STA 504 Homework #6

Due: Monday, October 23

Show you're your work to earn full credit. You are encouraged to work with your peers on assignments. The write-up must be your own.

Problem 1.

Consider the following gamma distribution

$$f(y) = ye^{-y}$$

with MGF

$$m_Y(t) = \frac{1}{(1-t)^2}.$$

Find the following numerical measures of the shape of this gamma distribution using the MGF to find all required moments. [Hint: review the example used in the class before attempting the following items]

First of all, we re-write the moment-generating function

$$m_Y(t) = \frac{1}{(1-t)^2} = (1-t)^{-2}.$$

For validating purposes, we write the given density in the standard form so the shape and scale parameters are explicitly shown in the density function.

$$f(y) = \frac{y^{2-1}}{1^2\Gamma(2)}e^{-\frac{y}{1}}$$

That is shape $\alpha = 2$ and scale $\beta = 1$.

1. Express $m_Y(t)$ in terms of moments.

$$E[Y^k] = \frac{d^k}{dt}m_Y(t)\Big|_{t=0}$$

2. E[Y]

$$E[Y] = E[Y^1] = \frac{d}{dt}m_Y(t)\Big|_{t=0} = -2(1-t)^{-3}(-1)|_{t=0} = 2$$

That is identical to the result obtained from the formula $E[Y] = \alpha\beta = 2 \times 1 = 2$

3. V[Y].

We use formula $V[Y] = E[Y^2] - \{E[Y]\}^2$. Note that

$$E[Y^2] = \frac{d^2}{dt^2}(1-t)^{-2}\Big|_{t=0} = \frac{d}{dt}2(1-t)^{-3}\Big|_{t=0} = 6(1-t)^{-4}|_{t=0} = 6$$

Therefore,

$$V[Y] = E[Y^2] - \{E[Y]\}^2 = 6 - 2^2 = 2.$$

This is also identical to the result obtained from the variance formula

$$V[Y] = \alpha\beta^2 = 2 \times 1^2 = 2$$

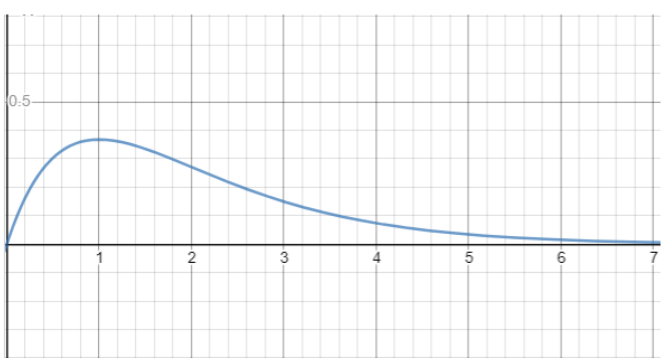
4. Skew[Y].

We need to find the third moment to find the coefficient of skewness.

$$E[Y^3] = \frac{d^3}{dt^3}(1-t)^{-2}\Big|_{t=0} = \frac{d}{dt}6(1-t)^{-4}\Big|_{t=0} = 24(1-t)^{-5}|_{t=0} = 24.$$

$$Skew[Y] = \frac{E[Y^3] - 3\mu E[Y^2] + 3\mu^2 E[Y] - \mu^3}{\sigma^3}$$

$$= \frac{24 - 3 \times 2 \times 6 + 3 \times 2^2 \times 2 - 2^3}{\left(\sqrt{2}\right)^3} = \frac{24 - 36 + 24 - 8}{2\sqrt{2}} = \sqrt{2}$$



As expected, the density curve is skewed to the right.

5. Kurt[Y].

The kurtosis coefficient requires the 4th moment of Y.

$$[Y^4] = \frac{d^4}{dt^4}(1-t)^{-2}\bigg|_{t=0} = \frac{d}{dt}24(1-t)^{-5}\bigg|_{t=0} = 120(1-t)^{-6}|_{t=0} = 120.$$

$$Kurt[Y] = \frac{E[Y^4] - 4\mu E[Y^3] + 6\mu^2 E[Y^2] - 3\mu^4}{\sigma^4}$$

$$= \frac{120 - 4 \times 2 \times 24 + 6 \times 4 \times 6 - 3 \times 2^4}{\left(\sqrt{2}\right)^4} = 6$$

The density curve of this special gamma distribution has the same kurtosis as normal distributions.

Caution: There are two different definitions of kurtosis. One is the one used in our lecture note (its sample version is called standard unbiased estimator of the kurtosis). The other definition is used in literature is our version minus 3 (a kind of centralized version). Its sample version yields a biased estimator of the kurtosis.