STA 504 Homework #3

Due: Monday, September 25

Show you're your work to earn full credit. You are encouraged to work with your peers on assignments. The write-up must be your own.

Note: The following questions are based on the definition of continuous random variables and their distribution functions [probability density function (pdf) and cumulative distribution function (CDF)]; conditions for a given function to be a valid pdf; the relationship between pdf and CDF, etc. You are expected to have a clear understanding of these basic concepts.

4.8 Suppose that Y has density function

$$f(y) = \begin{cases} ky(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of k that makes $f(y)$ a probability density function.

b Find $P(.4 \le Y \le 1)$.

c Find $P(.4 \le Y < 1)$.

d Find $P(Y \le .4 | Y \le .8)$.

e Find $P(Y < .4 | Y < .8)$.

Solution

Please keep in mind the following relationship between CDF and probability.

P(X < a) = F(a), P(X < a) = 1 – F(a), P(d < X < c) = F(c) – F(d)

a. The constant $k = 6$ is required so the density function integrates to 1.

$$1 = \int_0^1 ky(1-y)\,dy = k\int_0^1 (y-y^2)\,dy = k\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_0^1 = k\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{k}{6}$$

$$\Rightarrow k = 6$$

b. $P(.4 \le Y \le 1) = .648$.

$$P(0.4 \le Y \le 1) = \int_{0.4}^1 6(y-y^2) = 6\int_{0.4}^1 (y-y^2) = 6\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_{0.4}^1$$

$$= (3y^2-2y^3)\Big|_{0.4}^1 = 1-[3\times 0.4^2 - 2\times 0.4^3] \approx 0.648$$

c. Same as part b. above.

d. $P(Y \le .4 \mid Y \le .8) = P(Y \le .4)/P(Y \le .8) = .352/.896 = 0.393$.

$$P(Y \le 0.4 | Y \le 0.8) = \frac{P(Y \le 0.4 \cap Y \le 0.8)}{P(Y \le 0.8)} = \frac{P(Y \le 0.4)}{P(Y \le 0.8)} = \frac{F(0.4)}{F(0.8)}$$

$F(x) = \int_0^x 6(y-y^2)\,dy = 3y^2 - 2y^3.$

$F(0.4) = 3\times0.4^2 - 2\times0.4^3 = 3\times0.16 - 2\times0.064 = 0.48 - 0.128 = 0.352$

$F(0.8) = 3\times0.8^2 - 2\times0.8^3 = 3\times0.64 - 2\times0.512 = 1.92 - 1.024 = 0.896$

$P(Y \le 0.4 \mid Y \le 0.8) = \dfrac{0.352}{0.896} = 0.393.$

e. Same as part **d.** above.

4.17 The length of time required by students to complete a one-hour exam is a random variable with a density function given by

$$f(y) = \begin{cases} cy^2 + y, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find c.

b Find $F(y)$.

c Use $F(y)$ in part (b) to find $F(-1)$, $F(0)$, and $F(1)$.

d Find the probability that a randomly selected student will finish in less than half an hour.

e Given that a particular student needs at least 15 minutes to complete the exam, find the probability that she will require at least 30 minutes to finish.

Suggested Solution

Please keep in mind the following relationship between CDF and probability.

P(X < a) = F(a), P(X < a) = 1 – F(a), P(d < X < c) = F(c) – F(d)

a. $\int_0^1 (cy^2 + y)dy = \left[cy^3/3 + y^2/2\right]_0^2 = 1$, $c = 3/2$.

b. $F(y) = y^3/2 + y^2/2$ for $0 \le y \le 1$.

$F(y) = \int_0^y \left(\frac{3}{2}x^2 + x\right)dx = \left(\frac{1}{2}x^3 + \frac{x^2}{2}\right)\Big|_0^y = \frac{1}{2}y^3 + \frac{1}{2}y^2$

c. $F(-1) = 0$, $F(0) = 0$, $F(1) = 1$.

d. $P(Y < .5) = F(.5) = 3/16 = 0.1875$.

e. $P(Y \ge .5 \mid Y \ge .25) = P(Y \ge .5)/P(Y \ge .25) = 104/123 = 0.8455$.

4.21 If, as in Exercise 4.17, Y has density function

$$f(y) = \begin{cases} (3/2)y^2 + y, & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of Y.

Suggested Solution

$$E(Y) = \int_0^1 1.5y^3 + y^2)dy = \frac{3y^4}{8} + \frac{y^3}{3}\Big]_0^1 = 17/24 = .708.$$

$$E(Y) = \int_0^1 y\left(\tfrac{3}{2}y^2+y\right)dx = \int_0^1\left(\tfrac{3}{2}y^3+y^2\right)dy = \left(\tfrac{3}{2}\times\tfrac{y^4}{4}+\tfrac{y^3}{3}\right)\Big|_0^1 = \tfrac{3}{2}\times\tfrac{1}{4}+\tfrac{1}{3} = 0.708$$

$$E(Y^2) = \int_0^1 1.5y^4 + y^3)dy = \frac{3y^5}{10} + \frac{y^4}{4}\Big]_0^1 = 3/10 + 1/4 = .55.$$

So, $V(Y) = .55 - (.708)^2 = .0487$.

4.32 Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} (3/64)y^2(4-y), & 0 \le y \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the expected value and variance of weekly CPU time.

b The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

c Would you expect the weekly cost to exceed \$600 very often? Why?

Suggested Solution

a. $E(Y) = \frac{3}{64}\int_0^4 y^3(4-y)dy = \frac{3}{64}\left[y^4 - \frac{y^5}{5}\right]_0^4 = 2.4.$ V(Y) = .64.

b. $E(200Y) = 200(2.4) = \$480$, $V(200Y) = 200^2(.64) = 25{,}600$.

c. $P(200Y > 600) = P(Y > 3) = \frac{3}{64}\int_3^4 y^2(4-y)dy = .2616$, or about 26% of the time the cost will exceed \$600 (fairly common).

4.47 The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. The delivery time, Y, is uniformly distributed on the interval one to five days. The cost of a board failure and interruption includes the fixed cost c_0 of a new board and a cost that increases proportionally to Y^2. If C is the cost incurred, $C = c_0 + c_1 Y^2$.

a Find the probability that the delivery time exceeds two days.

b In terms of c_0 and c_1, find the expected cost associated with a single failed circuit board.

Suggested Solutions

The density for Y = delivery time is f(x) = 1/4, $1 \le y \le 5$. Also, $E(Y) = 3$, $V(Y) = 4/3$.

a. $P(Y > 2) = 3/4$.

b. $E(C) = E(c_0 + c_1 Y^2) = c_0 + c_1 E(Y^2) = c_0 + c_1[V(Y) + (E(Y))^2] = c_0 + c_1[4/3 + 9]$

4.48 Beginning at 12:00 midnight, a computer center is up for one hour and then down for two hours on a regular cycle. A person who is unaware of this schedule dials the center at a random time between 12:00 midnight and 5:00 A.M. What is the probability that the center is up when the person's call comes in?

Suggested Solutions

Let Y = time when the phone call comes in. Then, Y has a uniform distribution on the interval (0, 5) with a density function

$$f(x) = \begin{cases} \frac{1}{5}, & x \in [0,5] \\ 0, & otherwise \end{cases}.$$

The probability is $P(0 < Y < 1) + P(3 < Y < 4)$ = (1/5)(1-0) + (1/5)(4-3) = 2/5.