Linear Function of Random Variables and Conditional Expectations

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1 Introduction

This note discusses the expectation and variance of a special family of functions of random variables - a linear combination of random variables. **Sections 8 and 11 of chapter 5** in the textbook covers these topics.

2 Linear Combination of Two Random Variables

Let X and Y be two random variables with joint density function $f(x, y)$. Define $U = aX + bY$ (a and b are constants). Then

$$E[U] = E[aX + bY] = aE[X] + bE[Y]$$

The variance of U can be found in the following

$$\mathrm{V}[U] = E\left[(U - E[U])^2\right] = E\left[(a[X - E[X]] + b[Y - EY])^2\right]$$

$$= E\left[\left(a^2[X - E[X]]^2 + 2ab[X - E[X]][Y - EY] + b^2[Y - EY]^2\right)^2\right]$$

$$= a^2E\left[[X - E(X)]^2\right] + 2abE[X - E(X)][Y - E(Y)] + b^2E[Y - E(Y)]^2$$

$$= a^2\mathrm{V}(X) + 2ab\mathrm{COV}(X, Y) + b^2\mathrm{V}(Y)$$

We know that $\mathrm{V}[X] = E[X^2] - (E[X])^2$. Similarly, we have

$$\text{COV}(X, Y) = E\left[(X - E[X])(Y - E[Y])\right] = E\left[XY - E[X]Y - E[Y]X + E[X]E[Y]\right] = E[XY] - E[X]E[Y]$$

Special Case: If X and Y are uncorrelated, then $\text{COV}(X, Y) = 0$. Therefore,

$$\text{V}[U] = a^2\text{V}(X) + b^2\text{V}(Y).$$

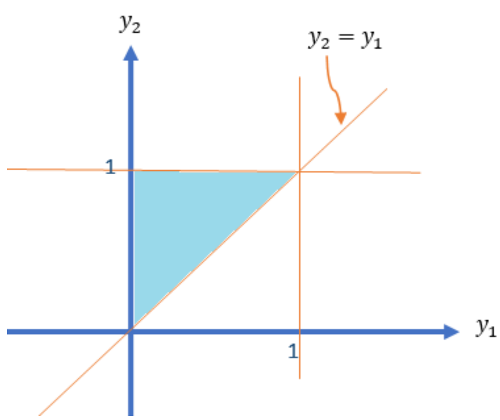
When the joint distribution of X and Y are given, we can find the mean and variance of the linear combination of X and Y.

Example 1: Consider the following joint density function of Y_1 and Y_2

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2) & 0 \le y_1 \le y_2 \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Find $\text{V}(Y_1 - 3Y_2)$.

Solution: We first draw the domain of the density function in the following.



Note that $\text{V}(Y_1 - 3Y_2) = \text{V}(Y_1) - 6\text{COV}(Y_1, Y_2) + 9\text{V}(Y_2)$

The above variances and the covariance can be expressed into the first and second moments: $E[Y_1], E[Y_2], E[Y_1^2], E[Y_2^2],$ and $E[Y_1Y_2]$.

$$E[Y_1] = \iint_{R_1} y_1 f(y_1, y_2) dA = \int_0^1 6y_1 \int_{y_1}^1 (1 - y_2) dy_2 dy_1$$

$$= \int_0^1 6y_1 (y_2 - y_2^2/2)\Big|_{y_1}^1 dy_1 = \int_0^1 6y_1(1/2 - y_1 + y_1^2/2) dy_1$$

$$= \int_0^1 (3y_1 - 6y_1^2 + 3y_1^3) dy_1 = \left[3y^2/2 - 2y_1^3 + 3y_2^4/4\right]\Big|_0^1 = 3/2 - 2 + 3/4 = 1/4.$$

$$E[Y_1^2] = \iint_{R_1} y_1^2 f(y_1, y_2) dA = \int_0^1 6y_1^2 \int_{y_1}^1 (1 - y_2) dy_2 dy_1$$

$$= \int_0^1 6y_1^2 (y_2 - y_2^2/2)\Big|_{y_1}^1 dy_1 = \int_0^1 6y_1^2(1/2 - y_1 + y_1^2/2) dy_1$$

$$= \int_0^1 (3y_1^2 - 6y_1^3 + 3y_1^4) dy_1 = \left[y_1^3 - \frac{3y_1^4}{2} + \frac{3y_1^5}{5} \right] \Bigg|_0^1 = -\frac{1}{2} + \frac{3}{5} = \frac{1}{10}$$

Therefore,

$$\mathrm{V}[\mathrm{Y}_1] = \frac{1}{10} - \frac{1}{16} = \frac{8}{80} - \frac{5}{80} = \frac{3}{80}$$

$$E[Y_2] =] = \iint_{R_1} y_2 f(y_1, y_2) dA = \int_0^1 \int_{y_1}^1 6y_2(1 - y_2) dy_2 dy_1$$

$$= \int_0^1 \int_{y_1}^1 (6y_2 - 6y_2^2) dy_2 dy_1 = \int_0^1 (3y_2^2 - 2y_2^3) \Bigg|_{y_1}^1 dy_1$$

$$= \int_0^1 (1 - 3y_1^2 + 2y_1^3) dy_1 = \left(y_1 - y_1^3 + \frac{y_1^4}{2} \right) \Bigg|_0^1 = \frac{1}{2}.$$

$$E[Y_2^2] = \iint_{R_1} y_2^2 f(y_1, y_2) dA = \int_0^1 \int_{y_1}^1 6y_2^2(1 - y_2) dy_2 dy_1$$

$$= \int_0^1 \int_{y_1}^1 (6y_2^2 - 6y_2^3) dy_2 dy_1 = \int_0^1 (2y_2^3 - \frac{3y_2^4}{2}) \Bigg|_{y_1}^1 dy_1$$

$$= \int_0^1 (\frac{1}{2} - 2y_1^3 + \frac{3y_1^4}{2}) dy_1 = \left(\frac{y_1}{2} - \frac{y_1^4}{2} + \frac{3y_1^5}{10} \right) \Bigg|_0^1 = \frac{3}{10}.$$

We obtain

$$\mathrm{V}[\mathrm{Y}_2] = E[Y_2^2] - (E[Y_2])^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$

Next, we calculate $E[Y_1 Y_2]$ in the following.

$$E[Y_1 Y_2] = \iint_{R_1} y_1 y_2 f(y_1, y_2) dA = \int_0^1 6y_1 \int_{y_1}^1 y_2(1 - y_2) dy_2 dy_1$$

$$= \int_0^1 6y_1 \left(\frac{y_2^2}{2} - \frac{y_2^3}{3} \right) \Bigg|_{y_1}^1 dy_1 = \int_0^1 6y_1 \left(\frac{1}{6} - \frac{y_1^2}{2} + \frac{y_1^3}{3} \right) dy_1$$

$$= \int_0^1 (y_1 - 3y_1^3 + 2y_1^4) dy_1 = \left(\frac{y_1^2}{2} - \frac{3y_1^4}{4} + \frac{2y_1^5}{5} \right) \Bigg|_0^1 = \frac{10 - 15 + 8}{20} = \frac{3}{20}.$$

The covariance between Y_1 and Y_2 is

$$\mathrm{COV}(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2] = \frac{3}{20} - \frac{1}{4} \times \frac{1}{2} = \frac{1}{40}$$

Finally, we calculate the variance

$$\mathrm{V}(Y_1 - 3Y_2) = \mathrm{V}(Y_1) - 6\mathrm{COV}(Y_1, Y_2) + 9\mathrm{V}(Y_2) = \frac{3}{80} - \frac{6 \times 1}{40} + \frac{9 \times 1}{20} = \frac{27}{80}.$$

3 General Linear Functions of RVs

In this section, we discuss expectations and variance (covariance) between to linear functions of two sets of random variables without using specific joint density functions of multiple random variables. The results are the generalizations of those discussed in the previous section.

3.1 General Linear Functions

As our discussion involves more random variables, we will use the following notations in the subsequent results.

Let $\{X_1, X_2, \cdots, X_m\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$ be two sets of random variables with

1. $E[X_i] = \mu_i$
2. $E[Y_j] = \xi_j$
3. $\mathrm{V}[X_i] = \sigma_i^2$
4. $\mathrm{V}[Y_j] = \delta_j^2$
5. $\mathrm{COV}(X_i, Y_j) = \phi_{ij}$ for $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$.

Furthermore, we denote

6. $\mathrm{COV}(X_i, X_k) = \theta_{ik}$ for $1 \le i, k \le I$ and $i \ne k$,
7. $\mathrm{COV}(Y_j, Y_l) = \omega_{jl}$ for $1 \le j, l \le I$ and $j \ne l$.

Define

$$U = \sum_{i=1}^{m} a_i X_i \quad \text{and} \quad V = \sum_{j=1}^{m} b_j Y_j$$

We have defined the expectation of a linear function of random variables in the following

$$E[U] = E[\sum_{i=1}^{m} a_i X_i] = \sum_{i=1}^{m} a_i E[X_i] = \sum_{i=1}^{m} a_i \mu_i.$$

$$\mathrm{V}[U] = E\left[\left(\sum_{i=1}^{m} a_i X_i - \sum_{i=1}^{m} a_i \mu_i\right)^2\right] = E\left[\left(\sum_{i=1}^{m} a_i (X_i - \mu_i)\right)^2\right]$$

Note that

$$\left(\sum_{i=1}^{m} a_i (X_i - \mu_i)\right)^2 = \sum_{i=1}^{m} a_i^2 (X_i - \mu_i)^2 + 2\sum_{j<k} a_i a_k (X_i - \mu_i)(X_k - \mu_k)$$

Therefore,

$$\mathrm{V}(U) = \sum_{i=1}^{m} a_i^2 E\left[(X_i - \mu_i)^2\right] + 2\sum_{i<k} a_i a_k E\left[(X_i - \mu_i)(X_k - \mu_k)\right] = \sum_{i=1}^{m} a_i^2 \sigma_i^2 + 2\sum_{i<k} a_i a_k \theta_{ik},$$

where θ_{ik} is the covariance between X_i and X_k. Similarly,

$$\mathrm{V}(V) = \sum_{j=1}^{n} b_j^2 E\left[(Y_j - \xi_j)^2\right] + 2\sum_{j<l} b_j b_l E\left[(Y_j - \xi_j)(Y_l - \xi_l)\right] = \sum_{j=1}^{n} b_j^2 \delta_j^2 + 2\sum_{j<l} b_j b_l \omega_{jl},$$

where ω_{jl} is the covariance between Y_j and Y_l. The covariance of U and V is given by

$$\mathrm{COV}(U,V) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \mathrm{COV}(X_i, Y_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \phi_{ij}.$$

A Special Case: When all X_i and Y_j $(1 \le i \le m, 1 \le j \le n)$ are mutually independent, all covariances are equal to 0. Therefore,

1. $\mathrm{V}[U] = \sum_{i=1}^{m} a_i^2 \sigma_i^2$.
2. $\mathrm{V}[V] = \sum_{j=1}^{m} b_j^2 \sigma_j^2$.
3. $\mathrm{COV}(U,V) = 0$.

In terms of computation, we still need to do double integrals if the joint distribution involves two continuous random variables.

3.2 Linear Functions of IID RVs

One of the basic assumptions of many statistical modeling is that the random sample is **i**ndependently taken from an **i**dentically **d**istributed population with a certain distribution. This sample is conventionally called the **iid** sample. Next, we will show why **i.i.d** sample is so important in mathematical statistics.

Let $\{X_1, X_2, \cdots, X_n\}$ be an **iid** sample from a population with density function $f(x)$ with mean μ and standard deviation σ. If $f(x) is specified, it could be any distribution such as binomial, Poison, uniform, normal, or gamma distributions, etc. If we make an inference about the population mean, the best point estimate is the sample mean (will show this in STA506) which is the average of the sample

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{X_1}{n} + \frac{X_2}{n} + \cdots + \frac{X_n}{n}$$

The above estimate of the population mean is a linear function of the **iid** random variables with equal weight $(\frac{1}{n})$. The primary interest is to know the mean and variance of $\bar{X}$.

$$\mu_{\bar{X}} = E\left[\bar{X}\right] = \frac{\sum_{i=1}^{n} E[X_i]}{n} = \frac{\sum_{i=1}^{n} \mu}{n} = \frac{n\mu}{n} = \mu$$

Since the sample is an **iid** sample, the covariance of any pairs of X_i $(1 \le i \le n)$ is equal to 0. So we can calculate the variance of $\bar{X}$ in the following

$$\sigma_{\bar{X}}^2 = \mathrm{V}(\bar{X}) = \sum_{i=1}^{n} \mathrm{V}\left(\frac{X_i}{n}\right) = \sum_{i=1}^{n} \frac{\mathrm{V}[X_i]}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

The distribution of $\bar{X}$ is called the sampling distribution of sample means. The mean and the variance of the sampling distribution are given by μ and σ^2/n respectively.

1. $n > 30$, by the *Central Limit Theorem*,

$$\bar{X} \to_{\text{approx.}} N\left(\mu, \frac{\sigma^2}{n}\right)$$

2. If the population is normally distributed, then

$$\bar{X} \to_{\text{exact}} N\left(\mu, \frac{\sigma^2}{n}\right)$$

Question: If$\{X_1, X_2, \cdots, X_n\}$ **is not** an **i.i.d** sample from a population with density function $f(x)$ with mean μ and standard deviation σ. What are the mean and variance? Do we still have the above two neat results?

We can consider the the above questions in different scenarios:

- independent but not identical distributions.
- identical but not independent distributions.

4 Conditional Expectation

When two random variables are not independent, the mean and variance of one variable are dependent on the other variables. For example, Let X = income and Y = level of eduxation. Assume the possible values of Y are: high school diploma, associate degree, four-year college degree, and graduate degree. Then the expected income of each sub-population defined based on the level of education could be different depending on the education.

We may be interested in the expectations such as $E\left[X|Y = \text{four-year college degree}\right]$. This is the logic of linear regression modeling.

We discuss conditional expectations in this section.

Definition: If Y_1 and Y_2 are any two random variables, the conditional expectation of $g(Y_1)$, given that $Y_2 = y_2$, is defined to be

$$E[g(Y_1)|Y_2 = y_2)] = \int_{-\infty}^{\infty} g(y_1)f(y_1|y_2)dy_1$$

if Y_1 and Y_2 are jointly continuous and

$$E[g(Y_1)|Y_2 = y_2] = \sum_{\text{all } y_1} g(y_1)p(y_1|y_2)$$

if Y_1 and Y_2 are jointly discrete.

From the above definition, we can see that the conditional expectation of $g(X)$ given $Y = y$ is the expectation of the conditional distribution of $X|Y = y$.

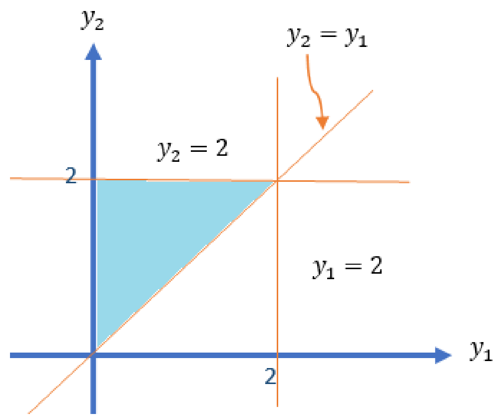
Example 2: A soft-drink machine has a random amount Y_2 in supply at the beginning of a given day and dispenses a random amount Y_1 during the day (with measurements in gallons). It is not resupplied during the day, and hence $Y_1 \leq Y_2$. It has been observed that Y_1 and Y_2 have a joint density given by

$$f(y_1, y_2) = \begin{cases} 1/2 & \text{if } 1/2, 0 \le y_1 \le y_2 \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find the conditional expectation of the amount of sales, Y_1, given that $Y_2 = 1.5$.

Solution: We first find the marginal density function of Y_2 to define the conditional density $Y_1|Y_2 = y_2$ before calculate the expectation.

As usual, we sketch the domain of the density function in the following



$$f_{Y_2}(y_2) = \int_1^{y_2} f(y_1, y_2) dy_1 = \frac{1}{2} \int_0^{y_2} dy_1 = \frac{y_2}{2}, \text{ for } 0 \le y_2 \le 2.$$

Therefore, the conditional density function of $Y_1|Y_2 = y_2$ is given by

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f(y_1, y_2)}{f_{Y_1}(y_1)} = \frac{1/2}{y_2/2} = \frac{1}{y_2}, \text{ for } 0 \le y_1 \le y_2 \le 2.$$

Is the above expression a valid density function?

It is a valid density (**caution:** y_2 is considered as a constant)! To see this

$$\int_0^{y_2} \frac{1}{y_2} dy_1 = \frac{1}{y_2} \times y_1|_0^{y_2} = \frac{1}{y_2}(y_2 - 0) = 1.$$

Hence, the conditional expectation of $Y_1|Y_2 = 1.5$ is given by

$$E[Y_1|Y_2 = 1.5] = \int_0^{y_2} y_1 f_{Y_1|Y_2}(y_1|y_2) dy_1$$

$$= \int_0^{1.5} y_1 \times \frac{1}{1.5} = \frac{y_1^2}{3}\bigg|_0^{1.5} = \frac{1.5^2}{3} = 0.75.$$

This means if the soft-drink machine contains 1.5 gallons at the start of the day, the expected amount to be sold that day is 0.75 gallons.

The above example shows the conditional expectation of Y_1 given $Y_2 = y_2$ (i.e., conditioning on a specific value). If instead of conditioning on a specific value, we condition on the random variable, then we have

$$E[Y_1] = E_{Y_2}[E_{Y_1|Y_2}(Y_1|Y_2)]$$

That is,

$$E[Y_1] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} y_1 f_{y_1|y_2}(y_1, y_2) dy_1 \right\} f_{Y_2}(y_2) dy_2.$$

We can also similarly define conditional variance.

$$\mathrm{V}(Y_1|Y_2 = y_2) = E[Y_1^2|Y_2 = y_2] - (E[Y_1|Y_2 = y_2])^2$$