STA 504 Homework #9

Due: Tuesday, November 3

This set of homework problems focuses on setting up the integral limits of a double integral. You are expected to draw the integral region on the y_1-y_2 coordinate plane and based on the shape of the region set up the integral limits.

The following linked page helps set up integral limits of double integrals.

http://tutorial.math.lamar.edu/Classes/CalcIII/DIGeneralRegion.aspx

Read the following example carefully and provide a similar level of detail (graphs and algebra) in your work.

Problem 1.

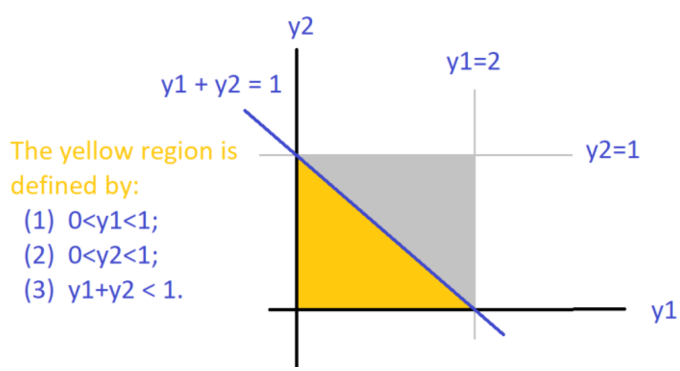
Let Y_1 and Y_2 denote the proportions of two different types of components in a sample from a mixture of chemicals used as an insecticide. Suppose that Y_1 and Y_2 have the joint density function given by

$$f(y_1, y_2) = \begin{cases} 2, & 0 \le y_1 \le 1, 0 \le y_2 \le 1, 0 \le y_1 + y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

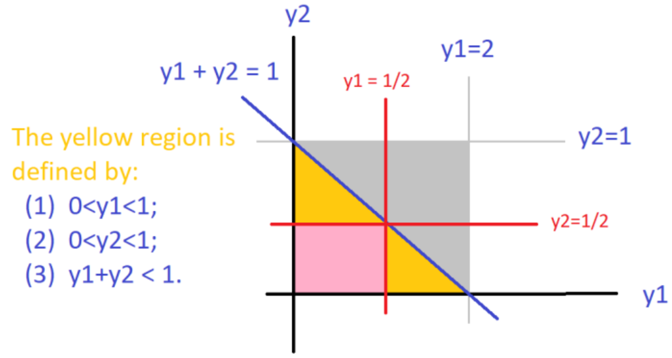
(Notice that $Y_1 + Y_2 \le 1$ because the random variables denote proportions within the same sample.) Find

a $P(Y_1 \le 1/2, Y_2 \le 1/2)$.

Solution: Before finding the probabilities in a and b, we first sketch the region (domain) given in the density function in the following (the yellow region is the domain of the density function



We the two constraints y1 < ½ and y2 < ½ to the original yellow region (the big yellow triangle) and get the following square (pink). That is,



We integrate the density over the pink square region as follows

$$\iint_{pink\ square} f(y_1, y_2)\, dxdy = \int_0^{\frac{1}{2}} \left[\int_0^{\frac{1}{2}} f(y_1, y_2)\, dy_1 \right] dy_2 = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

Problem 2.

The joint density function of Y_1 and Y_2 is given by

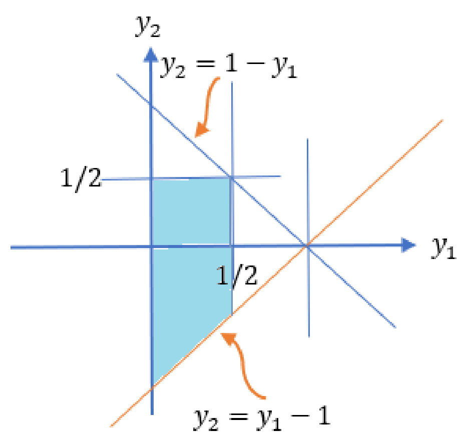
$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \le y_2 \le 1 - y_1, 0 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find $F(1/2, 1/2)$.

b Find $F(1/2, 2)$.

c Find $P(Y_1 > Y_2)$.

Solution. The region on which the probability is defined is sketched in the following. It is a type I region



$$F\left(\frac{1}{2}, \frac{1}{2}\right) = P\left[Y_1 < \frac{1}{2}, Y_2 < \frac{1}{2}\right] = \int_0^{\frac{1}{2}} \int_{y_1 - 1}^{\frac{1}{2}} 30y_1y_2^2\, dy_2\, dy_1$$

$$= \int_0^{\frac{1}{2}} 10y_1 y_2^3 \Big|_{y_2=y_1-1}^{y_2=\frac{1}{2}} dy_1 = \int_0^{\frac{1}{2}} 10y_1 \left[\frac{1}{8} - (y_1-1)^3\right] dy_1$$

$$= 10 \int_0^{\frac{1}{2}} \left[-y_1^4 + 3y_1^3 - 3y_1^2 + \frac{9y_1}{8}\right] dy_1$$

$$= 10 \left[-\frac{y_1^5}{5} + \frac{3y_1^4}{4} - y_1^3 + \frac{9y_1^2}{16}\Big|_0^{\frac{1}{2}}\right] = 10\left[-\frac{1}{5\times 32} + \frac{3}{64} - \frac{1}{8} + \frac{9}{64}\right]$$

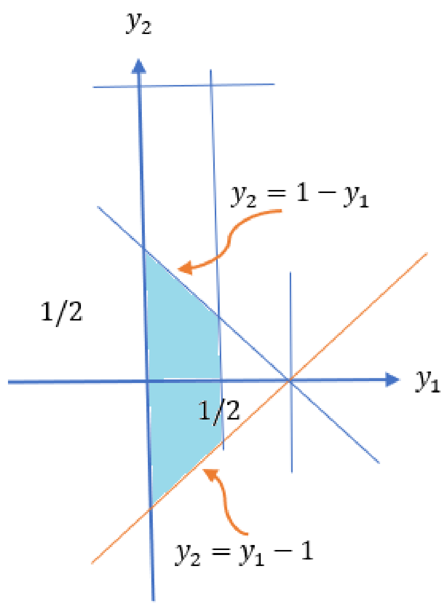
$$= -\frac{2}{32} + \frac{15}{32} - \frac{5}{4} + \frac{45}{32} = \frac{-2+15-40+45}{32} = \frac{18}{32} = \frac{9}{16}$$

(b).

$$F\left(\frac{1}{2}, 2\right) = P\left[Y_1 < \frac{1}{2}, Y_2 < 1\right] = \int_0^{\frac{1}{2}} \int_{y_1-1}^{1-y_1} 30y_1 y_2^2 \, dy_2 \, dy_1$$

$$= \int_0^{\frac{1}{2}} 10y_1 \int_{y_1-1}^{1-y_1} 3y_2^2 \, dy_2 \, dy_1 = \int_0^{\frac{1}{2}} 10y_1 \times y_2^3 \Big|_{y_1-1}^{1-y_1} dy_1$$

$$= \int_0^{\frac{1}{2}} 10y_1 \times [(1-y_1)^3 - (y_1-1)^3] \, dy_1 = \int_0^{\frac{1}{2}} 20y_1(1-y_1)^3 \, dy_1$$



$$= -\int_0^{\frac{1}{2}} 5y_1 \, d(1-y_1)^4 = -5y_1(1-y_1)^4\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} 5(1-y_1)^4 \, dy_1$$

$$= -\frac{5}{32} - (1-y_1)^5\Big|_0^{\frac{1}{2}} = -\frac{5}{32} - \frac{1}{32} + 1 = \frac{-6+32}{32} = \frac{13}{16}$$

(c). The region associated with the desired probability is the sky-blue region in the following figure. There are different ways to find this probability. We can integrate over the sky-blue region by splitting it in two regular regions (R1 type II and R2 is both type I and type II). An easy way is to integrate over R0 and then subtract it from 1 (integrating over the domain yields 1). We take the short cut.

$$\iint_{R_0} f(y_1, y_2)dA = \int_0^{\frac{1}{2}} \int_{y_1}^{1-y_1} 30y_1y_2^2\, dy_2\, dy_1$$

$$= \int_0^{\frac{1}{2}} 10y_1[(1-y_1)^3 - y_1^3]\, dy_1$$

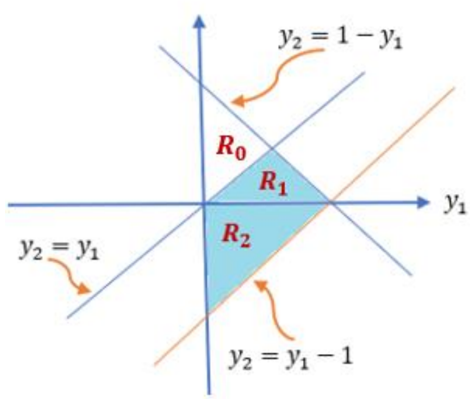
$$= 10\int_0^{\frac{1}{2}} [y_1 - 3y_1^2 + 3y_1^3 - 2y_1^4]\, dy_1$$

$$= 10\left[\frac{y_1^2}{2} - y_1^3 + \frac{3y_1^4}{4} - \frac{2y_1^5}{5}\right]\Bigg|_0^{\frac{1}{2}}$$

$$= 10\left[\frac{1}{8} - \frac{1}{8} + \frac{3}{64} - \frac{2}{5\times 32}\right] = \frac{15}{32} - \frac{4}{32} = \frac{11}{32}$$

Therefore,

$$P[Y_1 > Y_2] = 1 - \frac{11}{32} = \frac{21}{32}$$



Problem 3.

Suppose that the random variables Y_1 and Y_2 have joint probability density function $f(y_1, y_2)$ given by

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2,\ y_1 + y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

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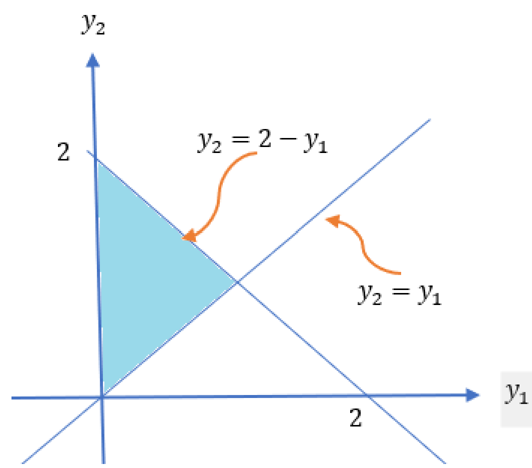
a Verify that this is a valid joint density function.

b What is the probability that $Y_1 + Y_2$ is less than 1?

(a) . The domain of the density is sketched below.



$$\iint_R f(y_1, y_2)dA = \int_0^1 \int_{y_1}^{2-y_1} 6y_1^2 y_2\, dy_2\, dy_1$$

$$= \int_0^1 3y_1^2\, y_2^2 |_{y_1}^{2-y_1} dy_1 = \int_0^1 3y_1^2\, [(2-y_1)^2 - y_1^2\,]\, dy_1$$

$$= \int_0^1 12\, [y_1^2 - \ y_1^3]\ dy_1 = 12\left[\frac{y_1^3}{3} - \frac{y_1^4}{4}\right]\Bigg|_0^1 = 12\left[\frac{1}{3} - \frac{1}{4}\right] = 1.$$

The given function is a valid joint probability density function.

(b). The subregion used in the definition of the probability is sketched in the following figure (gold region). It is both type I and type II region.

$$P[Y_1 +\ Y_2 < 1] = \iint_R f(y_1, y_2)\, dA = \int_0^{1/2} \int_{y_1}^{1-y_1} 6y_1^2 y_2\, d\, y_2 dy_1$$

$$= \int_0^{1/2} 3y_1^2\, y_2^2 |_0^{1-y_1} dy_1 = \int_0^{1/2} 3y_1^2\, [(1-y_1)^2 - y_1^2] dy_1$$

$$= \int_0^{1/2} 3\,(y_1^2 - 2y_1^3)dy_1 = 3\left(\frac{y_1^3}{3} - \frac{y_1^4}{2}\right)\Bigg|_0^{1/2}$$
$$= 3\left(\frac{1/8}{3} - \frac{1/16}{2}\right) = 3 \times \frac{2/8 - 3/16}{6} = \frac{1}{32}$$

[Caution: the answer in the solution manual is incorrect.]