-5, 5]$ in the follwoing function call.

```
# call the function
error.matrix = Secant.Method(fn = example01.func,        # input function
                             TOL = 10^(-8),              # error tolerance
                             max.iter = 15,              # max allowed iterations
                             x1 = -5,                    # initial value #1
                             x2 = 5)                     # initial value #2
pander(error.matrix)
```

Iteration	Est.root	Abs.error
1	-1.387	6.387
2	-1.689	0.3018
3	-2.277	0.5877
4	-2.113	0.1635
5	-2.135	0.02192
6	-2.136	0.001122
7	-2.136	9.03e-06
8	-2.136	3.524e-09

The error plot is given by

```
Error = error.matrix$Abs.error
nitr = length(Error)
plot(1:nitr, Error, type = "l", lwd = 2, col = "blue",
             main="Error Plot",
             xlim = c(0,nitr+1),
             ylim = c(0, max(Error)),
             xlab = "Iteration Numbers",
             ylab = "Absolute Error",
             cex.main = 0.8,
             col.main = "darkred"
        )
```



Practice Exercise: find the solution to $0.8(x + 0.5)^3 - 1 = 0$ 0n $[0.25, 2.75]$.

3 Error Analysis

Let $e_n = x_n - p$, then $e_n - e_{n-1} = x_n - x_{n-1}$. From the definition of the secant method we have

$$e_{n+1} = e_n + x_{n+1} - x_n = e_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n).$$

With some algebraic manipulation, we can express e_{n+1} as

$$e_{n+1} = \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} e_n e_{n-1}.$$

Note that

$$\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \approx \frac{1}{f'(p)}$$

After expanding $f(x_n)$ and $f(x_{n-1})$ at p, we have

$$\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} \approx \frac{f''(p)}{2}$$

Therefore,

$$e_{n+1} \approx \frac{f''(p)}{2f'(p)} e_n e_{n-1}$$

Consequently,

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n e_{n-1}} = \frac{f''(p)}{2f'(p)} = C_0$$

To find the order of convergence, we assume that $e_{n+1} = C_n e_n^\alpha$ where $\lim_{n\to\infty} C_n = C$. Then

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n e_{n-1}} = \lim_{n\to\infty} \frac{C[Ce_{n-1}^\alpha]^\alpha}{Ce_{n-1}^\alpha e_{n-1}} = \lim_{n\to\infty} C^\alpha e_{n-1}^{\alpha^2-\alpha-1} = C_0.$$

This implies that

$$\alpha^2 - \alpha - 1 = 0$$

The positive root of the above equation is

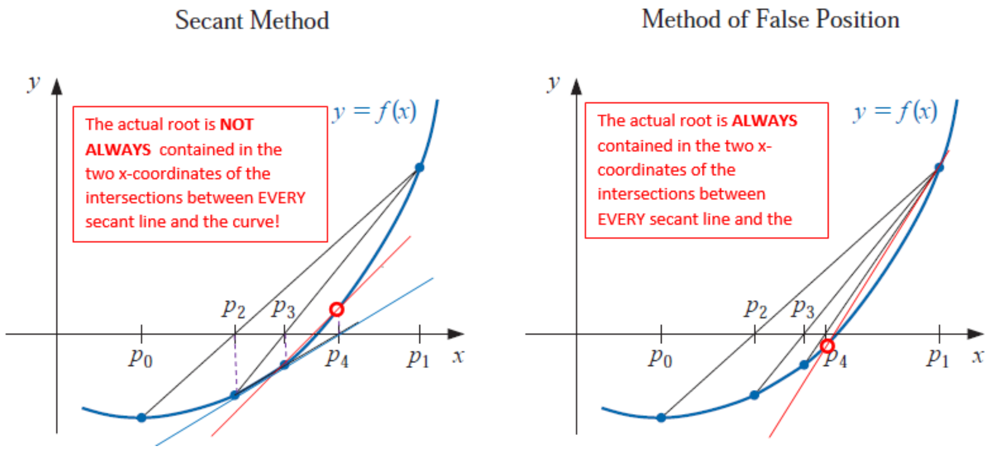
$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.62$$

Therefore, the convergence order for the secant method is between linear and quadratic orders – we call this **super-linear** convergence!

4 False Position Method

The method of False Position (also called Regula Falsi) generates approximations using the x-coordinates of successive secant lines defined prior approximated roots in such a ways that the actual roots is always $[x_n, x_{n-1}]$ in the n-th iteration. **Programmatically, it is the secant method with an additional control statement.**

> First choose initial approximations p_0 and p_1 with $f(p_0) \times f(p_1) < 0$. The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. To decide which secant line to use to compute p_3, consider $f(p_2) \times f(p_1)$, or more correctly $\text{sgn}f(p_2) \times \text{sgn}f(p_1)$
>
> - If $\text{sgn}f(p_2) \times \text{sgn}f(p_1) < 0$, then p_1 and p_2 bracket a root. Choose p_3 as the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$.
> - If not, choose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on p_0 and p_1.



The following is an animated graph showing the process of search the root using false position method.

https://github.com/pengdsci/MAT325/raw/main/w05/img/w05-regula.gif

Finally, the order of convergence of **false position method** is same as the **secant method**.

5 Chapter 2 Homework - Part II.

Section 2.3. If you use MAT

Problem 2 [page 75]. *Use either R or MATLAB to do this problem.*

Problem 3(a) [page 75]. *Do this problem manually first and then call an R/MATLAB function we developed to verify your results.*

Problem 17(b,c), [Page 76]. *Call the functions we developed in class. If you use MATLAB, please include the the MATLAB script with your annotations (line-by-line).*

Problem 19 (**optional**)