STA 504 Mathematical Statistics with Calculus Review

Midterm Exam #2

11/25/2024

Due: **9:00 AM, 11/26/2024** (Tuesday)

Please Print: ________________ __________________
(First Name) (Last Name)

Instructions

- This is an open-book test. Textbooks and notes can be used. However, you must complete this exam independently. All forms of collaboration are NOT allowed.
- Please show your detailed work to earn full credit.
- Partial credit will be granted to the key steps that reflect your correct reasoning even if your numerical answer is incorrect.

Problem 1.

Consider two discrete random variables X and Y whose values are r and s respectively and suppose that the joint probability distribution is given by:

X (r $\downarrow$) \ Y (s $\rightarrow$)	0	1	2	3	$\mathrm{P}(X=r)$ $\downarrow$
0	$\frac{0}{48}$	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{6}{48}$
1	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{10}{48}$
2	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{14}{48}$
3	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{6}{48}$	$\frac{18}{48}$
$\mathrm{P}(Y=s)$ $\rightarrow$	$\frac{6}{48}$	$\frac{10}{48}$	$\frac{14}{48}$	$\frac{18}{48}$	

Answer the following questions based on the above distribution table.

1. Are X and Y independent?

Solution. Since $P[X=0, Y=0] = \frac{0}{48} \neq \left(\frac{6}{48}\right) \times \left(\frac{6}{48}\right) = P[X=0] \times P[Y=0]$, X and Y are dependent.

2. $E[X + Y]$

Solution. Since the two marginal distributions are identical, $E[X+Y] = E[X] + E[Y] = 2E[X]$. We only find $E[X] = 0 \times \left(\frac{6}{48}\right) + 1 \times \left(\frac{10}{48}\right) + 2 \times \left(\frac{14}{48}\right) + 3 \times \left(\frac{18}{48}\right) = \frac{92}{48}$.
Therefore, $E[X+Y] = 2 \times \left(\frac{92}{48}\right) = \frac{23}{6}$

3. $E[XY]$

Solution. $E[XY] = (1 \times 1)\left(\frac{2}{48}\right) + (1 \times 2)\left(\frac{3}{48}\right) + (1 \times 3)\left(\frac{4}{48}\right) + (2 \times 1)\left(\frac{3}{48}\right) + (2 \times 2)\left(\frac{4}{48}\right) + (2 \times 3)\left(\frac{5}{48}\right) + (3 \times 1)\left(\frac{4}{48}\right) + (3 \times 2)\left(\frac{5}{48}\right) + (3 \times 3)\left(\frac{6}{48}\right) = \frac{168}{48} = 3.5.$

4. $\mathrm{COV}(X, Y)$

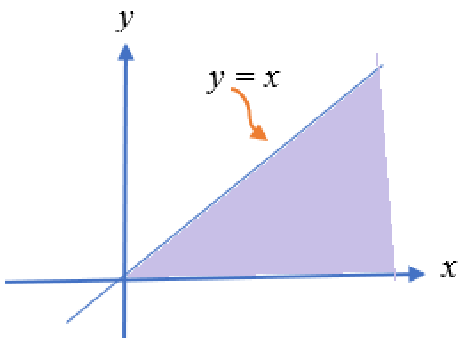
Solution. $COV(X, Y) = E[XY] - E[X]E[Y] = \frac{168}{48} - \frac{92}{48} \times \frac{92}{48} = -\frac{25}{144}.$

Problem 2.

Let X be the total time that a customer spends at a bank, and Y the time she spends waiting in line. Assume that X and Y have joint density

$$f(x,y) = \begin{cases} 4\mathrm{e}^{-2\mathrm{x}}, & 0 \le y \le x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Sketch the domain or related regions whenever appropriate.



1. Find the marginal density functions of X and Y.

Solution: $f(x) = \int_0^x f(x,y)dy = \int_0^x 4\mathrm{e}^{-2\mathrm{x}} dy = 4x\mathrm{e}^{-2\mathrm{x}}$.

$$f(y) = \int_y^\infty f(x,y)dx = \int_y^\infty 2^2\mathrm{e}^{-2\mathrm{x}}dx = -2 \times \mathrm{e}^{-2\mathrm{x}}\big|_y^\infty = 2\mathrm{e}^{-2y}$$

Recall that the gamma density function

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-\beta x}$$

That is,

$$\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-\beta x}\, dx = 1.$$

This means that

$$f(x) = 4x\mathrm{e}^{-2\mathrm{x}} = \frac{2^2}{\Gamma(2)} x^{2-1}\mathrm{e}^{-2x}$$

is a gama distribution with $\beta = 2$ and $\alpha = 2$.

$$f(y) = 2\mathrm{e}^{-2y} = \frac{2^1}{\Gamma(1)} x^{1-1}\mathrm{e}^{-2x}$$

is the gama distribution with $\beta = 2$ and $\alpha = 1$.

2. Are X and Y independent?

Solution: Since $f(x,y) = 4\mathrm{e}^{-2\mathrm{x}} \ne 4x\mathrm{e}^{-2\mathrm{x}} \times 2\mathrm{e}^{-2\mathrm{x}}$, random variables X and Y are dependent.

3. Find out the mean service time: $E[X-Y]$.

Solution: $E[X-Y] = \iint_D (x-y)f(x,y)dA = \int_0^\infty \int_0^x (x-y)4\mathrm{e}^{-2x}dydx$

$$= \int_0^\infty \int_0^x 4x\mathrm{e}^{-2x}dydx - \int_0^\infty \int_0^x 4y\mathrm{e}^{-2x}dydx$$

$$= \int_0^\infty 4x^2\mathrm{e}^{-2x}dx - 0.5\int_0^\infty 4x^2\mathrm{e}^{-2x}dx$$

$$= 0.5\int_0^\infty 4x^2\mathrm{e}^{-2x}dx$$

Note that

$4x^2\mathrm{e}^{-2x} = \frac{2^3}{\Gamma(3)}x^{3-1}\mathrm{e}^{-2x}$ is the gamma density with $\beta = 2$ and $\alpha = 3$.

That is,

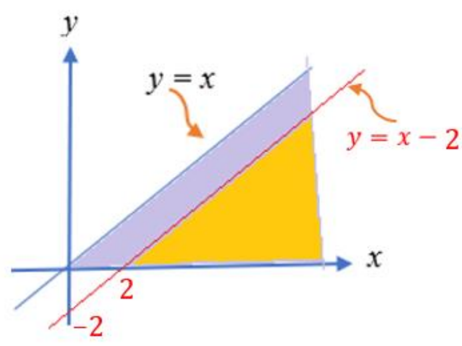
$$\int_0^\infty 4x^2\mathrm{e}^{-2x}\,dx = 1.$$

Therefore,

$$E[X-Y] = 0.5\int_0^\infty 4x^2\mathrm{e}^{-2x}dx = 0.5.$$

4. Find the probability $P[X-Y > 2]$

The yellow region in the following figure is defined by $X-Y > 2$.



$P[X-Y>\lambda] = \iint_{\text{orange}} f(x,y)dA$

$$= \int_2^\infty \left[\int_0^{x-2} 4\mathrm{e}^{-2x}dy\right]dx = \int_2^\infty 4\mathrm{e}^{-2x}\left[\int_0^{x-2} dy\right]dx$$

$$= \int_2^\infty 4(x-2)\mathrm{e}^{-2x}\,dx = \int_2^\infty 4x\mathrm{e}^{-2x}\,dx - \int_2^\infty 8\mathrm{e}^{-2x}\,dx$$

Note that

$$\int_2^\infty 4x\mathrm{e}^{-2x}\,dx = -\int_2^\infty 2x\mathrm{e}^{-2x}\,d(-2x) = -\int_2^\infty 2x\,d\mathrm{e}^{-2x}$$

$$= -2x\mathrm{e}^{-2x}|_2^{\infty} + \int_2^{\infty} \mathrm{e}^{-2x}\, d2x = 4\mathrm{e}^{-2^2} - \mathrm{e}^{-2x}|_{\lambda 2}^{\infty} = 4\mathrm{e}^{-4} + \mathrm{e}^{-4} = 5\mathrm{e}^{-4}$$

$$\int_2^{\infty} 8\mathrm{e}^{-2x}\, dx = -4\int_2^{\infty} \mathrm{e}^{-2x}\, d(-2x) = -4\mathrm{e}^{-2x}|_2^{\infty} = 4\mathrm{e}^{-4}$$

$$P[X - Y > 2] = 5\mathrm{e}^{-2} - 4\mathrm{e}^{-2} = \mathrm{e}^{-4}.$$

5. Find the variance of $X - Y$.

Solution: Since $COV(X,Y) = E[XY] - E[X]E[Y]$. We first calculate the involved moments in the following using the marginal density functions derived in part 1 and the joint density as well.

$$\begin{aligned} E[X] &= \int_0^{\infty} x4\mathrm{e}^{-2x}\, dx = -2\int_0^{\infty} x^2 d\mathrm{e}^{-2x} \\ &= -2\mathrm{e}^{-2x}|_0^{\infty} + 2\int_0^{\infty} 2x\mathrm{e}^{-2x} dx \\ &= \int_0^{\infty} 4x\mathrm{e}^{-2x} dx = 1. \text{ Since } 4x\mathrm{e}^{-2x} \text{ is the gamma density function.} \end{aligned}$$

$$\begin{aligned} E[Y] &= \int_0^{\infty} y2\mathrm{e}^{-2y}\, dy = -\int_0^{\infty} yd\mathrm{e}^{-2y} \\ &= -y\mathrm{e}^{-2x}|_0^{\infty} + \int_0^{\infty} \mathrm{e}^{-2y}\, dy = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E[XY] &= \iint xyf(x,y)dA = \int_0^{\infty}\int_0^{x} xy\, 4\mathrm{e}^{-2x} dy\, dx \\ &= \int_0^{\infty} x4\mathrm{e}^{-2x} \times \frac{y^2}{2}\Big|_0^x dx \\ &= \frac{1}{2}\int_0^{\infty} 4x^3\mathrm{e}^{-2x}\, dx = 2\int_0^{\infty} x^3\mathrm{e}^{-2x}\, dx \end{aligned}$$

Note that the above integrand $x^3\mathrm{e}^{-2x}$ can be rewritten as

$$x^3\mathrm{e}^{-2x} = \left(\frac{\Gamma(4)}{2^4}\right)\left(\frac{2^4}{\Gamma(4)}\right)x^{4-1}\mathrm{e}^{-2x}.$$

Therefore,

$$E[XY] = 2\int_0^{\infty} x^3\mathrm{e}^{-2x}\,\mathrm{dx} = 2\left(\frac{\Gamma(4)}{2^4}\right)\int_0^{\infty}\left(\frac{2^4}{\Gamma(4)}\right)x^{4-1}\mathrm{e}^{-2x}\, dx$$

$$= 2 \times \frac{\Gamma(4)}{2^4} \times 1 = \frac{3\times 2\times 1}{2^3} = \frac{3}{4}$$

That is,

$$COV(X,Y) = E[XY] - E[X]E[Y] = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

Note that

$$f(x) = 4x\mathrm{e}^{-2\mathrm{x}} = \frac{2^2}{\Gamma(2)} x^{2-1}\mathrm{e}^{-2x}$$

is a gama distribution with $\beta = 2$ and $\alpha = 2$.

$$f(y) = 2e^{-2y} = \frac{2^1}{\Gamma(1)} x^{1-1} e^{-2x}$$

is the gama distribution with $\beta = 2$ and $\alpha = 1$.

The variance of X and Y are given by

$$V(X) = \frac{\alpha}{\beta^2} = \frac{2}{2^2} = \frac{1}{2} \text{ and } V(X) = \frac{\alpha}{\beta^2} = \frac{1}{2^2} = \frac{1}{4}$$

Therefore,

$$V(X-Y) = V(X) + V(Y) - 2\text{cov}(X,Y) = \frac{1}{2} + \frac{1}{4} - 2\frac{1}{4} = \frac{1}{4}$$

6. Find the correlation coefficient between X and Y.

   Solution: Since the $f(x) = 4xe^{-2x}$ and $f(y) = 2e^{-2y}$ are both gamma densities. The variance of X and Y are given by

$$Var[X] = \frac{\alpha}{\beta^\alpha} = \frac{2-1}{2^2} = \frac{1}{4}$$

   and

$$Var[Y] = \frac{\alpha}{\beta^\alpha} = \frac{1}{2^1} = \frac{1}{2}.$$

   The correlation coefficient

$$\rho = \frac{COV(X,Y)}{\sqrt{V[X]V[Y]}} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{4} \times \frac{1}{2}}} = \frac{\sqrt{2}}{2}$$

7. Given that waiting time $Y = 2$, what is $E[X|\, Y = 2]$?

   Solution: By the definition, the conditional density function of $X|Y = y$ is given by

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{4e^{-2x}}{2e^{-2y}} = 2e^{-2(x-y)}$$

   where $0 \le y \le x < \infty$.

   The conditional expectation is given by

$$E[X|\, Y = 2] = \int_2^\infty 2xe^{-2(x-2)}\, dx = -\int_2^\infty xe^{-2(x-2)}\, d[-2x]$$

$$= -\int_2^\infty xe^{-2(x-2)}\, d[-2(x-2)] = -\int_2^\infty x de^{-2(x-2)}$$

$$= -xe^{-2(x-2)}\Big|_2^\infty + \int_2^\infty e^{-2(x-2)} dx$$

$$= 2 - \frac{1}{2}\int_2^\infty e^{-2(x-2)} d[-2(x-2)]$$

$$= 2 - \frac{1}{2}e^{-2(x-2)}\Big|_2^\infty = 2 + \left[0 - (-\frac{1}{2})\right] = \frac{5}{2}.$$