Order Statistics

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1 Introduction and Motivation

This note discusses the method of transformation for finding the probability distributions of functions of random variables in both univariate and multivariate cases. **Section 7 of chapter 6** in the textbook covers these topics.

A Motivational Example: Consider a n-component reliability system in which the component lifetimes $\{X_1, X_2, \cdots, X_n\}$ are exponential random variables with rate parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively. We can define the **order statistics** in the following

$X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$

$X_{(2)} =$ the 2nd smallest of $X_1, X_2, \cdots, X_n$

$X_{(3)} =$ the 3rd smallest of $X_1, X_2, \cdots, X_n$

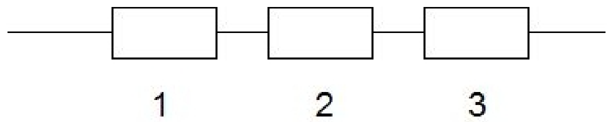
$\cdots\cdots\cdots\cdots\cdots\cdots$

$X_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$

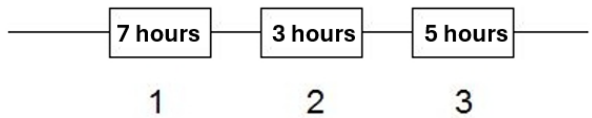
The general objective is to find the distribution of $X_{(i)}$ for $i = 1, 2, \cdots, n$. Since order statistic $X_{(i)}$ is defined on the set of all existing random variables $\{X_1, X_2, \cdots, X_n\}$, So $X_{(i)}$ is a function of $\{X_1, X_2, \cdots, X_n\}$. In this note, we will discuss some special order statistics.

2 Distribution of Minimum Statistic $X_{(1)}$

In a reliability system, a series system needs all of its components to function for the system itself to be functional. Assuming the serial system has n independent components with corresponding lifetimes $\{X_1, X_2, \cdots, X_n\}$. In this situation, the lifetime of a serial system is $X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$. We next derive the distribution of $X_{(1)}$ so we can calculate the mean, variance, and other numeric measures of $X_{(1)}$.



For example, we add the lifetime of each component in the above serial system in the following



The lifetime of the system is

$$\text{system lifetime} = \min\{7, 3, 5\} = 4.$$

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Next, we use an example to show how to characterize the distribution of the statistic $X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$ where the independent component $\{X_1, X_2, \cdots, X_n\} \to f_i(x)$.

Example 1 Consider an **independent** n-component series system in which the component lifetimes $\{X_1, X_2, \cdots, X_n\}$ are exponential random variables with rate parameters rates λ_i for $i = 1, 2, \cdots, \lambda_n$. Let Y denote the lifetime that the system fails. What is the distribution of Y?

Solution: Since the density function of i-th component's lifetime is given by

$$f_i(x) = \lambda_i e^{-\lambda_i x}, \quad \text{for} \quad x > 0.$$

Its CDF is given by

$$F_i(x) = 1 - e^{-\lambda_i x}.$$

Using the CDF method, we derive the distribution of $Y_{(1)}$ as follows.

$$f_{Y_{(1)}}(y) = P[Y_{(1)} \le y] = P[\min\{X_1, X_2, \cdots, X_n\} \le y] = 1 - P[\min\{X_1, X_2, \cdots, X_n\} > y]$$

Since the smallest lifetime is bigger than y, therefore, every X_i is greater than Y. Equivalently, event $\min\{X_1, X_2, \cdots, X_n\} > y$ is identical to $\{X_1 > y \cap X_2 > y \cap \cdots \cap X_n > y\}$. Using the assumption that the components' lifetimes are independent, we have

$$P[\min\{X_1, X_2, \cdots, X_n\} > y] = P[X_1 > y \cap X_2 > y \cap \cdots \cap X_n > y]$$

$$= P[X_1 > y] \times P[X_2 > y] \times \cdots \times P[X_n > y] = \big(1 - P[X_1 \le y]\big) \times \big(1 - P[X_2 > y]\big) \times \cdots \times \big(1 - P[X_n > y]\big)$$

$$= \big(1 - [1 - e^{-\lambda_1 y}]\big) \times \big(1 - [1 - e^{-\lambda_2 y}]\big) \times \cdots \times \big(1 - [1 - e^{-\lambda_n y}]\big) = e^{-\lambda_1 y} \times e^{-\lambda_2 y} \times \cdots \times e^{-\lambda_n y} = e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n) y}.$$

Hence, the CDF of $Y_{(1)}$ is given by

$$F_{Y_{(1)}}(y) = 1 - e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)y}.$$

The corresponding PDF is given by

$$f_{Y_{(1)}}(y) = (\lambda_1 + \lambda_2 + \cdots + \lambda_n)e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)y}.$$

We can see that the minimum statistics is also an exponential distribution with rate $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Example 2: We now look at a numerical example. Consider a 3-component **series system**: where each component has an exponential lifetime with rates $0.2, 0.3$, and 0.5, respectively. Find the probability that the system fails in one unit of time.

Solution: From the result of the above example, the time to failure of the series system is $Y_{(1)}$ that has distribution

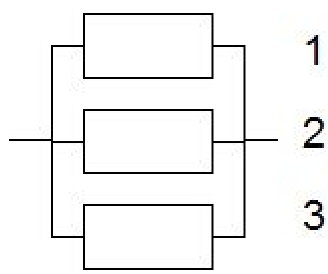
$$F_{Y_{(1)}}(y) = 1 - e^{-(0.1+0.2+0.5)y} = 1 - e^{-y}.$$

The probability that the system fails in one unit of time is given by

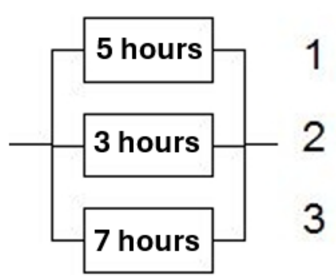
$$P(Y_{(1)} < 1) = 1 - e^{-1} \approx 0.632.$$

3 Distribution of Minimum Statistic $X_{(m)}$

We also use a reliability system as an example. A parallel system is one that *needs only one of its components to function in order for the system itself to be functional* . Assuming the parallel system has n independent components with corresponding lifetimes $\{X_1, X_2, \cdots, X_n\}$. In this situation, the lifetime of a serial system is $X_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$. We next derive the distribution of $X_{(n)}$ so we can calculate the mean, variance, and other numeric measures of $X_{(n)}$.



For ease of illustration, we assign a lifetime to each individual component in the following.



The lifetime of the system is

$$\text{system lifetime} = \max\{7, 3, 5\} = 7.$$

The following example demonstrates how to find the distribution of the system lifetime based on a given liftime distributions of the individual components in the parallel system.

Example 3 Consider an **independent** n-component **parallel** system in which the component lifetimes $\{X_1, X_2, \cdots, X_n\}$ are exponential random variables with rate parameters rates λ_i for $i = 1, 2, \cdots, \lambda_n$. Let Y denote the lifetime that the system fails. What is the distribution of Y?

Solution: Since the density function of i-th component's lifetime is given by

$$f_i(x) = \lambda_i e^{-\lambda_i x}, \quad \text{for} \quad x > 0.$$

Its CDF is given by

$$F_i(x) = 1 - e^{-\lambda_i x}.$$

Using the CDF method, we derive the distribution of $Y_{(1)}$ as follows.

$$f_{Y_{(n)}}(y) = P[Y_{(n)} \le y] = P[\max\{X_1, X_2, \cdots, X_n\} \le y]$$

Since the largest lifetime is less than y, therefore, every X_i is less than Y. Equivalently, event $\max\{X_1, X_2, \cdots, X_n\} \le y$ is identical to $\{X_1 \le y \cap X_2 \le y \cap \cdots \cap X_n \le y\}$. Using the assumption that the components' lifetimes are independent, we have

$$P[\max\{X_1, X_2, \cdots, X_n\} \le y] = P[X_1 \le y \cap X_2 \le y \cap \cdots \cap X_n \le y]$$

$$= P[X_1 \le y] \times P[X_2 \le y] \times \cdots \times P[X_n \le y] = [1 - e^{-\lambda_1 y}] \times [1 - e^{-\lambda_2 y}] \times \cdots \times [1 - e^{-\lambda_n y}].$$

Therefore,

$$F_{Y_{(n)}}(y) = [1 - e^{-\lambda_1 y}] \times [1 - e^{-\lambda_2 y}] \times \cdots \times [1 - e^{-\lambda_n y}].$$

Example 4: We now modify the system we discussed in **Example 2**. Consider a 3-component **parallel system**: where each component has an exponential lifetime with rates $0.2, 0.3$, and 0.5, respectively. Find the probability that the system fails in one unit of time.

Solution: We use the CDF derived in the above **example 3**

$$F_{Y_{(3)}}(y) = [1 - e^{-0.2y}] \times [1 - e^{-0.3y}] \times [1 - e^{-0.5y}].$$

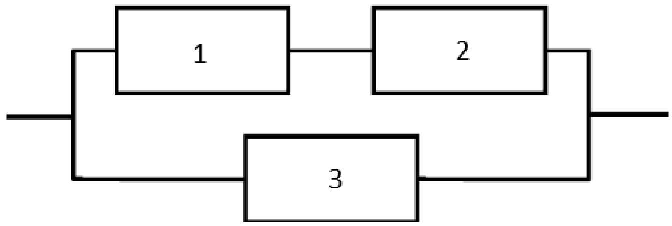
The probability that the system fails in one unit of time is given by

$$P(Y_{(1)} < 1) = [1 - e^{-0.2}] \times [1 - e^{-0.3}] \times [1 - e^{-0.5}] \approx 0.0185.$$

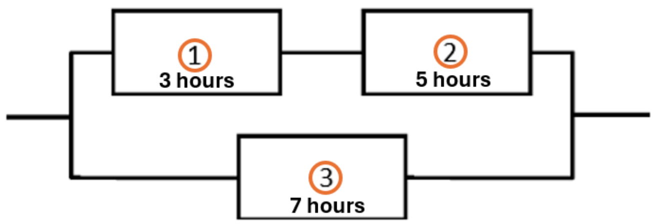
This means the probability the parallel system's lifetime is less than one unit of time is 1.85%, that much less 63.2% for a series system.

4 Combined Reliability System

We have discussed the simplest reliability systems: series and parallel systems. The actual reliability systems usually consist of combined series and parallel components. For example, the following figure depicts a system with both series and parallel components.



We can similarly assign lifetime to individual component in the above system and get



The lifetime of the hybrid system is given by

$$\text{system lifetime} = \max\{\min\{3, 5\}, 7\} = 7.$$

We now use the derived CDF in Sections 1 and 2 to derive the lifetime distribution of the combined system shown in the above system.

Example 5: Consider the above combined system. Let $\{X_1, X_2, X_3\}$ be the random variables representing lifetimes of three **independent components** in the above system respectively. Assume also that all three systems follow the same exponential distributions with rates λ_1, λ_2, and λ_3, respectively. Find the probability distribution of lifetime.

Solution: First we know that the lifetime of the above combined three-component system $Y = \max\{\min\{X_1, X_2\}, X_3\}$ (think about why?). The CDF of Y is defined to be

$$F_Y(y) = P[Y \le y] = P[\max\{\min\{X_1, X_2\}, X_3\} \le y] = P[\min\{X_1, X_2\} \le y \cap X_3 \le y]$$

$$= P[\min\{X_1, X_2\} \le y] \times P[X_3 \le y] = (1 - P[\min\{X_1, X_2\} > y]) \times P[X_3 \le y]$$

$$= (1 - P[X_1 > y] \times P[X_2\} > y]) \times P[X_3 \le y]$$

$$= \left(1 - [1 - e^{-\lambda_1 y}] \times [1 - e^{-\lambda_2 y}]\right) \times \left(1 - \lambda_2 e^{-\lambda_3 y}\right)$$

$$= \left(e^{-\lambda_1 y} + e^{-\lambda_2 y} - e^{-(\lambda_1 + \lambda_2) y}\right)\left(1 - e^{-\lambda_3 y}\right).$$

Next, we modify examples 2 and 4 with numerical rates.

Example 6: Consider the above 3-component **combined system**: where each component has an exponential lifetime with rates $\lambda_1 = 0.2, \lambda_2 = 0.3$ and $\lambda_3 = 0.5$, respectively. Find the probability that the system fails in one unit of time.

Solution: Using the above derived CDF of the combined system, we

$$P[Y \leq 1] = \left(e^{-0.2} + e^{-0.3} - e^{-(0.2+0.3)}\right)\left(1 - e^{-0.5}\right) \approx 0.375.$$

The probability that the combined system fails in one unit of time is about 37.5%, as expected, that is between 1.85% (for the parallel system) and 63.2% (for the series system).