STA 504 Mathematical Statistics with Calculus Review

Midterm Exam #2

11/19/2022

Please Print: ________________ __________________
(First Name) (Last Name)

Instructions

- This is an open-book test. Textbook and notes can be used. However, you must complete this exam independently. All forms of collaborations are NOT allowed.
- You may use a calculator for the exam.
- Please show your detailed work to earn full credit.
- Partial credit will be granted to the key steps that reflect your correct reasoning even if your numerical answer is incorrect.

Problem 1.

Consider two discrete random variables X and Y whose values are r and s respectively and suppose that the probability of the event {X = r} ∩ {Y = s} is given by:

$$f(s,t) = \begin{cases} \dfrac{\mathrm{r+s}}{48}, & 0 \le r,s \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

The above probability distribution can be tabulated in the following

X \ Y ($s \rightarrow$), $r \downarrow$	0	1	2	3	$P(X=r)$
0	$\frac{0}{48}$	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{6}{48}$
1	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{10}{48}$
2	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{14}{48}$
3	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{6}{48}$	$\frac{18}{48}$
$P(Y=s)$	$\frac{6}{48}$	$\frac{10}{48}$	$\frac{14}{48}$	$\frac{18}{48}$	

Find the expectation of

1. Are X and Y independent?

Solution. Since $P[X=0,Y=0] = \frac{0}{48} \neq \left(\frac{6}{48}\right) \times \left(\frac{6}{48}\right) = P[X=0] \times P[Y=0]$, X and Y are dependent.

2. $E[X + Y]$

Solution. Since the two marginal distributions are identical, $E[X + Y] = E[X] + E[Y] = 2E[X]$. We only find $E[X] = 0 \times \left(\frac{6}{48}\right) + 1 \times \left(\frac{10}{48}\right) + 2 \times \left(\frac{14}{48}\right) + 3 \times \left(\frac{18}{48}\right) = \frac{92}{48}$.
Therefore, $E[X+Y] = 2 \times \left(\frac{92}{48}\right) = \frac{23}{6}$

3. $E[XY]$

Solution. $E[XY] = (1 \times 1)\left(\frac{2}{48}\right) + (1 \times 2)\left(\frac{3}{48}\right) + (1 \times 3)\left(\frac{4}{48}\right) + (2 \times 1)\left(\frac{3}{48}\right) + (2 \times 2)\left(\frac{4}{48}\right) + (2 \times 3)\left(\frac{5}{48}\right) + (3 \times 1)\left(\frac{4}{48}\right) + (3 \times 2)\left(\frac{5}{48}\right) + (3 \times 3)\left(\frac{6}{48}\right) = \frac{168}{48} = 3.5$.

4. $COV(X,Y)$

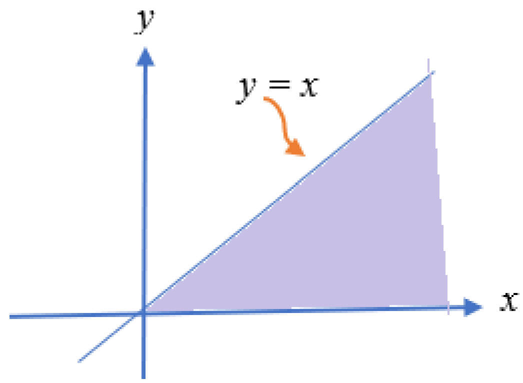
Solution. $COV(X,Y) = E[XY] - E[X]E[Y] = \frac{168}{48} - \frac{92}{48} \times \frac{92}{48} = -\frac{25}{144}$.

Problem 2.

Let X be the total time that a customer spends at a bank, and Y the time she spends waiting in line. Assume that X and Y have joint density

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda x}, & 0 \le y \le x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Sketch the domain or related regions whenever needed.



1. Find the marginal density functions of X and Y.

Solution: $f(x) = \int_0^x f(x,y)dy = \int_0^x \lambda^2 e^{-\lambda x} dy = \lambda^2 x e^{-\lambda x}$

$f(y) = \int_y^\infty f(x,y)dx = \int_y^\infty \lambda^2 e^{-\lambda x} dx = -\lambda \times e^{-\lambda x}|_y^\infty = \lambda e^{-\lambda y}$

2. Are X and Y independent?

Solution: Since $f(x,y) = \lambda^2 e^{-\lambda x} \neq \lambda^2 x e^{-\lambda x} \times \lambda e^{-\lambda x}$, random variables X and Y are dependent.

3. Find out the mean service time: $E[T] = E[X - Y]$.

Solution: $E[T] = \iint_D (x-y) f(x,y) dA = \int_0^\infty \int_0^x (x-y)\lambda^2 e^{-\lambda x} dy dx$

$$= \int_0^\infty \int_0^x x\lambda^2 e^{-\lambda x} dy dx - \int_0^\infty \int_0^x y\lambda^2 e^{-\lambda x} dy dx = \int_0^\infty \lambda^2 x^2 e^{-\lambda x} dx - 0.5 \int_0^\infty \lambda^2 x^2 e^{-\lambda x} dx$$

$= 0.5 \int_0^\infty \lambda^2 x^2 e^{-\lambda x} dx = -0.5\lambda[\, x^2 e^{-\lambda x}|_0^\infty - \int_0^\infty 2x e^{-\lambda x} dx] = \int_0^\infty \lambda x e^{-\lambda x} dx$

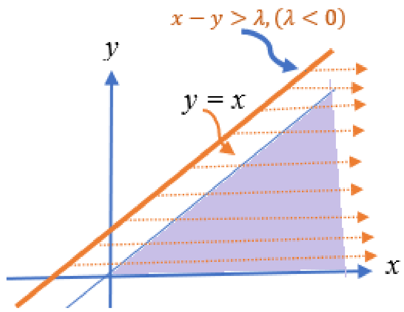
$= -\int_0^\infty x d e^{-\lambda x} = -x e^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda x} dx = -\frac{1}{\lambda}\int_0^\infty e^{-\lambda x} d(-\lambda x) = \frac{1}{\lambda}$

4. Find the probability $P[T > \lambda]$

Solution: We calculate the probability in two cases

Case 1. $\lambda \le 0$

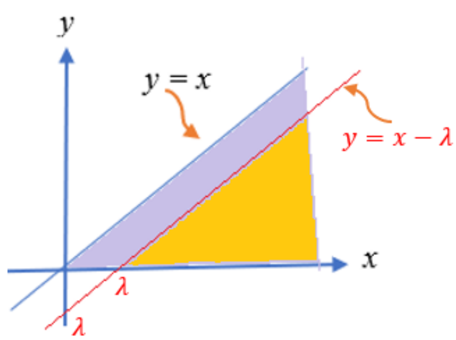
$T > \lambda$ implies $X - Y > \lambda$. This defines a region that contains the domain as a sub-region. The following figure depicts the region defined by $T > \lambda$ and the domain. You can test location of the defined region by taking a special value at $\lambda = -1$.



Since $T > \lambda$ contain the domain, therefore, P[$T > \lambda$] = 1.

Case 2. $\lambda < 0$

The yellow region in the following figure is defined by $T > \lambda$.



$$P[T > \lambda] = P[X - Y > \lambda] = \iint_{\text{orange}} f(x,y)dA$$
$$= \int_\lambda^\infty \left[\int_0^{x-\lambda} \lambda^2 e^{-\lambda x} dy\right] dx = \int_\lambda^\infty \lambda^2 e^{-\lambda x}\left[\int_0^{x-\lambda} dy\right] dx$$
$$= \int_\lambda^\infty (x-\lambda)\lambda^2 e^{-\lambda x}\, dx = \int_\lambda^\infty \lambda^2 x e^{-\lambda x}\, dx - \int_\lambda^\infty \lambda^3 e^{-\lambda x}\, dx$$

Note that

$$\int_\lambda^\infty \lambda^2 x e^{-\lambda x}\, dx = -\int_\lambda^\infty \lambda x e^{-\lambda x}\, d(-\lambda x) = -\int_\lambda^\infty \lambda x\, de^{-\lambda x}$$
$$= -\lambda x e^{-\lambda x}\Big|_\lambda^\infty + \int_\lambda^\infty e^{-\lambda x}\, d\lambda x = \lambda^2 e^{-\lambda^2} - e^{-\lambda x}\Big|_\lambda^\infty = \lambda^2 e^{-\lambda^2} + e^{-\lambda^2}$$

and

$$\int_\lambda^\infty \lambda^3 e^{-\lambda x}\, dx = -\lambda^2 \int_\lambda^\infty e^{-\lambda x}\, d(-\lambda x) = -\lambda^2 e^{-\lambda x}\Big|_\lambda^\infty = \lambda^2 e^{-\lambda^2}$$

$$P[T > \lambda] = P[X - Y > \lambda] = \lambda^2 e^{-\lambda^2} + e^{-\lambda^2} - \lambda^2 e^{-\lambda^2} = e^{-\lambda^2}.$$

5. Find the correlation coefficient between X and Y.

Solution: Since $COV(X,Y) = E[XY] - E[X]E[Y]$. We first calculate the involved moments in the following using the marginal density functions derived in part 1 and the joint density as well.

$$E[X] = \int_0^\infty x\lambda^2 x e^{-\lambda x}\,dx = -\lambda\int_0^\infty x^2 de^{-\lambda x} = -\lambda x^2 e^{-\lambda x}|_0^\infty + \lambda\int_0^\infty 2xe^{-\lambda x}dx$$
$$= \frac{2}{\lambda}\int_0^\infty \lambda^2 x e^{-\lambda x}dx = \frac{2}{\lambda}\times 1 = \frac{2}{\lambda}.$$

$$E[Y] = \int_0^\infty y\lambda e^{-\lambda y}\,dy = -\int_0^\infty y de^{-\lambda y} = -ye^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda y}\,dy = \frac{1}{\lambda}$$

$$E[XY] = \iint xyf(x,y)dA = \int_0^\infty\int_0^x xy\,\lambda^2 e^{-\lambda x}dy\,dx = \int_0^\infty x\lambda^2 e^{-\lambda x}\times\frac{y^2}{2}\Big|_0^x dx$$
$$= \frac{1}{2}\int_0^\infty \lambda^2 x^3 e^{-\lambda x}\,dx = -\frac{1}{2}\int_0^\infty \lambda x^3\,d\left(e^{-\lambda x}\right)$$
$$= -\frac{1}{2}\left[\lambda x^3 e^{-\lambda x}\Big|_0^\infty - \int_0^\infty 3\lambda x^2 e^{-\lambda x}dx\right]$$
$$= \frac{3}{2\lambda}\int_0^\infty \lambda^2 x^2 e^{-\lambda x}\,dx = \frac{2}{\lambda}\times\frac{3}{2\lambda} = \frac{3}{\lambda^2}.$$

$$COV[X,Y] = E[XY] - E[X]E[Y] = \frac{3}{\lambda^2} - \frac{2}{\lambda}\times\frac{1}{\lambda} = \frac{1}{\lambda^2}$$

To find the Pearson correlation coefficient between X and Y, we also need V[X] and V[Y]. Next, we calculate the second moment of X and Y.

$$E[X^2] = \int_0^\infty x^2\lambda^2 x e^{-\lambda x}\,dx = \int_0^\infty \lambda^2 x^3 e^{-\lambda x}\,dx = 2\times\frac{3}{\lambda^2} = \frac{6}{\lambda^2}$$
$$E[Y^2] = \int_0^\infty y^2\lambda e^{-\lambda y}\,dx = \int_0^\infty \lambda y^2 e^{-\lambda y}\,dy = \frac{1}{\lambda}\frac{2\lambda}{3}\left\{\frac{3}{2\lambda}\int_0^\infty \lambda^2 y^2 e^{-\lambda y}\,dy\right\} = \frac{2}{3}\times\frac{3}{\lambda^2} = \frac{2}{\lambda^2}$$

We used part of the results in the calculation of E[XY] .

$$V[X] = \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \frac{2}{\lambda^2},$$
$$V[Y] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

The correlation coefficient

$$\rho = \frac{COV(X,Y)}{\sqrt{V[X]V[Y]}} = \frac{\frac{1}{\lambda^2}}{\sqrt{\frac{2}{\lambda^2}\times\frac{1}{\lambda^2}}} = \frac{\sqrt{2}}{2}$$

6. Find the variance of service time T.

Solution: We have calculated all expectations needed in the variance of T in the following

$$V[T] = V[X - Y] = V[X] + V[Y] - 2COV(X,Y)$$
$$= \frac{2}{\lambda^2} + \frac{1}{\lambda^2} - \frac{2}{\lambda^2} = \frac{1}{\lambda^2}.$$

7. Find the conditional density function of $X|Y = y$.

Solution: By the definition, the conditional density function of $X|Y = y$ is given by

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{\lambda^2 e^{-\lambda x}}{\lambda e^{-\lambda y}} = \lambda e^{-\lambda(x-y)}$$

where $0 \leq y \leq x < \infty$.

8. Given that waiting time $Y = \lambda$, what is $E[X|\, Y = \lambda]$?

Solution: $E[X|\, Y = \lambda] = \int_{\lambda}^{\infty} x\lambda e^{-\lambda(x-\lambda)}\, dx = -\int_{\lambda}^{\infty} x d e^{-\lambda(x-\lambda)}$

$$= -x\lambda e^{-\lambda(x-\lambda)}\Big|_{\lambda}^{\infty} + \int_{\lambda}^{\infty} e^{-\lambda(x-\lambda)} dx = \lambda - \frac{1}{\lambda} e^{-\lambda(x-\lambda)}\Big|_{\lambda}^{\infty} = \lambda + \frac{1}{\lambda}.$$