STA 504 Homework #13

Due: Monday, December 11

Using transformation methods to find the following distributions.

Problem 1.

Let continuous random vector (X_1, X_2) have joint probability density function

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \frac{1}{4} e^{-(x_1 + x_2)/2}, 0 < x_1 < \infty, 0 < x_2 < \infty; \\ 0, \quad \text{elsewhere.} \end{cases}$$

Define $Y_1 = \frac{(X_1 - X_2)}{2}$ and $Y_2 = \frac{(X_1 + X_2)}{2}$. Find the joint probability distribution of Y_1 and Y_2.

Solution: $Y_1 = f(X_1, X_2) = \frac{(X_1 - X_2)}{2}$ and $Y_2 = g(X_1, X_2) = \frac{(X_1 + X_2)}{2}$. Therefore,

$$x_1 = f^{-1}(y_1, y_2) = y_1 + y_2 \text{ and } x_2 = g^{-1}(y_1, y_2) = y_2 - y_1$$

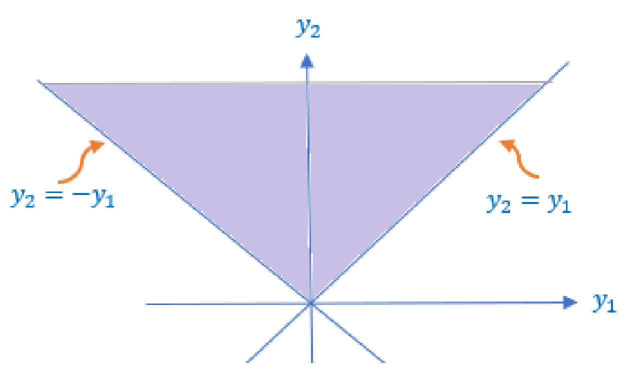
This means $0 < y_1 + y_2 < \infty$ and $0 < y_2 - y_1 < \infty$.

$$|J| = \begin{vmatrix} \frac{\partial f^{-1}(y_1, y_2)}{\partial y_1} & \frac{\partial f^{-1}(y_1, y_2)}{\partial y_2} \\ \frac{\partial g^{-1}(y_1, y_2)}{\partial y_1} & \frac{\partial g^{-1}(y_1, y_2)}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{4} e^{-(y_1 + y_2 + y_2 - y_1)/2} |2|, 0 < y_1 + y_2 < \infty, 0 < y_2 - y_1 < \infty; \\ 0, \quad \text{elsewhere.} \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-y_2}, 0 < y_1 + y_2 < \infty, 0 < y_2 - y_1 < \infty; \\ 0, \quad \text{elsewhere.} \end{cases}$$

The domain of $f_{Y_1,Y_2}(y_1, y_2)$ given by



Problem 2.

Let continuous random vector (X_1, X_2) have the joint probability density function

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 10x_1x_2^2, 0 < x_1 < x_2 < 1; \\ 0, \quad \text{elsewhere.} \end{cases}$$

Let $Y_1 = X_1/X_2$ and $Y_2 = X_2$. Find the joint probability distribution of Y_1 and Y_2.

Solution: $Y_1 = f(X_1, X_2) = X_1/X_2$ and $Y_2 = g(X_1, X_2) = X_2$. Therefore,

$$x_1 = f^{-1}(y_1, y_2) = y_1y_2 \text{ and } x_2 = g^{-1}(y_1, y_2) = y_2$$

This means $0 < y_1 < 1$ and $0 < y_2 < 1$.

$$|J| = \begin{vmatrix} \frac{\partial f^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial f^{-1}(y_1,y_2)}{\partial y_2} \\ \frac{\partial g^{-1}(y_1,y_2)}{\partial y_1} & \frac{\partial g^{-1}(y_1,y_2)}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 10y_1y_2y_2^2\,|y_2|, 0 < y_1 < 1, 0 < y_2 < 1; \\ 0, \quad \text{elsewhere.} \end{cases}$$

$$= \begin{cases} 10y_1y_2^4, 0 < y_1 < 1, 0 < y_2 < 1; \\ 0, \quad \text{elsewhere.} \end{cases}$$

The domain if the above density function is given by