STA 504 Mathematical Statistics with Calculus Review

Midterm Exam #2

11/25/2024

Due: **9:00 AM, 11/26/2024** (Tuesday)

Please Print: ______________ ________________
(First Name) (Last Name)

Instructions

- This is an open-book test. Textbooks and notes can be used. However, you must complete this exam independently. All forms of collaboration are NOT allowed.
- Please show your detailed work to earn full credit.
- Partial credit will be granted to the key steps that reflect your correct reasoning even if your numerical answer is incorrect.

Take-home Exam Honor Code

I understand that academic integrity is fundamental to the value of my education and the educational institution. By completing this take-home exam, I pledge to uphold the following:

- **Honesty**: I will complete the exam independently, without seeking unauthorized help from others.
- **Integrity**: I will not share or distribute any part of the exam or its answers to any other students, either during or after the exam.
- **Responsibility**: I will adhere to all instructions, deadlines, and guidelines set for the exam.
- **Respect**: I understand that any violation of the Honor Code will be taken seriously and may result in academic consequences, as outlined by the WCU's policies.

By submitting this exam, I affirm that I have adhered to the principles outlined in this Honor Code.

Your Signature: ______________________________

Problem 1.

Consider two discrete random variables X and Y whose values are r and s respectively and suppose that the joint probability distribution is given by:

		Y		s	$\rightarrow$		
		0	1	2	3		
X	0	$\frac{0}{48}$	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{6}{48}$	
	1	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{10}{48}$	$P(X = r)$
r	2	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{14}{48}$	$\downarrow$
$\downarrow$	3	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{6}{48}$	$\frac{18}{48}$	
		$\frac{6}{48}$	$\frac{10}{48}$	$\frac{14}{48}$	$\frac{18}{48}$		
		$P(Y = s)$			$\rightarrow$		

Answer the following questions based on the above distribution table.

1. Are X and Y independent?

Solution. Since $P[X = 0, Y = 0] = \frac{0}{48} \neq \left(\frac{6}{48}\right) \times \left(\frac{6}{48}\right) = P[X = 0] \times P[Y = 0]$, X and Y are dependent.

2. $E[X + Y]$

Solution. Since the two marginal distributions are identical, $E[X + Y] = E[X] + E[Y] = 2E[X]$.
We only find $E[X] = 0 \times \left(\frac{6}{48}\right) + 1 \times \left(\frac{10}{48}\right) + 2 \times \left(\frac{14}{48}\right) + 3 \times \left(\frac{18}{48}\right) = \frac{92}{48}$.
Therefore, $E[X + Y] = 2 \times \left(\frac{92}{48}\right) = \frac{23}{6}$

3. $E[XY]$

Solution. $E[XY] = (1 \times 1)\left(\frac{2}{48}\right) + (1 \times 2)\left(\frac{3}{48}\right) + (1 \times 3)\left(\frac{4}{48}\right) + (2 \times 1)\left(\frac{3}{48}\right) + (2 \times 2)\left(\frac{4}{48}\right) + (2 \times 3)\left(\frac{5}{48}\right) + (3 \times 1)\left(\frac{4}{48}\right) + (3 \times 2)\left(\frac{5}{48}\right) + (3 \times 3)\left(\frac{6}{48}\right) = \frac{168}{48} = 3.5.$

4. $\mathrm{COV}(X, Y)$

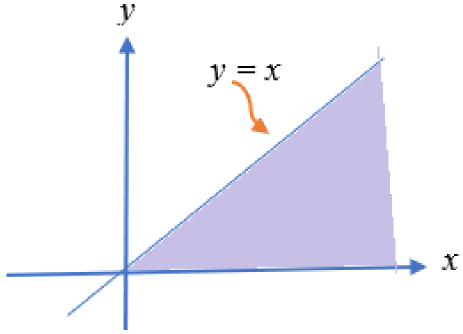
Solution. $COV(X, Y) = E[XY] - E[X]E[Y] = \frac{168}{48} - \frac{92}{48} \times \frac{92}{48} = -\frac{25}{144}$.

Problem 2.

Let X be the total time that a customer spends at a bank, and Y the time she spends waiting in line. Assume that X and Y have joint density

$$f(x,y) = \begin{cases} 4\mathrm{e}^{-2\mathrm{x}}, & 0 \le y \le x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Sketch the domain or related regions whenever appropriate.



1. Find the marginal density functions of X and Y.

   Solution: $f(x) = \int_0^x f(x,y)dy = \int_0^x 4\mathrm{e}^{-2\mathrm{x}}dy = 4x\mathrm{e}^{-2\mathrm{x}}$.

   $$f(y) = \int_y^\infty f(x,y)dx = \int_y^\infty 2^2\mathrm{e}^{-2\mathrm{x}}dx = -2 \times \mathrm{e}^{-2\mathrm{x}}\big|_y^\infty = 2\mathrm{e}^{-2y}$$

   Recall that the gamma density function

   $$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-\beta x}$$

   That is,

   $$\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-\beta x}\, dx = 1.$$

   This means that

   $$f(x) = 4x\mathrm{e}^{-2\mathrm{x}} = \frac{2^2}{\Gamma(2)} x^{2-1}\mathrm{e}^{-2x}$$

   is a gama distribution with $\beta = 2$ and $\alpha = 2$.

   $$f(y) = 2\mathrm{e}^{-2y} = \frac{2^1}{\Gamma(1)} x^{1-1}\mathrm{e}^{-2x}$$

   is the gama distribution with $\beta = 2$ and $\alpha = 1$.

2. Are X and Y independent?

   Solution: Since $f(x,y) = 4\mathrm{e}^{-2\mathrm{x}} \ne 4x\mathrm{e}^{-2\mathrm{x}} \times 2\mathrm{e}^{-2\mathrm{x}}$, random variables X and Y are dependent.

3. Find out the mean service time: $E[X - Y]$.

Solution: $E[X-Y]=\iint_D (x-y)f(x,y)dA=\int_0^\infty \int_0^x (x-y)4\mathrm{e}^{-2x}dydx$

$$=\int_0^\infty \int_0^x 4x\mathrm{e}^{-2x}dydx-\int_0^\infty \int_0^x 4y\mathrm{e}^{-2x}dydx$$

$$=\int_0^\infty 4x^2\mathrm{e}^{-2x}dx-0.5\int_0^\infty 4x^2\mathrm{e}^{-2x}dx$$

$$=0.5\int_0^\infty 4x^2\mathrm{e}^{-2x}dx$$

Note that

$$4x^2\mathrm{e}^{-2x}=\frac{2^3}{\Gamma(3)}x^{3-1}\mathrm{e}^{-2x} \text{ is the gamma density with } \beta=2 \text{ and } \alpha=3.$$

That is,

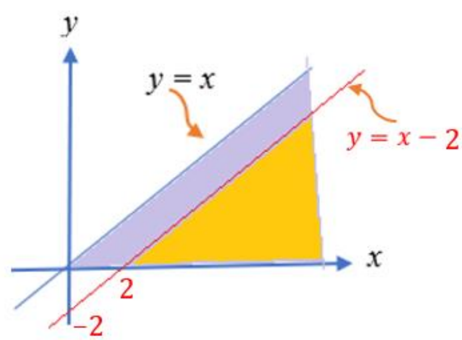
$$\int_0^\infty 4x^2\mathrm{e}^{-2x}\,dx=1.$$

Therefore,

$$E[X-Y]=0.5\int_0^\infty 4x^2\mathrm{e}^{-2x}dx=0.5.$$

4. Find the probability $P[X-Y\ >\ 2]$

The yellow region in the following figure is defined by $X-Y\ >\ 2$.



$$P[X-Y>\lambda]=\iint_{\text{orange}} f(x,y)dA$$

$$=\int_2^\infty \left[\int_0^{x-2} 4\mathrm{e}^{-2x}dy\right]dx=\int_2^\infty 4\mathrm{e}^{-2x}\left[\int_0^{x-2} dy\right]dx$$

$$=\int_2^\infty 4(x-2)\mathrm{e}^{-2x}\,dx=\int_2^\infty 4x\mathrm{e}^{-2x}\,dx-\int_2^\infty 8\mathrm{e}^{-2x}\,dx$$

Note that

$$\int_2^\infty 4x\mathrm{e}^{-2x}\,dx=-\int_2^\infty 2x\mathrm{e}^{-2x}\,d(-2x)=-\int_2^\infty 2x\,d\mathrm{e}^{-2x}$$

$$=-2x\mathrm{e}^{-2x}|_2^\infty+\int_2^\infty \mathrm{e}^{-2x}\,d2x=4\mathrm{e}^{-2^2}-\mathrm{e}^{-2x}|_{\lambda 2}^\infty=4\mathrm{e}^{-4}+\mathrm{e}^{-4}=5\mathrm{e}^{-4}$$

$$\int_2^\infty 8e^{-2x}\,dx = -4\int_2^\infty e^{-2x}\,d(-2x) = -4e^{-2x}|_2^\infty = 4e^{-4}$$

$$P[X-Y>2] = 5e^{-2} - 4e^{-2} = e^{-4}.$$

5. Find the variance of $X - Y$.

Solution: Since $COV(X,Y) = E[XY] - E[X]E[Y]$. We first calculate the involved moments in the following using the marginal density functions derived in part 1 and the joint density as well.

$$\begin{aligned} E[X] &= \textstyle\int_0^\infty x4e^{-2x}\,dx = -2\int_0^\infty x^2 de^{-2x} \\ &= -2e^{-2x}|_0^\infty + 2\int_0^\infty 2xe^{-2x}dx \\ &= \textstyle\int_0^\infty 4xe^{-2x}dx = 1. \text{ Since } 4xe^{-2x} \text{ is the gamma density function.} \end{aligned}$$

$$\begin{aligned} E[Y] &= \textstyle\int_0^\infty y2e^{-2y}\,dy = -\int_0^\infty yde^{-2y} \\ &= -ye^{-2x}|_0^\infty + \int_0^\infty e^{-2y}\,dy = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E[XY] &= \textstyle\iint xyf(x,y)dA = \int_0^\infty \int_0^x xy\,4e^{-2x}dy\,dx \\ &= \int_0^\infty x4e^{-2x} \times \left.\frac{y^2}{2}\right|_0^x dx \\ &= \textstyle\frac{1}{2}\int_0^\infty 4x^3e^{-2x}\,dx = 2\int_0^\infty x^3e^{-2x}\,dx \end{aligned}$$

Note that the above integrand x^3e^{-2x} can be rewritten as

$$x^3e^{-2x} = \left(\frac{\Gamma(4)}{2^4}\right)\left(\frac{2^4}{\Gamma(4)}\right)x^{4-1}e^{-2x}.$$

Therefore,

$$E[XY] = 2\int_0^\infty x^3e^{-2x}\,\mathrm{dx} = 2\left(\frac{\Gamma(4)}{2^4}\right)\int_0^\infty \left(\frac{2^4}{\Gamma(4)}\right)x^{4-1}e^{-2x}\,dx$$

$$= 2 \times \frac{\Gamma(4)}{2^4} \times 1 = \frac{3\times 2\times 1}{2^3} = \frac{3}{4}$$

That is,

$$COV(X,Y) = E[XY] - E[X]E[Y] = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

Note that

$$f(x) = 4xe^{-2x} = \frac{2^2}{\Gamma(2)}x^{2-1}e^{-2x}$$

is a gama distribution with $\beta = 2$ and $\alpha = 2$.

$$f(y) = 2e^{-2y} = \tfrac{2^1}{\Gamma(1)}x^{1-1}e^{-2x}$$

is the gama distribution with $\beta = 2$ and $\alpha = 1$.

The variance of X and Y are given by

$$V(X) = \frac{\alpha}{\beta^2} = \frac{2}{2^2} = \frac{1}{2} \text{ and } V(X) = \frac{\alpha}{\beta^2} = \frac{1}{2^2} = \frac{1}{4}$$

Therefore,

$$V(X - Y) = V(X) + V(Y) - 2\text{cov}(X, Y) = \frac{1}{2} + \frac{1}{4} - 2\frac{1}{4} = \frac{1}{4}$$

6. Find the correlation coefficient between X and Y.

   Solution: Since the $f(x) = 4x\mathrm{e}^{-2x}$ and $f(y) = 2\mathrm{e}^{-2y}$ are both gamma densities. The variance of X and Y are given by

$$Var[X] = \frac{\alpha}{\beta^\alpha} = \frac{2-1}{2^2} = \frac{1}{4}$$

   and

$$Var[Y] = \frac{\alpha}{\beta^\alpha} = \frac{1}{2^1} = \frac{1}{2}.$$

   The correlation coefficient

$$\rho = \frac{COV(X,Y)}{\sqrt{V[X]V[Y]}} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{4} \times \frac{1}{2}}} = \frac{\sqrt{2}}{2}$$

7. Given that waiting time $Y = 2$, what is $E[X|\, Y = 2]$?

Solution: By the definition, the conditional density function of $X|Y = y$ is given by

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{4\mathrm{e}^{-2x}}{2\mathrm{e}^{-2y}} = 2\mathrm{e}^{-2(x-y)}$$

where $0 \le y \le x < \infty$.

The conditional expectation is given by

$$E[X|\, Y = 2] = \int_2^\infty 2x\mathrm{e}^{-2(x-2)}\, dx = -\int_2^\infty x\mathrm{e}^{-2(x-2)}\, d[-2x]$$

$$= -\int_2^\infty x\mathrm{e}^{-2(x-2)}\, d[-2(x-2)] = -\int_2^\infty x d\mathrm{e}^{-2(x-2)}$$

$$= -x\mathrm{e}^{-2(x-2)}\Big|_2^\infty + \int_2^\infty \mathrm{e}^{-2(x-2)} dx$$

$$= 2 - \frac{1}{2}\int_2^\infty \mathrm{e}^{-2(x-2)} d[-2(x-2)]$$

$$= 2 - \frac{1}{2}\mathrm{e}^{-2(x-2)}\Big|_2^\infty = 2 + \left[0 - (-\frac{1}{2})\right] = \frac{5}{2}.$$