STA 504 Homework #8

Due: Tuesday, November 7

Answer the following questions based on the given joint density function in problems 1 - 4.

(i)   Find the marginal densities $f(x)$ and $f(y)$.

(ii)   Check if the variables X and Y are independent.

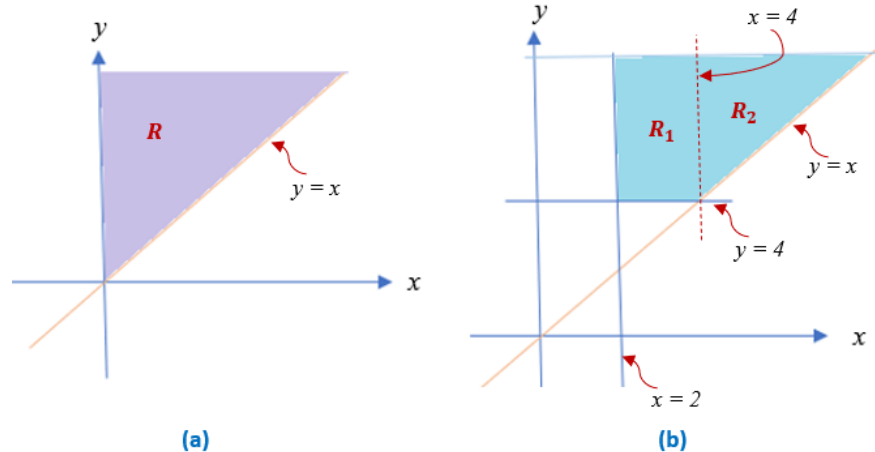(iii)   find the conditional probability distribution function of X, given $Y = y$

Problem 1.

Let the continuous random vector (X, Y) have joint probability density function

$$f(x, y) = e^{-y}, 0 < x < y < \infty.$$

In addition to the (i)-(iii), compute the $P(X \geq 2; Y \geq 4)$.

Solution: We first sketch the domain of the joint density function in the left panel (a) of the following figure. It is both type I and II region.

(a)   (b)

(i). Marginal density functions.

$$f_X(x) = \int_x^\infty e^{-y} \, dy = -e^{-y}|_x^\infty = e^{-x}$$

$$f_Y(y) = \int_0^y e^{-y} \, dx = e^{-y}x|_0^y = ye^{-y}$$

The above two densities represent exponential and gamma distributions respectively.

(ii) Since $f_X(x)f_Y(y) = ye^{-x-y} \neq e^{-y} = f(x,y)$, random variable X and Y are dependent on each other.

(iii). $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}, \quad x < y < \infty$

(iv). $P(X > 2, Y > 4)$ is the volume between the sky-blue region and the surface of the density function. However, the sky-blue region is irregular. We can partition it into two regular regions: one type I and one type II (see (b) of the above figure). Therefore,

$$P(X > 2, Y > 4) = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

$$= \int_2^4 \int_4^\infty e^{-y}\, dy\, dx + \int_4^\infty \int_4^y e^{-y}\, dx\, dy = \int_2^4 e^{-4}\, dx + \int_4^\infty (y-4)e^{-y}\, dy$$

$$= (4-2)e^{-4} - \int_4^\infty (y-4)\, de^{-y} = 2e^{-4} - (y-4)e^{-y}\big|_4^\infty + \int_4^\infty e^{-y}\, dy$$

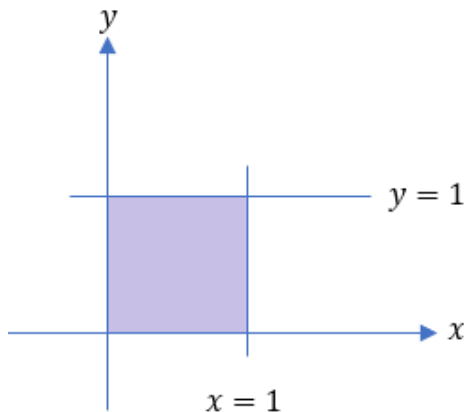$$= 2e^{-4} - e^{-y}\big|_4^\infty = 2e^{-4} + (0 - e^{-4}) = 2e^{-4}$$

Problem 2.

Let X and Y be a random vector with joint probability density function

$$f(x,y) = \begin{cases} 6xy(2 - x - y), & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then find the conditional probability distribution function of X, given $Y = y$, where $0 < y < 1$.

Solution. We first sketch the domain in the following.

(i). Marginal density functions.

$$f_X(x) = \int_0^1 6xy(2 - x - y)\,dy = \int_0^1 (12xy - 6x^2y - 6xy^2)\,dy$$
$$= (6xy^2 - 3x^2y^2 - 2xy^3)|_{y=0}^{y=1} = 6x - 3x^2 - 2x = 4x - 3x^2$$

$$f_Y(y) = \int_0^1 6xy(2 - x - y)\,dx = \int_0^1 (12xy - 6x^2y - 6xy^2)\,dx$$

$$= (6x^2y - 2x^3y - 3x^2y^2)|_{x=0}^{x=1} = 6y - 3y^2 - 2y = 4y - 3y^2$$

(ii) Since $f_X(x)f_Y(y) = (6x - 3x^2 - 2x)(6y - 3y^2 - 2y) \neq 6xy(2 - x - y) = f(x,y)$, random variable X and Y are dependent on each other.
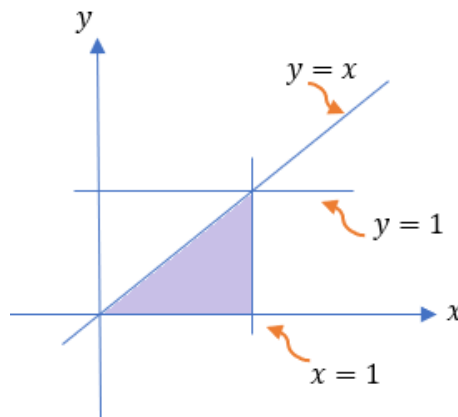
(iii). $f(x|y) = \dfrac{f(x,y)}{f(y)} = \dfrac{6xy(2 - x - y)}{4y - 3y^2} = \dfrac{6x(2 - x - y)}{4 - 3y}, \quad x < y < \infty$

Problem 3.

Let *X* and *Y* be a random vector with joint probability density function.
$$f(x,y) = \begin{cases} 1/x, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution. The domain of the density is sketched below.

(i). Marginal density functions.

$$f_X(x) = \int_0^x \frac{1}{x} dy = \frac{1}{x} \times y\big|_{y=0}^{y=x} = 1$$

$$f_Y(y) = \int_y^1 \frac{1}{x} dx = ln(x)\big|_y^1 = -\ln(y)$$

(ii). Since $f_X(x)f_Y(y) = 1(-\ln(y)) \neq 1/x = f(x,y)$, random variable X and Y are dependent on each other.

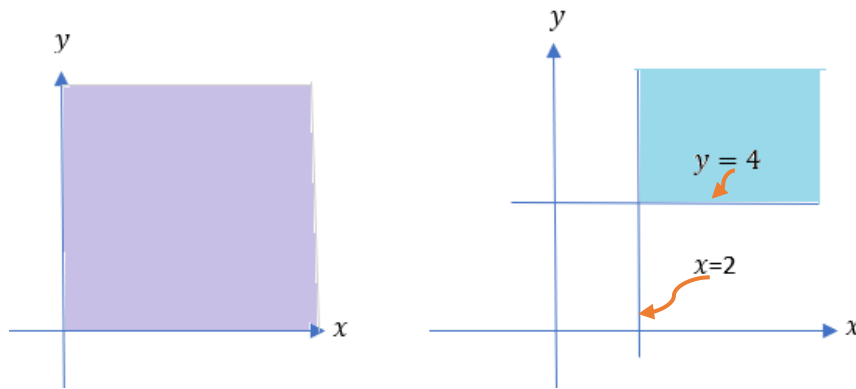(iii). $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{1/x}{-\ln(y)} = -\frac{1}{-x\ln(y)}, \quad 0 < y < x < 1.$

Problem 4.

Let X and Y have density

$$f(x,y) = \begin{cases} xe^{-x(1+y)}, & \text{if } y > 0, x > 0 \\ 0, & \text{otherwise} \end{cases}$$

In addition to the (i)-(iii), compute the $P(X \geq 2; Y \geq 4)$.

Solution: We sketch the domain in the following.

(i). Marginal density functions.

$$f_X(x) = \int_0^\infty xe^{-x(1+y)} \, dy = -\int_0^\infty xe^{-x(1+y)} \, d[-x(1+y)] = e^{-x(1+y)}\big|_{y=0}^{y=\infty} = e^{-x}$$

$$f_Y(y) = \int_0^\infty xe^{-x(1+y)}\,dx = -\frac{1}{(1+y)}\int_0^\infty xe^{-x(1+y)}\,d[-x(1+y)]$$

$$= -\frac{1}{(1+y)}\int_0^\infty x\,de^{-x(1+y)} = -\frac{1}{(1+y)}\left[xe^{-x(1+y)}\Big|_{x=0}^{x=\infty} - \int_0^\infty e^{-x(1+y)}\,dx\right]$$

$$= \frac{1}{(1+y)}\left[-\frac{1}{(1+y)}\int_0^\infty e^{-x(1+y)}\,d(-x(1+y))\right] = -\frac{1}{(1+y)^2}e^{-x(1+y)}\Big|_{x=0}^{x=\infty} = \frac{1}{(1+y)^2}$$

(ii). Since $f_X(x)f_Y(y) = e^{-x}\frac{1}{(1+y)^2} \neq xe^{-x(1+y)} = f(x,y)$, random variable X and Y are dependent on each other.

(iii). $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{xe^{-x(1+y)}}{\frac{1}{(1+y)^2}} = x(1+y)^2 e^{-x(1+y)}, \quad 0 < y < x < 1.$

(iv). $P(X > 2, Y > 4)$ is the volume between the sky-blue region and the surface of the density function. However, the sky-blue region is irregular. We can partition it into two regular regions: one type I and one type II (see (b) of the above figure). Therefore,

$$P(X > 2, Y > 4) = \iint_{R_1} f(x,y)dA = \int_2^\infty \int_4^\infty xe^{-x(1+y)}\,dy\,dx$$

$$= \int_2^\infty \int_4^\infty e^{-x(1+y)}\,dx(1+y)\,dx = -\int_2^\infty e^{-x(1+y)}\Big|_{y=4}^{y=\infty}\,dx$$

$$= \int_2^\infty e^{-5x}\,dx = -\frac{e^{-5x}}{5}\Big|_{x=2}^{x=\infty} = \frac{e^{-10}}{5}$$