

STA 504 Homework #9

Due: Monday, November 11

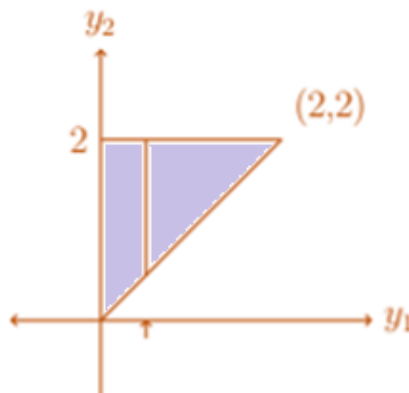
Problem 1.

Let Y_1 and Y_2 be continuous random variables with pdf:

$$f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \frac{3y_1^2}{4}, & \text{if } 0 \leq y_1 \leq y_2 \leq 2. \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Compute the expectation of Y_2 .
- (b) What is the expectation of $Y_1 \times Y_2$
- (c). Find the correlation coefficient of Y_1 and Y_2
- (d). Are Y_1 and Y_2 independent?

Solution: Note the domain of the density function is



(a).

$$\begin{aligned} E[Y_2] &= \iint_R y_2 f_{Y_1 Y_2}(y_1, y_2) dA = \int_0^2 \int_{y_1}^2 y_2 \frac{3y_1^2}{4} dy_2 dy_1 \\ &= \int_0^2 \frac{3y_1^2}{4} \int_{y_1}^2 y_2 dy_2 dy_1 = \int_0^2 \frac{3y_1^2}{4} \frac{4 - y_1^2}{2} dy_1 = \left(\frac{3}{8}\right) \int_0^2 (4y_1^2 - y_1^4) dy_1 \end{aligned}$$

$$= \left(\frac{3}{8}\right) \left[\frac{4y_1^3}{3} - \frac{y_1^5}{5} \right]_0^2 = \left(\frac{3}{8}\right) \left[\frac{4 \times 8}{3} - \frac{32}{5} \right] = \left(\frac{3}{8}\right) \left(\frac{32 \times 2}{15} \right) = \frac{8}{5}$$

(b)

$$\begin{aligned} E[Y_1 Y_2] &= \iint_R y_1 y_2 f_{Y_1 Y_2}(y_1, y_2) dA = \int_0^2 \int_{y_1}^2 y_1 y_2 \frac{3y_1^2}{4} dy_2 dy_1 \\ &= \int_0^2 \frac{3y_1^3}{4} \int_{y_1}^2 y_2 dy_2 dy_1 = \int_0^2 \frac{3y_1^3}{4} \frac{4 - y_1^2}{2} dy_1 = \left(\frac{3}{8}\right) \int_0^2 (4y_1^3 - y_1^5) dy_1 \\ &= \left(\frac{3}{8}\right) \left[y_1^4 - \frac{y_1^6}{6} \right]_0^2 = \left(\frac{3}{8}\right) \left[16 - \frac{64}{6} \right] = \left(\frac{3}{8}\right) \left(\frac{32}{6} \right) = 2 \end{aligned}$$

(c). We need several moments to find the covariance.

$$\begin{aligned} E[Y_1] &= \iint_R y_1 f_{Y_1 Y_2}(y_1, y_2) dA = \int_0^2 \int_{y_1}^2 \frac{3y_1^3}{4} dy_2 dy_1 \\ &= \int_0^2 \frac{3y_1^3}{4} \int_{y_1}^2 1 dy_2 dy_1 = \int_0^2 \frac{3y_1^3}{4} (2 - y_1) dy_1 = \left(\frac{3}{4}\right) \int_0^2 (2y_1^3 - y_1^4) dy_1 \\ &= \left(\frac{3}{4}\right) \left[\frac{y_1^4}{2} - \frac{y_1^5}{5} \right]_0^2 = \left(\frac{3}{4}\right) \left(\frac{5 \times 16 - 2 \times 32}{10} \right) = \left(\frac{3}{4}\right) \frac{16}{10} = \frac{6}{5} \end{aligned}$$

$$\begin{aligned} E[Y_1^2] &= \iint_R y_1^2 f_{Y_1 Y_2}(y_1, y_2) dA = \int_0^2 \int_{y_1}^2 \frac{3y_1^4}{4} dy_2 dy_1 \\ &= \int_0^2 \frac{3y_1^4}{4} \int_{y_1}^2 1 dy_2 dy_1 = \int_0^2 \frac{3y_1^4}{4} (2 - y_1) dy_1 = \left(\frac{3}{4}\right) \int_0^2 (2y_1^4 - y_1^5) dy_1 \\ &= \left(\frac{3}{4}\right) \left[\frac{2y_1^5}{5} - \frac{y_1^6}{6} \right]_0^2 = \left(\frac{3}{4}\right) \left(\frac{64}{5} - \frac{64}{6} \right) = \left(\frac{3}{4}\right) \frac{64}{30} = \frac{8}{5} \end{aligned}$$

$$E[Y_2^2] = \iint_R y_2^2 f_{Y_1 Y_2}(y_1, y_2) dA = \int_0^2 \frac{3y_1^2}{4} \int_{y_1}^2 y_2^2 dy_2 dy_1$$

$$\begin{aligned}
&= \int_0^2 \frac{3y_1^2}{4} \left(\frac{8}{3} - \frac{y_1^3}{3} \right) dy_1 = \left(\frac{3}{12} \right) \int_0^2 (8y_1^2 - y_1^5) dy_1 \\
&= \left(\frac{3}{12} \right) \left[\frac{8y_1^3}{3} - \frac{y_1^6}{6} \right]_0^2 = \left(\frac{1}{4} \right) \left(\frac{64}{3} - \frac{64}{6} \right) = \left(\frac{3}{12} \right) \frac{64}{30} = \frac{8}{3}
\end{aligned}$$

Note that

$$COV(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2] = 2 - \frac{6}{5} \times \frac{8}{5} = \frac{2}{25}$$

$$V[Y_1] = E[Y_1^2] - (E[Y_1])^2 = \frac{8}{5} - \left(\frac{6}{5} \right)^2 = \frac{4}{25}$$

$$V[Y_2] = E[Y_2^2] - (E[Y_2])^2 = \frac{8}{3} - \left(\frac{8}{5} \right)^2 = \frac{8}{75}$$

$$\rho = \frac{COV(Y_1, Y_2)}{\sqrt{V[Y_1]V[Y_2]}} = \frac{\frac{2}{25}}{\sqrt{\frac{4}{25} \times \frac{8}{75}}} = \frac{\sqrt{6}}{4} > 0$$

(d). Since the Pearson correlation coefficient is positive, Y_1 and Y_2 are linearly correlated. Therefore, Y_1 and Y_2 are NOT independent.

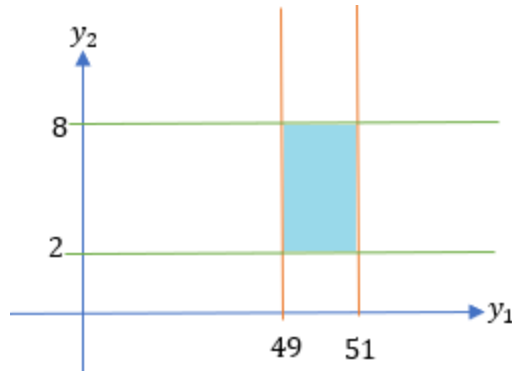
Problem 2.

A machine fills a potato chip bag. Although each bag should weigh 50 grams and contain 5 milligrams of salt, in fact, because of differing machines, the weight and amount of salt placed in each bag varies. The bivariate density function for this machine is

$$f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \frac{1}{12}, & \text{if } 49 \leq y_1 \leq 51, 2 \leq y_2 \leq 8. \\ 0, & \text{otherwise.} \end{cases}$$

- Compute the variance of Y_1 and Y_2 respectively.
- What is the expectation of $Y_1 - Y_2$
- Find the covariance of Y_1 and Y_2
- Are Y_1 and Y_2 independent?

Solution: The domain of the density function is given by



(a). We first find the two marginal densities in the following

$$f_2(y_2) = \int_{49}^{51} \left(\frac{1}{12}\right) dy_1 = \left(\frac{1}{12}\right) \times (51 - 49) = \frac{1}{6}, \quad \text{for } 2 < y_2 < 8.$$

$$f_1(y_1) = \int_2^8 \left(\frac{1}{12}\right) dy_2 = \left(\frac{1}{12}\right) \times (8 - 2) = \frac{1}{2}, \quad \text{for } 49 < y_1 < 51.$$

Therefore, both marginal distributions are uniform [We learn this continuous distribution in Chapter 4]. Recall that for a uniform random variable X defined on the interval $[a, b]$, the expectation and variance is given by

$$E[X] = (b + a)/2 \text{ and } V[X] = (b - a)^2/12$$

$$V[Y_2] = \frac{(8 - 2)^2}{12} = 3$$

$$V[Y_1] = \frac{(51 - 49)^2}{12} = \frac{1}{3}$$

(b). We use the results in (a)

$$E[Y_1 - Y_2] = E[Y_1] - E[Y_2] = \frac{49 + 51}{2} - \frac{2 + 8}{2} = 50 - 5 = 45.$$

(c). Since from (a), we $f_1(y_1) \times f_2(y_2) = f_{Y_1 Y_2}(y_1, y_2)$. Therefore, Y_1 and Y_2 are independent. On the other hand, independence implies an uncorrelated relationship. Hence, the covariance of Y_1 and Y_2 is 0.

(d). Answered in (c).