STA 504 Homework #3

Due: Monday, September 25

Show you're your work to earn full credit. You are encouraged to work with your peers on assignments. The write-up must be your own.

Note: Following questions are based on the definition of continuous random variables and their distribution functions [probability density function (pdf) and cumulative distribution function (CDF)]; conditions for a given function to be a valid pdf; the relationship between pdf and CDF, etc. You are expected to have a clear understanding of these basic concepts.

4.8 Suppose that Y has density function

$$f(y) = \begin{cases} ky(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of k that makes $f(y)$ a probability density function.

b Find $P(.4 \le Y \le 1)$.

c Find $P(.4 \le Y < 1)$.

d Find $P(Y \le .4 | Y \le .8)$.

e Find $P(Y < .4 | Y < .8)$.

Solution

Please keep in mind the following relationship between CDF and probability.

P(X < a) = F(a), P(X < a) = 1 – F(a), P(d < X < c) = F(c) – F(d)

a. The constant k = 6 is required so the density function integrates to 1.

b. $P(.4 \le Y \le 1) = .648$.

c. Same as part b. above.

d. $P(Y \le .4 \mid Y \le .8) = P(Y \le .4)/P(Y \le .8) = .352/.896 = 0.393$.'

e. Same as part **d.** above.

4.17 The length of time required by students to complete a one-hour exam is a random variable with a density function given by

$$f(y) = \begin{cases} cy^2 + y, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find c.

b Find $F(y)$.

c Use $F(y)$ in part (b) to find $F(-1)$, $F(0)$, and $F(1)$.

d Find the probability that a randomly selected student will finish in less than half an hour.

e Given that a particular student needs at least 15 minutes to complete the exam, find the probability that she will require at least 30 minutes to finish.

Suggested Solution

a. $\int_0^1 (cy^2 + y)dy = \left[cy^3/3 + y^2/2\right]_0^2 = 1$, $c = 3/2$.

b. $F(y) = y^3/2 + y^2/2$ for $0 \le y \le 1$.

c. $F(-1) = 0$, $F(0) = 0$, $F(1) = 1$.

d. $P(Y < .5) = F(.5) = 3/16$.

e. $P(Y \ge .5 \mid Y \ge .25) = P(Y \ge .5)/P(Y \ge .25) = 104/123$.

4.21 If, as in Exercise 4.17, Y has density function

$$f(y) = \begin{cases} (3/2)y^2 + y, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of Y.

Suggested Solution

$$E(Y) = \int_0^1 1.5y^3 + y^2)dy = \tfrac{3y^4}{8} + \tfrac{y^3}{3}\Big]_0^1 = 17/24 = .708.$$

$$E(Y^2) = \int_0^1 1.5y^4 + y^3)dy = \tfrac{3y^5}{10} + \tfrac{y^4}{4}\Big]_0^1 = 3/10 + 1/4 = .55.$$

So, $V(Y) = .55 - (.708)2 = .0487$.

4.32 Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} (3/64)y^2(4 - y), & 0 \le y \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the expected value and variance of weekly CPU time.

b The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

c Would you expect the weekly cost to exceed \$600 very often? Why?

Suggested Solution

a. $E(Y)=\frac{3}{64}\int_0^4 y^3(4-y)dy=\frac{3}{64}\left[y^4-\frac{y^5}{5}\right]_0^4=2.4.$ V(Y) = .64.

b. $E(200Y) = 200(2.4) = \$480$, $V(200Y) = 200^2(.64) = 25{,}600$.

c. $P(200Y > 600) = P(Y > 3) = \frac{3}{64}\int_3^4 y^2(4-y)dy=.2616$, or about 26% of the time the cost will exceed \$600 (fairly common).

4.47 The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. The delivery time, Y, is uniformly distributed on the interval one to five days. The cost of a board failure and interruption includes the fixed cost c_0 of a new board and a cost that increases proportionally to Y^2. If C is the cost incurred, $C = c_0 + c_1Y^2$.

a Find the probability that the delivery time exceeds two days.

b In terms of c_0 and c_1, find the expected cost associated with a single failed circuit board.

Suggested Solutions

The density for Y = delivery time is $f(x) = 1/4$, $1 \le y \le 5$. Also, $E(Y) = 3$, $V(Y) = 4/3$.

a. $P(Y > 2) = 3/4$.

b. $E(C) = E(c_0 + c_1Y^2) = c_0 + c_1E(Y^2) = c_0 + c_1[V(Y) + (E(Y))^2] = c_0 + c_1[4/3 + 9]$

4.48 Beginning at 12:00 midnight, a computer center is up for one hour and then down for two hours on a regular cycle. A person who is unaware of this schedule dials the center at a random time between 12:00 midnight and 5:00 A.M. What is the probability that the center is up when the person's call comes in?

Suggested Solutions

Let Y = time when the phone call comes in. Then, Y has a uniform distribution on the interval (0, 5) with density function

$$f(x) = \begin{cases} \frac{1}{5}, & x \in [0,5] \\ 0, & otherwise \end{cases}.$$

The probability is $P(0 < Y < 1) + P(3 < Y < 4) = (1/5)(1-0) + (1/5)(4-3) = 2/5$.