5: Bootstrapping Multiple Linear Regression Model

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Contents

1 Introduction

In this note, we introduce two versions of bootstrap procedures to generate bootstrap samples to estimate the confidence intervals of the coefficients of the regression model identified last week. For reference, I include the code that creates the analytic data set for the final model and the summarized statistics of the model as well.

```
realestate0 <- read.csv("https://raw.githubusercontent.com/pengdsci/sta321/main/ww03/w03-Realestate.csv
realestate <- realestate0[, -1]
# longitude and latitude will be used to make a map in the upcoming analysis.
lat <- realestate$Latitude
lon <- realestate$Longitude
##
geo.group <- (lon > 121.529) & (lat > 24.96)     # define the geo.group variable
                                                 # top-right region = TRUE, other region = FALSE
realestate$geo.group <- as.character(geo.group)  # convert the logical values to character values.
realestate$sale.year <- as.character(realestate$TransactionYear) # convert transaction year to dummy.
realestate$Dist2MRT.kilo <- (realestate$Distance2MRT)/1000   # re-scale distance: foot -> kilo feet
final.data = realestate[, -c(1,3,5,6)]          # keep only variables to be used in the candidate model
final.data$logAreaPrice = log(final.data$PriceUnitArea)  #
## the final model
log.price <- lm(log(PriceUnitArea) ~ HouseAge + NumConvenStores + sale.year +
                 Dist2MRT.kilo  + geo.group, data = final.data)
log.price02 <- lm(logAreaPrice ~ HouseAge + NumConvenStores + sale.year +
                 Dist2MRT.kilo  + geo.group, data = final.data)
cmtrx <- summary(log.price)$coef
cmtrx02 <- summary(log.price02)$coef
kable(cmtrx, caption = "Inferential Statistics of Final Model")
```

Table 1: Inferential Statistics of Final Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	3.5724282	0.0443235	80.599030	0.0000000
HouseAge	-0.0075712	0.0010097	-7.498507	0.0000000
NumConvenStores	0.0274872	0.0049096	5.598667	0.0000000
sale.year2013	0.0805519	0.0244373	3.296272	0.0010655
Dist2MRT.kilo	-0.1445122	0.0137541	-10.506820	0.0000000
geo.groupTRUE	0.1825871	0.0347151	5.259583	0.0000002

The explicit expression of the final model is given by

$$\log(price) = 3.5723 - 0.0076 \times HouseAge + 0.0275 \times NumConvenStores +$$
$$0.0805 \times Sale.year2013 - 0.1445 \times Dist2MRT.kilo + 0.1826 \times geo.groupTRUE$$

As another example of the interpretation of the regression coefficient, we choose the coefficient associated with **geo.group**. In the output, you see the name of the dummy variable with the suffix `TRUE`, **geo.groupTRUE**. The suffix **TRUE** indicates that the dummy variable represents the category 'TRUE' of the category variable **geo.group**. The associated coefficient reflects the **mean** difference between the category `TRUE` and the baseline category `FALSE`. In R, the default baseline category is the lowest value of the categorical variable (in alphabetical order).

Let's consider **the set of all houses** that are in the same conditions except the regions (region `TRUE` and region `FALSE`) and the sale prices.

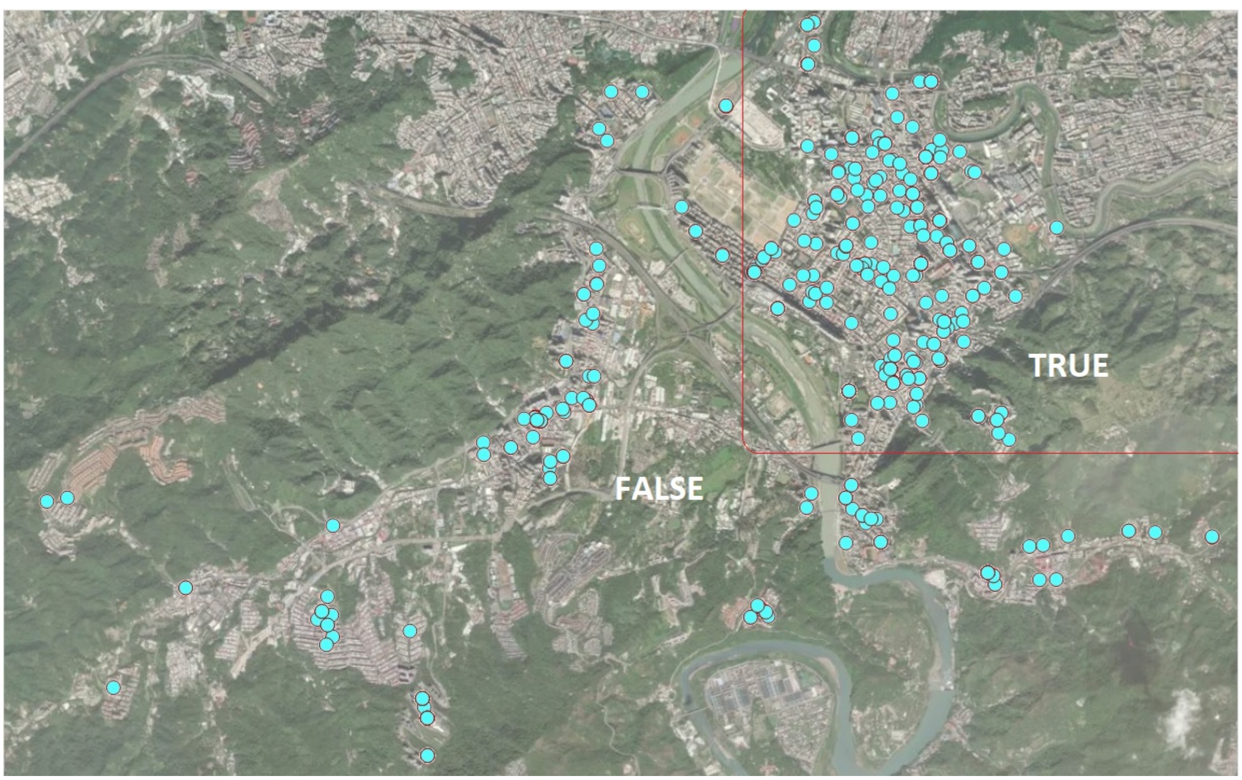


Figure 1: Location of Houses for Sale

Next, we explain the estimated regression coefficient of 0.1826. Let p_{TRUE} be the mean price of a house in the region TRUE and p_{FALSE} be the mean price of houses in the region FALSE. Then

$$\log(p_{TRUE}) - \log(p_{FALSE}) = 0.1826 \to \log(p_{TRUE}/p_{FALSE}) = 0.1826 \to p_{TRUE} = 1.20 p_{FALSE}$$

We re-express the above equation can be re-written as

$$p_{TRUE} - p_{FALSE} = 0.206 p_{FALSE} \to \frac{p_{TRUE} - p_{FALSE}}{p_{FALSE}} = 0.20 = 20\%.$$

That is, the average house sales price in the TRUE region (top right corner on the map) is about 20% higher than that in the FALSE region. We can similarly interpret other regression coefficients.

2 Bootstrap Cases (BOOT.C)

In this section, we use bootstrapping cases to find the confidence intervals for the coefficients in the final regression model. The method was used in bootstrap simple linear regression (SLR) in week #3. The following code finds the confidence interval.

```
log.price = lm(log(PriceUnitArea) ~ HouseAge + NumConvenStores + sale.year +
                 Dist2MRT.kilo  + geo.group, data = final.data)
##
B = 1000       # choose the number of bootstrap replicates.
##
num.p = dim(model.frame(log.price))[2]  # returns number of parameters in the model
smpl.n = dim(model.frame(log.price))[1] # sample size
## zero matrix to store bootstrap coefficients
coef.mtrx = matrix(rep(0, B*num.p), ncol = num.p)
##
for (i in 1:B){
  bootc.id = sample(1:smpl.n, smpl.n, replace = TRUE) # fit final model to the bootstrap sample
  log.price.btc = lm(log(PriceUnitArea) ~ HouseAge + NumConvenStores + sale.year +
                       Dist2MRT.kilo  + geo.group, data = final.data[bootc.id,])
  coef.mtrx[i,] = coef(log.price.btc)    # extract coefs from bootstrap regression model
}
```

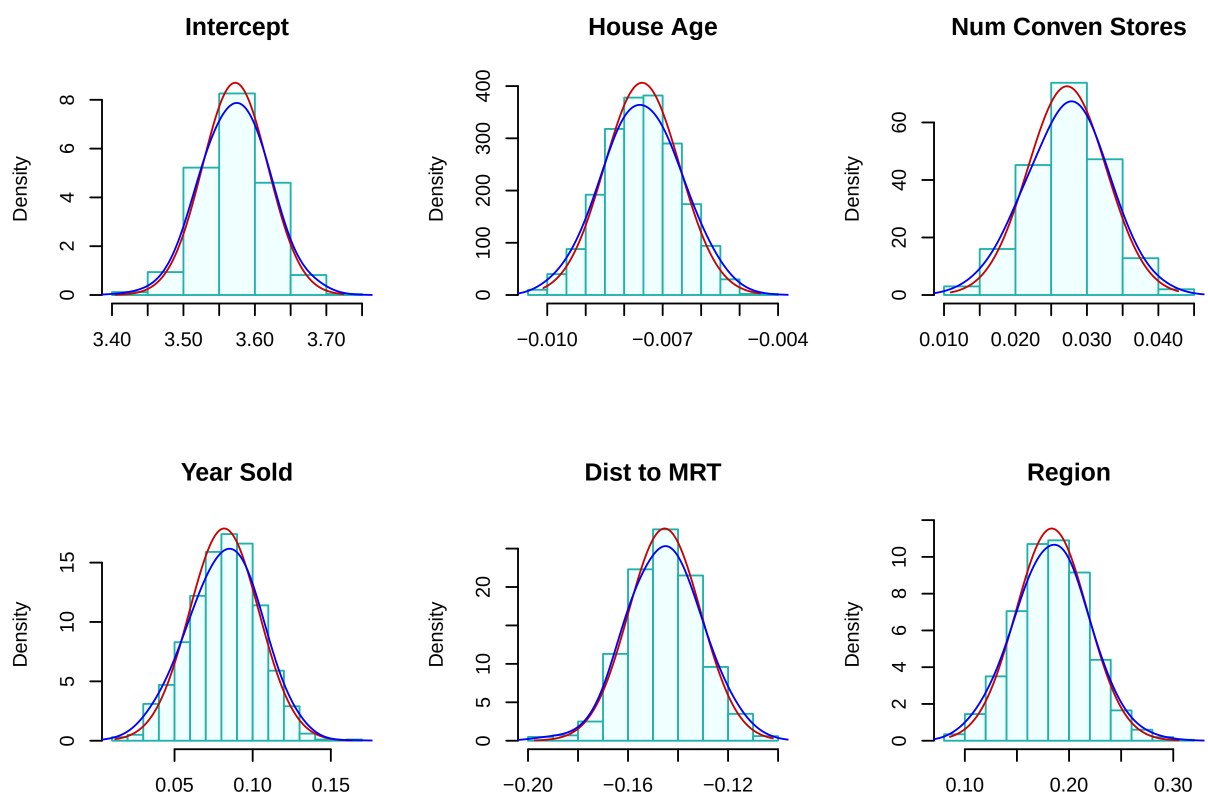
We define an R function to make histograms of the bootstrap regression coefficients in the following. I will also use this function to make histograms for the residual bootstrap estimated regression coefficients as well.

```
boot.hist = function(cmtrx, bt.coef.mtrx, var.id, var.nm){
  ## bt.coef.mtrx = matrix for storing bootstrap estimates of coefficients
  ## var.id = variable ID (1, 2, ..., k+1)
  ## var.nm = variable name on the hist title, must be the string in the double quotes
  ## coefficient matrix of the final model
  ## Bootstrap sampling distribution of the estimated coefficients
  x1.1 <- seq(min(bt.coef.mtrx[,var.id]), max(bt.coef.mtrx[,var.id]), length=300 )
  y1.1 <- dnorm(x1.1, mean(bt.coef.mtrx[,var.id]), sd(bt.coef.mtrx[,var.id]))
  # height of the histogram - use it to make a nice-looking histogram.
  highestbar = max(hist(bt.coef.mtrx[,var.id], plot = FALSE)$density)
  ylimit <- max(c(y1.1,highestbar))
  hist(bt.coef.mtrx[,var.id], probability = TRUE, main = var.nm, xlab="",
       col = "azure1",ylim=c(0,ylimit), border="lightseagreen")
  lines(x = x1.1, y = y1.1, col = "red3")
  lines(density(bt.coef.mtrx[,var.id], adjust=2), col="blue")
```

```
  #legend("topright", c(""))
}
```

The following histograms of the bootstrap estimates of regression coefficients represent the sampling distributions of the corresponding estimates in the final model.

```
par(mfrow=c(2,3))  # histograms of bootstrap coefs
boot.hist(bt.coef.mtrx=coef.mtrx, var.id=1, var.nm ="Intercept" )
boot.hist(bt.coef.mtrx=coef.mtrx, var.id=2, var.nm ="House Age" )
boot.hist(bt.coef.mtrx=coef.mtrx, var.id=3, var.nm ="Num Conven Stores" )
boot.hist(bt.coef.mtrx=coef.mtrx, var.id=4, var.nm ="Year Sold" )
boot.hist(bt.coef.mtrx=coef.mtrx, var.id=5, var.nm ="Dist to MRT" )
boot.hist(bt.coef.mtrx=coef.mtrx, var.id=6, var.nm ="Region" )
```



Two normal-density curves were placed on each of the histograms.

- The red **density curve** uses the estimated regression coefficients and their corresponding standard error in the output of the regression procedure. The p-values reported in the output are based on the red curve.

*The **blue curve** is a non-parametric data-driven estimate of the density of bootstrap sampling distribution. The bootstrap confidence intervals of the regressions are based on these non-parametric bootstrap sampling distributions.

We can see from the above histograms that the two density curves in all histograms are close to each other. we would expect that significance test results and the corresponding bootstrap confidence intervals are consistent.

Next, we find 95% bootstrap confidence intervals of each regression coefficient and combined them with the output of the final model.

```
num.p = dim(coef.mtrx)[2]  # number of parameters
btc.ci = NULL
btc.wd = NULL
for (i in 1:num.p){
  lci.025 = round(quantile(coef.mtrx[, i], 0.025, type = 2),8)
  uci.975 = round(quantile(coef.mtrx[, i],0.975, type = 2 ),8)
  btc.wd[i] =  uci.975 - lci.025
  btc.ci[i] = paste("[", round(lci.025,4),", ", round(uci.975,4),"]")
 }
#as.data.frame(btc.ci)
kable(as.data.frame(cbind(formatC(cmtrx,4,format="f"), btc.ci.95=btc.ci)),
      caption = "Regression Coefficient Matrix")
```

Table 2: Regression Coefficient Matrix

	Estimate	Std. Error	t value	Pr(>|t|)	btc.ci.95
(Intercept)	3.5724	0.0443	80.5990	0.0000	[3.4843 , 3.6646]
HouseAge	-0.0076	0.0010	-7.4985	0.0000	[-0.0095 , -0.0056]
NumConvenStores	0.0275	0.0049	5.5987	0.0000	[0.0159 , 0.0376]
sale.year2013	0.0806	0.0244	3.2963	0.0011	[0.0349 , 0.1251]
Dist2MRT.kilo	-0.1445	0.0138	-10.5068	0.0000	[-0.1729 , -0.1169]
geo.groupTRUE	0.1826	0.0347	5.2596	0.0000	[0.1154 , 0.2526]

We can see from the above table of summarized statistics, the significance tests of regression coefficients based on the p-values and the corresponding 95% confidence intervals are consistent.

3 Bootstrap Residuals (BOOT.R)

In this section, we introduce bootstrap residual methods to estimate bootstrap confidence intervals. The idea is straightforward and is summarized in the following.

3.1 Fitted Model

Assume that the fitted regression model is given by

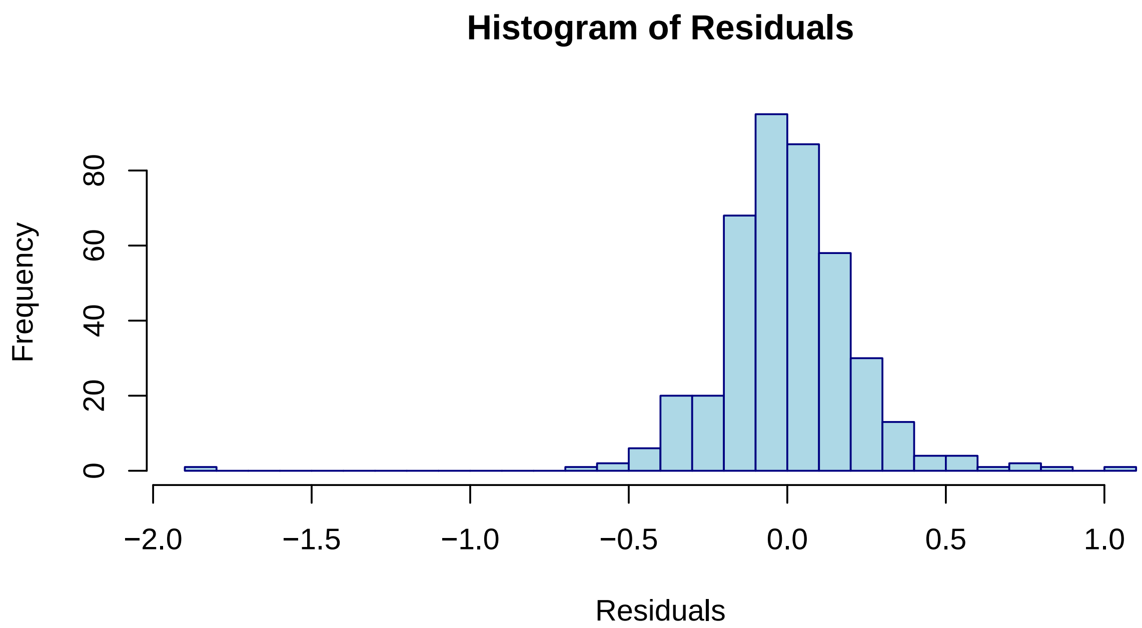
$$\begin{array}{ccl} y_1 & = & \hat{\beta}_0 + \hat{\beta}_1 x_{11} + \hat{\beta}_2 x_{12} + \cdots + \hat{\beta}_k x_{1k} + e_1 \\ y_2 & = & \hat{\beta}_0 + \hat{\beta}_1 x_{21} + \hat{\beta}_2 x_{22} + \cdots + \hat{\beta}_k x_{2k} + e_2 \\ y_3 & = & \hat{\beta}_0 + \hat{\beta}_1 x_{31} + \hat{\beta}_2 x_{32} + \cdots + \hat{\beta}_k x_{3k} + e_3 \\ \vdots & \vdots & \qquad\qquad\qquad \vdots \\ y_n & = & \hat{\beta}_0 + \hat{\beta}_1 x_{n1} + \hat{\beta}_2 x_{n2} + \cdots + \hat{\beta}_k x_{nk} + e_n \end{array}$$

where $\{e_1, e_2, \cdots, e_n\}$ is the set of residuals obtained from the final model. $\{x_{i1}, x_{i2}, \cdots, x_{ik}\}$ is the i-th record from the data, and $\{\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_k\}$

The distribution of the residuals is depicted in the following histogram.

```
hist(sort(log.price$residuals),n=40,
     xlab="Residuals",
     col = "lightblue",
```

```
    border="navy",
    main = "Histogram of Residuals")
```



The above histogram reveals the same information as we saw in the residual plot in the last note: (1) one out-lier; (2). The distribution is skewed to the right.

3.2 Residual Bootstrap Samples

The residual bootstrap sample of y is defined in the following:

- Take a **bootstrap sample** from the set of residuals $\{e_1, e_2, \cdots, e_n\}$, denoted by $\{e_1^*, e_2^*, \cdots, e_n^*\}$.
- The residual bootstrap sample of $\{y_1^*, y_2^*, \cdots, y_n^*\}$ is defined by

$$
\begin{array}{rcl}
y_1^* & = & \hat{\beta}_0 + \hat{\beta}_1 x_{11} + \hat{\beta}_2 x_{12} + \cdots + \hat{\beta}_k x_{1k} + e_1^* \\
y_2^* & = & \hat{\beta}_0 + \hat{\beta}_1 x_{21} + \hat{\beta}_2 x_{22} + \cdots + \hat{\beta}_k x_{2k} + e_2^* \\
y_3^* & = & \hat{\beta}_0 + \hat{\beta}_1 x_{31} + \hat{\beta}_2 x_{32} + \cdots + \hat{\beta}_k x_{3k} + e_3^* \\
\vdots & \vdots & \vdots \\
y_n^* & = & \hat{\beta}_0 + \hat{\beta}_1 x_{n1} + \hat{\beta}_2 x_{n2} + \cdots + \hat{\beta}_k x_{nk} + e_n^*
\end{array}
$$

The above definition implies that the residual bootstrap is equal to the **fitted value + bootstrap residuals**.

- The resulting **residual bootstrap sample** is given by

$$
\begin{array}{ccccc}
y_1^* & x_{11} & x_{12} & \cdots & x_{1k} \\
y_2^* & x_{21} & x_{22} & \cdots & x_{2k} \\
y_3^* & x_{31} & x_{32} & \cdots & x_{3k} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
y_n^* & x_{n1} & x_{n2} & \cdots & x_{nk}
\end{array}
$$

- We fit the final model to the **residual bootstrap sample** and denote the bootstrap estimates of regression coefficients in the following

$$\{\hat{\beta}_0^*, \hat{\beta}_1^*, \cdots, \hat{\beta}_k^*\}$$

- Repeat the above steps B times, we obtain the following bootstrap estimates

$$\begin{array}{cccc}
\hat{\beta}_0^{1*} & \hat{\beta}_1^{1*} & \cdots & \hat{\beta}_k^{1*} \\
\hat{\beta}_0^{2*} & \hat{\beta}_1^{2*} & \cdots & \hat{\beta}_k^{2*} \\
\hat{\beta}_0^{3*} & \hat{\beta}_1^{3*} & \cdots & \hat{\beta}_k^{3*} \\
\vdots & \vdots & \vdots & \vdots \\
\hat{\beta}_0^{b*} & \hat{\beta}_1^{b*} & \cdots & \hat{\beta}_k^{b*} \\
\vdots & \vdots & \vdots & \vdots \\
\hat{\beta}_0^{B*} & \hat{\beta}_1^{B*} & \cdots & \hat{\beta}_k^{B*}
\end{array}$$

The residual bootstrap confidence intervals of regression coefficients can be estimated from the above bootstrap coefficients.

3.3 Implementation of Residual Bootstrap Regression

The following code generates bootstrap confidence intervals of regression coefficients.

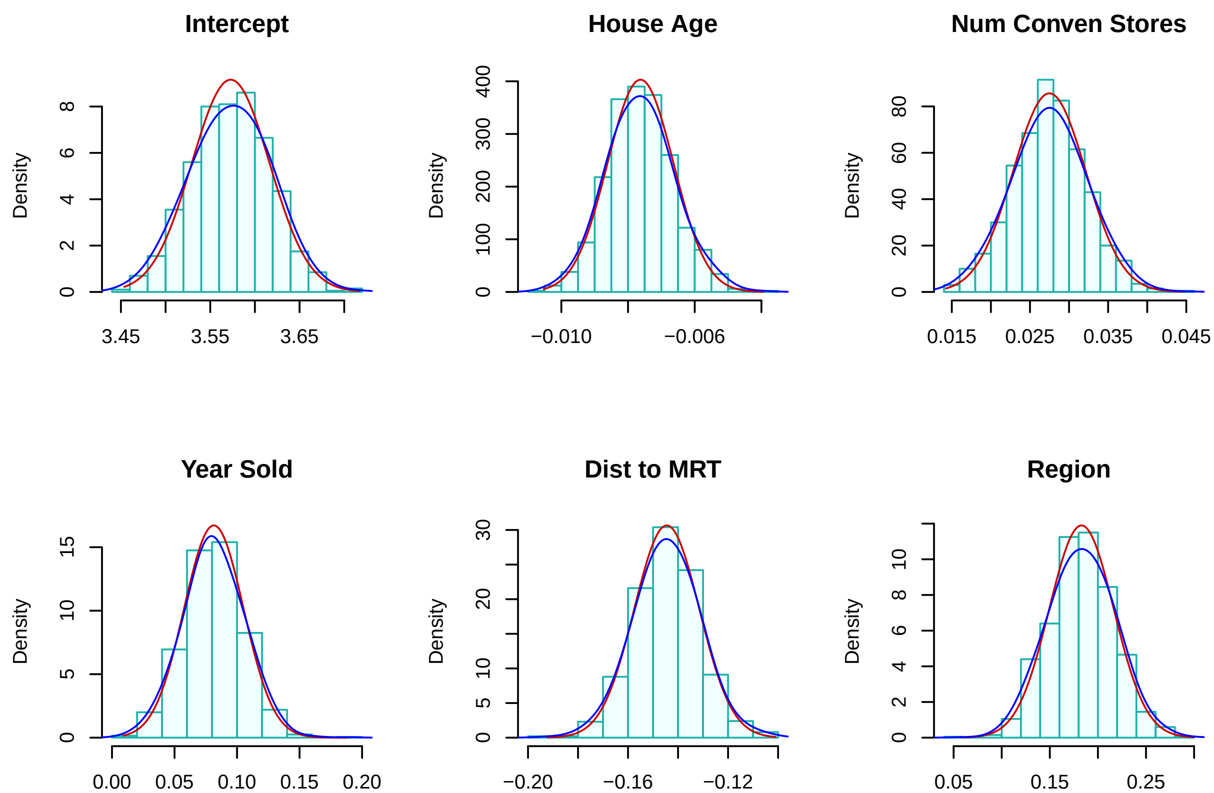
```
## Final model
log.price <- lm(log(PriceUnitArea) ~ HouseAge + NumConvenStores + sale.year +
                 Dist2MRT.kilo  + geo.group, data = final.data)
model.resid = log.price$residuals
##
B=1000
num.p = dim(model.matrix(log.price))[2]   # number of parameters
samp.n = dim(model.matrix(log.price))[1]  # sample size
btr.mtrx = matrix(rep(0,6*B), ncol=num.p) # zero matrix to store boot coefs
for (i in 1:B){
  ## Bootstrap response values
  bt.lg.price = log.price$fitted.values +
        sample(log.price$residuals, samp.n, replace = TRUE)  # bootstrap residuals
  # replace PriceUnitArea with bootstrap log price
  final.data$bt.lg.price =  bt.lg.price   #  send the boot response to the data
  btr.model = lm(bt.lg.price ~ HouseAge + NumConvenStores + sale.year +
                 Dist2MRT.kilo  + geo.group, data = final.data)   # b
  btr.mtrx[i,]=btr.model$coefficients
}
```

Next, I make histograms of the residual bootstrap estimates of the regression coefficients.

```
boot.hist = function(bt.coef.mtrx, var.id, var.nm){
  ## bt.coef.mtrx = matrix for storing bootstrap estimates of coefficients
  ## var.id = variable ID (1, 2, ..., k+1)
  ## var.nm = variable name on the hist title, must be the string in the double quotes
  ## Bootstrap sampling distribution of the estimated coefficients
  x1.1 <- seq(min(bt.coef.mtrx[,var.id]), max(bt.coef.mtrx[,var.id]), length=300 )
  y1.1 <- dnorm(x1.1, mean(bt.coef.mtrx[,var.id]), sd(bt.coef.mtrx[,var.id]))
  # height of the histogram - use it to make a nice-looking histogram.
  highestbar = max(hist(bt.coef.mtrx[,var.id], plot = FALSE)$density)
  ylimit <- max(c(y1.1,highestbar))
```

```
  hist(bt.coef.mtrx[,var.id], probability = TRUE, main = var.nm, xlab="",
       col = "azure1",ylim=c(0,ylimit), border="lightseagreen")
  lines(x = x1.1, y = y1.1, col = "red3")        # normal density curve
  lines(density(bt.coef.mtrx[,var.id], adjust=2), col="blue")     # loess curve
}
```

```
par(mfrow=c(2,3))  # histograms of bootstrap coefs
boot.hist(bt.coef.mtrx=btr.mtrx, var.id=1, var.nm ="Intercept" )
boot.hist(bt.coef.mtrx=btr.mtrx, var.id=2, var.nm ="House Age" )
boot.hist(bt.coef.mtrx=btr.mtrx, var.id=3, var.nm ="Num Conven Stores" )
boot.hist(bt.coef.mtrx=btr.mtrx, var.id=4, var.nm ="Year Sold" )
boot.hist(bt.coef.mtrx=btr.mtrx, var.id=5, var.nm ="Dist to MRT" )
boot.hist(bt.coef.mtrx=btr.mtrx, var.id=6, var.nm ="Region" )
```



The residual bootstrap sampling distributions of each estimated regression coefficient. The normal and LOESS curves are close to each other. This also indicated that the inference of the significance of variables based on p-values and residual bootstrap will yield the same results.

The 95% residual bootstrap confidence intervals are given in the following

```
#
num.p = dim(coef.mtrx)[2]   # number of parameters
btr.ci = NULL
btr.wd = NULL
for (i in 1:num.p){
```

```
  lci.025 = round(quantile(btr.mtrx[, i], 0.025, type = 2),8)
  uci.975 = round(quantile(btr.mtrx[, i],0.975, type = 2 ),8)
  btr.wd[i] = uci.975 - lci.025
  btr.ci[i] = paste("[", round(lci.025,4),", ", round(uci.975,4),"]")
}
#as.data.frame(btc.ci)
kable(as.data.frame(cbind(formatC(cmtrx,4,format="f"), btr.ci.95=btr.ci)),
      caption = "Regression Coefficient Matrix with 95% Residual Bootstrap CI")
```

Table 3: Regression Coefficient Matrix with 95% Residual Bootstrap CI

	Estimate	Std. Error	t value	Pr(>|t|)	btr.ci.95
(Intercept)	3.5724	0.0443	80.5990	0.0000	[3.489 , 3.6569]
HouseAge	-0.0076	0.0010	-7.4985	0.0000	[-0.0095 , -0.0055]
NumConvenStores	0.0275	0.0049	5.5987	0.0000	[0.0179 , 0.0365]
sale.year2013	0.0806	0.0244	3.2963	0.0011	[0.0349 , 0.1285]
Dist2MRT.kilo	-0.1445	0.0138	-10.5068	0.0000	[-0.1707 , -0.1186]
geo.groupTRUE	0.1826	0.0347	5.2596	0.0000	[0.1199 , 0.2477]

As expected, the residual bootstrap confidence intervals yield the same results as p-values do. This is because the sample size is large enough so that the sampling distributions of estimated coefficients have sufficiently good approximations of normal distributions.

3.4 Combining All Inferential Statistics

Finally, we put all inferential statistics in a single table so we can compare these results.

```
kable(as.data.frame(cbind(formatC(cmtrx[,-3],4,format="f"), btc.ci.95=btc.ci,btr.ci.95=btr.ci)),
      caption="Final Combined Inferential Statistics: p-values and Bootstrap CIs")
```

Table 4: Final Combined Inferential Statistics: p-values and Bootstrap CIs

	Estimate	Std. Error	Pr(>|t|)	btc.ci.95	btr.ci.95
(Intercept)	3.5724	0.0443	0.0000	[3.4843 , 3.6646]	[3.489 , 3.6569]
HouseAge	-0.0076	0.0010	0.0000	[-0.0095 , -0.0056]	[-0.0095 , -0.0055]
NumConvenStores	0.0275	0.0049	0.0000	[0.0159 , 0.0376]	[0.0179 , 0.0365]
sale.year2013	0.0806	0.0244	0.0011	[0.0349 , 0.1251]	[0.0349 , 0.1285]
Dist2MRT.kilo	-0.1445	0.0138	0.0000	[-0.1729 , -0.1169]	[-0.1707 , -0.1186]
geo.groupTRUE	0.1826	0.0347	0.0000	[0.1154 , 0.2526]	[0.1199 , 0.2477]

The above table shows that

- All three methods yield the same results in terms of the significance of individual explanatory variables. The reason is that the final working model does not have a serious violation of the model assumption.

```
kable(cbind(btc.wd, btr.wd), caption="width of the two bootstrap confidence intervals")
```

Table 5: width of the two bootstrap confidence intervals

btc.wd	btr.wd
0.1803121	0.1679100
0.0038962	0.0039633
0.0217122	0.0186318
0.0902490	0.0936306
0.0560047	0.0521207
0.1371911	0.1277816

- The widths of residual bootstrap and case-bootstrap confidence intervals are similar to each other. See the above table for the widths of each confidence interval.