9. Newton Interpolation

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Lecture Note for MAT325 Numerical Analysis

Contents

1 Introduction

Let $f(x)$ be a function whose values are known or can be calculated at a set of points (nodes) $\{x_0.x_1, \cdots, x_n\}$. Assume that these points are distinct, but NOT necessarily be ordered on the real line. There exists a polynomial $p_n(x)$ of degree at most n that interpolates $f(x)$ at $n+1$ nodes:

$$p_n(x) = f(x_i), \quad 0 \le i \le n.$$

We have discussed the Lagrange form of the interpolation polynomial. In this note, we introduce Newton's form of interpolation polynomial.

2 Some Definitions

We introduce several definitions related to the divided differences and how to calculate the dived difference manually and programmatically.

2.1 Definitions

The Newton form basis polynomials is defined in the following

$$\begin{array}{lcl} q_0(x) & = & a \\ q_1(x) & = & x - x_0 \\ q_2(x) & = & (x-x_0)(x-x_1) \\ \cdots & \cdots & \cdots \\ q_n(x) & = & (x-x_0)(x-x_1)\cdots(x-x_{n-1}) \end{array}$$

The $q_0(x), q_1(x), \cdots, 1_n(x)$ is basis of span $\{x^0, x^1, x^2, \cdots, x^n\}$.

The Newton Form Polynomial is defined as

$$p_n(x) = \sum_{i=0}^{n} c_i q_i(x).$$

Divided Differences are defined based on the coordinates of a set of given points on the curve of $f(x)$.

x	x_0	x_1	x_2	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

For $i, j, k = 0, 1, 2, \cdots, n$,

- The zero-th order *divided difference* is $f[x_i] = f(x_i)$.
- The second order *divided difference* based on distinct nodes x_i and x_j is defined as

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}.$$

  - This is the slope of the secant line that passes the two given points on the curve of $f(x)$.
  - It is also used to approximate the derivative of the function over interval $[x_i, x_j]$ (assuming $x_i < x_j$) if it exists.

- The third order *divided difference* based on the distinct nodes $x_i < x_j < x_k$ is defined as

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

- The high order *divided difference* based on nodes $x_0 < x_1 < \cdots < x_n$ is similarly defined as

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_0, x_1, \cdots, x_{n-1}] - f[x_1, \cdots, x_n]}{x_0 - x_n}$$

2.2 Calculation of Divided Difference

The Newton interpolation is defined based on the divided difference. We first look at the structure of the divided difference and then develop an algorithm to compute the divided difference programmatically.

- *Iterative Nature of Divided Differences*: Consider the following given points on the curve of $f(x)$

x	x_0	x_1	x_2	x_3
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

- Three *first-order divided differences* are defined based on the given 4 nodes. The first one based

$$f[x_0,x_1]=\frac{f(x_0)-f(x_1)}{x_0-x_1},\quad f[x_1,x_2]=\frac{f(x_1)-f(x_2)}{x_1-x_2},\quad f[x_2,x_3]=\frac{f(x_2)-f(x_3)}{x_2-x_3},$$

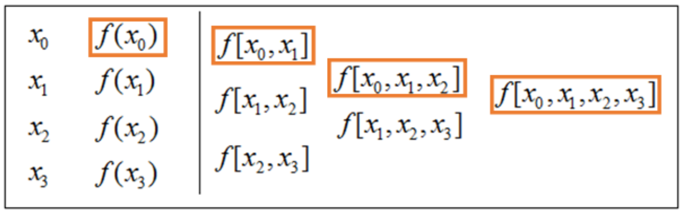
- Two second order *divided differences* are defined by

$$f[x_0,x_1,x_2]=\frac{f[x_0,x_1]-f[x_1,x_2]}{x_0-x_2}=\frac{\frac{f(x_0)-f(x_1)}{x_0-x_1}-\frac{f(x_1)-f(x_2)}{x_1-x_2}}{x_0-x_2},$$

$$f[x_1,x_2,x_3]=\frac{f[x_1,x_2]-f[x_2,x_3]}{x_1-x_3}=\frac{\frac{f(x_1)-f(x_2)}{x_1-x_2}-\frac{f(x_2)-f(x_3)}{x_2-x_3}}{x_1-x_3}$$

- The third-order divided difference is defined by

$$f[x_0,x_1,x_2,x_3]=\frac{f[x_0,x_1,x_2]-f[x_1,x_2,x_3]}{x_0-x_3}$$

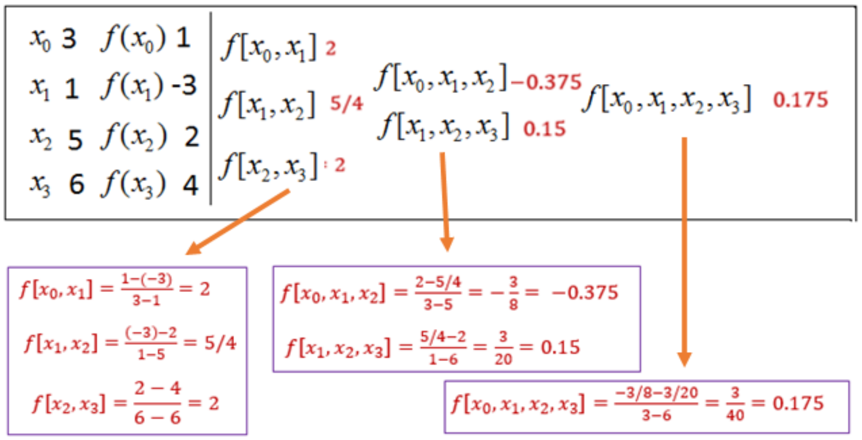


x_0	$\boxed{f(x_0)}$	$\boxed{f[x_0,x_1]}$	$\boxed{f[x_0,x_1,x_2]}$	$\boxed{f[x_0,x_1,x_2,x_3]}$
x_1	$f(x_1)$	$f[x_1,x_2]$	$f[x_1,x_2,x_3]$	
x_2	$f(x_2)$	$f[x_2,x_3]$		
x_3	$f(x_3)$			

Example 1: Calculate all divided differences based on the following given data table

x	3	1	5	6
$f(x)$	1	-3	2	4

Based definition, the divided differences are calculated and summarized in the following table.



x_0	3	$f(x_0)$	1	$f[x_0,x_1]$ 2	$f[x_0,x_1,x_2]$ −0.375	$f[x_0,x_1,x_2,x_3]$ 0.175
x_1	1	$f(x_1)$	-3	$f[x_1,x_2]$ 5/4	$f[x_1,x_2,x_3]$ 0.15	
x_2	5	$f(x_2)$	2	$f[x_2,x_3]$ 2		
x_3	6	$f(x_3)$	4			

$$f[x_0,x_1]=\frac{1-(-3)}{3-1}=2$$

$$f[x_1,x_2]=\frac{(-3)-2}{1-5}=5/4$$

$$f[x_2,x_3]=\frac{2-4}{6-6}=2$$

$$f[x_0,x_1,x_2]=\frac{2-5/4}{3-5}=-\frac{3}{8}=-0.375$$

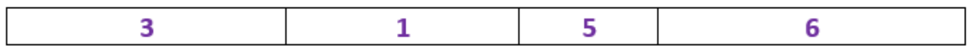
$$f[x_1,x_2,x_3]=\frac{5/4-2}{1-6}=\frac{3}{20}=0.15$$

$$f[x_0,x_1,x_2,x_3]=\frac{-3/8-3/20}{3-6}=\frac{3}{40}=0.175$$

2.3 Algorithm of Calculating Divided Differences

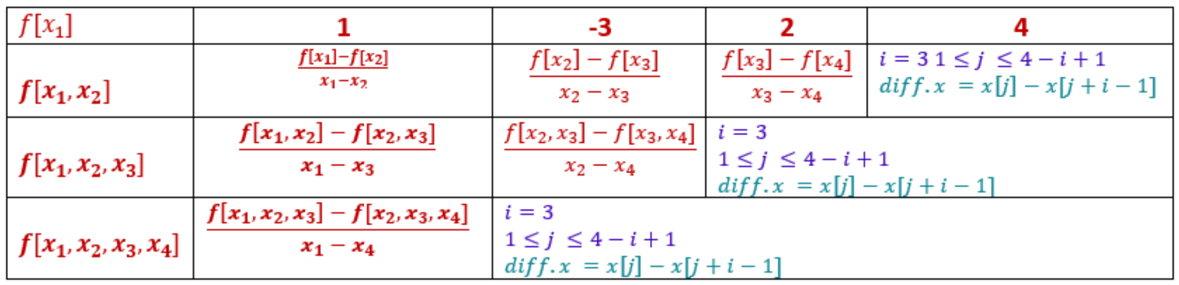
We re-organize the above calculation in the following matrix and use the logic to develop the pseudo-code for calculating the divided differences.

Input Nodes $x[i]$



3	1	5	6

Divided Difference Matrix $A[i.j]$



$f[x_1]$	1	-3	2	4
$f[x_1,x_2]$	$\frac{f[x_1]-f[x_2]}{x_1-x_2}$	$\frac{f[x_2]-f[x_3]}{x_2-x_3}$	$\frac{f[x_3]-f[x_4]}{x_3-x_4}$	$i=3$ $1\le j\ \le 4-i+1$ $diff.x\ = x[j]-x[j+i-1]$
$f[x_1,x_2,x_3]$	$\frac{f[x_1,x_2]-f[x_2,x_3]}{x_1-x_3}$	$\frac{f[x_2,x_3]-f[x_3,x_4]}{x_2-x_4}$	$i=3$ $1\le j\ \le 4-i+1$ $diff.x\ = x[j]-x[j+i-1]$	
$f[x_1,x_2,x_3,x_4]$	$\frac{f[x_1,x_2,x_3]-f[x_2,x_3,x_4]}{x_1-x_4}$	$i=3$ $1\le j\ \le 4-i+1$ $diff.x\ = x[j]-x[j+i-1]$		

Divided Algorithm

```
INPUT: vec.x              (nodes)
       vec.y (or fn() )
       pred.x

OUTPUT: pn(pred.x)

STEP 1: Define a square zero matrix (array): A
STEP 2: Store vec.y in the first row of A
STEP 3: FOR i from 2 to ncol DO:      (Caution: 2nd row!)
            FOR j from 1 to (ncol - i + 1) DO:
                denominator = vec.x[j] - vec.x[j+i-1]
                numerator = A[i-1,j] - A[i-1, j+1]      (using previous row of A)
                A[i,j] = numerator / denominator        (next order divided difference)
            ENDFOR
        ENDFOR
STEP 4: RETURN A
```

```
Divided.Dif = function(
        vec.x,          # input nodes:
        vec.y = NULL,   # one of vec.y and fn must be given
        fn = NULL,
        pred.x          # scalar x for predicting pn(pred.x)
         ){
   n = length(vec.x)
   if (length(vec.y) == 0) vec.y = fn(vec.x) #
   node.x = vec.x
   A = matrix(c(rep(0,n^2)), nrow = n, ncol = n, byrow = TRUE)
   A[1,] = vec.y      # fill the first row with vec.y
   #
   for(i in 2:(n)){
     for(j in 1:(n-i+1)){
      denominator = vec.x[j] - vec.x[j+1+(i-2)]
      numerator = A[i-1,j]- A[i-1,j+1]
      A[i,j] = numerator/denominator
      }
    }
  A
 }
```

```
pander(Divided.Dif(
        vec.x = c(3,1,5,6),     # input nodes:
        vec.y = c(1,-3,2,4),    # one of vec.y and fn must be given
        fn = NULL
        )
)
```

1	-3	2	4
2	1.25	2	0
-0.375	0.15	0	0
0.175	0	0	0

Example 2: (Example 1 of Burden and Faires' textbook, 9th edition, page 127) Complete the divided difference table for the following data.

x	y
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

We use the above function to calculate the divided differences.

```
pander(Divided.Dif(
        vec.x = c(1, 1.3, 1.6, 1.9, 2.2),
        vec.y = c(0.7651977, 0.6200860, 0.4554022, 0.2818186, 0.1103623),
        fn = NULL
        )
     )
```

0.7652	0.6201	0.4554	0.2818	0.1104
-0.4837	-0.5489	-0.5786	-0.5715	0
-0.1087	-0.04944	0.01182	0	0
0.06588	0.06807	0	0	0
0.001825	0	0	0	0

The resulting table is the same as the one obtained in the table of the textbook (except for rounding errors).

3 Newton Interpolation Polynomial

From the definition of the divided difference, for any x and x_0, we have

$$f[x, x_0] = \frac{f(x) - f(x_0)}{x - x_0}$$

Solving for $f(x)$, we have

$$f(x) = f(x_0) + f[x, x_0](x - x_0).$$

Similarly, for x_0 and x_1, the second order divided difference is given by

$$f[x, x_0, x_1] = \frac{f[x, x_0] - f[x_0, x_1]}{x - x_1}$$

Therefore,

$$f[x, x_0] = f[x_0, x_1] + f[x, x_0, x_1](x - x_1).$$

That is,

$$f(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x, x_0, x_1](x - x_0)(x - x_1).$$

Repeating the above calculation, we have

$$f(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x, x_0, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})$$

3.1 Definition of Newton Interpolation

Based on the above derivation, we define **Newton Form Interpolation Polynomial** to be of the following form

$$N_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})$$

We can see that $N_n(x)$ passes all interpolating points:

$$N_n(x_0) = f(x_0)$$

$$N_n(x_1) = f(x_0) + f[x_0, x_1](x_1 - x_0) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0) = f(x_1)$$

In general,

$$N_n(x_k) = f(x_k), \quad \text{for} \quad 0 \le k \le n.$$

3.2 Newton Interpolation Algorithm

The associated **divided differences** in the above **Newton Interpolation Polynomial** are returned in the first column of the function `Divided.Dif()`. To write the algorithmic function

Newton Interpolation Algorithm 2

```
INPUT:  vec.x
        vec.y or fn
        pred.x     (for prediction)
OUTPUT: pred.y     (pn(pred.x))

STEP 1: initialization:
        Divided.Dif          (function call)
        Pn = 0

STEP 2: FOR i from 1 to n DO:
          cumProd = 1             (initial value for the cumulative product)
          FOR j from 1 to i DO:
              cumProd = cumProd*(pred.x-vec.x[j])
          ENDFOR
          Pn = Pn + Divided.Dif[i]*cumProd
        ENDFOR
STEP 3: RETURN(Pn)
```

Next, we write an R function to implement the Newton interpolation polynomial.

```r
NewtonInterpolation = function( vec.x,             # input interpolation nodes
                                vec.y = NULL,
                                fn = NULL,         # either vec.y or fn must be provided
                                pred.x             # single value of x for prediction
                               ){
   if(length(vec.y) ==0) vec.y = fn(vec.x)
   DivDif = Divided.Dif(vec.x, vec.y)[,1]       # the values in the first column of the div dif matrix
   n = length(vec.x)
   Nn = vec.y[1]                     # f[xo]
   for (i in 1:(n-1)){               # Must be n - 1 according to the last term in the polynomial
     cumProd = 1                     # initial value to calculate the cumulative product
     for(j in 1:i){                  # forward difference formula
       cumProd = cumProd*(pred.x-vec.x[j])    # updating the cumulative product in the inner loop
     }
     Nn = Nn + DivDif[i+1]*cumProd     # adding high order terms alliteratively to the Nn(x)
   }
  Nn                                   # return the value of the Newton polynomial
}
```

Example 3 (Continuation of *example 2*). We evaluate the Newton interpolation polynomial $p_4(x)$ at $x = 1.1, 1.6$ and 2.0, respectively. Recall the given data points are

x	y
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Solution: We use the above R function to evaluate the function at the two given nodes.

```
pred.1.6 = NewtonInterpolation(vec.x = c(1, 1.3, 1.6, 1.9, 2.2),
                    vec.y = c(0.7651977, 0.6200860, 0.4554022, 0.2818186, 0.1103623),
                    pred.x = 1.6)
pred.1.1 = NewtonInterpolation(vec.x = c(1, 1.3, 1.6, 1.9, 2.2),
                    vec.y = c(0.7651977, 0.6200860, 0.4554022, 0.2818186, 0.1103623),
                    pred.x = 1.1)
pred.2.0 = NewtonInterpolation(vec.x = c(1, 1.3, 1.6, 1.9, 2.2),
                    vec.y = c(0.7651977, 0.6200860, 0.4554022, 0.2818186, 0.1103623),
                    pred.x = 2.0)

pander(cbind(pred.1.6=pred.1.6, pred.1.1=pred.1.1, pred.2.0 = pred.2.0))
```

pred.1.6	pred.1.1	pred.2.0
0.4554	0.7196	0.2239

The results are the same as those obtained in the textbook Burden and Faires (9th edition, page 131).

R Function: Vector Enabled Newton Interpolated Polynomial

Next, we modify the previous R function to take a vector of input x-values for prediction just like other R functions.

```
##################################################################################
##  Newton Interpolated Polynomial Approximation: vector-enabled input
##################################################################################

Newton.Interpolation = function( vec.x,            # input interpolation nodes
                                vec.y = NULL,
                                fn = NULL,        # either vec.y or fn must be provided
                                pred.x            # VECTOR INPUT!!!
                              ){
   if(length(vec.y) ==0) vec.y = fn(vec.x)
   DivDif = Divided.Dif(vec.x, vec.y)[,1]       # the values in the first column of the div dif matrix
   n = length(vec.x)
   ############
   m = length(pred.x)
   NV = rep(0, m)                 # values of Nn(pred.x)
   for(k in 1:m) {
   ################
   Nn = vec.y[1]                  # f[xo]
   for (i in 1:(n-1)){            # Must be n - 1 according to the last term in the polynomial
     cumProd = 1                  # initial value to calculate the cumulative product
     for(j in 1:i){               # forward difference formula
       cumProd = cumProd*(pred.x[k]-vec.x[j])   # updating the cumulative product in the inner loop
     }
      Nn = Nn + DivDif[i+1]*cumProd    # adding high order terms alliteratively to the Nn(x)
    }
    NV[k] = Nn                                  # return the value the Newton polynomial
   }
 NV
}
```

```r
pred.x = c(1.6, 1.1, 2.0)    # pred.x is the argument is a local variable!
pred.NIP = Newton.Interpolation(vec.x = c(1, 1.3, 1.6, 1.9, 2.2),
                     vec.y = c(0.7651977, 0.6200860, 0.4554022, 0.2818186, 0.1103623),
                     pred.x = c(1.6, 1.1, 2.0))
pander(cbind(pred.x = pred.x, pred.NIP=pred.NIP))
```

pred.x	pred.NIP
1.6	0.4554
1.1	0.7196
2	0.2239

4 Error Analysis

Using the generalized Rolle theorem repeatedly on the expression of the divided difference, have the following result.

Theorem: Let $x_0, \cdots, x_{n-1}$, x be $n+1$ distinct points. Let $a = \min(x_0, \cdots, x_{n-1}, x)$ and $b = \max(x_0, \cdots, x_{n-1}, x)$. Assume that $f(x)$ has a continuous derivative of order n in the interval (a, b). Then

$$f[x_0, x_1, \cdots, x_{n-1}, x] = \frac{f^{(n)}(\xi)}{n!}$$

where $\xi \in (a, b)$.

Proof: Let $P_{n+1}(y)$ be the interpolated polynomial at y for given nodes $\{x_0, x_1, x_2, \cdots, x_{n-1}, x\}$ such that $P_{n+1}(x_i) = f(x_i)$ for $i = 0, 1, 2, \cdots, n-1$, and $P_{n+1}(x) = f(x)$. From the construction of Newton interpolating polynomial $P_n(x)$, we know that

$$P_n(y) = P_{n-1}(x) + f[x_0, x_1, x_2, \cdots, x_{n-1}, x](x - x_0)\cdots(x - x_{n-1})$$

Apparently,

$$f(x) = P_n(x) = P_{n-1}(x) + f[x0, x1, x2, \cdots, x_{n-1}, x](x - x_0)\cdots(x - x_{n-1})$$

Using the theorem introduced in the last unit and the uniqueness of the app, we have

$$f(x) = P_{n-1}(x) + \frac{f^{(n)}(\xi)}{n!}(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

Using the same arguments in the Lagrange interpolation polynomial, we can establish the error bound for the Newton form interpolation polynomials.

5 Some Remarks of Newton Interpolation

First of all, both Lagrange and Newton interpolation polynomials introduced earlier can be viewed as a linear combination of basis polynomials $\{x^0, x^1, x^2, x^2, \cdots, x^n\}$. In fact, Lagrange and Newton interpolation polynomials are two different algebraic representations of the same polynomial.

That is, $P_n(x)$ and $N_n(x)$ be Lagrange and Newton interpolation polynomials based on $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. We express both $P_n(x)$ and $N_n(x)$ in terms of basis polynomials $\{x^0, x^1, x^2, x^2, \cdots, x^n\}$ as

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad \text{and} \quad N_n(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

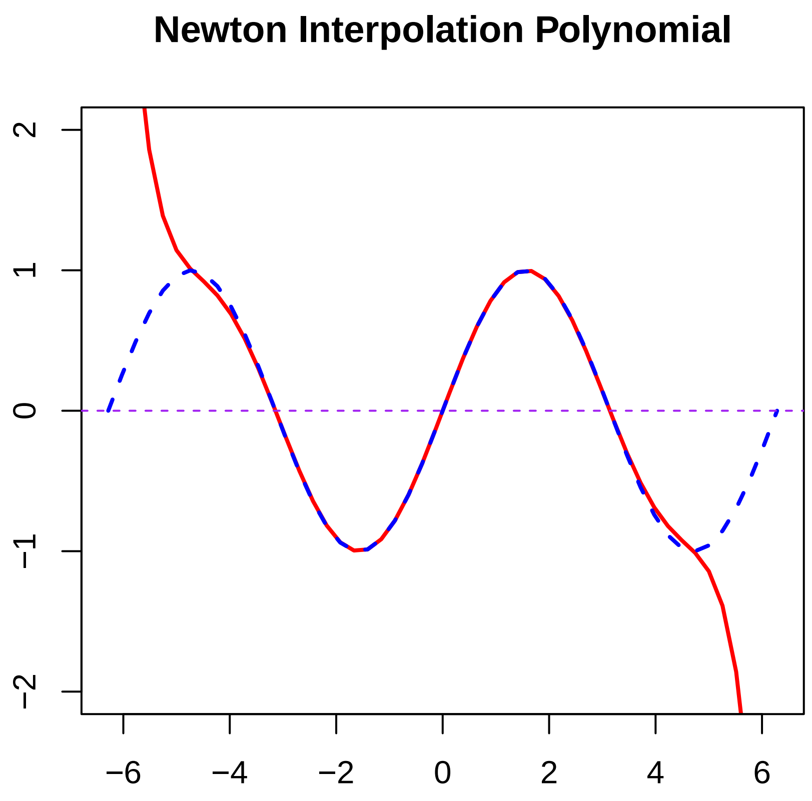
Since $P_n(x_i) = y_i = N_n(x_i)$ for $i = x_0, x_1, \cdots, x_n$, we can easily show that $a_j = b_j$ for $j = 0, 1, 2, \cdots, n$. Therefore, the error bounds of both Lagrange and Newton interpolation polynomials are the same.

For the Newton form interpolation polynomial, we only need to add more terms if additional interpolation nodes are added to the existing ones. Since the divided differences are independent of the newly added nodes. But for the Lagrange form, we need to restart the program. This is a unique feature of the Newton interpolation polynomial.

Example 4. Use the Newton Interpolation polynomial to approximate $f(x) = \sin(x)$ with interpolating nodes $x = -\pi, -0.75\pi, -0.5\pi, 0.25\pi, 0, 0.25\pi, 0.5\pi, 0.75\pi, \pi$.

Solution: We plot both $f(x) = \sin(x)$ and $N_8(x)$ over interval $[-2\pi, 2\pi]$ and see the performance of the approximation.

```
nodes = c(-1.5*pi, -pi, -0.5*pi, -0.25*pi, 0, 0.25*pi, 0.5*pi,  pi, 1.5*pi)
####
xx = seq(-2*pi, 2*pi, length = 50)
yy = sin(xx)
Nn = Newton.Interpolation(vec.x = nodes,
                          vec.y = sin(nodes),
                          pred.x = xx)
plot(xx, Nn, xlab = "", ylab = "", type = "l", lwd = 2, col = "red", ylim = c(-2,2),
        main = "Newton Interpolation Polynomial")
lines(xx, yy, lty =2, col = "blue", lwd = 2)
abline(h = 0, lty = 2, col = "purple")
```


Newton Interpolation Polynomial