

STA 504 Mathematical Statistics with Calculus Review

Midterm Exam #2

11/20/2023

Due: 9:00 AM, 11/21/2023 (Tuesday)

Please Print: _____
(First Name) (Last Name)

Instructions

- This is an open-book test. Textbooks and notes can be used. However, you must complete this exam independently. All forms of collaboration are NOT allowed.
- Please show your detailed work to earn full credit.
- Partial credit will be granted to the key steps that reflect your correct reasoning even if your numerical answer is incorrect.

Problem 1.

Consider two discrete random variables X and Y whose values are r and s respectively and suppose that the joint probability distribution is given by:

		Y				$s \rightarrow$
		0	1	2	3	
X	0	$\frac{0}{48}$	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{6}{48}$
	1	$\frac{1}{48}$	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{10}{48}$
	2	$\frac{2}{48}$	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{14}{48}$
	3	$\frac{3}{48}$	$\frac{4}{48}$	$\frac{5}{48}$	$\frac{6}{48}$	$\frac{18}{48}$
r						$P(X = r)$
\downarrow						\downarrow
		$\frac{6}{48}$	$\frac{10}{48}$	$\frac{14}{48}$	$\frac{18}{48}$	
		$P(Y = s)$				\rightarrow

Answer the following questions based on the above distribution table.

1. Are X and Y independent?

Solution. Since $P[X = 0, Y = 0] = \frac{0}{48} \neq \left(\frac{6}{48}\right) \times \left(\frac{6}{48}\right) = P[X = 0] \times P[Y = 0]$, X and Y are dependent.

2. $E[X + Y]$

Solution. Since the two marginal distributions are identical, $E[X + Y] = E[X] + E[Y] = 2E[X]$. We only find $E[X] = 0 \times \left(\frac{6}{48}\right) + 1 \times \left(\frac{10}{48}\right) + 2 \times \left(\frac{14}{48}\right) + 3 \times \left(\frac{18}{48}\right) = \frac{92}{48}$. Therefore, $E[X + Y] = 2 \times \left(\frac{92}{48}\right) = \frac{23}{6}$.

3. $E[XY]$

Solution. $E[XY] = (1 \times 1) \left(\frac{2}{48}\right) + (1 \times 2) \left(\frac{3}{48}\right) + (1 \times 3) \left(\frac{4}{48}\right) + (2 \times 1) \left(\frac{3}{48}\right) + (2 \times 2) \left(\frac{4}{48}\right) + (2 \times 3) \left(\frac{5}{48}\right) + (3 \times 1) \left(\frac{4}{48}\right) + (3 \times 2) \left(\frac{5}{48}\right) + (3 \times 3) \left(\frac{6}{48}\right) = \frac{168}{48} = 3.5$.

4. $\text{COV}(X, Y)$

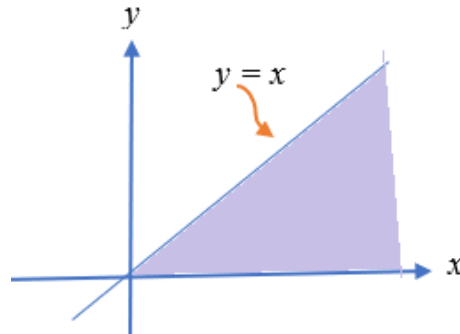
Solution. $\text{COV}(X, Y) = E[XY] - E[X]E[Y] = \frac{168}{48} - \frac{92}{48} \times \frac{92}{48} = -\frac{25}{144}$.

Problem 2.

Let X be the total time that a customer spends at a bank, and Y the time she spends waiting in line. Assume that X and Y have joint density

$$f(x, y) = \begin{cases} 4e^{-2x}, & 0 \leq y \leq x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Sketch the domain or related regions whenever appropriate.



1. Find the marginal density functions of X and Y .

Solution: $f(x) = \int_0^x f(x, y) dy = \int_0^x 4e^{-2x} dy = 4xe^{-2x}.$

$$f(y) = \int_y^\infty f(x, y) dx = \int_y^\infty 2^2 e^{-2x} dx = -2 \times e^{-2x} \Big|_y^\infty = 2e^{-2y}$$

Recall that the gamma density function

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

That is,

$$\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = 1.$$

This means that

$$f(x) = 4xe^{-2x} = \frac{2^2}{\Gamma(2)} x^{2-1} e^{-2x}$$

is a gamma distribution with $\beta = 2$ and $\alpha = 2$.

$$f(y) = 2e^{-2y} = \frac{2^1}{\Gamma(1)} x^{1-1} e^{-2x}$$

is the gamma distribution with $\beta = 2$ and $\alpha = 1$.

2. Are X and Y independent?

Solution: Since $f(x, y) = 4e^{-2x} \neq 4xe^{-2x} \times 2e^{-2x}$, random variables X and Y are dependent.

3. Find out the mean service time: $E[X - Y]$.

$$\begin{aligned}
 \text{Solution: } E[X - Y] &= \iint_D (x - y)f(x, y)dA = \int_0^\infty \int_0^x (x - y)4e^{-2x}dydx \\
 &= \int_0^\infty \int_0^x 4xe^{-2x}dydx - \int_0^\infty \int_0^x 4ye^{-2x}dydx \\
 &= \int_0^\infty 4x^2e^{-2x}dx - 0.5 \int_0^\infty 4x^2e^{-2x}dx \\
 &= 0.5 \int_0^\infty 4x^2e^{-2x}dx
 \end{aligned}$$

Note that

$$4x^2e^{-2x} = \frac{2^3}{\Gamma(3)} x^{3-1}e^{-2x} \text{ is the gamma density with } \beta = 2 \text{ and } \alpha = 3.$$

That is,

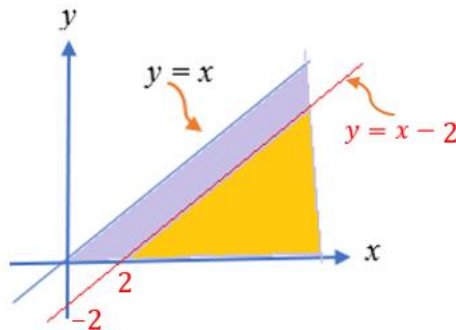
$$\int_0^\infty 4x^2e^{-2x} dx = 1.$$

Therefore,

$$E[X - Y] = 0.5 \int_0^\infty 4x^2e^{-2x}dx = 0.5.$$

4. Find the probability $P[X - Y > 2]$

The yellow region in the following figure is defined by $X - Y > 2$.



$$\begin{aligned}
 P[X - Y > \lambda] &= \iint_{\text{orange}} f(x, y)dA \\
 &= \int_2^\infty \left[\int_0^{x-2} 4e^{-2x}dy \right] dx = \int_2^\infty 4e^{-2x} \left[\int_0^{x-2} dy \right] dx \\
 &= \int_2^\infty 4(x - 2)e^{-2x} dx = \int_2^\infty 4xe^{-2x} dx - \int_2^\infty 8e^{-2x} dx
 \end{aligned}$$

Note that

$$\int_2^\infty 4xe^{-2x} dx = - \int_2^\infty 2xe^{-2x} d(-2x) = - \int_2^\infty 2x de^{-2x}$$

$$= -2xe^{-2x} \Big|_2^\infty + \int_2^\infty e^{-2x} d2x = 4e^{-2} - e^{-2x} \Big|_2^\infty = 4e^{-4} + e^{-4} = 5e^{-4}$$

$$\int_2^\infty 8e^{-2x} dx = -4 \int_2^\infty e^{-2x} d(-2x) = -4e^{-2x} \Big|_2^\infty = 4e^{-4}$$

$$P[X - Y > 2] = 5e^{-2} - 4e^{-2} = e^{-4}.$$

5. Find the variance of $X - Y$.

Solution: Since $COV(X, Y) = E[XY] - E[X]E[Y]$. We first calculate the involved moments in the following using the marginal density functions derived in part 1 and the joint density as well.

$$\begin{aligned} E[X] &= \int_0^\infty x4e^{-2x} dx = -2 \int_0^\infty x^2 de^{-2x} \\ &= -2e^{-2x} \Big|_0^\infty + 2 \int_0^\infty 2xe^{-2x} dx \\ &= \int_0^\infty 4xe^{-2x} dx = 1. \text{ Since } 4xe^{-2x} \text{ is the gamma density function.} \end{aligned}$$

$$\begin{aligned} E[Y] &= \int_0^\infty y2e^{-2y} dy = - \int_0^\infty y de^{-2y} \\ &= -ye^{-2y} \Big|_0^\infty + \int_0^\infty e^{-2y} dy = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E[XY] &= \iint xyf(x, y) dA = \int_0^\infty \int_0^x xy 4e^{-2x} dy dx \\ &= \int_0^\infty x4e^{-2x} \times \frac{y^2}{2} \Big|_0^x dx \\ &= \frac{1}{2} \int_0^\infty 4x^3 e^{-2x} dx = 2 \int_0^\infty x^3 e^{-2x} dx \end{aligned}$$

Note that the above integrand $x^3 e^{-2x}$ can be rewritten as

$$x^3 e^{-2x} = \left(\frac{\Gamma(4)}{2^4} \right) \left(\frac{2^4}{\Gamma(4)} \right) x^{4-1} e^{-2x}.$$

Therefore,

$$\begin{aligned} E[XY] &= 2 \int_0^\infty x^3 e^{-2x} dx = 2 \left(\frac{\Gamma(4)}{2^4} \right) \int_0^\infty \left(\frac{2^4}{\Gamma(4)} \right) x^{4-1} e^{-2x} dx \\ &= 2 \times \frac{\Gamma(4)}{2^4} \times 1 = \frac{3 \times 2 \times 1}{2^3} = \frac{3}{4} \end{aligned}$$

That is,

$$COV(X, Y) = E[XY] - E[X]E[Y] = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

6. Find the correlation coefficient between X and Y .

Solution: Since the $f(x) = 4xe^{-2x}$ and $f(y) = 2e^{-2y}$ are both gamma densities. The variance of X and Y are given by

$$\text{Var}[X] = \frac{\alpha}{\beta^\alpha} = \frac{2-1}{2^2} = \frac{1}{4}$$

and

$$\text{Var}[Y] = \frac{\alpha}{\beta^\alpha} = \frac{1}{2^1} = \frac{1}{2}.$$

The correlation coefficient

$$\rho = \frac{\text{COV}(X,Y)}{\sqrt{V[X]V[Y]}} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{4} \times \frac{1}{2}}} = \frac{\sqrt{2}}{2}$$

7. Given that waiting time $Y = 2$, what is $E[X | Y = 2]$?

Solution: By the definition, the conditional density function of $X|Y = y$ is given by

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{4e^{-2x}}{2e^{-2y}} = 2e^{-2(x-y)}$$

where $0 \leq y \leq x < \infty$.

The conditional expectation is given by

$$\begin{aligned} E[X | Y = 2] &= \int_2^\infty 2xe^{-2(x-2)} dx = - \int_2^\infty xde^{-2(x-2)} \\ &= -2xe^{-2(x-2)} \Big|_2^\infty + \int_2^\infty e^{-2(x-2)} dx = 2 - \frac{1}{2}e^{-2(x-2)} \Big|_2^\infty = 2 + \frac{1}{2} = \frac{5}{2}. \end{aligned}$$