

## STA 504 Homework #6

**Due: Monday, October 28**

This set of homework problems focuses on setting up the integral limits of a double integral. You are expected to draw the integral region on the  $y_1$ - $y_2$  coordinate plane and based on the shape of the region set up the integral limits.

The following linked page helps set up integral limits of double integrals.

<http://tutorial.math.lamar.edu/Classes/CalcIII/DIGeneralRegion.aspx>

Read the following example carefully and provide a similar level of detail (graphs and algebra) in your work.

### Problem 1.

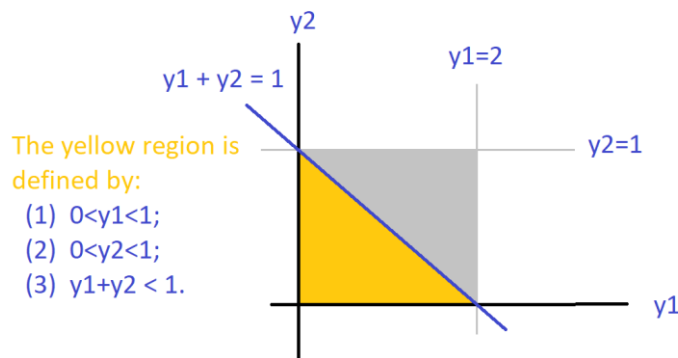
Let  $Y_1$  and  $Y_2$  denote the proportions of two different types of components in a sample from a mixture of chemicals used as an insecticide. Suppose that  $Y_1$  and  $Y_2$  have the joint density function given by

$$f(y_1, y_2) = \begin{cases} 2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

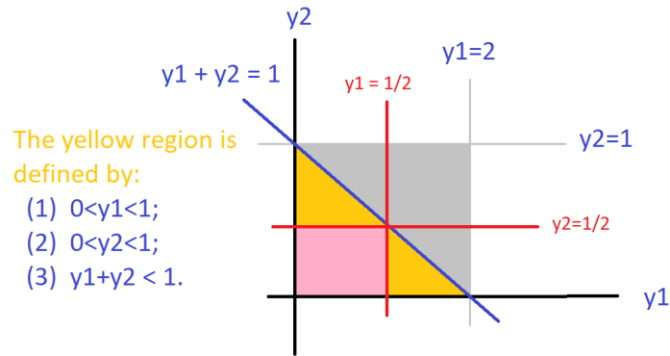
(Notice that  $Y_1 + Y_2 \leq 1$  because the random variables denote proportions within the same sample.) Find

a  $P(Y_1 \leq 1/2, Y_2 \leq 1/2)$ .

**Solution:** Before finding the probabilities in a and b, we first sketch the region (domain) given in the density function in the following (the yellow region is the domain of the density function)



We the two constraints  $y_1 < 1/2$  and  $y_2 < 1/2$  to the original yellow region (the big yellow triangle) and get the following square (pink). That is,



We integrate the density over the pink square region as follows.

$$\iint_{\text{pink square}} f(y_1, y_2) dy_1 dy_2 = \int_0^{\frac{1}{2}} \left[ \int_0^{\frac{1}{2}} f(y_1, y_2) dy_1 \right] dy_2 = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

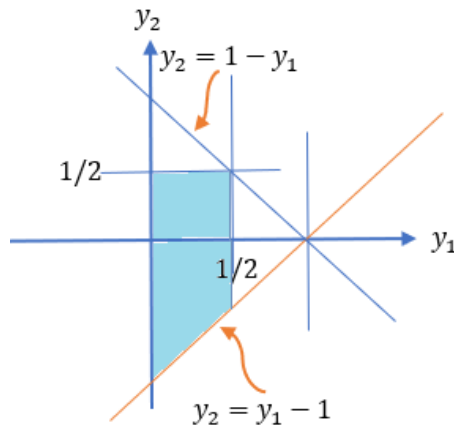
### Problem 2.

The joint density function of  $Y_1$  and  $Y_2$  is given by

$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \leq y_2 \leq 1 - y_1, 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- a Find  $F(1/2, 1/2)$ .
- b Find  $F(1/2, 2)$ .
- c Find  $P(Y_1 > Y_2)$ .

**Solution.** The region on which the probability is defined is sketched in the following. It is a type I region.

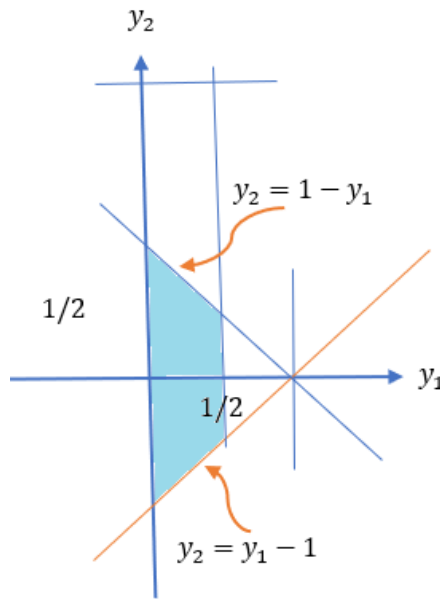


$$F\left(\frac{1}{2}, \frac{1}{2}\right) = P\left[Y_1 < \frac{1}{2}, Y_2 < \frac{1}{2}\right] = \int_0^{\frac{1}{2}} \int_{y_1-1}^{\frac{1}{2}} 30y_1y_2^2 dy_2 dy_1$$

$$\begin{aligned}
&= \int_0^{\frac{1}{2}} 10y_1 y_2^3 \Big|_{y_2=y_1-1}^{y_2=\frac{1}{2}} dy_1 = \int_0^{\frac{1}{2}} 10y_1 \left[ \frac{1}{8} - (y_1 - 1)^3 \right] dy_1 \\
&= 10 \int_0^{\frac{1}{2}} \left[ -y_1^4 + 3y_1^3 - 3y_1^2 + \frac{9y_1}{8} \right] dy_1 \\
&= 10 \left[ -\frac{y_1^5}{5} + \frac{3y_1^4}{4} - y_1^3 + \frac{9y_1^2}{16} \right]_0^{\frac{1}{2}} = 10 \left[ -\frac{1}{5 \times 32} + \frac{3}{64} - \frac{1}{8} + \frac{9}{64} \right] \\
&= -\frac{2}{32} + \frac{15}{32} - \frac{5}{4} + \frac{45}{32} = \frac{-2 + 15 - 40 + 45}{32} = \frac{18}{32} = \frac{9}{16}
\end{aligned}$$

(b).

$$\begin{aligned}
F\left(\frac{1}{2}, 2\right) &= P\left[Y_1 < \frac{1}{2}, Y_2 < 1\right] = \int_0^{\frac{1}{2}} \int_{y_1-1}^{1-y_1} 30y_1 y_2^2 dy_2 dy_1 \\
&= \int_0^{\frac{1}{2}} 10y_1 \int_{y_1-1}^{1-y_1} 3y_2^2 dy_2 dy_1 = \int_0^{\frac{1}{2}} 10y_1 \times y_2^3 \Big|_{y_1-1}^{1-y_1} dy_1 \\
&= \int_0^{\frac{1}{2}} 10y_1 \times [(1-y_1)^3 - (y_1-1)^3] dy_1 = \int_0^{\frac{1}{2}} 20y_1(1-y_1)^3 dy_1
\end{aligned}$$



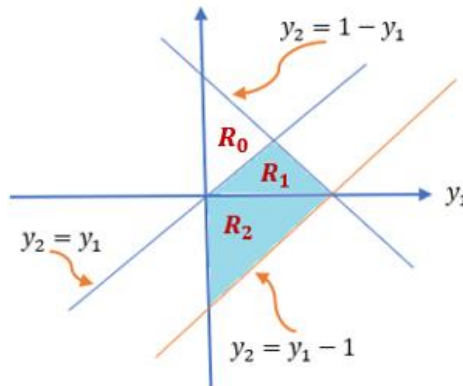
$$\begin{aligned}
&= - \int_0^{\frac{1}{2}} 5y_1 d(1-y_1)^4 = -5y_1(1-y_1)^4 \Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} 5(1-y_1)^4 dy_1 \\
&= -\frac{5}{32} - (1-y_1)^5 \Big|_0^{\frac{1}{2}} = -\frac{5}{32} - \frac{1}{32} + 1 = \frac{-6 + 32}{32} = \frac{13}{16}
\end{aligned}$$

(c). The region associated with the desired probability is the sky-blue region in the following figure. There are different ways to find this probability. We can integrate over the sky-blue region by splitting it into two regular regions (R1 is type II and R2 are both type I and type II). An easy way is to integrate over R0 and then subtract it from 1 (integrating over the domain yields 1). We take the shortcut.

$$\begin{aligned}
 \iint_{R_0} f(y_1, y_2) dA &= \int_0^{\frac{1}{2}} \int_{y_1}^{1-y_1} 30y_1y_2^2 dy_2 dy_1 \\
 &= \int_0^{\frac{1}{2}} 10y_1[(1-y_1)^3 - y_1^3] dy_1 \\
 &= 10 \int_0^{\frac{1}{2}} [y_1 - 3y_1^2 + 3y_1^3 - 2y_1^4] dy_1 \\
 &= 10 \left[ \frac{y_1^2}{2} - y_1^3 + \frac{3y_1^4}{4} - \frac{2y_1^5}{5} \right]_0^{\frac{1}{2}} \\
 &= 10 \left[ \frac{1}{8} - \frac{1}{8} + \frac{3}{64} - \frac{2}{5 \times 32} \right] = \frac{15}{32} - \frac{4}{32} = \frac{11}{32}
 \end{aligned}$$

Therefore,

$$P[Y_1 > Y_2] = 1 - \frac{11}{32} = \frac{21}{32}$$



### Problem 3.

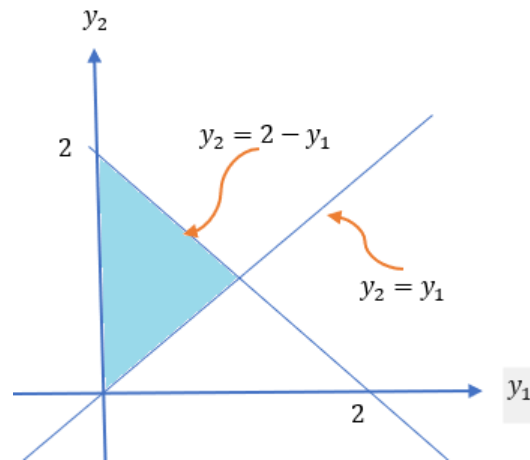
Suppose that the random variables  $Y_1$  and  $Y_2$  have joint probability density function  $f(y_1, y_2)$  given by

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \leq y_1 \leq y_2, y_1 + y_2 \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

1/2

- Verify that this is a valid joint density function.
- What is the probability that  $Y_1 + Y_2$  is less than 1?

(a) . The domain of the density is sketched below.



$$\begin{aligned} \iint_R f(y_1, y_2) dA &= \int_0^1 \int_{y_1}^{2-y_1} 6y_1^2 y_2 dy_2 dy_1 \\ &= \int_0^1 3y_1^2 y_2^2 \Big|_{y_1}^{2-y_1} dy_1 = \int_0^1 3y_1^2 [(2-y_1)^2 - y_1^2] dy_1 \\ &= \int_0^1 12 [y_1^2 - y_1^3] dy_1 = 12 \left[ \frac{y_1^3}{3} - \frac{y_1^4}{4} \right]_0^1 = 12 \left[ \frac{1}{3} - \frac{1}{4} \right] = 1. \end{aligned}$$

The given function is a valid joint probability density function.

(b). The subregion used in the definition of the probability is sketched in the following figure (gold region). It is of both type I and type II region.

$$\begin{aligned} P[Y_1 + Y_2 < 1] &= \iint_R f(y_1, y_2) dA = \int_0^{1/2} \int_{y_1}^{1-y_1} 6y_1^2 y_2 dy_2 dy_1 \\ &= \int_0^{1/2} 3y_1^2 y_2^2 \Big|_{y_1}^{1-y_1} dy_1 = \int_0^{1/2} 3y_1^2 [(1-y_1)^2 - y_1^2] dy_1 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{1/2} 3(y_1^2 - 2y_1^3) dy_1 = 3 \left( \frac{y_1^3}{3} - \frac{y_1^4}{2} \right) \bigg|_0^{1/2} \\
 &= 3 \left( \frac{1/8}{3} - \frac{1/16}{2} \right) = 3 \times \frac{2/8 - 3/16}{6} = \frac{1}{32}
 \end{aligned}$$

[Caution: the answer in the solution manual is incorrect.]

