STA 504 Topic #10

Practice Exercise

Problem 1.

Let Y_1 and Y_2 be continuous random variables with pdf:

$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} \frac{3y_1^2}{4}, \text{if } 0 \le y_1 \le y_2 \le 2. \\ 0, \quad \text{otherwise.} \end{cases}$$

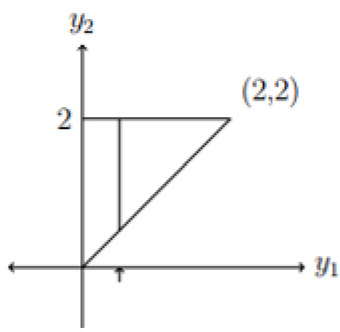
(a) Compute the conditional expectation of Y_2 given $Y_1 = y_1$.

(b) Given that $Y_1 = 1/2$, what is the expectation of Y_2

(c). Compute the conditional expectation of Y_2 given Y_1

Solution:

The domain of the density function is given by



(a). The marginal density function is given by

$$f_{Y_1}(y_1) = \int_{y_1}^{2} \frac{3y_1^2}{4} dy_2 = \frac{3y_1^2(2 - y_1)}{4}.$$

Therefore,

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{3y_1^2}{4} \Big/ \frac{3y_1^2(2 - y_1)}{4} = \frac{1}{2 - y_1}.$$

for $0 \le y_1 \le y_2 \le 2$ *and* 0, otherwise. The conditional expectation

$$E[Y_2|Y_1 = y_1] = \int_{y_1}^{2} y_2 \times \frac{1}{2 - y_1} dy_2 = \frac{1}{2 - y_1} \times \frac{2^2 - y_1^2}{2} = \frac{2 + y_1}{2}.$$

(b).

$$E[Y_2|Y_1 = 1/2] = \frac{2 + 1/2}{2} = \frac{5}{4}.$$

(c). As shown in (a),

$$f_{Y_2|Y_1}(y_2|Y_1) = \frac{1}{2 - Y_1}$$

for $0 \le y_1 \le y_2 \le 2$ *and* 0, otherwise. Therefore,

$$E[Y_2|Y_1] = \frac{2 + Y_1}{2}$$

Problem 2.

For the daily output of an industrial operation, let Y_1 denote the amount of sales and Y_2, the costs, in thousands of dollars. Assume that the density functions for Y_1 and Y_2 are given by

$$f_1(y_1) = \begin{cases} \left(\frac{1}{6}\right) y_1^3 e^{-y_1}, & y_1 > 0; \\ 0, & y_1 > 0. \end{cases}$$

and

$$f_2(y_2) = \begin{cases} \left(\frac{1}{2}\right) e^{-y_2/2}, & y_2 > 0; \\ 0, & y_2 > 0. \end{cases}$$

The daily profit is given by $U = Y_1 - Y_2$.

(1) Find $E(U)$.

Solution: Note that the Y_1 is a gamma distribution ($\alpha = 4, \beta = 1$) and Y_2 is an exponential distribution ($\alpha = 1, \beta = 1/2$).

$$E[U] = E[Y_1] - E[Y_2] = 4/1 - 1/(\frac{1}{2}) = 2$$

(2) Assuming that Y_1 and Y_2 are independent, find $V(U)$.

Solution: $E[U] = V[Y_1] - V[Y_2] = \frac{4}{1^2} - \frac{1}{\left(\frac{1}{2}\right)^2} = 0$

Problem 3.

Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \left(\frac{1}{8}\right) y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_1 > 0; \\ 0, & \text{elsewhere.} \end{cases}$$

The cost C of replacing the two components depends upon their length of life at failure and is given by $C = 50 + 2Y_1 + 4Y_2$.

(1). Find $E(C)$ and $V(C)$.

Solution: We first find the two marginal densities.

$$f_1(y_1) = \int_0^\infty \left(\frac{1}{8}\right) y_1 e^{-(y_1+y_2)/2}\, dy_2 = \left(\frac{1}{4}\right) y_1 e^{-y_1/2} \int_0^\infty \left(\frac{1}{2}\right) e^{-y_2/2}\, dy_2 = \left(\frac{1}{4}\right) y_1 e^{-y_1/2}$$

$$f_2(y_2) = \int_0^\infty \left(\frac{1}{8}\right) y_1 e^{-(y_1+y_2)/2}\, dy_1 = \left(\frac{1}{2}\right) e^{-y_1/2} \int_0^\infty \left(\frac{1}{4}\right) y_1 e^{-y_1/2}\, dy_1 = \left(\frac{1}{2}\right) e^{-y_2/2}$$

Since $f_1(y_1) \times f_2(y_2) = f(y_1, y_2)$, Y_1 and Y_2 are independent. Note also that Y_1 is a gamma distribution with shape $\alpha = 2$ and scale $\beta = \frac{1}{2}$ and Y_2 is an exponential with scale , $\beta = \frac{1}{2}$.
Therefore, $E[Y_1] = \frac{\alpha}{\beta} = \frac{2}{\frac{1}{2}} = 4$, $E[Y_2] = \frac{1}{\beta} = 2$; $V[Y_1] = \frac{\alpha}{\beta^2} = \frac{2}{1/4} = 8$, and $[Y_1] = \frac{1}{\beta^2} = \frac{1}{1/4} = 4$.

Hence, $E[C] = 50 + 2E[Y_1] + 4E[Y_2] = 50 + 2 \times 4 + 4 \times 2 = 66.$

(2). Let $U = Y_1 - Y_2$ and $W = Y_1 + Y_2$. Find $COV(U, W)$

Solution: Note that

$$\begin{aligned} \boldsymbol{COV}(U_1, W_2) &= \boldsymbol{COV}[Y_1 - Y_2, Y_1 + Y_2] \\ &= \boldsymbol{COV}[Y_1, Y_1 + Y_2] - \boldsymbol{COV}[Y_2, Y_1 + Y_2] \\ &= \boldsymbol{COV}[Y_1, Y_1] + \boldsymbol{COV}[Y_1, Y_2] - \boldsymbol{COV}[Y_2, Y_1] - \boldsymbol{COV}[Y_2, Y_2] \\ &= \boldsymbol{V}[Y_1] + \mathbf{0} - \mathbf{0} - \boldsymbol{V}[Y_2] = \mathbf{8} - \mathbf{4} = \mathbf{4}. \end{aligned}$$