STA 504 Homework #12

Due: Monday, December 02

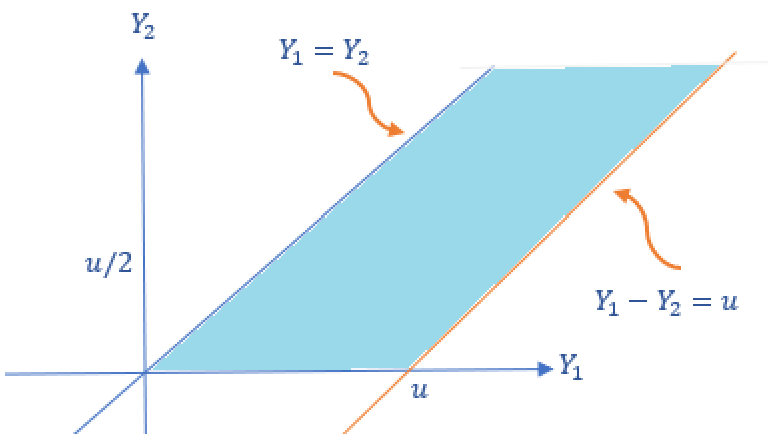
Problem 1.

The total time from arrival to completion of service at a fast-food outlet, Y_1, and the time spent waiting in line before arriving at the service window, Y_2, with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, 0 \le y_2 \le y_1 < \infty \\ 0, \quad elsewhere. \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find the probability density function for U.

Solution: Note that $F_U(u) = P[U \le u] = P[Y_1 - Y_1 \le u]$. We add constraint $Y_1 - Y_2 \le u$ to the domain $0 \le y_2 \le y_1 < \infty$ and obtain region in the following.



$$P[Y_1 - Y_1 \le u] = \int_0^\infty \int_{y_2}^{y_2+u} e^{-y_1}\, dy_1 dy_2$$

$$= -\int_0^\infty e^{-y_1}|_{y_2}^{y_2+u} dy_2 = -\int_0^\infty (e^{-y_2-u} - e^{-y_2}) dy_2$$

$$= -(e^{-u} - 1)\int_0^\infty e^{-y_2} dy_2 = 1 - e^{-u}$$

U is an exponential distribution with the following CDF

$$F_U(u) = \begin{cases} 1 - e^{-u}, \quad \text{for u} > 0 \\ 0, \quad \text{elsewhere.} \end{cases}$$

Problem 2.

Let v denote the volume of a three-dimensional figure. Let Y denote the number of particles observed in volume v and assume that Y has a Poisson distribution with mean λv. The particles might represent pollution particles in air, bacteria in water, or stars in the heavens. If a point is chosen at random within the volume v, then the distance R to the nearest particle has the probability density function given by

$$f(r) = \begin{cases} 4\lambda\pi r^2 e^{-(4/3)\lambda\pi r^3}, & r > 0. \\ 0, & \text{elsewhere}, \end{cases}$$

Find the density function of $U = R^3$ has an exponential distribution.

Solution: Note that $r > 0$ implies $u > 0$. By the definition of CDF, we have

$$F_U(u) = P[U \le u] = P[R^3 \le u] = P\left[R \le \sqrt[3]{u}\right]$$

$$= \int_0^{\sqrt[3]{u}} 4\lambda\pi r^2 e^{-(4/3)\lambda\pi r^3}\, dr.$$

Taking derivative of $F_U(u)$, we have

$$\frac{dF_U(u)}{du} = f_U(u) = \frac{d}{du}\int_0^{\sqrt[3]{u}} 4\lambda\pi r^2 e^{-\left(\frac{4}{3}\right)\lambda\pi r^3}\, dr$$

$$= 4\lambda\pi u^{\frac{2}{3}} e^{-\left(\frac{4}{3}\right)\lambda\pi u}\left(\sqrt[3]{u}\right)' = \frac{4}{3}\lambda\pi u^{\frac{2}{3}} e^{-\left(\frac{4}{3}\right)\lambda\pi u} u^{-\frac{2}{3}}$$

$$= \frac{4}{3}\lambda\pi e^{-\left(\frac{4}{3}\right)\lambda\pi u}.$$

Therefore, U is an exponential distribution with density function

$$f_U(u) = \begin{cases} \frac{4}{3}\lambda\pi e^{-\left(\frac{4}{3}\right)\lambda\pi u}, & \text{for u} > 0 \\ 0, & \text{elsewhere.} \end{cases}$$