# STA 504 Homework #10
# Due: Monday, November 18

## Problem 1.

Let $Y_1$ and $Y_2$ be continuous random variables with pdf:

$$f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \frac{3y_1^2}{4}, \text{if } 0 \le y_1 \le y_2 \le 2. \\ 0, \quad \text{otherwise.} \end{cases}$$

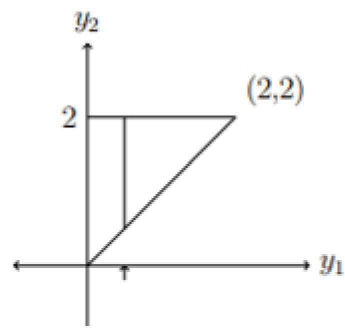
(a) Compute the conditional expectation of $Y_2$ given $Y_1 = y_1$.

(b) Given that $Y_1 = 1/2$, what is the expectation of $Y_2$

(c). Compute the conditional expectation of $Y_2$ given $Y_1$

**Solution:**

The domain of the density function is given by




(a). The marginal density function is given by

$$f_{Y_1}(y_1) = \int_{y_1}^{2} \frac{3y_1^2}{4} dy_2 = \frac{3y_1^2(2 - y_1)}{4}.$$

Therefore,

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{3y_1^2}{4} \Big/ \frac{3y_1^2(2 - y_1)}{4} = \frac{1}{2 - y_1}.$$

for $0 \le y_1 \le y_2 \le 2$ *and* $0$, otherwise. The conditional expectation

$$E[Y_2|Y_1 = y_1] = \int_{y_1}^{2} y_2 \times \frac{1}{2 - y_1} dy_2 = \frac{1}{2 - y_1} \times \frac{2^2 - y_1^2}{2} = \frac{2 + y_1}{2}.$$

(b).

$$E[Y_2|Y_1 = 1/2] = \frac{2 + 1/2}{2} = \frac{5}{4}.$$

(c). As shown in (a),

$$f_{Y_2|Y_1}(y_2|Y_1) = \frac{1}{2 - Y_1}$$

for $0 \leq y_1 \leq y_2 \leq 2$ *and* $0$, otherwise. Therefore,

$$E[Y_2|Y_1] = \frac{2 + Y_1}{2}$$

## Problem 2.

For the daily output of an industrial operation, let $Y_1$ denote the amount of sales and $Y_2$, the costs, in thousands of dollars. Assume that the density functions for $Y_1$ and $Y_2$ are given by

$$f_1(y_1) = \begin{cases} \left(\frac{1}{6}\right) y_1^3 e^{-y_1}, & y_1 > 0; \\ 0, & y_1 < 0. \end{cases}$$

and

$$f_2(y_2) = \begin{cases} \left(\frac{1}{2}\right) e^{-y_2/2}, & y_2 > 0; \\ 0, & y_2 < 0. \end{cases}$$

The daily profit is given by $U = Y_1 - Y_2$.

**(1)** Find $E(U)$.

**Solution**: We write the given density form in the following

$$f_1(y_1) = \begin{cases} \left(\frac{1}{6}\right) y_1^3 e^{-y_1}, & y_1 > 0; \\ 0, & y_1 < 0. \end{cases} = \begin{cases} \frac{1^4}{\Gamma(4)} y_1^{4-1} e^{-1y_1}, & y_1 > 0; \\ 0, & y_1 < 0. \end{cases}$$

This means that $Y_1$ is a gamma distribution with $\alpha = 4$ and $\beta = 1$. Similarly,

$$f_2(y_2) = \begin{cases} \left(\frac{1}{2}\right) e^{-\frac{y_2}{2}}, & y_2 > 0; \\ 0, & y_2 < 0. \end{cases} = \begin{cases} \frac{(\frac{1}{2})^1}{\Gamma(1)} y_2^{1-1} e^{-(1/2)y_2}, & y_2 > 0; \\ 0, & y_2 < 0. \end{cases}$$

$Y_2$ is an exponential distribution, a special gamma distribution with $\alpha = 1$ and $\beta = 1/2$.

Note that the expectation of a gamma distribution is given by $\frac{\alpha}{\beta}$. Therefore,

$$E[U] = E[Y_1] - E[Y_2] = 4/1 - 1/(\frac{1}{2}) = 2.$$

**(2)** Assuming that $Y_1$ and $Y_2$ are independent, find $V(U)$.

**Solution**: Note the variance of a gamma distribution with shape $\alpha$ and $\beta$ is given by $\frac{\alpha}{\beta^2}$. Since $Y_1$ and $Y_2$ are independent, $\text{cov}(Y_1, Y_2) = 0$,

$$V[U] = V[Y_1] + V[Y_2] - \text{cov}(Y_1, Y_2) = \frac{4}{1^2} + \frac{1}{\left(\frac{1}{2}\right)^2} = 8.$$

## Problem 3.

$Y_1$ and $Y_2$ denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of $Y_1$ and $Y_2$ is

$$f(y_1, y_2) = \begin{cases} \left(\frac{1}{8}\right) y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_1 > 0; \\ 0, & \text{elsewhere.} \end{cases}$$

The cost $C$ of replacing the two components depends upon their length of life at failure and is given by $C = 50 + 2Y_1 + 4Y_2$.

**(1).** Find $E(C)$ and $V(C)$.

**Solution:** We first find the two marginal densities.

$$f_1(y_1) = \int_0^\infty \left(\frac{1}{8}\right) y_1 e^{-\frac{y_1+y_2}{2}} dy_2 = \left(\frac{1}{4}\right) y_1 e^{-\frac{y_1}{2}} \int_0^\infty \left(\frac{1}{2}\right) e^{-\frac{y_2}{2}} dy_2 = \left(\frac{1}{4}\right) y_1 e^{-\frac{y_1}{2}}$$

$$f_2(y_2) = \int_0^\infty \left(\frac{1}{8}\right) y_1 e^{-(y_1+y_2)/2} dy_1 = \left(\frac{1}{2}\right) e^{-y_1/2} \int_0^\infty \left(\frac{1}{4}\right) y_1 e^{-y_1/2} dy_1 = \left(\frac{1}{2}\right) e^{-y_2/2}$$

Since $f_1(y_1) \times f_2(y_2) = f(y_1, y_2)$, $Y_1$ and $Y_2$ are independent.

Note that $f_1(y_1)$ and $f_2(y_2)$ can be re-expressed for the forms of gamma density functions in the following.

$$f_1(y_1) = \frac{(1/2)^2}{\Gamma(2)} {y_1}^{2-1} e^{-(\frac{1}{2})y_1}$$

and

$$f_2(y_2) = \frac{(1/2)^1}{\Gamma(1)} y_2^{1-1} e^{-(\frac{1}{2})y_2},$$

Respectively.That is, $Y_1$ is a gamma distribution with shape $\alpha = 2$ and scale $\beta = \frac{1}{2}$and $Y_2$ is an exponential with scale, $\beta = \frac{1}{2}$. Therefore,

$$E[Y_1] = \frac{\alpha}{\beta} = \frac{2}{\frac{1}{2}} = 4,\; E[Y_2] = \frac{1}{\beta} = \frac{1}{\frac{1}{2}} = 2;$$

and

$$V[Y_1] = \frac{\alpha}{\beta^2} = \frac{2}{1/4} = 8,\; V[Y_2] = \frac{1}{\beta^2} = \frac{1}{1/4} = 4.$$

Hence**,**

$$E[C] = 50 \; + 2E[Y_1] + 4E[Y_2] = 50 + 2 \times 4 + 4 \times 2 = 66,$$

and

$$V[C] = 0 \; + 2^2 V[Y_1] + 4^2 V[Y_2] = 4 \times 8 + 16 \times 4 = 32 + 64 = 96.$$

**(2).** Let $U \; = \; Y_1 - \; Y_2$ and $W \; = \; Y_1 + \; Y_2$. Find $COV(U, W)$

**Solution:** Note that

$$\begin{aligned}
\boldsymbol{COV}(U_1, W_2) &= \boldsymbol{COV}[Y_1 + (-\,Y_2)\,, Y_1 + \; Y_2] \\
&= \; \boldsymbol{COV}[Y_1\,, Y_1 + \; Y_2] + \boldsymbol{COV}[-Y_2\,, Y_1 + \; Y_2] \\
&= \; \boldsymbol{COV}[Y_1\,, Y_1 + \; Y_2] - \boldsymbol{COV}[Y_2\,, Y_1 + \; Y_2] \\
&= \boldsymbol{COV}[Y_1\,, Y_1] + \boldsymbol{COV}[Y_1\,, Y_2] - \boldsymbol{COV}[\,Y_2\,, Y_1] - \boldsymbol{COV}[\,Y_2\,, Y_2] \\
&= \boldsymbol{V}[Y_1] + \mathbf{0} - \mathbf{0} - \boldsymbol{V}[\,Y_2\,] = \mathbf{8} - \mathbf{4} = \mathbf{4}.
\end{aligned}$$