STA 504 Homework #8

Due: Tuesday, November 7

Answer the following questions based on the given joint density function in problems 1 - 4.

(i) Find the marginal densities $f(x)$ and $f(y)$.

(ii) Check if the variables X and Y are independent.

(iii) find the conditional probability distribution function of X, given $Y = y$

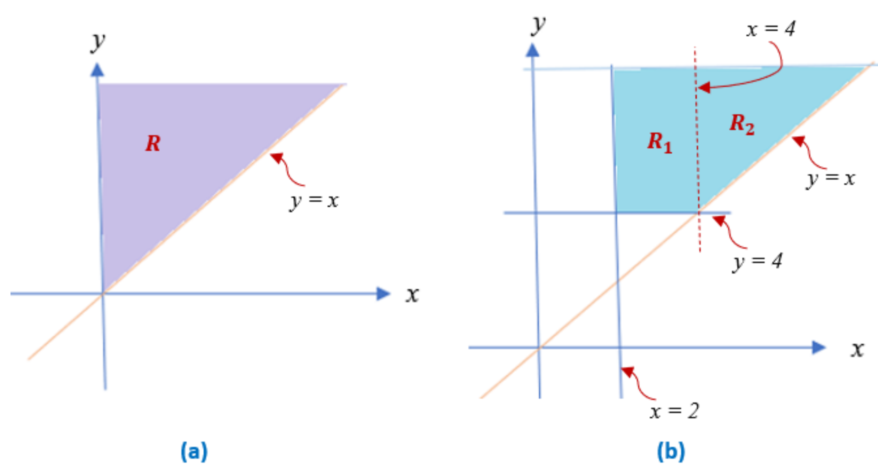
Problem 1.

Let the continuous random vector (X, Y) have joint probability density function

$$f(x,y) = e^{-y}, 0 < x < y < \infty.$$

In addition to the (i)-(iii), compute the $P(X \geq 2; Y \geq 4)$.

Solution: We first sketch the domain of the joint density function in the left panel (a) of the following figure. It is both type I and II regions.



(i). Marginal density functions.

$$f_X(x) = \int_x^{\infty} e^{-y}\, dy = -e^{-y}|_x^{\infty} = e^{-x}$$

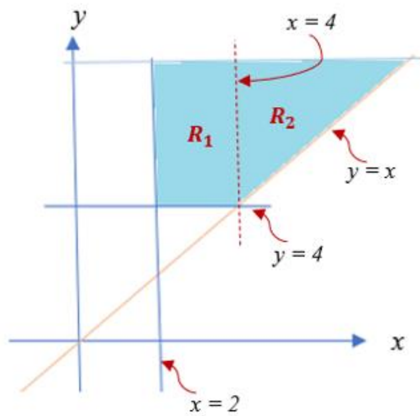
$$f_Y(y) = \int_0^{y} e^{-y}\, dx = e^{-y}x|_0^{y} = ye^{-y}$$

The above two densities represent exponential and gamma distributions respectively.

(ii) Since $f_X(x)f_Y(y) = ye^{-x-y} \neq e^{-y} = f(x,y)$, random variables X and Y are dependent on each other.

(iii). $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}, \quad x < y < \infty$

(iv). $P(X > 2, Y > 4)$ is the volume between the sky-blue region and the surface of the density function. However, the sky-blue region is irregular. We can partition it into two regular regions:



(b)

one type I and one type II (see (b) of the above figure). Therefore,

$$P(X > 2, Y > 4) = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

$$= \int_2^4 \int_4^\infty e^{-y}\, dy\, dx + \int_4^\infty \int_4^y e^{-y}\, dx\, dy = \int_2^4 e^{-4}\, dx + \int_4^\infty (y-4)e^{-y}\, dy$$

$$= (4-2)e^{-4} + \int_4^\infty (y-4)e^{-y}\, dy = 2e^{-4} + \int_4^\infty (y-4)e^{-y}\, dy$$

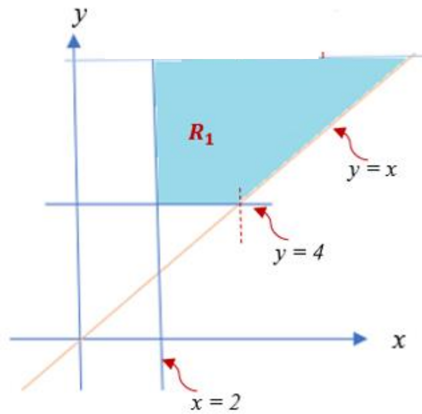
Note that

$$\int_4^\infty (y-4)e^{-y}\, dy = -\int_4^\infty (y-4)e^{-y}\, d(-y) = -\int_4^\infty (y-4)de^{-y}$$

$$= -\{(y-4)e^{-y}|_4^\infty - \int_4^\infty e^{-y}\, d(y-4)\}$$

$$= -\left\{0 + \int_4^\infty e^{-y}\, d(-y)\right\} = -e^{-y}|_4^\infty = -(0 - e^{-4}) = e^{-4}\,.$$

Therefore, $P(X > 2, Y > 4) = 2e^{-4} + \int_4^\infty (y-4)e^{-y}\, dy = 2e^{-4} + e^{-4} = 3e^{-4}$



If we don’t cut the original region into two subregions, we can set up the limits of the iterative integral as

$$P(X > 2, Y > 4) = \iint_{R_1} f(x,y)dA = \int_4^\infty \left[\int_2^y e^{-y}\, dx\right] dy$$

$$= \int_4^\infty \left[e^{-y} \int_2^y 1\, dx\right] dy = \int_4^\infty [e^{-y}(y-2)]\, dy$$

$$= -\int_4^\infty [e^{-y}(y-2)]\, d(-y) = -\int_4^\infty [y-2]\, de^{-y}$$

$$= -\left\{(y-2)e^{-y}|_4^\infty - \int_4^\infty e^{-y}\, d(y-2)\right\}$$

$$= -\left\{-2e^{-4} - \int_4^\infty e^{-y}\, dy\right\} = 2e^{-4} + \int_4^\infty e^{-y}\, d$$

$$= 2e^{-4} - \int_4^\infty e^{-y}\, d(-y) = 2e^{-4} - e^{-y}|_4^\infty = 3e^{-4}.$$

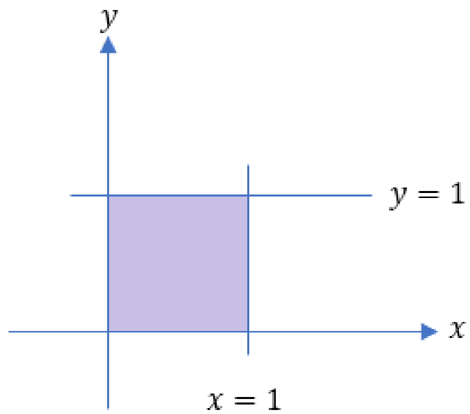
Problem 2.

Let X and Y be a random vector with a joint probability density function

$$f(x,y) = \begin{cases} 6xy(2 - x - y), & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then find the conditional probability distribution function of X, given $Y = y$, where $0 < y < 1$.

Solution. We first sketch the domain in the following.



(i). Marginal density functions.

$$f_X(x) = \int_0^1 6xy(2 - x - y)\,dy = \int_0^1 (12xy - 6x^2y - 6xy^2)\,dy$$
$$= (6xy^2 - 3x^2y^2 - 2xy^3)|_{y=0}^{y=1} = 6x - 3x^2 - 2x = 4x - 3x^2$$

$$f_Y(y) = \int_0^1 6xy(2 - x - y)\,dx = \int_0^1 (12xy - 6x^2y - 6xy^2)\,dx$$

$$= (6x^2y - 2x^3y - 3x^2y^2)|_{x=0}^{x=1} = 6y - 3y^2 - 2y = 4y - 3y^2$$

(ii) Since $f_X(x)f_Y(y) = (6x - 3x^2 - 2x)(6y - 3y^2 - 2y) \neq 6xy(2 - x - y) = f(x,y)$, random variables X and Y are dependent on each other.

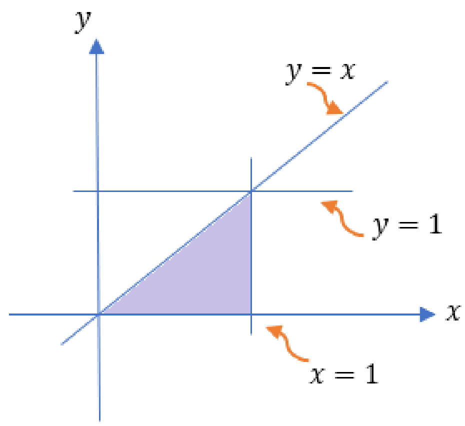
(iii). $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{6xy(2-x-y)}{4y-3y^2} = \frac{6x(2-x-y)}{4-3y}, \quad x < y < \infty$

Problem 3.

Let *X* and *Y* be a random vector with a joint probability density function.

$$f(x,y) = \begin{cases} 1/x, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution. The domain of the density is sketched below.



(i). Marginal density functions.

$$f_X(x) = \int_0^x \frac{1}{x} dy = \frac{1}{x} \times y|_{y=0}^{y=x} = 1$$

$$f_Y(y) = \int_y^1 \frac{1}{x} dx = ln(x)|_y^1 = -\ln(y)$$

(ii). Since $f_X(x) f_Y(y) = 1(-\ln(y)) \neq 1/x = f(x,y)$, random variables X and Y are dependent on each other.

(iii). $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{1/x}{-\ln(y)} = -\frac{1}{-x\ln(y)}, \quad 0 < y < x < 1$.

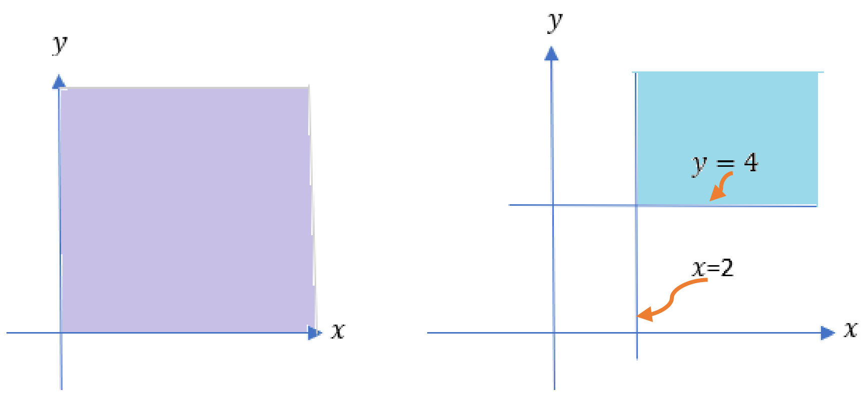
Problem 4.

Let X and Y have density

$$f(x,y) = \begin{cases} xe^{-x(1+y)}, & \text{if } y > 0, x > 0 \\ 0, & \text{otherwise} \end{cases}$$

In addition to the (i)-(iii), compute the $P(X \geq 2;\ Y \geq 4)$.

Solution: We sketch the domain in the following.



(i). Marginal density functions.

$$f_X(x) = \int_0^{\infty} xe^{-x(1+y)}\,dy = -\int_0^{\infty} xe^{-x(1+y)}\,d[-x(1+y)] = e^{-x(1+y)}\Big|_{y=0}^{y=\infty} = e^{-x}$$

$$f_Y(y) = \int_0^{\infty} xe^{-x(1+y)}\,dx = -\frac{1}{(1+y)}\int_0^{\infty} xe^{-x(1+y)}\,d[-x(1+y)]$$
$$= -\frac{1}{(1+y)}\int_0^{\infty} x\,de^{-x(1+y)} = -\frac{1}{(1+y)}\left[xe^{-x(1+y)}\Big|_{x=0}^{x=\infty} - \int_0^{\infty} e^{-x(1+y)}\,dx\right]$$
$$= \frac{1}{(1+y)}\left[-\frac{1}{(1+y)}\int_0^{\infty} e^{-x(1+y)}\,d\big(-x(1+y)\big)\right] = -\frac{1}{(1+y)^2}e^{-x(1+y)}\Big|_{x=0}^{x=\infty} = \frac{1}{(1+y)^2}$$

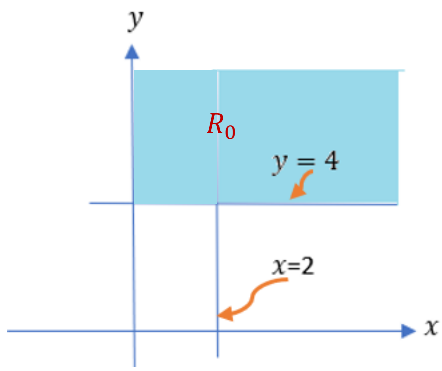
(ii). Since $f_X(x)f_Y(y) = e^{-x}\frac{1}{(1+y)^2} \neq\ xe^{-x(1+y)} = f(x,y)$, the random variables X and Y are dependent on each other.

(iii). $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{xe^{-x(1+y)}}{\frac{1}{(1+y)^2}} = x(1+y)^2e^{-x(1+y)},\quad 0\ <\ y < \ x\ <\ 1\,.$

(iv). $P(X>2, Y>4)$ is the volume between the sky-blue region and the surface of the density function. However, the sky-blue region is irregular. We can partition it into two regular regions: one type I and one type II (see (b) of the above figure). Therefore,

$$P(X>2, Y>4) = \iint_{R_1} f(x,y)dA = \int_2^{\infty}\left[\int_4^{\infty} xe^{-x(1+y)}\,dy\right]dx$$
$$= \int_2^{\infty}\left[e^{-x}\int_4^{\infty} xe^{-xy}\,dy\right]dx = \int_2^{\infty}\left[e^{-x}\int_4^{\infty} e^{-xy}\,d(xy)\right]dx$$
$$= \int_2^{\infty}\left[-e^{-x}\int_4^{\infty} e^{-xy}\,d(-xy)\right]dx = \int_2^{\infty}\left[-e^{-x}\int_4^{\infty} e^{-xy}\,d(-xy)\right]dx$$

$$= \int_2^{\infty}[-e^{-x}](e^{-xy}|_4^{\infty})\,dx = \int_2^{\infty}[-e^{-x}](0-e^{-4x}|)\,dx = \int_2^{\infty} e^{-5x}\,dx = -\frac{e^{-5x}}{5}\Bigg|_{x=2}^{x=\infty} = \frac{e^{-10}}{5}.$$



$$P(Y > 4) = \iint_{R_0} f(x,y)dA = \int_0^\infty \left[\int_4^\infty xe^{-x(1+y)}\,dy\right]dx = \cdots = \int_0^\infty e^{-5x}\,dx = \frac{1}{5}$$

Therefore,

$$P(X > 2|\,Y > 4) = \frac{P(X > 2, Y > 4)}{P(Y > 4)} = e^{-10}.$$