STA 504 Homework #4

Due: Monday, September 30

Show you're your work to earn full credit. You are encouraged to work with your peers on assignments. The write-up must be your own.

4.60 A normally distributed random variable has density function

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}, \qquad -\infty < y < \infty.$$

Using the fundamental properties associated with any density function, argue that the parameter σ must be such that $\sigma > 0$.

Suggested Solution

The parameter σ must be positive, otherwise, the density function could obtain a negative value (a violation).

4.61 What is the median of a normally distributed random variable with mean μ and standard deviation σ?

Suggested Solution

Since the density function is symmetric about the parameter μ, $P(Y < \mu) = P(Y > \mu) = .5$. Thus, μ is the median of the distribution, regardless of the value of σ.

4.62 If Z is a standard normal random variable, what is

a $P(Z^2 < 1)$?

b $P(Z^2 < 3.84146)$?

Subtitle

Suggested Solution

a. $P(Z^2 < 1) = P(-1 < Z < 1) = .6826$.

b. $P(Z^2 < 3.84146) = P(-1.96 < Z < 1.96) = .95$.

Problem 4.

The lifetime (in hours) Y of an electronic component is a random variable with density function given by

$$f(y) = \begin{cases} \frac{1}{100} e^{-y/100}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Based on the above distribution answer the following questions.

1. Show that f(y) is a valid density function.

$$\int_0^\infty \frac{1}{100} e^{-y/100}\, dy = -\int_0^\infty e^{-\frac{y}{100}}\, d\left(-\frac{y}{100}\right) = -e^{-\frac{y}{100}}\Big|_0^\infty = -\left(e^{-\frac{\infty}{100}} - e^0\right) = -(0-1) = 1$$

Therefore, f(y) is a valid density function.

2. Derive the cumulative distribution function (CDF) of the lifetime variable.

For y > 0, $F(y) = \int_0^y \frac{1}{100} e^{-\frac{x}{100}}\, dx = -e^{-\frac{x}{100}}\Big|_0^y = -\left(e^{-\frac{y}{100}} - e^0\right) = 1 - e^{-\frac{y}{100}}$

Therefore,

$$F(y) = \begin{cases} 1 - e^{-\frac{y}{100}}, & y \geq 0; \\ 0, & y < 0. \end{cases}$$

3. Derive the expectation and variance of Y.

$$E[Y] = \int_0^\infty y \frac{1}{100} e^{-\frac{y}{100}}\, dy = -\int_0^\infty y e^{-\frac{y}{100}}\, d\left(-\frac{y}{100}\right) = -\int_0^\infty y\, d\left(e^{-\frac{y}{100}}\right)$$
$$= -y e^{-\frac{y}{100}}\Big|_0^\infty + \int_0^\infty e^{-\frac{y}{100}}\, dy = 0 + 100 \int_0^\infty \frac{1}{100} e^{-\frac{y}{100}}\, dy$$
$$= 100 \times 1 = 100.$$

Note that

$$V[Y] = E[Y^2] - [E[Y]]^2 = E[Y^2] - 100^2$$

$$E[Y^2] = \int_0^\infty y^2 \frac{1}{100} e^{-\frac{y}{100}}\, dy = -\int_0^\infty y^2\, e^{-\frac{y}{100}}\, d\left(-\frac{y}{100}\right) = -\int_0^\infty y^2\, d\left(e^{-\frac{y}{100}}\right)$$

$$= -y^2 e^{-\frac{y}{100}}\Big|_0^\infty + \int_0^\infty e^{-\frac{y}{100}}\, dy^2 = 0 + \int_0^\infty 2y e^{-\frac{y}{100}}\, dy$$
$$= 200 \int_0^\infty y \frac{1}{100} e^{-\frac{y}{100}}\, dy = 200E[Y] = 200 \times 100 = 20000$$

Therefore, $V[Y] = E[Y^2] - 100^2 = 20000 - 10000 = 10000 = 100^2.$

4. Find P(Y > 150) using the CDF derived in part 2.

$$P(Y\ >\ 150) = 1 - F(150) = 1 - \left(1 - e^{-\frac{150}{100}}\right) = e^{-1.5} \approx 0.2231$$

5. Find P(Y > 150 | Y > 100) using CDF.

$$P(Y\ >\ 150\,|\,Y\ >\ 100) = \frac{P[Y\ >\ 150\ \cap\ Y\ >\ 100]}{P[Y > 100]}$$

$$== \frac{P[Y\ >\ 150\ ]}{P[Y > 100]} = \frac{e^{-1.5}}{e^{-1}} = e^{-0.5} \approx 0.6065.$$