STA 504 Homework #9

Due: Monday, November 14

Problem 1.

Let Y_1 and Y_2 be continuous random variables with pdf:

$$f_{Y_1Y_2}(y_1,y_2) = \begin{cases} \frac{3y_1^2}{4}, \text{if } 0 \le y_1 \le y_2 \le 2. \\ 0, \quad \text{otherwise.} \end{cases}$$

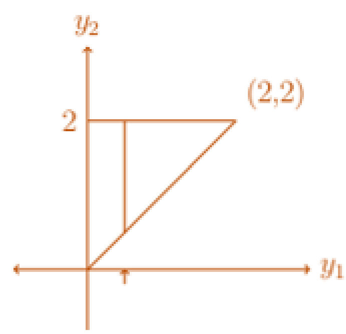
(a) Compute the expectation of Y_2.

(b) What is the expectation of $Y_1 \times Y_2$

(c). Find the correlation coefficient of Y_1 and Y_2

(d). Are Y_1 and Y_2 independent?

Solution: Note the domain of the density function is



(a).

$$E[Y_2] = \iint_R y_2 f_{Y_1Y_2}(y_1,y_2)dA = \int_0^2 \int_{y_1}^2 y_2 \frac{3y_1^2}{4} dy_2\, dy_1$$

$$= \int_0^2 \frac{3y_1^2}{4} \int_{y_1}^2 y_2\, dy_2\, dy_1 = \int_0^2 \frac{3y_1^2}{4} \frac{4-y_1^2}{2} dy_1 = \left(\frac{3}{8}\right) \int_0^2 (4y_1^2 - y_1^4) dy_1$$

$$= \left(\frac{3}{8}\right) \left[\frac{4y_1^3}{3} - \frac{y_1^5}{5}\right]\Bigg|_0^2 = \left(\frac{3}{8}\right) \left[\frac{4 \times 8}{3} - \frac{32}{5}\right] = \left(\frac{3}{8}\right) \left(\frac{32 \times 2}{15}\right) = \frac{8}{5}$$

(b)

$$E[Y_1Y_2] = \iint_R y_1y_2 f_{Y_1Y_2}(y_1, y_2)dA = \int_0^2 \int_{y_1}^2 y_1y_2 \frac{3y_1^2}{4} dy_2\, dy_1$$

$$= \int_0^2 \frac{3y_1^3}{4} \int_{y_1}^2 y_2\, dy_2\, dy_1 = \int_0^2 \frac{3y_1^3}{4} \frac{4 - y_1^2}{2} dy_1 = \left(\frac{3}{8}\right) \int_0^2 \left(4y_1^3 - y_1^5\right)dy_1$$

$$= \left(\frac{3}{8}\right)\left[y_1^4 - \frac{y_1^6}{6}\right]\Bigg|_0^2 = \left(\frac{3}{8}\right)\left[16 - \frac{64}{6}\right] = \left(\frac{3}{8}\right)\left(\frac{32}{6}\right) = 2$$

(c). We need several moments to find the covariance.

$$E[Y_1] = \iint_R y_1 f_{Y_1Y_2}(y_1, y_2)dA = \int_0^2 \int_{y_1}^2 \frac{3y_1^3}{4} dy_2\, dy_1$$

$$= \int_0^2 \frac{3y_1^3}{4} \int_{y_1}^2 1\, dy_2\, dy_1 = \int_0^2 \frac{3y_1^3}{4} (2 - y_1)\, dy_1 = \left(\frac{3}{4}\right) \int_0^2 (2y_1^3 - y_1^4)dy_1$$

$$= \left(\frac{3}{4}\right)\left[\frac{y_1^4}{2} - \frac{y_1^5}{5}\right]\Bigg|_0^2 = \left(\frac{3}{4}\right)\left(\frac{5 \times 16 - 2 \times 32}{10}\right) = \left(\frac{3}{4}\right)\frac{16}{10} = \frac{6}{5}$$

$$E[Y_1^2] = \iint_R y_1^2 f_{Y_1Y_2}(y_1, y_2)dA = \int_0^2 \int_{y_1}^2 \frac{3y_1^4}{4} dy_2\, dy_1$$

$$= \int_0^2 \frac{3y_1^4}{4} \int_{y_1}^2 1\, dy_2\, dy_1 = \int_0^2 \frac{3y_1^4}{4} (2 - y_1)\, dy_1 = \left(\frac{3}{4}\right) \int_0^2 (2y_1^4 - y_1^5)dy_1$$

$$= \left(\frac{3}{4}\right)\left[\frac{2y_1^5}{5} - \frac{y_1^6}{6}\right]\Bigg|_0^2 = \left(\frac{3}{4}\right)\left(\frac{64}{5} - \frac{64}{6}\right) = \left(\frac{3}{4}\right)\frac{64}{30} = \frac{8}{5}$$

$$E[Y_2^2] = \iint_R y_2^2 f_{Y_1Y_2}(y_1, y_2)dA = \int_0^2 \frac{3y_1^2}{4} \int_{y_1}^2 y_2^2\, dy_2\, dy_1$$

$$= \int_0^2 \frac{3y_1^2}{4} \left(\frac{8}{3} - \frac{y_1^3}{3}\right) dy_1 = \left(\frac{3}{12}\right) \int_0^2 (8y_1^2 - y_1^5)dy_1$$

$$= \left(\frac{3}{12}\right)\left[\frac{8y_1^3}{3} - \frac{y_1^6}{6}\right]\Bigg|_0^2 = \left(\frac{1}{4}\right)\left(\frac{64}{3} - \frac{64}{6}\right) = \left(\frac{3}{12}\right)\frac{64}{30} = \frac{8}{3}$$

Note that

$$COV(Y_1, Y_2) = E[Y_1Y_2] - E[Y_1]E[Y_2] = 2 - \frac{6}{5} \times \frac{8}{5} = \frac{2}{25}$$

$$V[Y_1] = E[Y_1^2] - (E[Y_1])^2 = \frac{8}{5} - \left(\frac{6}{5}\right)^2 = \frac{4}{25}$$

$$V[Y_2] = E[Y_2^2] - (E[Y_2])^2 = \frac{8}{3} - \left(\frac{8}{5}\right)^2 = \frac{8}{75}$$

$$\rho = \frac{COV(Y_1, Y_2)}{\sqrt{V[Y_1]V[Y_2]}} = \frac{\frac{2}{25}}{\sqrt{\frac{4}{25} \times \frac{8}{75}}} = \frac{\sqrt{6}}{4} > 0$$

(d). Since the Pearson correlation coefficient is positive, Y_1 *and* Y_2 are linearly correlated. Therefore, Y_1 *and* Y_2 are NOT independent.

Problem 2.

A machine fills potato chip bag. Although each bag should weigh 50 grams each and contain 5 milligrams of salt, in fact, because of differing machines, weight and amount of salt placed in each bag varies. Bivariate density function for this machine is

$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} \frac{1}{12}, & \text{if } 49 \le y_1 \le 51, 2 \le y_2 \le 8. \\ 0, & \text{otherwise.} \end{cases}$$

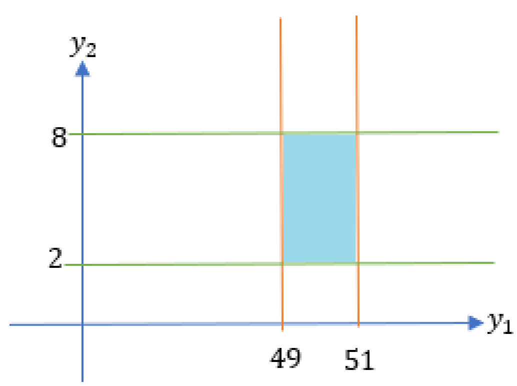
(a) Compute the variance of Y_1 and Y_2 respectively.

(b) What is the expectation of $Y_1 - Y_2$

(c). Find the covariance of Y_1 and Y_2

(d). Are Y_1 and Y_2 independent?

Solution: The domain of the density function is given by



(a). We first find the two marginal densities in the following

$$f_2(y_2) = \int_{49}^{51} \left(\frac{1}{12}\right) dy_1 = \left(\frac{1}{12}\right) \times (51 - 49) = \frac{1}{6}, \qquad \text{for } 2 < \ y_2 < 8 \ .$$

$$f_1(y_1) = \int_{2}^{8} \left(\frac{1}{12}\right) dy_2 = \left(\frac{1}{12}\right) \times (8 - 2) = \frac{1}{2}, \qquad \text{for } 49 < \ y_1 < 51 \ .$$

Therefore, both marginal distributions are uniform [We learn this continuous distribution in Chapter 4]. Recall that the for a uniform random variable X defined on interval [a, b], the expectation and variance is given by

$$E[X] \ = \ (b \ + \ a)/2 \text{ and } V[X] \ = \ (b - a)^2/12$$

$$V[Y_2] = \frac{(8-2)^2}{12} = 3$$

$$V[Y_1] = \frac{(51-49)^2}{12} = \frac{1}{3}$$

(b). We use the results in (a)

$$E[Y_1 - Y_2] = E[Y_1] - E[Y_2] = \ \frac{49+51}{2} - \frac{2+8}{2} = 50 - 5 = 45.$$

(c). Since from (a), we $f_1(y_1) \times f_2(y_2) = f_{Y_1Y_2}(y_1, y_2)$. Therefore, Y_1 and Y_2 are independent. On the other hand, independence implies uncorrelated relationship, Hence, the covariance of Y_1 and Y_2 is 0.

(d). Answered in (c).