THE ATIYAH-BOTT FORMULA AND CONNECTIVITY IN CHIRAL KOSZUL DUALITY

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ABSTRACT. We prove a family of results regarding connectivity in the theory of chiral Koszul duality. This provides new examples of Koszul duality being an equivalence, even when the base category is not pro-nilpotent in the sense of [FG11]. Based on ideas sketched in [Gai11], we show that these results also offer a simpler alternative to one of the two main steps in the proof of the Atiyah-Bott formula given in [GL14] and [Gai15].

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1. Introduction

1.1. **History.** Let X be a smooth and complete curve, and G a simply-connected semi-simple algebraic group over an algebraically closed field k.[1] Then we know that

$$C^*(BG, \Lambda) \simeq \operatorname{Sym} V$$

for some finite dimensional vector space V, where Λ is $\mathbb{Q}_\ell$ when $k = \overline{\mathbb{F}}_p$ ($\ell \neq p$), and Λ is any field of characteristic 0 when k has characteristic 0.

Let Bun_G denote the moduli stack of principal G-bundles over X. In the differential geometric setting, i.e. when $k = \mathbb{C}$, the cohomology ring of Bun_G was computed by Atiyah and Bott in [AB83].

Theorem 1.1.1 (Atiyah-Bott)**.** *We have the following equivalence*

$$C^*(\mathrm{Bun}_G, \Lambda) = \operatorname{Sym}_\Lambda(C^*(X, V \otimes \omega_X)),$$

where ω_X is the dualizing sheaf of X.

In the recent work [GL14], Gaitsgory and Lurie gave a purely algebro-geometric proof of the theorem above in the framework of étale cohomology (see also [Gai15] for an alternative perspective). In the case where X and G come from objects over $k = \mathbb{F}_q$, the isomorphism in Theorem 1.1.1 was proved to be compatible with the Frobenius actions on both sides. The Grothendieck-Lefschetz trace formula for Bun_G then gives an expression for the number of k-points on Bun_G and hence, confirms the conjecture of Weil that the Tamagawa number of G is 1.

Following ideas suggested in [Gai11], this paper aims to provide an alternative (and simpler) proof of one of the two main steps in the original proofs, as given in [GL14] and [Gai15]. This is possible due to a family of new results regarding connectivity in the theory of chiral Koszul duality proved in this paper.

1.2. **Prerequisites and guides to the literature.** For the reader's convenience, we include a quick review of the necessary background as well as pointers to the existing literature in §2. The readers who are unfamiliar with the language used in the introduction are encouraged to take a quick look at §2 before returning to the current section.

1.3. **A sketch of Gaitsgory and Lurie's method.** We will now provide a sketch of the method employed by [GL14] and [Gai15]. In both cases, the proofs utilize the theory of factorization algebras. Broadly speaking, there are two main steps: non-abelian Poincaré duality and Verdier duality on the Ran space.

[1] This corresponds to the case of constant group $G \times X$ over X. For simplicity's sake, we will restrict ourselves to this case in the introduction.

1.3.1. *Non-abelian Poincaré duality.* For the first step, one constructs a factorizable sheaf $\mathcal{A}$ on $\operatorname{Ran} X$ from $f_! \omega_{\mathrm{Gr}_{\operatorname{Ran} X}}$ where f is the natural map

$$f : \mathrm{Gr}_{\operatorname{Ran} X} \to \operatorname{Ran} X,$$

and $\mathrm{Gr}_{\operatorname{Ran} X}$ is the factorizable affine Grassmannian. The crucial observation is that the natural map

$$\mathrm{Gr}_{\operatorname{Ran} X} \to \mathrm{Bun}_G$$

has homologically contractible fibers, and hence, we get an equivalence

$$C^*_c(\mathrm{Bun}_G, \omega_{\mathrm{Bun}_G}) \simeq C^*_c(\operatorname{Ran} X, \mathcal{A}). \tag{1.3.2}$$

1.3.3. *Verdier duality.* The right hand side of (1.3.2) is, however, not directly computable. If one thinks of factorizable sheaves on $\operatorname{Ran} X$ as E_2-algebras, then one reason that makes it hard to compute the factorization homology of $\mathcal{A}$ is the fact that it's not necessarily commutative (i.e. not E_∞). $\mathcal{A}$, however, also has a commutative co-algebra structure, via the diagonal map[2]

$$\mathrm{Gr} \to \mathrm{Gr} \times \mathrm{Gr}\,.$$

Thus, its Verdier dual $D_{\operatorname{Ran} X}\mathcal{A}$ naturally has the structure of a commutative algebra. In fact, it's proved that $D_{\operatorname{Ran} X}\mathcal{A}$ is a commutative factorization algebra.

1.3.4. *Computing the Verdier dual.* One can prove something even better: $D_{\operatorname{Ran} X}\mathcal{A}$ is isomorphic to the commutative factorization algebra $\mathcal{B}$ coming from $C^*(BG)$. Indeed, a natural map from one to the other is given by a certain pairing between $\mathcal{A}$ and $\mathcal{B}$. Since these are factorizable, showing that this map is an equivalence amounts to showing that its restriction to X is also an equivalence. This is now a purely local problem, and hence, for example, one can reduce it to the case of $\mathbb{P}^1$ to prove it.

1.3.5. *Conclusion.* Recall that

$$\mathcal{B} \simeq C^*(BG) \simeq \operatorname{Sym} V$$

is a free commutative algebra, where V is some explicit chain complex that we can compute. But factorization homology with coefficients in a free commutative factorization algebra is easy to compute. Hence, we conclude

$$\begin{aligned} C^*(\mathrm{Bun}_G, \mathbb{Q}_\ell) &\simeq C^*_c(\mathrm{Bun}_G, \omega_{\mathrm{Bun}_G})^\vee \\ &\simeq C^*_c(\operatorname{Ran} X, \mathcal{A})^\vee \\ &\simeq C^*_c(\operatorname{Ran} X, D_{\operatorname{Ran} X}\mathcal{A}) \\ &\simeq C^*_c(\operatorname{Ran} X, \mathcal{B}) \\ &\simeq C^*_c(\operatorname{Ran} X, \operatorname{Sym} V) \\ &\simeq \operatorname{Sym} C^*_c(X, V). \end{aligned} \tag{1.3.6}$$

1.4. **What does this paper do?** The main difference between [GL14] and [Gai15] is in the use of Verdier duality on the Ran space.[3] The latter greatly simplifies and clarifies the former by formally introducing the concept of Verdier duality on a general prestack and then applying it to the case of the Ran space.

Since the Ran space is a big object,[4] its technical properties in relation to factorization homology and factorizability are difficult to establish. More precisely, it takes a lot of work to prove the (innocent looking) equivalence (1.3.6) and to a somewhat lesser extent, the fact that $D_{\operatorname{Ran} X}\mathcal{A}$ is factorizable. This results in a rather complicated technical heart of [Gai15].

In this paper, we prove a series of new results regarding connectivity in the theory of chiral Koszul duality. These are interesting in their own rights, since they give new examples of Koszul duality being an equivalence, even when the base category is not pro-nilpotent in the sense of [FG11].

Based on the ideas sketched in [Gai11], the results proved in this paper also further simplify the second step of the proof. More precisely, these results could be used to replace all of §8, §9, and part of §12 and §15 of [Gai15].

[2] We are eliding a minor, but technical, point about unital vs. non-unital here.

[3] [Gai15] doesn't reprove non-abelian Poincaré duality.

[4] In the terminology of [Gai15], it's not finitary.

1.5. **An outline of our results.** We will now state the main results proved in this paper.

Many results that we prove require connectivity assumptions that are somewhat cumbersome to state. Since these are merely technical conditions irrelevant to the discussion of the general method, we will gloss over them in this section.

Remark 1.5.1. Many results in this paper could be proved in a more general setting. We avoid doing so to keep the presentation simple. We will, however, provide remarks about this throughout the text.

1.5.2. *Koszul duality for* Lie *and* ComCoAlg. Let $\mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)$ and $\mathrm{Lie}^\star(\mathrm{Ran}\, X)$ denote the categories of commutative co-algebra objects and Lie algebra objects in $\mathrm{Shv}(\mathrm{Ran}\, X)$ with respect to the $\otimes^\star$-monoidal structure. The theory of Koszul duality developed in [FG11] gives a pair of adjoint functors[5]

$$\mathrm{Chev} : \mathrm{Lie}^\star(\mathrm{Ran}\, X) \rightleftarrows \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X) : \mathrm{Prim}[-1]. \tag{1.5.3}$$

which restricts to a pair of adjoint functors

$$\mathrm{Chev} : \mathrm{Lie}^\star(X) \rightleftarrows \mathrm{coFact}^\star(X) : \mathrm{Prim}[-1],$$

where $\mathrm{coFact}^\star(X)$ is the category of commutative factorization co-algebras on X.

Even though the pair of adjoint functors above are not mutually inverses of each other in general, they are when we impose certain connectivity constraints on both sides.

Theorem 1.5.4 (Theorem 3.3.3)**.** *We have the following commutative diagram*

$$\begin{tikzcd}
\mathrm{Lie}^\star(\mathrm{Ran}\, X)^{\leq c_L} \arrow[r, equal, "\mathrm{Chev}", "\mathrm{Prim}[-1]"'] & \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}} \\
\mathrm{Lie}^\star(X)^{\leq c_L} \arrow[u, hook] \arrow[r, equal, "\mathrm{Chev}", "\mathrm{Prim}[-1]"'] & \mathrm{coFact}^\star(X)^{\leq c_{cA}} \arrow[u, hook]
\end{tikzcd}$$

where $\leq c_L$ and $\leq c_{cA}$ denote some connectivity constraints, and where Chev *and* Prim[−1] *are the functors coming from Koszul duality.*

1.5.5. *Koszul duality for* coLie *and* ComAlg. Let $\mathrm{ComAlg}^\star(\mathrm{Ran}\, X)$ and $\mathrm{coLie}^\star(\mathrm{Ran}\, X)$ denote the categories of commutative algebra objects and co-Lie algebra objects in $\mathrm{Shv}(\mathrm{Ran}\, X)$ with respect to the $\otimes^\star$-monoidal structure. As above, we have the following pair of adjoint functors[6]

$$\mathrm{coPrim}[1] : \mathrm{ComAlg}^\star(\mathrm{Ran}\, X) \rightleftarrows \mathrm{coLie}^\star(\mathrm{Ran}\, X) : \mathrm{coChev}.$$

Unlike the case of $\mathrm{Lie}^\star$ and $\mathrm{ComCoAlg}^\star$, for a co-Lie algebra $\mathfrak{g} \in \mathrm{coLie}^\star(X)$,

$$\mathrm{coChev}(\mathfrak{g}) \in \mathrm{ComAlg}^\star(\mathrm{Ran}\, X)$$

doesn't necessarily live inside $\mathrm{Fact}^\star(X)$. However, we have the following

Theorem 1.5.6 (Theorem 4.1.3)**.** *Restricted to the full subcategory $\mathrm{coLie}^\star(X)^{\geq 1}$, where we are using the perverse t-structure on X, the functor* coChev *factors through* $\mathrm{Fact}^\star$*, i.e.*

$$\begin{tikzcd}
\mathrm{coLie}^\star(X)^{\geq 1} \arrow[rr, "\mathrm{coChev}"] \arrow[dr, "\mathrm{coChev}"'] & & \mathrm{ComAlg}^\star(\mathrm{Ran}\, X) \\
& \mathrm{Fact}^\star(X) \arrow[ur, hook] &
\end{tikzcd}$$

[5] Strictly speaking, we are using the category $\mathrm{ComCoAlg}^{\text{ind-nilp}}$ of ind-nilpotent commutative co-algebras. However, we will see easily that, subject to an appropriate connectivity assumption of sheaves on $\mathrm{Ran}\, X$, this category coincides with the category ComCoAlg.

[6] See also footnote 5.

1.5.7. *Interaction between* coChev *and factorization homology.* In [FG11], it's proved that the functor of taking factorization homology:

$$C_c^* : \mathrm{Shv}(\mathrm{Ran}\, X) \to \mathrm{Vect}$$

commutes with Chev. This is because Chev is computed as a colimit, and moreover, C_c^* has the following two useful properties:

(i) C_c^* is symmetric monoidal with respect to the $\otimes^\star$-monoidal structure on $\mathrm{Shv}(\mathrm{Ran}\, X)$ and the usual monoidal structure on Vect.
(ii) C_c^* is continuous.

The functor coChev, however, is constructed as a limit, so we need some extra conditions to make it behave nicely with C_c^*.

Theorem 1.5.8 (Theorem 5.1.2)**.** *Let* $\mathfrak{g} \in \mathrm{coLie}^\star(X)^{\geq c_{c_L}}$*, where* $\geq c_{cL}$ *denotes some co-connectivity constraint. Then we have a natural equivalence*

$$C_c^*(\mathrm{Ran}\, X, \mathrm{coChev}\, \mathfrak{g}) \simeq \mathrm{coChev}(C_c^*(\mathrm{Ran}\, X, \mathfrak{g})).$$

1.5.9. Chev, coChev *and Verdier duality.* Unsurprisingly, the functors Chev and coChev mentioned above are linked via the Verdier duality functor on $\mathrm{Ran}\, X$.

Theorem 1.5.10 (Theorem 5.3.1)**.** *Let* $\mathfrak{g} \in \mathrm{Lie}^\star(X)^{\leq -1}$*, where we are using the perverse* t*-structure on* X*. Then we have the following natural equivalence*

$$D_{\mathrm{Ran}\, X}\, \mathrm{Chev}\, \mathfrak{g} \simeq \mathrm{coChev}(D_X\, \mathfrak{g}).$$

Remark 1.5.11. As we shall see, the connectivity constraint $\mathrm{Lie}^\star(X)^{\leq -1}$ is less strict than the connectivity constraint $\mathrm{Lie}^\star(X)^{\leq c_L}$ required by Theorem 1.5.4.

As a corollary of Theorem 1.5.6, we know that when $\mathfrak{g} \in \mathrm{Lie}^\star(X)^{\leq c_L}$,

$$D_{\mathrm{Ran}\, X}\, \mathrm{Chev}\, \mathfrak{g} \simeq \mathrm{coChev}(D_X\, \mathfrak{g})$$

is factorizable.

1.6. **Relation to the Atiyah-Bott formula.**

1.6.1. The initial observation is that the sheaf $\mathcal{A}$ mentioned above lies in the essential image of Chev, i.e.

$$\mathcal{A} \simeq \mathrm{Chev}(\mathfrak{a}), \qquad \text{for some } \mathfrak{a} \in \mathrm{Lie}^\star(X)^{\leq c_L}.$$

This is a direct result of Theorem 1.5.4 and the fact that $\mathcal{A}$ satisfies this connectivity constraint on the $\mathrm{ComCoAlg}^\star$ side.

1.6.2. As in [Gai15], we have a pairing

$$\mathcal{A} \boxtimes \mathcal{B} \to \delta_! \omega_{\mathrm{Ran}\, X},$$

which induces a map

$$\mathcal{B} \to D_{\mathrm{Ran}\, X} \mathcal{A},$$

compatible with the commutative algebra structures on both sides. Thus, we get a map

$$\mathcal{B} \to D_{\mathrm{Ran}\, X}\, \mathrm{Chev}(\mathfrak{a}) \simeq \mathrm{coChev}(D_X\, \mathfrak{a}),$$

which we want to be an equivalence. Since both sides are factorizable, it suffices to show that they are over X, which is now a local problem, and the same proof as in [Gai15] applies.

1.6.3. *Conclusion.* Let $V \in \mathrm{Vect}$ such that $\mathrm{Sym}(V \otimes \omega_X) \simeq \mathcal{B}$ where Sym is taken inside $\mathrm{Shv}(\mathrm{Ran} X)$ using the $\otimes^\star$-monoidal structure. Then, we have

$$\begin{aligned}
\mathrm{Sym}\, C^*_c(X, V \otimes \omega_X) &\simeq C^*_c(\mathrm{Ran} X, \mathrm{Sym}(V \otimes \omega_X)) \\
&\simeq C^*_c(\mathrm{Ran} X, \mathcal{B}) \\
&\simeq C^*_c(\mathrm{Ran} X, \mathrm{coChev}_{\mathrm{Ran} X} D_X \mathfrak{a}) \\
&\simeq \mathrm{coChev}(C^*_c(X, D_X \mathfrak{a})) \\
&\simeq \mathrm{coChev}(C^*_c(X, \mathfrak{a})^\vee) \\
&\simeq \mathrm{Chev}(C^*_c(X, \mathfrak{a}))^\vee \\
&\simeq C^*_c(\mathrm{Ran} X, \mathrm{Chev}\, \mathfrak{a})^\vee \\
&\simeq C^*_c(\mathrm{Ran} X, \mathcal{A})^\vee \\
&\simeq C^*_c(\mathrm{Bun}_G, \omega_{\mathrm{Bun}_G})^\vee \\
&\simeq C^*(\mathrm{Bun}_G, \mathbb{Q}_\ell).
\end{aligned}$$

Remark 1.6.4. It is interesting to note that many technical results about Verdier duality are proved only for the case of curves in [Gai15], while results stated here about Koszul duality are for arbitrary dimension (even though in the end, they serve a similar purpose regarding the Atiyah-Bott formula). This is in part because [Gai15] works with more general sheaves on the Ran space, whereas we mostly concern ourselves with sheaves of special shapes, i.e. they are all of the form $\mathrm{Chev}\, \mathfrak{g}$ or $\mathrm{coChev}\, \mathfrak{g}$.

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2. Preliminaries

In this section, we will set up the language and conventions used throughout the paper. Since the material covered here are used in various places, the readers should feel free to skip it and backtrack when necessary.

The mathematical content in this section has already been treated elsewhere. Hence, results are stated without any proof, and we will do our best to provide the necessary references. It is important to note that it is not our aim to be exhaustive. Rather, we try to familiarize the readers with the various concepts and results used in the text, as well as to give pointers to the necessary references for the background materials.

2.1. **Notation and conventions.**

2.1.1. *Category theory.* We will use DGCat to denote the $(\infty, 1)$-category of stable infinity categories, $\mathrm{DGCat}_{\mathrm{pres}}$ to denote the full subcategory of DGCat consisting of presentable categories, and $\mathrm{DGCat}_{\mathrm{pres,cont}}$ the (non-full) subcategory of $\mathrm{DGCat}_{\mathrm{pres}}$ where we restrict to continuous functors, i.e. those commuting with colimits. Spc will be used to denote the category of spaces, or more precisely, ∞-groupoids.

The main references for this subject are [Lur15] and [Lur14]. For a slightly different point of view, see also [GR].

2.1.2. *Algebraic geometry.* Throughout this paper, k will be an algebraically closed ground field. We will denote by Sch the ∞-category obtained from the ordinary category of separated schemes of finite type over k. All our schemes will be objects of Sch. In most cases, we will use the calligraphic font to denote prestacks, for eg. $\mathcal{X}, \mathcal{Y}$ etc., and the usual font to denote schemes, for eg. X, Y etc.

2.1.3. *t-structures.* Let $\mathcal{C}$ be a stable infinity category, equipped with a t-structure. Then we have the following diagram of adjoint functors

$$\mathcal{C}^{\leq 0} \underset{\mathrm{tr}_{\leq 0}}{\overset{i_{\leq 0}}{\rightleftarrows}} \mathcal{C} \underset{i_{\geq 1}}{\overset{\mathrm{tr}_{\geq 1}}{\rightleftarrows}} \mathcal{C}^{\geq 1}$$

We use $\tau_{\leq 0}$ and $\tau_{\geq 1}$ to denote

$$\tau_{\leq 0} = i_{\leq 0} \circ \mathrm{tr}_{\leq 0} : \mathcal{C} \to \mathcal{C}$$

and

$$\tau_{\geq 1} = i_{\geq 1} \circ \mathrm{tr}_{\geq 1} : \mathcal{C} \to \mathcal{C}$$

respectively.

Shifts of these functors, for e.g. $\tau_{\geq n}$ and $\tau_{\leq n}$, are defined in the obvious ways.

2.2. **Prestacks.** The theory of sheaves on prestacks has been developed in [GL14] and [Gai15]. In this subsection and the next, we will give a brief review of this theory, including the definition of the category of sheaves as well as various pull and push functors. We will state them as facts, without any proof, which (unless otherwise specified), could all be found in [Gai15].

2.2.1. A prestack is a contravariant functor from Sch to Spc, i.e. a prestack $\mathcal{Y}$ is a functor

$$\mathcal{Y} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Spc}.$$

Let PreStk be the ∞-category of prestacks. Then by Yoneda's lemma, we have a fully-faithful embedding

$$\mathrm{Sch} \hookrightarrow \mathrm{PreStk}.$$

2.2.2. *Properties of prestacks.* Due to categorical reasons, any prestack $\mathcal{Y}$ can be written as a colimit of schemes

$$\mathcal{Y} \simeq \operatorname*{colim}_{i \in I} Y_i.$$

2.2.3. A prestack is said to be is a pseudo-scheme if it could be written as a colimit of schemes, where all morphisms are proper.

2.2.4. A prestack is pseudo-proper if it could be written as a colimit of proper schemes. It is straightforward to see that pseudo-proper prestacks are pseudo-schemes.

2.2.5. A prestack is said to be finitary if it could be expressed as a finite colimit of schemes.

2.2.6. We also have relative versions of the definitions above in an obvious manner. Namely, we can speak of a morphism $f : \mathcal{Y} \to S$, where $\mathcal{Y}$ is a prestack and S is a scheme, being pseudo-schematic (resp. pseudo-proper, finitary).

2.2.7. More generally, a morphism

$$f : \mathcal{Y}_1 \to \mathcal{Y}_2$$

is said to be pseudo-schematic (resp. pseudo-proper, finitary) if for any scheme S, equipped with a morphism $S \to \mathcal{Y}_2$, the morphism f_S in the following pull-back diagram

$$\begin{tikzcd} S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \arrow[r] \arrow[d, "f_S"'] & \mathcal{Y}_1 \arrow[d] \\ S \arrow[r] & \mathcal{Y}_2 \end{tikzcd}$$

is pseudo-schematic (resp. pseudo-proper, finitary).

2.3. **Sheaves on prestacks.** As we mentioned above, proofs of all the results in mentioned in this section, unless otherwise specified, could be found in [Gai15].

2.3.1. *Sheaves on schemes.* We will adopt the same conventions as in [Gai15], except that for simplicity, we will restrict ourselves to the "constructible setting." Namely, for a scheme S,

(i) when the ground field is $\mathbb{C}$, and Λ is an arbitrary field of characteristic 0, we take $\mathrm{Shv}(S)$ to be the ind-completion of the category of constructible sheaves on S with Λ-coefficients.
(ii) for any ground field k in general, and $\Lambda = \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell$ with $\ell \neq \mathrm{char}\, k$, we take $\mathrm{Shv}(S)$ to be the ind-completion of the category of constructible ℓ-adic sheaves on S with Λ-coefficients. See also [GL14, §4], [LZ12], and [LZ14].

The theory of sheaves on schemes is equipped with the various pairs of adjoint functors

$$f_! \dashv f^! \qquad \text{and} \qquad f^* \dashv f_*$$

for any morphism

$$f : S_1 \to S_2$$

between schemes. Moreover, we also have box-product $\boxtimes$ and hence, also $\otimes$ and $\overset{!}{\otimes}$.

2.3.2. Throughout the text, we will use the perverse t-structure on $\mathrm{Shv}(S)$, when S is a scheme.

2.3.3. We will also use Vect to denote the category of sheaves on a point, i.e. Vect denotes the (infinity derived) category of chain complexes in vector spaces over Λ.

2.3.4. *Sheaves on prestacks.* For a prestack $\mathcal{Y}$, the category $\mathrm{Shv}(\mathcal{Y})$ is defined by

$$\mathrm{Shv}(\mathcal{Y}) = \lim_{S \in (\mathrm{Sch}^{\mathrm{op}}_{/\mathcal{Y}})} \mathrm{Shv}(S),$$

where the transition functor we use is the shriek-pullback.

Thus, an object $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ is the same as the following data

(i) A sheaf $\mathcal{F}_{S,y} \in \mathrm{Shv}(S)$ for each $S \in \mathrm{Sch}$ and $y : S \to \mathcal{Y}$ (i.e. $y \in \mathcal{Y}(S)$).
(ii) An equivalence of sheaves $\mathcal{F}_{S',f(y)} \to f^! \mathcal{F}_{S,y}$ for each morphism of schemes $f : S' \to S$.

Moreover, we require that this assignment satisfies a homotopy-coherent system of compatibilities.

2.3.5. More formally, one can define $\mathrm{Shv}(\mathcal{Y})$ as the right Kan extension of

$$\mathrm{Shv} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{pres,cont}}$$

along the Yoneda embedding

$$\mathrm{Sch}^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}.$$

Thus, by formal reasons, the functor

$$\mathrm{Shv} : \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{pres,cont}}$$

preserves limits. In other words, we have

$$\mathrm{Shv}(\operatorname*{colim}_i \mathcal{Y}_i) \simeq \lim_i \mathrm{Shv}(\mathcal{Y}_i).$$

In particular, if a prestack

$$\mathcal{Y} \simeq \operatorname*{colim}_{i \in I} Y_i$$

is a colimit of schemes, then

$$\mathrm{Shv}(\mathcal{Y}) \simeq \lim_{i \in I} \mathrm{Shv}(Y_i).$$

2.3.6. Now, if we replace all the transition functors by their left adjoints, namely the !-pushforward, then we have a diagram

$$I^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{pres,cont}},$$

and we have a natural equivalence

$$\mathrm{Shv}(\mathcal{Y}) \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathrm{Shv}(Y_i)$$

where the colimit is taken inside $\mathrm{DGCat}_{\mathrm{pres,cont}}$.

2.3.7. Let

$$\mathcal{Y} = \operatorname{colim}_i Y_i$$

be a prestack, and denote

$$\mathrm{ins}_i : Y_i \to \mathcal{Y}$$

the canonical map. Then, for any sheaf $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$, we have the following natural equivalence

$$\mathcal{F} \simeq \operatorname{colim}_i \mathrm{ins}_{i!} \mathrm{ins}_i^! \mathcal{F} \tag{2.3.8}$$

2.3.9. $f_! \dashv f^!$. Let

$$f : \mathcal{Y}_1 \to \mathcal{Y}_2$$

be a morphism between prestacks. Then by restriction, we get a functor

$$f^! : \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1),$$

which commutes with both limits and colimits. In particular, $f^!$ admits a left adjoint $f_!$.[7]

The functor $f_!$ is generally not computable. However, there are a couple of cases where it is.

2.3.10. The first instance is when the target of f is a scheme

$$f : \mathcal{Y} \to S,$$

and suppose that

$$\mathcal{Y} \simeq \operatorname{colim}_i Y_i.$$

Then, by (2.3.8), we have

$$f_!\mathcal{F} \simeq \operatorname{colim} f_! \mathrm{ins}_{i!} \mathrm{ins}_i^! \mathcal{F} \simeq \operatorname{colim} f_{i!} \mathrm{ins}_i^! \mathcal{F}.$$

where

$$f_i : Y_i \to \mathcal{Y} \to S$$

is just a morphism between schemes.

2.3.11. The second case is where f is pseudo-proper, then $f_!$ satisfies the base change theorem with respect to the $(-)^!$-pullback. Namely, for any pull-back diagram of prestacks

$$\begin{tikzcd} \mathcal{Y}'_1 \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{Y}_1 \arrow[d, "f"] \\ \mathcal{Y}'_2 \arrow[r, "g"] & \mathcal{Y}_2 \end{tikzcd}$$

and any sheaf $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$, we have a natural equivalence

$$g^! f_! \mathcal{F} \simeq f_! g^! \mathcal{F}.$$

Thus, in particular, if we have a pull-back diagram

$$\begin{tikzcd} S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \arrow[r, "i_S"] \arrow[d, "f_S"'] & \mathcal{Y}_1 \arrow[d, "f"] \\ S \arrow[r, "i_S"] & \mathcal{Y}_2 \end{tikzcd}$$

where S is a scheme, then

$$i_S^! f_! \mathcal{F} \simeq f_{S!} i_S^! \mathcal{F}$$

and as discussed above, $f_{S!}$ could be computed as an explicit colimit.

[7]It also admits a right adjoint. However, we do not make use of it in this paper.

2.3.12. Let $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$. Then we denote by

$$C^*_c(\mathcal{Y}, \mathcal{F}) = s_! \mathcal{F},$$

where

$$s : \mathcal{Y} \to \mathrm{Spec}\, k$$

is the structural map of $\mathcal{Y}$ to a point.

2.3.13. In case where $\mathcal{F} \simeq \omega_{\mathcal{Y}}$ is the dualizing sheaf on $\mathcal{Y}$ (characterized by the property that its $(-)^!$-pullback to any scheme is the dualizing sheaf on that scheme), then we write

$$C_*(\mathcal{Y}) = C^*_c(\mathcal{Y}, \omega_{\mathcal{Y}}),$$

and

$$C^{\mathrm{red}}_*(\mathcal{Y}) = \mathrm{Fib}(C_*(\mathcal{Y}) \to \Lambda).$$

2.3.14. $f^* \dashv f_*$. When

$$f : \mathcal{Y}_1 \to \mathcal{Y}_2$$

is a schematic morphism between prestacks, one can also define a pair of adjoint functors (see [Gai15] where the functor f_* is defined, and [Ho15] where the adjunction is constructed)

$$f^* : \mathrm{Shv}(\mathcal{Y}_2) \rightleftarrows \mathrm{Shv}(\mathcal{Y}_1) : f_*.$$

2.3.15. The behavior of f_* is easy to describe, due to the fact that f_* satisfies the base change theorem with respect to the $(-)^!$-pullback functor. Namely, suppose $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y}_1)$ and we have a pullback square where S_2 (and hence, S_1) is a scheme

$$\begin{tikzcd} S_1 \arrow[r, "g"] \arrow[d, "f_S"'] & \mathcal{Y}_1 \arrow[d, "f"] \\ S_2 \arrow[r, "g"] & \mathcal{Y}_2 \end{tikzcd}$$

Then, the pullback could be described in classical terms, since

$$g^! f_* \mathcal{F} \simeq f_{S*} g^! \mathcal{F},$$

where f_S is just a morphism between schemes.

2.3.16. The functor f^* is slightly more complicated to describe. However, when

$$f : \mathcal{Y}_1 \to \mathcal{Y}_2$$

is étale, which is the case where we need, we have a natural equivalence (see [Ho15, Prop. 2.7.3])

$$f^! \simeq f^*. \tag{2.3.17}$$

2.3.18. We will also need the following fact in the definition of commutative factorizable co-algebras: let

$$\mathcal{U} \xrightarrow{f} \mathcal{Z} \xrightarrow{g} \mathcal{X}$$

be morphisms between prestacks, where g is finitary pseudo-proper, f and $h = g \circ f$ are schematic. Then we have a natural equivalence (see [Ho15, Prop. 2.10.4])

$$g_! \circ f_* \simeq (g \circ f)_* \simeq h_*. \tag{2.3.19}$$

2.3.20. *Monoidal structure.* The theory of sheaves on prestacks discussed so far naturally inherits the box-tensor structure from the theory of sheaves on schemes. Namely, let $\mathcal{F}_i \in \mathrm{Shv}(\mathcal{Y}_i)$ where $\mathcal{Y}_i$'s are prestacks, for $i = 1, 2$. Then, for any pair of schemes S_1, S_2 equipped with maps

$$f_i : S_i \to \mathcal{Y}_i,$$

we have

$$(f_1 \times f_2)^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \simeq f_1^! \mathcal{F}_1 \boxtimes f_2^! \mathcal{F}_2.$$

Pulling back along the diagonal

$$\delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$$

for any prestack $\mathcal{Y}$, we get the $\overset{!}{\otimes}$-symmetric monoidal structure on $\mathcal{Y}$ in the usual way. More explicitly, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Shv}(\mathcal{Y})$, we define

$$\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2 = \delta^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

2.4. **The** Ran **space/prestack.** The Ran space (or more precisely, prestack) of a scheme plays a central role in this paper. The Ran space, along with various objects on it, was first studied in the seminal book [BD04] in the case of curves, and was generalized to higher dimensions in [FG11]. In what follows, we will quickly review the main definitions and results. For proofs, unless otherwise specified, we refer the reader to [Gai15] and [FG11]. The topologically inclined reader could also find an intuitive introduction in [Ho15, §1].

2.4.1. For a scheme $X \in \mathrm{Sch}$, we will use $\mathrm{Ran}\, X$ to denote the following prestack: for each scheme $S \in \mathrm{Sch}$,

$$(\mathrm{Ran}\, X)(S) = \{\text{non-empty finite subsets of } X(S)\}$$

Alternatively, one has

$$\mathrm{Ran}\, X \simeq \operatorname*{colim}_{I \in \mathrm{fSet}^{\mathrm{surj,op}}} X^I$$

where $\mathrm{fSet}^{\mathrm{surj}}$ denotes the category of non-empty finite sets, where morphisms are surjections.

Using the fact that X is separated, one sees easily that $\mathrm{Ran}\, X$ is a pseudo-scheme. Moreover, when X is proper, $\mathrm{Ran}\, X$ is pseudo-proper.

2.4.2. *The $\otimes^\star$ monoidal structure.* There is a special monoidal structure on $\mathrm{Ran}\, X$ which we will use throughout the text: the $\otimes^\star$-monoidal structure.

Consider the following map

$$\mathrm{union} : \mathrm{Ran}\, X \times \mathrm{Ran}\, X \to \mathrm{Ran}\, X$$

given by the union of non-empty finite subsets of X. One can check that union is finitary pseudo-proper. Given two sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(\mathrm{Ran}\, X)$, we define

$$\mathcal{F} \otimes^\star \mathcal{G} = \mathrm{union}_!(\mathcal{F} \boxtimes \mathcal{G}).$$

This defines the $\otimes^\star$-monoidal structure on $\mathrm{Shv}(\mathrm{Ran}\, X)$.

2.4.3. Since union is pseudo-proper, $\mathcal{F} \otimes^\star \mathcal{G}$ has an easy presentation. Namely, for

$$\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k \in \mathrm{Shv}(\mathrm{Ran}\, X),$$

and any non-empty finite set I, we have the following

$$(\mathcal{F}_1 \otimes^\star \mathcal{F}_2 \otimes^\star \cdots \otimes^\star \mathcal{F}_k)|_{\overset{\circ}{X}{}^I} \simeq \bigoplus_{I = \bigcup_{i=1}^k I_i} \Delta^!_{\sqcup_{i=1}^k I_i \twoheadrightarrow \cup_{i=1}^k I_i} (\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k)|_{(\prod \overset{\circ}{X}{}^{I_i})_{\mathrm{disj}}} \tag{2.4.4}$$

where $(\prod \overset{\circ}{X}{}^{I_i})_{\mathrm{disj}}$ denotes the open subscheme of $\prod_i X^{I_i}$ where no two "coordinates" are equal, and where

$$\Delta_{\sqcup_{i=1}^k I_i \twoheadrightarrow \cup_{i=1}^k I_i} : X^I \hookrightarrow \prod_i X^{I_i}$$

is the map induced by the surjection

$$\bigsqcup_{i=1}^k I_i \twoheadrightarrow \bigcup_{i=1}^k I_i \simeq I.$$

2.4.5. *Factorizable sheaves.* Using the $\otimes^\star$-monoidal structure on $\mathrm{Shv}(\mathrm{Ran}\, X)$, one can talk about various types of algebras/coalgebras in $\mathrm{Shv}(\mathrm{Ran}\, X)$. The ones that are of importance to us in this papers are

$$\mathrm{ComAlg}^\star(\mathrm{Ran}\, X), \mathrm{Lie}^\star(\mathrm{Ran}\, X), \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X), \text{ and } \mathrm{coLie}^\star(\mathrm{Ran}\, X).$$

As the name suggests, these are used, respectively, to denote the categories of commutative algebras, Lie algebras, commutative co-algebras and co-Lie algebras in $\mathrm{Shv}(\mathrm{Ran}\, X)$ with respect to the $\otimes^\star$-monoidal structure defined above.

2.4.6. We use $\mathrm{Lie}^\star(X)$ and $\mathrm{coLie}^\star(X)$ to denote the full subcategories of $\mathrm{Lie}^\star(\mathrm{Ran}\, X)$ and $\mathrm{coLie}^\star(\mathrm{Ran}\, X)$ respectively, consisting of objects whose supports are inside the diagonal

$$\mathrm{ins}_X : X \hookrightarrow \mathrm{Ran}\, X$$

of $\mathrm{Ran}\, X$.

2.4.7. Let

$$j : (\mathrm{Ran}\, X)^n_{\mathrm{disj}} \to (\mathrm{Ran}\, X)^n$$

where $(\mathrm{Ran}\, X)^n_{\mathrm{disj}}$ is the open sub-prestack of $(\mathrm{Ran}\, X)^n$ defined by the following condition: for each scheme S, $(\mathrm{Ran}\, X)^n(S)$ consists of n non-empty subsets of $X(S)$, whose graphs are pair-wise disjoint.

2.4.8. Let

$$\mathcal{A} \in \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X).$$

Then, by definition, we have the following map (which is the co-multiplication of the commutative co-algebra structure)

$$\mathcal{A} \to \mathcal{A} \otimes^\star \mathcal{A} \otimes^\star \cdots \otimes^\star \mathcal{A} \simeq \mathrm{union}_!(\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A}).$$

Using the the unit map of the adjunction $j^* \dashv j_*$, we get the following map

$$\mathrm{union}_!(\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A}) \to \mathrm{union}_! j_* j^*(\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A}) \simeq (\mathrm{union} \circ j)_* j^!(\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A}),$$

where for the equivalence, we made use of (2.3.17) and (2.3.19).

Altogether, we get a map

$$\mathcal{A} \to (\mathrm{union} \circ j)_* j^!(\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A})$$

and hence, by adjunction and (2.3.17), we get a map

$$j^! \mathrm{union}^! \mathcal{A} \to j^!(\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A}). \tag{2.4.9}$$

Definition 2.4.10. $\mathcal{A}$ is a commutative factorization algebra if the map (2.4.9) is an equivalence for all n's. We use $\mathrm{coFact}^\star(X)$ to denote the full subcategory of $\mathrm{ComAlg}^\star(\mathrm{Ran}\, X)$ consisting of commutative factorization co-algebras.

2.4.11. Let

$$\mathcal{B} \in \mathrm{ComAlg}^\star(\mathrm{Ran}\, X).$$

Then, by definition, we have the following map (which is the multiplication of the commutative algebra structure)

$$\mathrm{union}_!(\mathcal{B} \boxtimes \mathcal{B} \boxtimes \cdots \boxtimes \mathcal{B}) \simeq \mathcal{B} \otimes^\star \mathcal{B} \otimes^\star \cdots \otimes^\star \mathcal{B} \to \mathcal{B}.$$

This induces the following map of sheaves

$$\mathcal{B} \boxtimes \cdots \boxtimes \mathcal{B} \to \mathrm{union}^! \mathcal{B}$$

on $(\mathrm{Ran}\, X)^n$, and hence, a map of sheaves

$$j^!(\mathcal{B} \boxtimes \cdots \boxtimes \mathcal{B}) \to j^! \mathrm{union}^! \mathcal{B}. \tag{2.4.12}$$

on $(\mathrm{Ran}\, X)^n_{\mathrm{disj}}$.

Definition 2.4.13. $\mathcal{B}$ is a commutative factorization algebra if the map (2.4.12) is an equivalence for all n's. We use $\mathrm{Fact}^\star(X)$ to denote the full subcategory of $\mathrm{ComAlg}^\star(\mathrm{Ran}\, X)$ consisting of commutative factorization algebras.

2.5. **Koszul duality.** In this subsection, we will quickly review various concepts and results in the theory of Koszul duality that are relevant to us. This theory, initially developed in [Qui69], illuminates the duality between commutative co-algebras and Lie algebras. It was further developed and generalized in the operadic setting in [GK94]. In the chiral/factorizable setting, the paper [FG11] provides us with necessary technical tools and language to carry out many topological arguments in the context of algebraic geometry. The results and definitions we review below could be found in [FG11] and [GR].

2.5.1. *Symmetric sequences.* Let Vect^Σ denote the category of symmetric sequences. Namely, its objects are collections

$$\mathcal{O} = \{\mathcal{O}(n), n \geq 1\},$$

where each $\mathcal{O}(n)$ is an object of Vect, acted on by the symmetric group Σ_n.

The infinity category Vect^Σ is equipped with a natural monoidal structure which makes the functor

$$\mathrm{Vect}^\Sigma \to \mathrm{Fun}(\mathrm{Vect}, \mathrm{Vect})$$

given by the following formula

$$\mathcal{O} \star V = \bigoplus_n (\mathcal{O}(n) \otimes V^{\otimes n})_{\Sigma_n}$$

symmetric monoidal.

2.5.2. *Operads and co-operads.* By an operad (resp. co-operad), we will mean an augmented associative algebra (resp. co-algebra) object in Vect^Σ, with respect to the monoidal structure described above. We use Op (resp. coOp) to denote the categories of operads (resp. co-operads).

In general, the Bar and coBar construction gives us the following pair of adjoint functors

$$\mathrm{Bar} : \mathrm{Op} \rightleftarrows \mathrm{coOp} : \mathrm{coBar}.$$

For an operad $\mathcal{O}$ (resp. co-operad $\mathcal{P}$), we also use $\mathcal{O}^\vee$ (resp. $\mathcal{P}^\vee$) to denote $\mathrm{Bar}(\mathcal{O})$ (resp. $\mathrm{coBar}(\mathcal{P})$).

Remark 2.5.3. In what follows, we will adopt the following convention: all our operads/co-operads will have the property that the augmentation map is an equivalence, when restricted to $\mathcal{O}(1)$ (resp. $\mathcal{P}(1)$). And under this restriction, one can show that the following unit map is an equivalence

$$\mathcal{O} \to \mathrm{coBar} \circ \mathrm{Bar}(\mathcal{O})$$

or in a slightly different notation

$$\mathcal{O} \to (\mathcal{O}^\vee)^\vee$$

when $\mathcal{O} \in \mathrm{Op}$ satisfying the assumption above.

2.5.4. *Algebras and co-algebras.* Let $\mathcal{C}$ be a stable presentable symmetric monoidal ∞-category compatibly tensored over Vect. Then, an operad $\mathcal{O}$ (resp. co-operad $\mathcal{P}$) naturally defines a monad (resp. comonad) on $\mathcal{C}$.

Thus, for an operad $\mathcal{O}$ (resp. co-operad $\mathcal{P}$), one can talk about the category of algebras $\mathcal{O}\text{-alg}(\mathcal{C})$ (resp. co-algebras $\mathcal{P}\text{-coalg}(\mathcal{C})$) in $\mathcal{C}$ with respect to the operad $\mathcal{O}$ (resp. co-operad $\mathcal{P}$).

As usual (as for any augmented monad), one has the following pairs of adjoint functors

$$\mathrm{Free}_{\mathcal{O}} : \mathcal{C} \rightleftarrows \mathcal{O}\text{-alg}(\mathcal{C}) : \mathrm{oblv}_{\mathcal{O}} \qquad \text{and} \qquad \mathrm{Bar}_{\mathcal{O}} : \mathcal{O}\text{-coalg}(\mathcal{C}) \rightleftarrows \mathcal{C} : \mathrm{triv}_{\mathcal{O}}$$

for an operad $\mathcal{O}$, and similarly, the following pairs of adjoint functors

$$\mathrm{oblv}_{\mathcal{P}} : \mathcal{P}\text{-coalg}(\mathcal{C}) \rightleftarrows \mathcal{C} : \mathrm{coFree}_{\mathcal{P}} \qquad \text{and} \qquad \mathrm{cotriv}_{\mathcal{P}} : \mathcal{C} \rightleftarrows \mathcal{P}\text{-coalg}(\mathcal{C}) : \mathrm{coBar}_{\mathcal{P}}$$

for a co-operad $\mathcal{P}$.

2.5.5. *Koszul duality.* The functors mentioned above could be lifted to get a pair of adjoint functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathcal{O}\text{-alg}(\mathcal{C}) \rightleftarrows \mathcal{P}\text{-coalg}(\mathcal{C}) : \mathrm{coBar}^{\mathrm{enh}} \tag{2.5.6}$$

where

$$\mathrm{oblv}_{\mathcal{P}} \circ \mathrm{Bar}^{\mathrm{enh}}_{\mathcal{O}} \simeq \mathrm{Bar}_{\mathcal{O}} \qquad \text{and} \qquad \mathrm{oblv}_{\mathcal{O}} \circ \mathrm{coBar}^{\mathrm{enh}}_{\mathcal{P}} \simeq \mathrm{coBar}_{\mathcal{P}}.$$

2.5.7. *Turning Koszul duality into an equivalence.* In general, the pair of adjoint functors at (2.5.6) is not an equivalence. One of the main achievements of [FG11] is to formulate a precise condition on the base category $\mathcal{C}$, namely the pro-nilpotent condition,[8] which turns (2.5.6) into an equivalence.

One of the main technical points of our paper is to prove another case where Koszul duality is still an equivalence, even when the categories involved are not pro-nilpotent.

The two main instances of Koszul duality that are important in this paper are the duality between Lie-algebras and ComCoAlg-algebras, and coLie-algebras and ComAlg-algebras.

2.5.8. *The case of* Lie *and* ComCoAlg. We have the following equivalence of co-operads (see [FG11]):

$$\mathrm{Lie}^\vee \simeq \mathrm{ComCoAlg}[1],$$

where

$$\mathrm{ComCoAlg}[1](n) \simeq k[n-1]$$

is equipped with the sign action of the symmetric group Σ_n.

2.5.9. Equivalently, the functor

$$[1] : \mathcal{C} \to \mathcal{C}$$

gives rise to an equivalence of categories

$$[1] : \mathrm{ComCoAlg}[1](\mathcal{C}) \simeq \mathrm{ComCoAlg}(\mathcal{C}).$$

2.5.10. This gives us the following diagram

$$\begin{tikzcd}
\mathrm{Lie}(\mathcal{C}) \arrow[r, shift left, "\mathrm{Bar}_{\mathrm{Lie}}"] \arrow[d, shift left, "{[1]}"] \arrow[dr, shift left, "\mathrm{Chev}"] & \mathrm{ComCoAlg}[1](\mathcal{C}) \arrow[l, shift left, "\mathrm{coBar}_{\mathrm{ComCoAlg}[1]}"] \arrow[d, shift left, "{[1]}"] \\
\mathrm{Lie}[-1](\mathcal{C}) \arrow[r, shift left, "\mathrm{Bar}_{\mathrm{Lie}[-1]}"] \arrow[u, shift left, "{[-1]}"] & \mathrm{ComCoAlg}(\mathcal{C}) \arrow[l, shift left, "\mathrm{coBar}_{\mathrm{ComCoAlg}}"] \arrow[u, shift left, "{[-1]}"] \arrow[ul, shift left, "{\mathrm{Prim}[-1]}"]
\end{tikzcd}$$

We usually use Chev to denote

$$[1] \circ \mathrm{Bar}_{\mathrm{Lie}} \simeq \circ \mathrm{Bar}_{\mathrm{Lie}[-1]} \circ [1]$$

and Prim$[-1]$ to denote

$$\mathrm{coBar}_{\mathrm{ComCoAlg}[1]} \circ [-1] \simeq [-1] \circ \mathrm{coBar}_{\mathrm{ComCoAlg}}. \tag{2.5.11}$$

2.5.12. *The case of* coLie *and* ComAlg. Dually, we have the following equivalence of co-operads

$$\mathrm{ComAlg}^\vee \simeq \mathrm{coLie}[1],$$

and similar to the above, the functor

$$[1] : \mathcal{C} \to \mathcal{C}$$

gives rise to an equivalence of categories

$$[1] : \mathrm{coLie}[1](\mathcal{C}) \simeq \mathrm{coLie}(\mathcal{C}).$$

[8]The interested reader could read more about this in [FG11], since we do not need this fact in the current work.

2.5.13. This gives us the following diagram

$$\begin{tikzcd}
\mathrm{ComAlg}(\mathcal{C}) \arrow[r, shift left, "\mathrm{Bar}_{\mathrm{ComAlg}}"] \arrow[d, shift left, "{[1]}"] & \mathrm{coLie}[1](\mathcal{C}) \arrow[l, shift left, "\mathrm{coBar}_{\mathrm{coLie}[1]}"] \arrow[d, shift left, "{[1]}"] \arrow[dl, shift left, "\mathrm{coChev}"] \\
\mathrm{ComAlg}[-1](\mathcal{C}) \arrow[r, shift left, "\mathrm{Bar}_{\mathrm{ComAlg}[-1]}"] \arrow[u, shift left, "{[-1]}"] \arrow[ur, shift left, "\mathrm{coPrim}[1]"] & \mathrm{coLie}(\mathcal{C}) \arrow[l, shift left, "\mathrm{coBar}_{\mathrm{coLie}}"] \arrow[u, shift left, "{[-1]}"]
\end{tikzcd}$$

As above, we usually use coChev to denote

$$[-1] \circ \mathrm{coBar}_{\mathrm{coLie}} \simeq \mathrm{coBar}_{\mathrm{coLie}[1]} \circ [-1]$$

and coPrim$[1]$ to denote

$$[1] \circ \mathrm{Bar}_{\mathrm{ComAlg}} \simeq \mathrm{Bar}_{\mathrm{ComAlg}[-1]} \circ [1].$$

3. Turning Koszul duality into an equivalence

The goal of this section is to prove Theorem 1.5.4. We will start by examining the special case where X is just a point, i.e. $\mathrm{Shv}(\mathrm{Ran}\, X) \simeq \mathrm{Shv}(X) \simeq \mathrm{Vect}$, and prove that Koszul duality provides a natural equivalence of categories

$$\mathrm{Chev} : \mathrm{Lie}(\mathrm{Vect}^{\leq -1}) \simeq \mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2}) : \mathrm{Prim}[-1].$$

Even though this case is not strictly needed in the proof of the general case, it is interesting in its own right, as it allows us to predict the correct connectivity condition needed in the general case, whose precise statement and proof are presented in the final subsection. We recommend the reader to first read the case of Vect, since it shares the same strategy as the main proof without the additional numerical complexity.

3.1. The case of Lie- and ComCoAlg-algebras inside Vect.

We will now prove the following

Theorem 3.1.1. Chev *and* Prim$[-1]$ *give rise to a pair of mutually inverse functors*

$$\mathrm{Chev} : \mathrm{Lie}(\mathrm{Vect}^{\leq -1}) \rightleftarrows \mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2}) : \mathrm{Prim}[-1]$$

Remark 3.1.2. Since Chev is defined as a colimit, it is easy to see that $\mathrm{Chev}|_{\mathrm{Lie}(\mathrm{Vect}^{\leq -1})}$ lands in the correct subcategory cut out by the connectivity assumption $\mathrm{Vect}^{\leq -2}$. However, a priori, the same is not obvious for Prim$[-1]$, being defined as a limit. It is, however, clear from the proof below that this in fact holds.

Remark 3.1.3. Unless otherwise specified, our functors will be automatically restricted to the subcategories with the appropriate connectivity conditions. For example, we will write Chev instead of $\mathrm{Chev}|_{\mathrm{Lie}(\mathrm{Vect}^{\leq -1})}$ in most cases.

Remark 3.1.4. Note that Theorem 3.1.1 can be proved more generally for a presentable symmetric monoidal stable infinity category with a t-structure satisfying some mild properties. The pair of operad and co-operad Lie and ComCoAlg could also be made more general. The curious readers could take a look at the remarks at the end of this subsection.

3.1.5. We follow a similar strategy as in [FG11]. Namely, to prove that Chev and Prim$[-1]$ are mutually inverse functors, it suffices to show that the left adjoint functor, Chev, is fully-faithful, and the right adjoint functor, Prim$[-1]$ is conservative.

We start with the following

Lemma 3.1.6. *The functor* $\mathrm{Prim}[-1]|_{\mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2})}$ *satisfies the following conditions*

(i) Prim$[-1]$ *commutes with sifted colimits.*
(ii) *The natural map*

$$\mathrm{Free}_{\mathrm{Lie}} \to \mathrm{Prim}[-1] \circ \mathrm{triv}_{\mathrm{ComCoAlg}}$$

is an equivalence.

As in [FG11, §4.1.8], this immediately implies the following corollary. For the sake of completeness, we include the proof here.

Corollary 3.1.7. $\mathrm{Chev}|_{\mathrm{Lie}(\mathrm{Vect}^{\le -1})}$ *is fully faithful.*

Proof. It suffices to show that the unit map

$$\mathrm{id} \to \mathrm{Prim}[-1] \circ \mathrm{Chev}$$

is an equivalence. Since $\mathrm{Prim}[-1]$ commutes with sifted colimits by part (i) of Lemma 3.1.6, it suffices to show that the following is an equivalence

$$\mathrm{Free}_{\mathrm{Lie}} \to \mathrm{Prim}[-1] \circ \mathrm{Chev} \circ \mathrm{Free}_{\mathrm{Lie}},$$

since any Lie-algebra could be written as a sifted colimit of the free ones.[9] However, we know that (even without the connectivity condition)

$$\mathrm{Chev} \circ \mathrm{Free}_{\mathrm{Lie}} \simeq \mathrm{triv}_{\mathrm{Lie}}$$

and hence, it suffices to show that

$$\mathrm{Free}_{\mathrm{Lie}} \to \mathrm{Prim}[-1] \circ \mathrm{Chev}.$$

But now, we are done due to part (ii) of Lemma 3.1.6. □

3.1.8. Before proving Lemma 3.1.6, we start with a couple of preliminary observations. In essence, the lemma is a statement about commuting limits and colimits. In a stable infinity category, if, for instance, the limit is a finite one, then one can always do that. In our situation, coBar is causing troubles because it is defined as an infinite limit.

The main idea of the proof is that when

$$c \in \mathrm{ComCoAlg}(\mathrm{Vect}^{\le -2}),$$

then even though

$$\mathrm{coBar}_{\mathrm{ComCoAlg}}(c)$$

is computed as an infinite limit, each of its cohomological degree will be controlled by finitely many of terms in the limit.

3.1.9. For brevity's sake, we will use $\mathcal{P}$ to denote the co-operad ComCoAlg. Recall that in general, for any

$$c \in \mathrm{ComCoAlg}(\mathrm{Vect}),$$

we have

$$\mathrm{coBar}_{\mathcal{P}}(c) = \mathrm{Tot}(\mathrm{coBar}^{\bullet}_{\mathcal{P}}(c))$$

where $\mathrm{coBar}^{\bullet}_{\mathcal{P}}(c)$ is a co-simplicial object.

Let

$$\mathrm{coBar}^{n}_{\mathcal{P}}(c) = \mathrm{Tot}(\mathrm{coBar}^{\bullet}_{\mathcal{P}}(c)|_{\Delta^{\le n}})$$

be the limit over the restriction of the co-simplicial object to $\Delta^{\le n}$. Then we have the following tower

$$c \simeq \mathrm{coBar}^{0}_{\mathcal{P}}(c) \leftarrow \mathrm{coBar}^{1}_{\mathcal{P}}(c) \leftarrow \cdots \leftarrow \mathrm{coBar}^{n}_{\mathcal{P}}(c) \leftarrow \cdots$$

and

$$\mathrm{coBar}_{\mathcal{P}}(c) \simeq \lim_{n} \mathrm{coBar}^{n}_{\mathcal{P}}(c).$$

Lemma 3.1.10. *Let*

$$c \in \mathrm{ComCoAlg}(\mathrm{Vect}^{\le -2}).$$

Then, for all $n \ge 0$, the following natural map

$$\mathrm{tr}_{\ge -2^{n+1}+n+1} \mathrm{coBar}^{n}_{\mathcal{P}}(c) \to \mathrm{tr}_{\ge -2^{n+1}+n+1} \mathrm{coBar}^{n-1}_{\mathcal{P}}(c).$$

is an equivalence.

[9] This fact applies to the category of algebras over any operad in general.

Proof. Let $F^n(c)$ denote the difference between $\mathrm{coBar}^n_{\mathcal{P}}(c)$ and $\mathrm{coBar}^{n-1}_{\mathcal{P}}(c)$,

$$F^n(c) = \mathrm{Fib}(\mathrm{coBar}^n_{\mathcal{P}}(c) \to \mathrm{coBar}^{n-1}_{\mathcal{P}}(c)).$$

Then for

$$c \in \mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2}),$$

we see that

$$F^n(c) \in \mathrm{Vect}^{\leq -2\cdot 2^n + n} \simeq \mathrm{Vect}^{\leq -2^{n+1}+n}.$$

Indeed, this is because of the fact that $c \in \mathrm{Vect}^{\leq -2}$ and in the direct sum

$$\mathrm{coBar}^{\bullet}_{\mathcal{P}}(c)([n]) = \bigoplus_{m\geq 1} \mathcal{P}^{\star n}(m) \otimes_{S_m} c^{\otimes m},$$

$m = 2^n$ is the first summand where we have non-degenerate "(co-)cells."

As a consequence,

$$\mathrm{tr}_{\geq -2^{n+1}+n+1} \mathrm{coBar}^n_{\mathcal{P}}(c) \to \mathrm{tr}_{\geq -2^{n+1}+n+1} \mathrm{coBar}^{n-1}_{\mathcal{P}}(c)$$

is an equivalence and we are done. □

Using the fact that infinite products preserve $\mathrm{Vect}^{\leq 0}$, the lemma above directly implies the following

Corollary 3.1.11. *Let*

$$c \in \mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2}).$$

Then, for any n, the following natural map

$$\mathrm{tr}_{\geq -n} \mathrm{coBar}_{\mathcal{P}}(c) \to \mathrm{tr}_{\geq -n} \mathrm{coBar}^m_{\mathcal{P}}(c)$$

is an equivalence for all $m \gg 0$, where the bound depends only on n.

Proof of Lemma 3.1.6. The proof is now simple. In fact, we will only prove part (i), as the other one is almost identical. Note that due to (2.5.11), what we prove about $\mathrm{coBar}_{\mathcal{P}}$ implies the corresponding statement of $\mathrm{Prim}[-1]$, up to a shift.

It suffices to show that for all n, we have

$$\mathrm{tr}_{\geq -n} \mathrm{coBar}_{\mathcal{P}}(\operatorname*{colim}_{\alpha} c_\alpha) \simeq \mathrm{tr}_{\geq -n} \operatorname*{colim}_{\alpha} \mathrm{coBar}_{\mathcal{P}}(c_\alpha)$$

where α runs over some sifted diagram. But now, from Corollary 3.1.11, for all $m \gg 0$, we have

$$\begin{aligned}\mathrm{tr}_{\geq -n} \mathrm{coBar}_{\mathcal{P}}(\operatorname*{colim}_{\alpha} c_\alpha) &\simeq \mathrm{tr}_{\geq -n} \mathrm{coBar}^m_{\mathcal{P}}(\operatorname*{colim}_{\alpha} c_\alpha) \simeq \mathrm{tr}_{\geq -n} \operatorname*{colim}_{\alpha} \mathrm{coBar}^m_{\mathcal{P}}(c_\alpha) \simeq \operatorname*{colim}_{\alpha} \mathrm{tr}_{\geq -n} \mathrm{coBar}^m_{\mathcal{P}}(c_\alpha)\\ &\simeq \operatorname*{colim}_{\alpha} \mathrm{tr}_{\geq -n} \mathrm{coBar}_{\mathcal{P}}(c_\alpha) \simeq \mathrm{tr}_{\geq -n} \operatorname*{colim}_{\alpha} \mathrm{coBar}_{\mathcal{P}}(c_\alpha).\end{aligned}$$

□

Remark 3.1.12. The cohomological estimate done above implies that

$$\mathrm{coBar}_{\mathrm{ComCoAlg}}(c) \in \mathrm{Lie}[-1](\mathrm{Vect}^{\leq -2}),$$

or equivalently

$$\mathrm{Prim}[-1](c) \in \mathrm{Lie}(\mathrm{Vect}^{\leq -1}),$$

when

$$c \in \mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2}).$$

Indeed, from Corollary 3.1.11, we know that for some $m \gg 0$,

$$\mathrm{tr}_{\geq -1} \mathrm{coBar}_{\mathcal{P}}(c) \simeq \mathrm{tr}_{\geq -1} \mathrm{coBar}^m_{\mathcal{P}}(c),$$

and moreover, a downward induction using Lemma 3.1.10 shows that

$$\mathrm{tr}_{\geq -1} \mathrm{coBar}^m_{\mathcal{P}}(c) \simeq \mathrm{tr}_{\geq -1} \mathrm{coBar}^0_{\mathcal{P}}(c) \simeq \mathrm{tr}_{\geq -1} c \simeq 0.$$

3.1.13. The following result will conclude the proof of Theorem 3.1.1.

Lemma 3.1.14. *The functor*
$$\mathrm{Prim}[-1] : \mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2}) \to \mathrm{Lie}(\mathrm{Vect}^{\leq -1})$$
is conservative.

Proof. It suffices to show that
$$\mathrm{coBar}_{\mathcal{P}} : \mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2}) \to \mathrm{Lie}[-1](\mathrm{Vect}^{\leq -2})$$
is conservative, and we will prove that by contradiction. Namely, let
$$f : c_1 \to c_2$$
be a morphism in $\mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2})$ such that f is not an equivalence. Suppose that
$$\mathrm{coBar}_{\mathcal{P}}(f) : \mathrm{coBar}_{\mathcal{P}}(c_1) \to \mathrm{coBar}_{\mathcal{P}}(c_2)$$
is an equivalence, we will derive a contradiction.

Let k be the smallest number such that
$$\mathrm{tr}_{\geq -k}(f) : \mathrm{tr}_{\geq -k}\, c_1 \to \mathrm{tr}_{\geq -k}\, c_2$$
is not an equivalence. Now, by Corollary 3.1.11, we know that there is some $m \gg 0$ such that
$$\mathrm{tr}_{\geq -k}\, \mathrm{coBar}_{\mathcal{P}}(c_i) \simeq \mathrm{tr}_{\geq -k}\, \mathrm{coBar}^m_{\mathcal{P}}(c_i)$$
for $i \in \{1, 2\}$. Thus, we know that
$$\mathrm{tr}_{\geq -k}\, \mathrm{coBar}^m_{\mathcal{P}}(c_1) \to \mathrm{tr}_{\geq -k}\, \mathrm{coBar}^m_{\mathcal{P}}(c_2)$$
is an equivalence.

By an estimate similar to that of Lemma 3.1.10, we see that
$$\mathrm{tr}_{\geq -k}\, F^n(c_1) \simeq \mathrm{tr}_{\geq -k}\, F^n(c_2)$$
for all $n \geq 1$. Indeed, the difference between $F^n(c_1)$ and $F^n(c_2)$ lies in cohomological degrees
$$\leq -2(2^n - 1) - k + n = -2^{n+1} - k + n + 2 < -k, \quad \forall n \geq 1.$$

And hence, a downward induction, starting from $n = m$, using the diagram
$$\begin{tikzcd}
F^n(c_1) \arrow[r] \arrow[d] & \mathrm{coBar}^n_{\mathcal{P}}(c_1) \arrow[r] \arrow[d] & \mathrm{coBar}^{n-1}_{\mathcal{P}}(c_1) \arrow[d] \\
F^n(c_2) \arrow[r] & \mathrm{coBar}^n_{\mathcal{P}}(c_2) \arrow[r] & \mathrm{coBar}^{n-1}_{\mathcal{P}}(c_2)
\end{tikzcd}$$
implies that
$$\tau_{\geq -k} c_1 \simeq \tau_{\geq -k} c_2,$$
which contradicts our original assumption. Hence, we are done. □

Remark 3.1.15. Note that the proof we gave above could be carried out in a more general setting. Namely, the only properties of Vect that we used are

(i) The symmetric monoidal structure is right exact (namely, it preserved $\mathrm{Vect}^{\leq 0}$).
(ii) The t-structure on Vect is left separated.
(iii) Infinite products preserve $\mathrm{Vect}^{\leq 0}$.

Remark 3.1.16. We can also replace the operad Lie by any operad $\mathcal{O}$ such that

(i) $\mathcal{O}$ is classical, i.e. it lies in the heart of the t-structure of Vect.
(ii) $\mathcal{O}^\vee[-1]$ is also classical.
(iii) $\mathcal{O}(1) \simeq \Lambda$ (as we already assume throughout this paper).

3.2. **Higher enveloping algebras.** This subsection serves as the topological analogue of the results proved in the next one. The main reference of this part is [GR].

3.2.1. Let
$$\mathfrak{g} \in \mathrm{Lie}(\mathrm{Vect}).$$
Then one can form its E_n-universal enveloping algebra
$$U_{E_n}(\mathfrak{g}) \in E_n(\mathrm{Vect})$$
by applying the following sequence of functors
$$\mathrm{Lie}(\mathrm{Vect}) \xrightarrow{\Omega^{\times n} \simeq [-n]} E_n(\mathrm{Lie}(\mathrm{Vect})) \xrightarrow{E_n(\mathrm{Chev})} E_n(\mathrm{ComCoAlg}(\mathrm{Vect})) \xrightarrow{\mathrm{oblv}_{\mathrm{ComCoAlg}}} E_n(\mathrm{Vect})$$
where $E_n(\mathrm{Lie}(\mathrm{Vect}))$ and $E_n(\mathrm{ComCoAlg}(\mathrm{Vect}))$ are categories of E_n-algebras with respect to the Cartesian monoidal structure on Lie(Vect) and ComCoAlg(Vect) respectively (note that the latter on is just the given by $\otimes$ in Vect).

3.2.2. It is proved in [GR] that $[-n]$ induces an equivalence
$$[-n] : \mathrm{Lie}(\mathrm{Vect}) \simeq E_n(\mathrm{Lie}(\mathrm{Vect})) : [n].$$
Moreover, we know from Theorem 3.1.1 that
$$E_n(\mathrm{Chev}) : E_n(\mathrm{Lie}(\mathrm{Vect}^{\leq -1})) \to E_n(\mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2})).$$
Thus, we get the following equivalence of categories
$$\mathrm{Lie}(\mathrm{Vect}^{\leq -n-1}) \simeq E_n(\mathrm{ComCoAlg}(\mathrm{Vect}^{\leq -2})). \tag{3.2.3}$$

3.2.4. The equivalence (3.2.3) is precisely what we are looking for in the context of factorization algebras on the Ran space in the following subsection. One part of the work is to find connectivity assumptions on $\mathrm{Shv}(\mathrm{Ran}\,X)$ which mirror those in $\mathrm{Vect}^{\leq -n-1}$ and $\mathrm{Vect}^{\leq -2}$ respectively.

3.3. The case of $\mathrm{Lie}^\star$- and $\mathrm{ComCoAlg}^\star$-algebras on $\mathrm{Ran}\,X$.

We come to the precise formulation and the proof of Theorem 1.5.4.

Definition 3.3.1. Let $\mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_{cA}}$ and $\mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_L}$ denote the full subcategory of $\mathrm{Shv}(\mathrm{Ran}\,X)$ consisting of sheaves $\mathcal{F}$ such that for all non-empty finite sets I,
$$\mathcal{F}|_{\overset{\circ}{X}{}^I} \in \mathrm{Shv}(\overset{\circ}{X}{}^I)^{\leq (-1-d)|I|-1},$$
and respectively,
$$\mathcal{F}|_{\overset{\circ}{X}{}^I} \in \mathrm{Shv}(\overset{\circ}{X}{}^I)^{\leq (-1-d)|I|}.$$
Here, we use the perverse t-structure, and X is a scheme of pure dimension d.

Notation 3.3.2. We will use
$$\mathrm{Lie}^\star(\mathrm{Ran}\,X)^{\leq c_L} \qquad \text{and} \qquad \mathrm{ComCoAlg}^\star(\mathrm{Ran}\,X)^{\leq c_{cA}}$$
to denote
$$\mathrm{Lie}^\star(\mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_L}) \qquad \text{and} \qquad \mathrm{ComCoAlg}^\star(\mathrm{Ran}\,X)^{\leq c_{cA}}$$
respectively.

With these connectivity assumptions in mind, we will prove the following

Theorem 3.3.3. *We have the following commutative diagram*
$$\begin{tikzcd} \mathrm{Lie}^\star(\mathrm{Ran}\,X)^{\leq c_L} \arrow[r, equal, "\mathrm{Chev}", "{\mathrm{Prim}[-1]}"'] & \mathrm{ComCoAlg}^\star(\mathrm{Ran}\,X)^{\leq c_{cA}} \\ \mathrm{Lie}^\star(X)^{\leq c_L} \arrow[u, hook] \arrow[r, equal, "\mathrm{Chev}", "{\mathrm{Prim}[-1]}"'] & \mathrm{coFact}^\star(X)^{\leq c_{cA}} \arrow[u, hook] \end{tikzcd} \tag{3.3.4}$$
where $\leq c_L$ and $\leq c_{cA}$ denote the connectivity constraints given in Definition 3.3.1, and where Chev *and* $\mathrm{Prim}[-1]$ *are the functors coming from Koszul duality.*

Remark 3.3.5. As in the case of Vect, our functors will be automatically restricted to the subcategories with the appropriate connectivity conditions, unless otherwise specified. For example, we will write Chev instead of $\mathrm{Chev}|_{\mathrm{Lie}^\star(\mathrm{Ran}\,X)^{\leq c_L}}$ in most cases.

Remark 3.3.6. As in the case of Vect, the pair of operad and co-operad Lie and ComCoAlg could be replaced by an operad $\mathcal{O}$ and its Koszul dual $\mathcal{O}^\vee$ such that[10]

(i) $\mathcal{O}$ is classical, i.e. it lies in the heart of the t-structure of Vect.
(ii) $\mathcal{O}^\vee[-1]$ is also classical.
(iii) $\mathcal{O}(1) \simeq \Lambda$ (as we already assume throughout this paper).

We start with a preliminary lemma, which ensures that the categories

$$\mathrm{Lie}^\star(\mathrm{Ran}\,X)^{\leq c_L} \qquad \text{and} \qquad \mathrm{ComCoAlg}^\star(\mathrm{Ran}\,X)^{\leq c_{cA}}$$

are actually well-defined.

Lemma 3.3.7. *The subcategories* $\mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_L}$ *and* $\mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_{cA}}$ *are preserved under the* $\otimes^\star$*-monoidal structure on* $\mathrm{Shv}(\mathrm{Ran}\,X)$*.*

Proof of Lemma 3.3.7. Recall from (2.4.4) that if

$$\mathcal{F}_1, \dots, \mathcal{F}_k \in \mathrm{Shv}(\mathrm{Ran}\,X),$$

then from the definition of $\otimes^\star$, we have

$$(\mathcal{F}_1 \otimes^\star \cdots \otimes^\star \mathcal{F}_k)|_{\overset{\circ}{X}{}^I} \simeq \bigoplus_{I = \cup_{i=1}^k I_i} \Delta^!_{\sqcup_{i=1}^k I_i \twoheadrightarrow \cup_{i=1}^k I_i} (\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k)|_{(\prod_{i=1}^k \overset{\circ}{X}{}^{I_i})_{\mathrm{disj}}}. \tag{3.3.8}$$

Now, suppose that

$$\mathcal{F}_1, \dots, \mathcal{F}_k \in \mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_L},$$

then we see that each summand in (3.3.8) lies in perverse cohomological degrees

$$\begin{aligned} &\leq (-1-d)\sum_{i=1}^k |I_i| + d\left(\sum_{i=1}^k |I_i| - |I|\right) \\ &\leq -\sum_{i=1}^k |I_i| - d|I| \\ &\leq (-1-d)|I|. \end{aligned}$$

Here, the first inequality is due to the fact that the map

$$\overset{\circ}{X}{}^I \to \prod_{i=1}^k (\overset{\circ}{X}{}^{I_i})_{\mathrm{disj}}$$

is a regular embedding, and that the (perverse) cohomological amplitude of the !-pullback along a regular embedding is equal to the codimension. Thus, this implies that

$$\mathcal{F}_1 \otimes^\star \cdots \otimes^\star \mathcal{F}_k \in \mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_L}.$$

Similarly, suppose that

$$\mathcal{F}_1, \dots, \mathcal{F}_k \in \mathrm{Shv}(\mathrm{Ran}\,X)^{\leq c_{cA}},$$

then each summand in (3.3.8) lies in perverse cohomological degrees

$$\begin{aligned} &\leq (-1-d)\sum_{i=1}^k |I_i| - k + d\left(\sum_{i=1}^k |I_i| - |I|\right) \\ &\leq -\sum_{i=1}^k |I_i| - k - d|I| \\ &\leq (-1-d)|I| - 1. \end{aligned} \tag{3.3.9}$$

[10]Note that for a general operad $\mathcal{O}$, only the first row of (3.3.4) makes sense.

Thus,

$$\mathcal{F}_1 \otimes^\star \cdots \otimes^\star \mathcal{F}_k \in \mathrm{Shv}(\mathrm{Ran}\, X)^{\leq c_{cA}},$$

which concludes the proof. □

3.3.10. Back to Theorem 3.3.3. First, we will prove the equivalence on the top row of (3.3.4). And then, we will show that it induces an equivalence between the corresponding sub-categories on the bottom row.

As in the case of Vect, to prove that Chev and Prim[−1] are mutually inverse functors, it suffices to show that Chev is fully-faithful, and Prim[−1] is conservative. The following lemma will help us achieve this goal.

Lemma 3.3.11. *The functor* $\mathrm{Prim}[-1]|_{\mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}}}$ *satisfies the following conditions (see Remark 3.3.5)*

(i) Prim[−1] *commutes with sifted colimits.*
(ii) The natural map

$$\mathrm{Free}_{\mathrm{Lie}} \to \mathrm{Prim}[-1] \circ \mathrm{triv}_{\mathrm{ComCoAlg}}$$

is an equivalence.

As in Corollary 3.1.7, this immediately implies the following

Corollary 3.3.12. $\mathrm{Chev}|_{\mathrm{Lie}^\star(\mathrm{Ran}\, X)^{\leq c_L}}$ *is fully faithful.*

3.3.13. In essence, the strategy we follow here is identical to that of the Vect case even though the actual execution might seem somewhat more involved. The main observation (which is new compared to the case of Vect) is that to prove the equivalences involved in Lemma 3.3.11, it suffices to prove them after after pulling back to $\overset{\circ}{X}{}^I$ for each non-empty finite set I.

3.3.14. In general, for any

$$\mathcal{A} \in \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}},$$

we have

$$\mathrm{coBar}_{\mathrm{ComCoAlg}}(\mathcal{A}) = \mathrm{Tot}(\mathrm{coBar}^\bullet_{\mathrm{ComCoAlg}}(\mathcal{A})),$$

where $\mathrm{coBar}^\bullet_{\mathrm{ComCoAlg}}(\mathcal{A})$ is a co-simplicial object.

Let

$$\mathrm{coBar}^n_{\mathrm{ComCoAlg}}(\mathcal{A}) = \mathrm{Tot}(\mathrm{coBar}^\bullet_{\mathrm{ComCoAlg}}(\mathcal{A})|_{\Delta^{\leq n}}).$$

Then, we have the following tower

$$\mathcal{A} \simeq \mathrm{coBar}^0_{\mathrm{ComCoAlg}}(\mathcal{A}) \leftarrow \mathrm{coBar}^1_{\mathrm{ComCoAlg}}(\mathcal{A}) \leftarrow \cdots$$

and

$$\mathrm{coBar}_{\mathrm{ComCoAlg}}(\mathcal{A}) \simeq \lim_n \mathrm{coBar}^n_{\mathrm{ComCoAlg}}(\mathcal{A}).$$

3.3.15. Let

$$F^n(\mathcal{A}) = \mathrm{Fib}(\mathrm{coBar}^n_{\mathrm{ComCoAlg}}(\mathcal{A}) \to \mathrm{coBar}^{n-1}_{\mathrm{ComCoAlg}}(\mathcal{A})),$$

and let I be a non-empty finite set. Using the same argument as in the case of Vect in combination with the cohomological estimate (3.3.9), we see that $F^n(\mathcal{A})|_{\overset{\circ}{X}{}^I}$ lives in cohomological degrees

$$\begin{aligned} &\leq (-1-d)\sum_{i=1}^{2^n} |I_i| - 2^n + d\left(\sum_{i=1}^{2^n} |I_i| - |I|\right) + n \\ &= -\sum_{i=1}^{2^n} |I_i| - 2^n - d|I| + n \\ &\leq -2^{n+1} - d|I| + n \end{aligned}$$

which goes to $-\infty$ when $n \to \infty$.

This gives us the following analog of Lemma 3.1.10.

Lemma 3.3.16. *Let*
$$\mathcal{A} \in \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}}.$$
Then, for any n and I, the following natural map
$$\mathrm{tr}_{\geq -2^{n+1}-d|I|+n+1}(\mathrm{coBar}^n_{\mathrm{ComCoAlg}}(\mathcal{A})|_{\overset{\circ}{X}{}^I}) \to \mathrm{tr}_{\geq -2^{n+1}-d|I|+n+1}(\mathrm{coBar}^{n-1}_{\mathrm{ComCoAlg}}(\mathcal{A})|_{\overset{\circ}{X}{}^I})$$
is an equivalence.

This implies the following result, which is parallel to Corollary 3.1.11.

Corollary 3.3.17. *Let*
$$\mathcal{A} \in \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}}.$$
Then, for any n and I, the following natural map
$$\mathrm{tr}_{\geq -n}(\mathrm{coBar}_{\mathrm{ComCoAlg}}(\mathcal{A})|_{\overset{\circ}{X}{}^I}) \to \mathrm{tr}_{\geq -n}(\mathrm{coBar}^m_{\mathrm{ComCoAlg}}(\mathcal{A})|_{\overset{\circ}{X}{}^I})$$
is an equivalence, when $m \gg 0$ depending only on n and I.

3.3.18. Now, Lemma 3.3.16 and Corollary 3.3.17 allow us to conclude the proof of Lemma 3.3.11, and hence, Corollary 3.3.12, as in the Vect case.

Remark 3.3.19. Note that when X is a point, namely when $d = \dim X = 0$, the cohomological estimates in Lemma 3.3.16 recover those of Lemma 3.1.10.

To finish with the top equivalence in (3.3.4), we need the following

Lemma 3.3.20. *The functor*
$$\mathrm{Prim}[-1] : \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}} \to \mathrm{Lie}^\star(\mathrm{Ran}\, X)^{\leq c_L}$$
is conservative.

Proof. It suffices to show that
$$\mathrm{coBar}_{\mathrm{ComCoAlg}} : \mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}} \to \mathrm{Lie}^\star[-1](\mathrm{Ran}\, X)^{\leq c_{cA}}$$
is conservative, and we will do so by contradiction. Namely, let
$$f : \mathcal{A}_1 \to \mathcal{A}_2$$
be a morphism in $\mathrm{ComCoAlg}^\star(\mathrm{Ran}\, X)^{\leq c_{cA}}$ that is not an equivalence. Suppose that
$$\mathrm{coBar}_{\mathrm{ComCoAlg}}(f) : \mathrm{coBar}_{\mathrm{ComCoAlg}}(\mathcal{A}_1) \to \mathrm{coBar}_{\mathrm{ComCoAlg}}(\mathcal{A}_2)$$
is an equivalence, we will derive a contradiction.

Let I be the smallest set such that the map
$$f|_{\overset{\circ}{X}{}^I} : \mathcal{A}_1|_{\overset{\circ}{X}{}^I} \to \mathcal{A}_2|_{\overset{\circ}{X}{}^I}$$
is not an equivalence. Let $k \geq 0$ be the smallest number such that
$$\mathrm{tr}_{\geq (-1-d)|I|-1-k}(\mathcal{A}_1|_{\overset{\circ}{X}{}^I}) \to \mathrm{tr}_{\geq (-1-d)|I|-1-k}(\mathcal{A}_2|_{\overset{\circ}{X}{}^I})$$
is not an equivalence.

By Corollary 3.3.17, we know that there exists some $m \gg 0$ such that
$$\mathrm{tr}_{\geq (-1-d)|I|-1-k}(\mathrm{coBar}_{\mathrm{ComCoAlg}}(\mathcal{A}_i)|_{\overset{\circ}{X}{}^I}) \simeq \mathrm{tr}_{\geq (-1-d)|I|-1-k}(\mathrm{coBar}^m_{\mathrm{ComCoAlg}}(\mathcal{A}_i)|_{\overset{\circ}{X}{}^I})$$
for $i \in \{1, 2\}$. Thus, we get the following equivalence
$$\mathrm{tr}_{\geq (-1-d)|I|-1-k}(\mathrm{coBar}^m_{\mathrm{ComCoAlg}}(\mathcal{A}_1)|_{\overset{\circ}{X}{}^I}) \simeq \mathrm{tr}_{\geq (-1-d)|I|-1-k}(\mathrm{coBar}^m_{\mathrm{ComCoAlg}}(\mathcal{A}_2)|_{\overset{\circ}{X}{}^I}).$$
But observe that if we let
$$F^n(\mathcal{A}_i) = \mathrm{Fib}(\mathrm{coBar}^n_{\mathrm{ComCoAlg}}(\mathcal{A}_i) \to \mathrm{coBar}^{n-1}_{\mathrm{ComCoAlg}}(\mathcal{A}_i))$$

then the difference between $F^n(\mathcal{A}_1)|_{\overset{\circ}{X}{}^I}$ and $F^n(\mathcal{A}_2)|_{\overset{\circ}{X}{}^I}$ lies in cohomological degrees

$$\begin{aligned}
&\leq (-1-d)|I|-1-k+(-1-d)\sum_{i=1}^{2^n-1}|I_i|-(2^n-1)+n+d\left(|I|+\sum_{i=1}^{2^n-1}|I_i|-|I|\right)\\
&\leq (-1-d)|I|-1-k-\sum_{i=1}^{2^n-1}|I_i|-2^n+1+n\\
&< (-1-d)|I|-1-k.
\end{aligned}$$

This implies that for $n \geq 1$,

$$\mathrm{tr}_{\geq(-1-d)|I|-1-k}(F^n(\mathcal{A}_1)|_{\overset{\circ}{X}{}^I}) \simeq \mathrm{tr}_{\geq(-1-d)|I|-1-k}(F^n(\mathcal{A}_2)|_{\overset{\circ}{X}{}^I}).$$

Thus, as in the case of Vect, a downward induction implies that

$$\mathrm{tr}_{\geq(-1-d)|I|-1-k}(\mathcal{A}_1|_{\overset{\circ}{X}{}^I}) \simeq \mathrm{tr}_{\geq(-1-d)|I|-1-k}(\mathcal{A}_2|_{\overset{\circ}{X}{}^I}),$$

which contradicts our original assumption, and we are done. □

3.3.21. Corollary 3.3.12 and Lemma 3.3.20 together prove the equivalence on the top row of diagram (3.3.4). For the equivalence in the bottom row, it suffices to show that for

$$\mathfrak{g} \in \mathrm{Lie}^\star(\mathrm{Ran}\,X)^{\leq c_L},$$

$\mathrm{Chev}(\mathfrak{g})$ is factorizable if and only if $\mathfrak{g} \in \mathrm{Lie}^\star(X)^{\leq c_L}$.

3.3.22. For the "if" direction, recall that as a consequence of [FG11, Thm. 6.4.2 and 5.2.1], we know that the functor

$$\mathrm{Chev} : \mathrm{Lie}^\star(X) \to \mathrm{ComCoAlg}^\star(\mathrm{Ran}\,X)$$

lands inside the full-subcategory $\mathrm{coFact}^\star(X)$ of factorizable co-algebras. We thus get a functor

$$\mathrm{Chev} : \mathrm{Lie}^\star(X)^{\leq c_L} \to \mathrm{coFact}^\star(X)^{\leq c_{cA}},$$

which settles the "if" direction.

3.3.23. For the "only if" direction, let

$$\mathfrak{g} \in \mathrm{Lie}^\star(\mathrm{Ran}\,X)^{\leq c_L}$$

whose support does not lie in X. We will show that $\mathrm{Chev}\,\mathfrak{g}$ is not factorizable.

Using the ass-gr$\circ$addFil trick (see §A), it suffices to prove for the case where $\mathfrak{g}$ is a trivial (i.e. abelian) Lie algebra. In that case, we know that

$$\mathrm{Chev}\,\mathfrak{g} = \mathrm{Sym}^{>0}(\mathfrak{g}[1]),$$

where Sym is taken using the $\otimes^\star$-monoidal structure.

Let I be the smallest set, with $|I| > 1$, such that $\mathfrak{g}|_{\overset{\circ}{X}{}^I} \not\simeq 0$. Now, it's easy to see that $\mathrm{Sym}^{>0}(\mathfrak{g}[1])$ fails the factorizability condition at $\overset{\circ}{X}{}^I$, which concludes the "only if" direction.

4. Factorizability of coChev

In this section, we will prove Theorem 1.5.6, which asserts that when

$$\mathfrak{g} \in \mathrm{coLie}^\star(X)$$

satisfies a certain co-connectivity constraint, the commutative algebra

$$\mathrm{coChev}(\mathfrak{g}) \in \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)$$

is factorizable.

Note that an analog of this result, where coChev is replaced by Chev, has been proved in [FG11] (and in fact, we used this result in the previous section). The main difficulties of the coChev case stem from the fact that, unlike Chev, coChev is defined as a limit, and most of the functors that we want it to interact with don't generally commute with limits.

As above, our main strategy is to introduce a certain co-connectivity condition to ensure that when one takes the limit of a diagram involving objects satisfying it, the answer, in some sense, converges instead of running off to infinity, so we still have a good control over it.

We start with the precise statement of the theorem. Then, after a quick digression on the various notions related to the convergence of a limit, we will present the main strategy. Finally, the proof itself will be given.

4.1. **The statement.** We start with the co-connectivity conditions.

Definition 4.1.1. Let $\mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n}$ denote the full subcategory of $\mathrm{Shv}(\mathrm{Ran}\,X)$ consisting of sheaves $\mathcal{F}$ such that for all non-empty finite sets I,

$$\mathcal{F}|_{\overset{\circ}{X}{}^I} \in \mathrm{Shv}(\overset{\circ}{X}{}^I)^{\geq n},$$

As before, we use the perverse t-structure.

Notation 4.1.2. We will use

$$\mathrm{coLie}^\star(\mathrm{Ran}\,X)^{\geq n} \qquad \text{and} \qquad \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq n}$$

to denote

$$\mathrm{coLie}^\star(\mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n}) \qquad \text{and} \qquad \mathrm{ComAlg}^\star(\mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n})$$

respectively.

We will prove the following

Theorem 4.1.3. *Restricted to the full subcategory* $\mathrm{coLie}^\star(X)^{\geq 1}$*, the functor* coChev *factors through* $\mathrm{Fact}^\star$*, i.e.*

$$\begin{tikzcd} \mathrm{coLie}^\star(X)^{\geq 1} \arrow[rr, "\mathrm{coChev}"] \arrow[dr, "\mathrm{coChev}"'] & & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X) \\ & \mathrm{Fact}^\star(X) \arrow[ur, hook] & \end{tikzcd}$$

In other words, $\mathrm{coChev}\,\mathfrak{g}$ *is factorizable when* $\mathfrak{g} \in \mathrm{coLie}^\star(X)^{\geq 1}$.

4.2. **Stabilizing co-filtrations and decaying sequences (a digression).** In this subsection, we describe a condition on co-filtered and graded objects which make them behave nicely with respect to taking limits.

Definition 4.2.1. Let $\mathcal{C}$ be a stable infinity category equipped with a t-structure. Then, a co-filtered object $c \in \mathcal{C}^{\mathrm{coFil}^{>0}}$ (see §B) is said to stabilize if for all n,

$$\mathrm{tr}_{\leq n}\, c_m \to \mathrm{tr}_{\leq n}\, c_{m+1}$$

is an equivalence for all $m \gg 0$.

A graded object in $c \in \mathcal{C}^{\mathrm{gr}^{>0}}$ is said to be decaying if for all n if

$$\mathrm{tr}_{\leq n}\, c_m \simeq 0$$

for all $m \gg 0$.

Notation 4.2.2. We use $\mathcal{C}^{\mathrm{coFil}^{>0},\mathrm{stab}}$ and $\mathcal{C}^{\mathrm{gr}^{>0},\mathrm{decay}}$ to denote the subcategories of $\mathcal{C}^{\mathrm{coFil}^{>0}}$ and $\mathcal{C}^{\mathrm{gr}^{>0}}$ consisting of stabilizing and decaying objects respectively.

We have the following lemmas, whose proofs are straightforward.

Lemma 4.2.3. *Let* $c \in \mathcal{C}^{\mathrm{coFil}^{>0}}$*. Then*

$$c \in \mathcal{C}^{\mathrm{coFil}^{>0},\mathrm{stab}}$$

if and only if

$$\text{ass-gr}\, c \in \mathcal{C}^{\mathrm{gr}^{>0},\mathrm{decay}}.$$

Lemma 4.2.4. *If* $c \in \mathcal{C}^{\mathrm{coFil}^{>0},\mathrm{stab}}$*, then for each* n*, the natural map*

$$\tau_{\leq n}\mathrm{oblv}_{\mathrm{coFil}}\, c \to \tau_{\leq n} c_m$$

is an equivalence when $m \gg 0$.

Proof. By throwing away finitely many terms at the beginning, without loss of generality, we can assume that the natural maps

$$\tau_{\leq n+1} c_i \to \tau_{\leq n+1} c_j, \qquad \forall i \geq j > 0$$

are all equivalences. Now, it suffices to show that the following map is an equivalence

$$\tau_{\leq n} \lim_i c_i \to \tau_{\leq n} c_1.$$

Equivalently, it suffices to show that

$$\mathrm{Fib}(\lim_i c_i \to c_1) \in \mathcal{C}^{\geq n+1}.$$

However,

$$\mathrm{Fib}(\lim_i c_i \to c_1) \simeq \lim_i (\mathrm{Fib}(c_i \to c_1)) \in \mathcal{C}^{\geq n+1}$$

because

$$\mathrm{Fib}(c_i \to c_1) \in \mathcal{C}^{\geq n+1}, \qquad \forall i.$$

Hence, we are done, since

$$i_{\geq n+1} : \mathcal{C}^{\geq n+1} \to \mathcal{C}$$

commutes with limits (see §2.1.3). □

Lemma 4.2.5. *The natural transformation*

$$\bigoplus \to \prod$$

between functors

$$\mathcal{C}^{\mathrm{gr}^{>0},\mathrm{decay}} \to \mathcal{C}$$

is an equivalence.

Proof. Note that

$$\prod_i c_i \simeq \lim_k \bigoplus_{i\leq k} c_i.$$

Moreover, since the sequence we are taking the limit over stabilizes, the result follows as a direct corollary of Lemma 4.2.4. □

4.2.6. The various definitions and observations above have straightforward analogues in the case of sheaves on the Ran space.

Definition 4.2.7. A co-filtered sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{coFil}^{>0}}$ is said to stabilize if for any non-empty finite set I,

$$\mathcal{F}|_{\overset{\circ}{X}{}^I} \in \mathrm{Shv}(\overset{\circ}{X}{}^I)^{\mathrm{coFil}^{>0},\mathrm{stab}}.$$

Similarly, a graded sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{gr}^{>0}}$ is said to be decaying if for any non-empty finite set I,

$$\mathcal{F}|_{\overset{\circ}{X}{}^I} \in \mathrm{Shv}(\overset{\circ}{X}{}^I)^{\mathrm{gr}^{>0},\mathrm{decay}}.$$

Notation 4.2.8. We use $\mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{coFil}^{>0},\mathrm{stab}}$ and $\mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{gr}^{>0},\mathrm{decay}}$ to denote the sub-categories of $\mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{coFil}^{>0}}$ and $\mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{gr}^{>0}}$ consisting of stabilizing and decaying objects, respectively.

It's straightforward to see that the following analogs of the lemmas above still hold in this setting.

Lemma 4.2.9. *Let* $\mathcal{F} \in \mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{coFil}^{>0}}$. *Then*

$$\mathcal{F} \in \mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{coFil}^{>0},\mathrm{stab}}$$

if and only if

$$\text{ass-gr}\, \mathcal{F} \in \mathrm{Shv}(\mathrm{Ran}\, X)^{\mathrm{gr}^{>0},\mathrm{decay}}.$$

Lemma 4.2.10. *If $\mathcal{F} \in \mathrm{Shv}(\mathrm{Ran}\,X)^{\mathrm{coFil}^{>0},\mathrm{stab}}$, then for each I and n, the natural map*[11]

$$\tau_{\leq n}\mathrm{oblv}_{\mathrm{coFil}}\mathcal{F}|_{\overset{\circ}{X}{}^I} \to \tau_{\leq n}\mathcal{F}_m|_{\overset{\circ}{X}{}^I}$$

is an equivalence when $m \gg 0$.

Lemma 4.2.11. *The natural transformation*

$$\bigoplus \to \prod$$

between functors

$$\mathrm{Shv}(\mathrm{Ran}\,X)^{\mathrm{gr}^{>0},\mathrm{decay}} \longrightarrow \mathrm{Shv}(\mathrm{Ran}\,X)$$

is an equivalence.

4.3. **Strategy.** To prove that $\mathrm{Chev}\,\mathfrak{g}$ is factorizable when $\mathfrak{g} \in \mathrm{Lie}^\star(X)$, [FG11] uses the addFil trick (see §A) to reduce to the case where $\mathfrak{g}$ is a trivial. In that case,

$$\mathrm{Chev}\,\mathfrak{g} \simeq \mathrm{Sym}^{>0}\,\mathfrak{g},$$

and the result can be seen directly. In the case of coChev, while the core strategy remains the same, it is more complicated to carry out since many commutative diagrams needed for the addFil trick to work (see (A.3.3)) don't commute in general in this new setting. The co-connectivity constraints are what needed to make these diagrams commute and hence, to allow us to reduce to the trivial case.

4.3.1. Now, suppose for the moment that we have the following commutative diagram, which is analogous to (A.3.3), except for the extra conditions

$$\begin{tikzcd}
\mathrm{coLie}^\star(X)^{\geq 1} \arrow[r, "\mathrm{coChev}"] \arrow[d, "\mathrm{addCoFil}"'] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2} \\
\mathrm{coLie}^\star(X)^{\geq 1,\mathrm{coFil}^{>0},\mathrm{stab}} \arrow[r, "\mathrm{coChev}_{\mathrm{coFil}}"] \arrow[d, "\text{ass-gr}"'] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2,\mathrm{coFil}^{>0},\mathrm{stab}} \arrow[u, "\mathrm{oblv}_{\mathrm{coFil}}"'] \arrow[d, "\text{ass-gr}"'] \\
\mathrm{coLie}^\star(X)^{\geq 1,\mathrm{gr}^{>0},\mathrm{decay}} \arrow[r, "\mathrm{coChev}_{\mathrm{gr}}"] \arrow[d, "\prod"'] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2,\mathrm{gr}^{>0},\mathrm{decay}} \arrow[d, "\prod"'] \\
\mathrm{coLie}^\star(X)^{\geq 1} \arrow[r, "\mathrm{coChev}"] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2}
\end{tikzcd} \tag{4.3.2}$$

Suppose also that $\mathrm{oblv}_{\mathrm{coFil}}$ preserves factorizability, and that ass-gr and $\prod$ are conservative with respect to factorizability. Then by the same reasoning as in the addFil trick, to prove that $\mathrm{coChev}\,\mathfrak{g}$ is factorizable, it suffices to assume that $\mathfrak{g}$ has a trivial coLie-structure. In that case,

$$\mathrm{coChev}\,\mathfrak{g} \simeq \mathrm{Sym}^{>0}(\mathfrak{g}[-1]),$$

and as in the Chev case, we are done.

4.3.3. The rest of this section will be devoted to the execution of the strategy outlined above.

[11]Note that $\mathrm{oblv}_{\mathrm{coFil}}$ commutes with restricting to $\overset{\circ}{X}{}^I$ for any non-empty, finite set I. Thus, the LHS is free of ambiguity.

4.4. **Well-definedness of functors.** Before proving that the diagram commutes, we need to first make sense of it. A priori, the functors written in the diagram are not necessarily well-defined. For instance, we haven't shown that all the four instances of coChev land in the correct target categories. Moreover, we also don't know that $\mathrm{oblv}_{\mathrm{coFil}}$, ass-gr, and $\prod$ preserve the algebra/co-algebra structures.

We start with the following straight-forward observation which settles the latter question.

Lemma 4.4.1. *For any n, the functors*

$$\mathrm{oblv}_{\mathrm{coFil}} : \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n,\mathrm{coFil}^{>0},\mathrm{stab}} \to \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n}$$
$$\text{ass-gr} : \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n,\mathrm{coFil}^{>0}} \to \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n,\mathrm{gr}^{>0}}$$
$$\prod \simeq \bigoplus : \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n,\mathrm{gr}^{>0},\mathrm{decay}} \to \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n}$$

are symmetric monoidal with respect to the $\otimes^\star$-monoidal structure on $\mathrm{Ran}\,X$.

4.4.2. We will now tackle the former question: namely, the various instances of the functor coChev appeared in (4.3.2) land in the correct target categories.

4.4.3. The top and bottom coChev are the same, and it's easy to see that they land in the correct category using the fact that the shriek-pullback functor is left exact and $\mathcal{C}^{\geq n}$ is preserved under limits for any stable infinity category $\mathcal{C}$ with a t-structure (since $i_{\geq n}$ commutes with limits, see §2.1.3).

4.4.4. By the same token, we know that the essential images of $\mathrm{coChev}_{\mathrm{coFil}}$ and $\mathrm{coChev}_{\mathrm{gr}}$ satisfy the co-connectivity assumption (i.e. live in (perverse) cohomological degree ≥ 1). Thus, it remains to show that they also satisfy the stab and decay conditions respectively.

First, observe that the assertion about ass-gr in Lemma 4.4.1, combined with the fact that ass-gr commutes with limits, gives us a weakened version of the middle square of (4.3.2).

Corollary 4.4.5. *We have the following commutative diagram*

$$\begin{tikzcd}
\mathrm{coLie}^\star(X)^{\geq 1,\mathrm{coFil}^{>0},\mathrm{stab}} \arrow[r, "\mathrm{coChev}_{\mathrm{coFil}}"] \arrow[d, "\text{ass-gr}"'] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2,\mathrm{coFil}^{>0}} \arrow[d, "\text{ass-gr}"'] \\
\mathrm{coLie}^\star(X)^{\geq 1,\mathrm{gr}^{>0},\mathrm{decay}} \arrow[r, "\mathrm{coChev}_{\mathrm{gr}}"] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2,\mathrm{gr}^{>0}}
\end{tikzcd}$$

Now, by Lemma 4.2.9, to show that $\mathrm{coChev}_{\mathrm{coFil}}$ and $\mathrm{coChev}_{\mathrm{gr}}$ satisfy the stab and decay conditions respectively, it suffices to show that $\mathrm{coChev}_{\mathrm{gr}}$ satisfies the decay condition. However, this is also a direct consequence of the fact that the shriek-pullback functor is left exact and $\mathcal{C}^{>n}$ is preserved under limits (for any stable infinity category $\mathcal{C}$ with a t-structure), and we are done.

4.5. **Commutative diagrams.** We will now proceed to prove that the diagram (4.3.2) commutes. First note that we have just settled the commutativity of the middle diagram of (4.3.2) at the end of the previous subsection.

4.5.1. The commutativity of the bottom diagram of (4.3.2) is clear if we know that $\prod$ is symmetric monoidal. However, by Lemma 4.2.11, we have

$$\prod \simeq \bigoplus$$

and we know that $\bigoplus$ is symmetric monoidal.

4.5.2. Finally, to show that the top diagram of (4.3.2) commutes, it suffices to show that the following diagram commutes

$$\begin{tikzcd}
\mathrm{coLie}^\star(X)^{\geq 1} \arrow[r, "\mathrm{coChev}"] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2} \\
\mathrm{coLie}^\star(X)^{\geq 1,\mathrm{coFil}^{>0},\mathrm{stab}} \arrow[r, "\mathrm{coChev}_{\mathrm{coFil}}"] \arrow[u, "\mathrm{oblv}_{\mathrm{coFil}}"] & \mathrm{ComAlg}^\star(\mathrm{Ran}\,X)^{\geq 2,\mathrm{coFil}^{>0},\mathrm{stab}} \arrow[u, "\mathrm{oblv}_{\mathrm{coFil}}"']
\end{tikzcd} \tag{4.5.3}$$

since the composition

$$\mathrm{coLie}^{\star}(X)^{\geq 1} \xrightarrow{\mathrm{addCoFil}} \mathrm{coLie}^{\star}(X)^{\geq 1,\mathrm{coFil}^{>0},\mathrm{stab}} \xrightarrow{\mathrm{oblv}_{\mathrm{coFil}}} \mathrm{coLie}^{\star}(X)^{\geq 1}$$

is the identity functor (see also §A.3.1). However, this is clear since the functor

$$\mathrm{oblv}_{\mathrm{coFil}} : \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n,\mathrm{coFil}^{>0},\mathrm{stab}} \to \mathrm{Shv}(\mathrm{Ran}\,X)^{\geq n}$$

commutes with limit for any n, and moreover it is symmetric monoidal with respect to the $\otimes^{\star}$-monoidal structure on $\mathrm{Shv}(\mathrm{Ran}\,X)$ by Lemma 4.4.1.

4.6. **Relation to factorizability.** It is easy to see that

$$\text{ass-gr} : \mathrm{ComAlg}^{\star}(\mathrm{Ran}\,X)^{\geq 2,\mathrm{coFil}^{>0},\mathrm{stab}} \to \mathrm{ComAlg}^{\star}(\mathrm{Ran}\,X)^{\geq 2,\mathrm{gr}^{>0},\mathrm{decay}}$$

reflects factorizability. Moreover, as we've discussed above, we have the equivalence

$$\prod \simeq \bigoplus$$

as functors

$$\mathrm{ComAlg}^{\star}(\mathrm{Ran}\,X)^{\geq 2,\mathrm{gr}^{>0},\mathrm{decay}} \to \mathrm{ComAlg}^{\star}(\mathrm{Ran}\,X)^{\geq 2}.$$

But now it's clear that $\prod$ reflects factorizability, since $\bigoplus$ does.

Finally, since

$$\mathrm{oblv}_{\mathrm{coFil}} : \mathrm{ComAlg}^{\star}(\mathrm{Ran}\,X)^{\geq 2,\mathrm{coFil}^{>0},\mathrm{stab}} \to \mathrm{ComAlg}^{\star}(\mathrm{Ran}\,X)^{\geq 2}$$

is compatible with $\boxtimes$ (for the same reason that it is compatible with $\otimes^{\star}$), and moreover $(-)^{!}$ commutes with limits (being a right adjoint), we see easily that $\mathrm{oblv}_{\mathrm{coFil}}$ preserves factorizability. Thus, we conclude the proof of Theorem 4.1.3.

4.7. **Relation to** $\mathrm{coLie}^{!}(X)$ **and** $\mathrm{ComAlg}^{!}(X)$**.** In this subsection, we will discuss the various links between objects defined on X such as $\mathrm{coLie}^{!}(X)$ and $\mathrm{ComAlg}^{!}(X)$ and objects defined on $\mathrm{Ran}\,X$ such as $\mathrm{coLie}^{\star}(\mathrm{Ran}\,X)$, $\mathrm{ComAlg}^{\star}(\mathrm{Ran}\,X)$ and $\mathrm{Fact}^{\star}(X)$. This subsection is not used anywhere in the paper. We include it here for the sake of completeness.

4.7.1. Recall that on a scheme X, there are two symmetric monoidal structures, $\otimes$ and $\overset{!}{\otimes}$. Thus, we could talk about various algebra/co-algebra objects defined on it

$$\mathrm{Lie}^{*}(X), \mathrm{coLie}^{!}(X), \mathrm{ComAlg}^{!}(X),$$

where $\mathrm{Lie}^{*}(X)$ (not to be confused with $\mathrm{Lie}^{\star}(X)$) is the category of Lie-algebra objects in $\mathrm{Shv}(X)$ with respect to the $\otimes$-monoidal structure, and $\mathrm{coLie}^{!}(X)$ (resp. $\mathrm{ComAlg}^{!}(X)$) is the category of coLie-algebra (resp. commutative algebra) objects in $\mathrm{Shv}(X)$ with respect to the $\overset{!}{\otimes}$-monoidal structure.

4.7.2. The following observations are straightforward, and are both based on the fact that the functors

$$\mathrm{ins}_X^{*} : \mathrm{Shv}(\mathrm{Ran}\,X)^{\otimes^{\star}} \to \mathrm{Shv}(X)^{\otimes}$$

and

$$\mathrm{ins}_X^{!} : \mathrm{Shv}(\mathrm{Ran}\,X)^{\otimes^{\star}} \to \mathrm{Shv}(X)^{\overset{!}{\otimes}}$$

are symmetric monoidal, where

$$\mathrm{ins}_X : X \to \mathrm{Ran}\,X$$

is the diagonal embedding.

Lemma 4.7.3. *We have a pair of adjoint functors*

$$\mathrm{ins}_X^{*} : \mathrm{Lie}^{\star}(\mathrm{Ran}\,X) \rightleftarrows \mathrm{Lie}^{*}(X) : \mathrm{ins}_{X*}$$

which induces an equivalence of categories

$$\mathrm{Lie}^{\star}(X) \simeq \mathrm{Lie}^{*}(X).$$

Lemma 4.7.4. *We have a pair of adjoint functors*

$$\mathrm{ins}_{X!} : \mathrm{coLie}^!(X) \rightleftarrows \mathrm{coLie}^\star(\mathrm{Ran}\, X) : \mathrm{ins}_X^!$$

which induces an equivalence of categories

$$\mathrm{coLie}^!(X) \simeq \mathrm{coLie}^\star(X).$$

4.7.5. We also have the following functor

$$\mathrm{ins}_X^! : \mathrm{ComAlg}^\star(\mathrm{Ran}\, X) \to \mathrm{ComAlg}^!(X)$$

which commutes with limits. Thus, we get a pair of adjoint functors

$$\mathrm{ins}_{X?} : \mathrm{ComAlg}^!(X) \rightleftarrows \mathrm{ComAlg}^\star(\mathrm{Ran}\, X) : \mathrm{ins}_X^! . \tag{4.7.6}$$

We have the following result from [GL14, Thm. 5.6.4].

Theorem 4.7.7. *The pair of adjoint functors in (4.7.6) induces an equivalence of categories*

$$\mathrm{ComAlg}^!(X) \simeq \mathrm{Fact}^\star(X).$$

4.7.8. The first link between $\mathrm{coLie}^!(X), \mathrm{coLie}^\star(X), \mathrm{ComAlg}^!(X), \mathrm{ComAlg}^\star(\mathrm{Ran}\, X)$ and $\mathrm{Fact}^\star(X)$ is given by the following diagram

$$\begin{tikzcd} \mathrm{coLie}^!(X) \arrow[d, "\mathrm{coChev}"'] & \mathrm{coLie}^\star(X) \arrow[l, "\mathrm{ins}_X^!", "\simeq"'] \arrow[d, "\mathrm{coChev}"'] \\ \mathrm{ComAlg}^!(X) & \mathrm{ComAlg}^\star(\mathrm{Ran}\, X) \arrow[l, "\mathrm{ins}_X^!"] \end{tikzcd} \tag{4.7.9}$$

whose commutativity is straightforward due to the fact that $\mathrm{ins}_X^!$ commutes with limits and that it's monoidal.

4.7.10. The second link, and also the more interesting one, is given by the following

Proposition 4.7.11. *We have the following commutative diagram*

$$\begin{tikzcd} \mathrm{coLie}^!(X)^{\geq 1} \arrow[r, "\mathrm{ins}_{X!}", "\simeq"'] \arrow[d, "\mathrm{coChev}"'] & \mathrm{coLie}^\star(X)^{\geq 1} \arrow[d, "\mathrm{coChev}"'] \\ \mathrm{ComAlg}^!(X) \arrow[r, "\mathrm{ins}_{X?}"] & \mathrm{Fact}^\star(X) \end{tikzcd}$$

Proof. For any $\mathfrak{g} \in \mathrm{coLie}^!(X)$, we have a natural map

$$\mathrm{ins}_{X?} \circ \mathrm{coChev} \to \mathrm{coChev} \circ \mathrm{ins}_{X!} .$$

of objects in $\mathrm{ComAlg}^\star(\mathrm{Ran}\, X)$. Now, we know from Theorem 4.7.7 that the LHS is factorizable. Moreover, when $\mathfrak{g} \in \mathrm{coLie}^!(X)^{\geq 1}$, we know from Theorem 4.1.3 that the RHS is also factorizable. Thus, to show that the map above is an equivalence when $\mathfrak{g} \in \mathrm{coLie}^!(X)^{\geq 1}$, it suffices to show that they are equivalence on the diagonal. However, that is clear from (4.7.9) and we are done. □

5. Interactions between various functors on the Ran space

In this section, we tie together the links between the various functors on the Ran spaces: Chev, coChev, $C_c^*(\mathrm{Ran}\, X, -)$, and D_{Ran}, the functor of Verdier duality on the Ran space.

5.1. $C_c^*(\mathrm{Ran}\, X, -)$ **and** coChev. In this subsection, we will prove Theorem 1.5.8, which gives us a criterion for the commutativity of the functor coChev and the functor $C_c^*(\mathrm{Ran}\, X, -)$. Note that it has been proved in [FG11] that Chev always commutes with $C_c^*(\mathrm{Ran}\, X, -)$. The main reason is that $C_c^*(\mathrm{Ran}\, X, -)$ is continuous and monoidal with respect to the $\otimes^\star$-monoidal structure on $\mathrm{Shv}(\mathrm{Ran}\, X)$ and the usual monoidal structure on Vect. As before, our main difficulty comes from the fact that coChev is defined as a limit, and for that to behave well with respect to $C_c^*(\mathrm{Ran}\, X, -)$, we need to impose a certain co-connectivity assumption.

5.1.1. Throughout this subsection, X will be assumed to be a proper scheme of pure dimension d.

Theorem 5.1.2. *For any $\mathfrak{g} \in \mathrm{coLie}^\star(X)^{\geq 1+d}$, the natural map*
$$C^*_c(\mathrm{Ran}\, X, \mathrm{coChev}\, \mathfrak{g}) \to \mathrm{coChev}(C^*_c(X, \mathfrak{g}))$$
is an equivalence.[12]

5.1.3. The proof of Theorem 5.1.2 is essentially the dual of the proofs of Lemma 3.1.6 and Lemma 3.3.11. There, we express the limit (i.e. $\mathrm{coBar}_{\mathcal{P}}$) as a sequential limit, and then establish a certain stability condition on the sequence we take the limit over. The main point is to show that for any n, $\mathrm{tr}_{\geq -n}$ of our limit is just $\mathrm{tr}_{\geq -n}$ of the terms when we go sufficiently far in the sequence. And at a finite step, commuting with a colimit is automatic.

5.1.4. Our current situation is the dual of that. Namely, we will express
$$C^*_c(\mathrm{Ran}\, X, -)$$
as the colimit of a sequence satisfying a certain stability condition, which allows us, after truncating on the right via $\mathrm{tr}_{\leq n}$ for each n, to commute it with the limit defining coChev.

5.1.5. We start with a general remark: in general, the limit (resp. colimit) of a diagram
$$\mathcal{K} \to \mathcal{C}$$
could be written as a sequential limit (resp. colimit) if we have a functor $\mathcal{K} \to \mathbb{Z}$. We can then use left (resp. right) Kan extension to produce a new diagram
$$\mathbb{Z} \to \mathcal{C},$$
and the original limit (resp. colimit) could be written as a sequential limit (resp. colimit) of this new diagram.

5.1.6. *Truncated Ran space.* Now we can apply the remark above to the case of the Ran space. For any scheme X and any positive integer n, we define
$$\mathrm{Ran}^{\leq n} X \simeq \operatorname*{colim}_{\substack{I \in \mathrm{fSet}^{\mathrm{surj}} \\ |I| \leq n}} X^I.$$
Then
$$\mathrm{Ran}\, X \simeq \mathrm{colim}\, \mathrm{Ran}^{\leq n} X \simeq \mathrm{colim}(X \to \mathrm{Ran}^{\leq 2} X \to \mathrm{Ran}^{\leq 3} X \to \cdots),$$
and hence, for any $\mathcal{F} \in \mathrm{Shv}(\mathrm{Ran}\, X)$,
$$C^*_c(\mathrm{Ran}\, X, \mathcal{F}) \simeq \operatorname*{colim}_n C^*_c(\mathrm{Ran}^{\leq n}, \mathcal{F}|_{\mathrm{Ran}^{\leq n} X}).$$

The following observation, which gives the link among the cohomology groups
$$C^*_c(\mathrm{Ran}^{\leq n} X, \mathcal{F}|_{\mathrm{Ran}^{\leq n} X})$$
for various n's, comes from [Gai15, Cor. 9.1.4].

Lemma 5.1.7. *We have the following natural equivalence*
$$C^*(\overset{\circ}{X}{}^I, \mathcal{F}|_{\overset{\circ}{X}{}^I})_{\Sigma_I} \simeq \mathrm{coFib}(C^*_c(\mathrm{Ran}^{\leq |I|-1} X, \mathcal{F}|_{\mathrm{Ran}^{\leq |I|-1} X}) \to C^*_c(\mathrm{Ran}^{\leq |I|} X, \mathcal{F}|_{\mathrm{Ran}^{\leq |I|} X})).$$

5.1.8. When
$$\mathfrak{g} \in \mathrm{coLie}^\star(X)^{\geq 1+d},$$
using the addCoFil trick (4.3.2), we can also express $\mathrm{coChev}\, \mathfrak{g}$ as a sequential limit
$$\mathrm{coChev}\, \mathfrak{g} \simeq \mathrm{oblv}_{\mathrm{coFil}}\, \mathrm{coChev}_{\mathrm{coFil}}\, \mathrm{addCoFil}\, \mathfrak{g} \simeq \lim_i (\mathrm{coChev}_{\mathrm{coFil}}\, \mathrm{addCoFil}\, \mathfrak{g})_i.$$
Where
$$(\mathrm{coChev}_{\mathrm{coFil}}\, \mathrm{addCoFil}\, \mathfrak{g})_i$$
is the i-th step in the co-filtration.

[12] Since $\mathrm{Supp}\, \mathfrak{g} \subset X \subset \mathrm{Ran}\, X$,
$$C^*_c(\mathrm{Ran}\, X, \mathfrak{g}) \simeq C^*_c(X, \mathfrak{g}).$$

5.1.9. For brevity's sake, we will denote

$$\mathrm{coChev}^i \mathfrak{g} = (\mathrm{coChev}_{\mathrm{coFil}} \mathrm{addCoFil}\, \mathfrak{g})_i$$

and so we have

$$\mathrm{coChev}\, \mathfrak{g} \simeq \lim_i \mathrm{coChev}^i \mathfrak{g}.$$

5.1.10. The advantage of using this co-filtration (instead of the usual one coming from the co-simplicial object defining coChev) lies in the fact that both the supports and cohomological estimates of coChev^i vary nicely with respect to i. Namely, for any non-negative integer i,

$$\mathrm{Supp}\, \mathrm{coChev}^i \mathfrak{g} \subset \mathrm{Ran}^{\leq i} X$$

and for all non-empty finite set I such that $|I| \leq i$,

$$(\mathrm{coChev}^i \mathfrak{g})|_{\overset{\circ}{X}{}^I}$$

lives in perverse cohomological degrees $\geq i(d+1)+1$. This gives us the following observations.

Lemma 5.1.11. *For any $\mathfrak{g} \in \mathrm{coLie}^\star(X)^{\geq 1+d}$ and any non-empty finite set I,*

$$(\mathrm{coChev}\, \mathfrak{g})|_{\overset{\circ}{X}{}^I}$$

lives in cohomological degrees $\geq (1+d)|I|+1$.

Corollary 5.1.12. *For any $\mathfrak{g} \in \mathrm{coLie}^\star(X)^{\geq 1+d}$ and any non-empty finite set I,*

$$C^*(\overset{\circ}{X}{}^I, (\mathrm{coChev}\, \mathfrak{g})|_{\overset{\circ}{X}{}^I})_{\Sigma_I}$$

lives in cohomological degrees $\geq |I|+1$.

Lemma 5.1.13. *For any $\mathfrak{g} \in \mathrm{coLie}^\star(X)^{\geq 1+d}$, any positive integer i, and any non-empty finite set I,*

$$C^*(\overset{\circ}{X}{}^I, (\mathrm{coChev}^i \mathfrak{g})|_{\overset{\circ}{X}{}^I})_{\Sigma_I}$$

lives in cohomological degrees $\geq \max(|I|+1, i+1)$.

With these observations, we are ready for the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. For each i, we know that $\mathrm{coChev}^i \mathfrak{g}$ is computed as a finite limit. Thus, we have the following natural equivalence

$$C^*_c(\mathrm{Ran}\, X, \mathrm{coChev}^i \mathfrak{g}) \simeq \mathrm{coChev}^i(C^*_c(\mathrm{Ran}\, X, \mathfrak{g})).$$

Taking the limit over i on both sides, we observe that it suffices to prove that

$$\lim_i C^*_c(\mathrm{Ran}\, X, \mathrm{coChev}^i \mathfrak{g}) \simeq C^*_c(\mathrm{Ran}\, X, \lim_i \mathrm{coChev}^i \mathfrak{g}).$$

For that, it suffices to show that for each m, we have an equivalence

$$\mathrm{tr}_{\leq m} \lim_i C^*_c(\mathrm{Ran}\, X, \mathrm{coChev}^i \mathfrak{g}) \simeq \mathrm{tr}_{\leq m} C^*_c(\mathrm{Ran}\, X, \lim_i \mathrm{coChev}^i \mathfrak{g}).$$

But now, for some $M \gg 0$, depending only on m, we have

$$\begin{aligned}
\mathrm{tr}_{\leq m} C^*_c(\mathrm{Ran}\, X, \lim_i \mathrm{coChev}^i \mathfrak{g}) &\simeq \mathrm{tr}_{\leq m} \operatorname*{colim}_n C^*_c(\mathrm{Ran}^{\leq n} X, \lim_i \mathrm{coChev}^i \mathfrak{g}|_{\mathrm{Ran}^{\leq n} X}) \\
&\simeq \mathrm{tr}_{\leq m} C^*_c(\mathrm{Ran}^{\leq M} X, \lim_i \mathrm{coChev}^i \mathfrak{g}|_{\mathrm{Ran}^{\leq M} X}) \qquad (5.1.14) \\
&\simeq \mathrm{tr}_{\leq m} \lim_i C^*_c(\mathrm{Ran}^{\leq M} X, \mathrm{coChev}^i \mathfrak{g}|_{\mathrm{Ran}^{\leq M} X}) \qquad (5.1.15) \\
&\simeq \lim_i \mathrm{tr}_{\leq m} C^*_c(\mathrm{Ran}^{\leq M} X, \mathrm{coChev}^i \mathfrak{g}|_{\mathrm{Ran}^{\leq M} X}) \\
&\simeq \lim_i \mathrm{tr}_{\leq m} C^*_c(\mathrm{Ran}\, X, \mathrm{coChev}^i \mathfrak{g}) \qquad (5.1.16) \\
&\simeq \mathrm{tr}_{\leq m} \lim_i C^*_c(\mathrm{Ran}\, X, \mathrm{coChev}^i \mathfrak{g}).
\end{aligned}$$

Here, we used Lemma 5.1.7 in both (5.1.14) and (5.1.16). Moreover, (5.1.14) and (5.1.16) use Corollary 5.1.12 and Lemma 5.1.13 respectively. Finally, (5.1.15) is due to the fact that

$$C^*_c(\mathrm{Ran}^{\leq M} X, -)$$

commutes with limits. Indeed, this functor is computed as a finite colimit of functors of the form

$$C^*_c(X^I, -).$$

But these functors commute with limits since X^I are all complete due to our assumption on X, i.e.

$$C^*_c(X^I, -) \simeq C^*(X^I, -).$$

□

5.2. **Verdier duality.** Before studying the link between Chev and coChev, we start with a quick recollection of Verdier duality on prestacks along with various useful properties. The main reference is [Gai15]. However, since we only use basic properties of D_{Ran}, we'll provide the complete proof in most cases.

5.2.1. Let $\mathcal{Y}$ be a prestack such that the diagonal map

$$\mathrm{diag}_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$$

is pseudo-proper. Given $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(\mathcal{Y})$, by a pairing between them, we shall mean a map

$$\mathcal{F} \boxtimes \mathcal{G} \to \mathrm{diag}_{\mathcal{Y}!}\, \omega_{\mathcal{Y}}.$$

We define the Verdier dual, $D_{\mathcal{Y}} \mathcal{G}$, of $\mathcal{G}$ to represent the functor

$$\mathcal{F} \mapsto \mathrm{Hom}(\mathcal{F} \boxtimes \mathcal{G}, \mathrm{diag}_{\mathcal{Y}!}\, \omega_{\mathcal{Y}}).$$

Namely, we have the following natural equivalence

$$\mathrm{Hom}(\mathcal{F}, D_{\mathcal{Y}} \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F} \boxtimes \mathcal{G}, \mathrm{diag}_{\mathcal{Y}!}\, \omega_{\mathcal{Y}}).$$

The following lemma is immediate from the definition.

Lemma 5.2.2. *Let $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$, such that*

$$\mathcal{F} \simeq \operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}_i.$$

Then

$$D_{\mathcal{Y}} \mathcal{F} \simeq \lim_{i \in \mathcal{I}^{\mathrm{op}}} D_{\mathcal{Y}} \mathcal{F}_i.$$

5.2.3. We will now study the link between Verdier duality and $\boxtimes$.

Proposition 5.2.4. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be finitary pseudo-schemes, and*

$$\mathcal{F}_i \in \mathrm{Shv}(\mathcal{Y}_i).$$

Then, we have a natural equivalence

$$D_{\mathcal{Y}_1} \mathcal{F}_1 \boxtimes D_{\mathcal{Y}_2} \mathcal{F}_2 \simeq D_{\mathcal{Y}_1 \times \mathcal{Y}_2} (\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

Proof. First, note that the result holds when both $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are schemes.

For the general case of finitary pseudo-schemes, we write

$$\mathcal{Y}_1 \simeq \operatorname*{colim}_i Y_{1i} \quad \text{and} \quad \mathcal{Y}_2 \simeq \operatorname*{colim}_j Y_{2j}.$$

Then,

$$\mathcal{F}_1 \simeq \operatorname*{colim}_i \mathrm{ins}_{1i!}\, \mathrm{ins}^!_{1i}\, \mathcal{F}_1 \quad \text{and} \quad \mathcal{F}_2 \simeq \operatorname*{colim}_j \mathrm{ins}_{2j!}\, \mathrm{ins}^!_{2j}\, \mathcal{F}_2.$$

Thus,

$$D_{\mathcal{Y}_1\times\mathcal{Y}_2}(\mathcal{F}_1\boxtimes\mathcal{F}_2)\simeq D_{\mathcal{Y}_1\times\mathcal{Y}_2}\operatorname*{colim}_{i,j}(\mathrm{ins}_{1i}\times\mathrm{ins}_{2j})_!(\mathrm{ins}_{1i}\times\mathrm{ins}_{2j})^!(\mathcal{F}_1\boxtimes\mathcal{F}_2)$$

$$\simeq\lim_{i,j}(\mathrm{ins}_{1i}\times\mathrm{ins}_{2j})_! D_{Y_{1i}\times Y_{2j}}(\mathrm{ins}^!_{1i}\mathcal{F}_1\boxtimes\mathrm{ins}^!_{2j}\mathcal{F}_2) \tag{5.2.5}$$

$$\simeq\lim_{i,j}(\mathrm{ins}_{1i}\times\mathrm{ins}_{2j})_!(D_{Y_{1i}}\mathrm{ins}^!_{1i}\mathcal{F}_1\boxtimes D_{Y_{2j}}\mathrm{ins}^!_{2j}\mathcal{F}_2) \tag{5.2.6}$$

$$\simeq(\lim_i\mathrm{ins}_{1i!}D_{Y_{1i}}\mathrm{ins}^!_{1i}\mathcal{F}_1)\boxtimes(\lim_j\mathrm{ins}_{2j!}D_{Y_{2j}}\mathrm{ins}^!_{1j}\mathcal{F}_2) \tag{5.2.7}$$

$$\simeq(D_{\mathcal{Y}_1}\operatorname*{colim}_i\mathrm{ins}_{1i!}\mathrm{ins}^!_{1i}\mathcal{F}_1)\boxtimes(D_{\mathcal{Y}_2}\mathrm{ins}_{2j!}\mathrm{ins}^!_{2j}\mathcal{F}_2) \tag{5.2.8}$$

$$\simeq D_{\mathcal{Y}_1}\mathcal{F}_1\boxtimes D_{\mathcal{Y}_2}\mathcal{F}_2.$$

Here,

– (5.2.6) is due to the fact that the statement we are trying to prove holds for the case of schemes.
– (5.2.7) is due to the fact that the limits we are taking are all finite (due to the finitary assumption).
– (5.2.5) and (5.2.8) are both due to Lemma 5.2.2 and Proposition 5.2.9 below.

□

Proposition 5.2.9. *Let $f:\mathcal{Y}_1\to\mathcal{Y}_2$ be a finitary pseudo-proper map between pseudo-schemes, each having a finitary diagonal. Then, the natural transformation*

$$f_!\circ D_{\mathcal{Y}_1}\to D_{\mathcal{Y}_2}\circ f_!$$

is an equivalence.

Proof. See [Gai15, Cor. 7.5.6]. □

Remark 5.2.10. One direct corollary of this proposition is the fact that for any sheaf $\mathcal{F}\in\mathrm{Shv}(X)$, we have the following natural equivalence

$$\delta_{X!}D_X\mathcal{F}\simeq D_{\mathrm{Ran}\,X}\delta_{X!}\mathcal{F}.$$

Corollary 5.2.11. *Let $\mathcal{F}_1,\mathcal{F}_2,\cdots,\mathcal{F}_k\in\mathrm{Shv}(\mathrm{Ran}\,X)$ with finite supports, i.e. there exists an n such that all the $\mathcal{F}_i$'s come from $\mathrm{Shv}(\mathrm{Ran}^{\leq n}X)$. Then, we have the following natural equivalence*

$$D_{\mathrm{Ran}\,X}(\mathcal{F}_1\otimes^\star\mathcal{F}_2\otimes\cdots\otimes^\star\mathcal{F}_k)\simeq(D_{\mathrm{Ran}\,X}\mathcal{F}_1)\otimes^\star(D_{\mathrm{Ran}\,X}\mathcal{F}_2)\otimes^\star\cdots\otimes^\star(D_{\mathrm{Ran}\,X}\mathcal{F}_k).$$

5.3. Chev, coChev, **and** $D_{\mathrm{Ran}\,X}$. We will now turn to Theorem 1.5.10, which provides the link between the two functors Chev and coChev via the functor of taking Verdier duality on the Ran space.

Theorem 5.3.1. *Let $\mathfrak{g}\in\mathrm{Lie}^\star(X)^{\leq -1}$. Then we have a natural equivalence*

$$\mathrm{coChev}(D_X\mathfrak{g})\simeq D_{\mathrm{Ran}\,X}\,\mathrm{Chev}(\mathfrak{g}),$$

of objects in $\mathrm{ComAlg}^\star(\mathrm{Ran}\,X)$, where $D_{\mathrm{Ran}\,X}$ is the functor of taking Verdier duality on $\mathrm{Ran}\,X$.

Note that this is the only place we use Verdier duality on the Ran space. However, we essentially use it in a rather minimal way: not much besides the definition itself.

Proof. We will employ ideas originated from the addFil and addCoFil tricks (see also §A). First, observe that for any $\mathfrak{g}\in\mathrm{Lie}^\star(X)$, we have a canonical equivalence

$$\mathrm{addCoFil}\,D_{\mathrm{Ran}\,X}\,\mathfrak{g}\simeq D_{\mathrm{Ran}\,X}\,\mathrm{addFil}\,\mathfrak{g}.$$

We denote

$$\mathrm{Chev}^i\,\mathfrak{g}\qquad\text{and}\qquad\mathrm{coChev}^i\,D_{\mathrm{Ran}\,X}\,\mathfrak{g}$$

to be the i-th piece in the filtration/co-filtration of

$$\mathrm{Chev}(\mathrm{addFil}\,\mathfrak{g})\qquad\text{and}\qquad\mathrm{coChev}(\mathrm{addCoFil}\,D_{\mathrm{Ran}\,X}\,\mathfrak{g})$$

respectively.

From §A and the top part of the commutative diagram (4.3.2), we have the following natural equivalences

$$\operatorname{Chev}\mathfrak{g} \simeq \operatorname*{colim}_i \operatorname{Chev}^i \mathfrak{g},$$
$$\operatorname{coChev}(D_{\operatorname{Ran}X}\mathfrak{g}) \simeq \lim_i \operatorname{coChev}^i(D_{\operatorname{Ran}X}\mathfrak{g}).$$

At the same time, by Lemma 5.2.2, we know that

$$D_{\operatorname{Ran}X} \operatorname*{colim}_i \operatorname{Chev}^i \mathfrak{g} \simeq \lim_i D_{\operatorname{Ran}X} \operatorname{Chev}^i \mathfrak{g}.$$

Thus, it suffices to show that

$$D_{\operatorname{Ran}X} \operatorname{Chev}^i \mathfrak{g} \simeq \operatorname{coChev}^i D_{\operatorname{Ran}X}\mathfrak{g}.$$

Now, it's an immediate consequence of Corollary 5.2.11. □

Corollary 5.3.2. *Let* $\mathfrak{g} \in \operatorname{Lie}^\star(X)^{\le -1}$. *Then*

$$D_{\operatorname{Ran}X} \operatorname{Chev}(\mathfrak{g})$$

is a factorizable commutative algebra on $\operatorname{Ran}X$.

Proof. This is a direct consequence of Theorem 5.3.1 and Theorem 1.5.6. □

5.4. coChev **and open embeddings.** We end the section with the following easy observation.

Proposition 5.4.1. *Let*

$$j : X' \to X$$

be an open embedding of schemes, which induces an open embedding of prestacks

$$j_{\operatorname{Ran}} : \operatorname{Ran}X' \to \operatorname{Ran}X.$$

Then for any $\mathfrak{g}' \in \operatorname{coLie}^\star(X')$*, we have the following natural equivalence*

$$(j_{\operatorname{Ran}})_* \operatorname{coChev}(\mathfrak{g}') \simeq \operatorname{coChev}(j_*\mathfrak{g}').$$

Proof (Sketch). The result is a direct consequence of the fact that f_*, being a right adjoint, commutes with limits for any schematic morphism f between prestacks. Moreover, if $f_i : X'_i \to X_i$ are open embeddings of schemes, and $\mathcal{F}_i \in \operatorname{Shv}(X'_i)$ for $i = 1, 2$, then we have a natural equivalence

$$(f_1 \times f_2)_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \simeq f_{1*}\mathcal{F}_1 \boxtimes f_{2*}\mathcal{F}_2.$$

□

6. An application to the Atiyah-Bott formula

We will now give an application of the results proved so far to the Atiyah-Bott formula. As mentioned in the introduction, these results allow us to simplify the second of the two main steps in the original proofs given in [GL14] and [Gai15]. In what follows, §6.1–§6.4 are intended to orient the readers with the existing results proved in [GL14] and [Gai15],[13] whereas the purpose of the last part, §6.5, is to explain how the results we've proved so far fit in with the rest.

[13] Namely, all the results stated in these subsections could be found in [GL14] or [Gai15]. The readers should be warned that we provide a mere overview of the development given in these two papers, with many technical points elided.

6.1. **The statement.** From now on, X is a smooth and complete curve over an algebraically closed field k, and G a smooth, fiber-wise connected group-scheme over X, whose generic fiber is semi-simple simply connected. Due to [GL14, Lem. 7.1.1 and Prop A.3.11], we can (and from now on we will) assume that G is semi-simple simply connected over an open dense subset

$$j : X' \hookrightarrow X,$$

and moreover, the fibers of G over any point in $X - X'$ are homologically trivial.

We will also use

$$j_{\mathrm{Ran}} : \mathrm{Ran}\, X' \to \mathrm{Ran}\, X$$

to denote the corresponding open embedding on the Ran space and

$$\Gamma_{j_{\mathrm{Ran}}} : \mathrm{Ran}\, X' \to \mathrm{Ran}\, X' \times \mathrm{Ran}\, X$$

to denote its graph.

6.1.1. Let G_0 be the split form of G. Then it is well-known that

$$C^*(BG_0) \simeq \mathrm{Sym}\, M_0 \tag{6.1.2}$$

is a free commutative algebra, for some $M_0 \in \mathrm{Vect}$. In the case of ℓ-adic sheaves in positive characteristic setting, this equivalence is compatible with the geometric Frobenius action, where

$$M_0 \simeq \bigoplus_e \Lambda[-2e](-e),$$

and e's are the exponents of G_0.

The assignment $G_0 \mapsto M_0$ is functorial with respect to automorphisms of G_0, and hence, for a general G (subject to the assumptions mentioned above), we get a local system

$$M \in \mathrm{Shv}(X'),$$

whose !-fiber at each geometric point $x \in X$ is equivalent to M_0.

Below is the statement of the Atiyah-Bott formula.

Theorem 6.1.3. *Let G, X as above. Then*

(a) We have an equivalence

$$C^*(\mathrm{Bun}_G) \simeq \mathrm{Sym}(C^*(X', M)).$$

(b) When $k = \overline{\mathbb{F}}_q$, and X and G are defined over $\mathbb{F}_q$, the above equivalence can be chosen to be compatible with the Frobenius actions.

6.2. BG **and the sheaf** $\mathcal{B}$.

6.2.1. The sheaf $\mathcal{B}$ that we will describe now encodes the reduced cohomology BG, the classifying stack of G. For each $I \in \mathrm{Ran}_X(S)$, let $D_I \subset S \times X$ be the corresponding Cartier divisor. Let BG_I denote the Artin stack classifying G-bundles over D_I and $f_I : BG_I \to S$ the forgetful map. Then, we define

$$\widetilde{\mathcal{B}}_{S,I} = D_S(\mathrm{Fib}(f_{I!} f_I^! \Lambda_S \to \Lambda_S)),$$

where D_S is the functor of taking Verdier duality on S. These sheaves, assembled together, give rise to a sheaf (see also [GL14, Prop. 5.4.3])

$$\widetilde{\mathcal{B}} \in \mathrm{Shv}(\mathrm{Ran}\, X).$$

6.2.2. Note that for any finite set of points $\{x_1, \dots, x_n\} \in (\mathrm{Ran}\, X)(k)$, the !-fiber of $\widetilde{\mathcal{B}}$ at this point is

$$\mathrm{coFib}\left(\Lambda \to \bigotimes_{i=1}^{n} C^*(BG_{x_i})\right). \tag{6.2.3}$$

6.2.4. Using a variant of the diagonal map

$$BG \to BG \times BG,$$

we can equip $\widetilde{\mathcal{B}}$ with the structure of an object in

$$\mathrm{ComAlg}^{\star}(\mathrm{Ran}\, X).$$

However, we see easily from (6.2.3) that $\widetilde{\mathcal{B}}$ is not factorizable. The functor TakeOut developed in [Gai15] allows us to remove all the extra components in it and construct out of it a new object $\mathcal{B} \in \mathrm{Fact}^{\star}(X)$ with the correct !-fibers at a point $\{x_1, \dots, x_n\} \in (\mathrm{Ran}\, X)(k)$

$$\bigotimes_{i=1}^{n} C^*_{\mathrm{red}}(BG_{x_i}).$$

Moreover, $\mathcal{B}$ has the same cohomology along $\mathrm{Ran}\, X$ as the original sheaf $\widetilde{\mathcal{B}}$ (see also [Gai15, Cor. 5.3.5])

$$C^*_c(\mathrm{Ran}\, X, \mathcal{B}) \simeq C^*_c(\mathrm{Ran}\, X, \widetilde{\mathcal{B}}).$$

6.2.5. $\mathcal{B}$ *and* Bun_G. For every $S \in \mathrm{Sch}$ and $I \in (\mathrm{Ran}\, X)(S)$, we have a map of prestacks over S by restricting the bundle to the divisor D_I

$$S \times \mathrm{Bun}_G \to BG_I. \tag{6.2.6}$$

This induces a map

$$\widetilde{\mathcal{B}}_{S,I} \to \omega_S \otimes C^*_{\mathrm{red}}(\mathrm{Bun}_G)$$

and hence, also a map

$$\widetilde{\mathcal{B}} \to \omega_{\mathrm{Ran}\, X} \otimes C^*_{\mathrm{red}}(\mathrm{Bun}_G).$$

Applying the functor $C^*_c(\mathrm{Ran}\, X, -)$ and using the fact that $\mathrm{Ran}\, X$ is homologically contractible, we get a map

$$C^*_c(\mathrm{Ran}\, X, \mathcal{B}) \simeq C^*_c(\mathrm{Ran}\, X, \widetilde{\mathcal{B}}) \to C^*_{\mathrm{red}}(\mathrm{Bun}_G). \tag{6.2.7}$$

6.2.8. Using (6.1.2) and the assumption we have on G, i.e. homologically contractible fibers outside of X', one gets an equivalence

$$\mathcal{B} \simeq (j_{\mathrm{Ran}\, X})_* \mathcal{B}' \simeq \mathrm{Sym}^{>0}(j_* M) \tag{6.2.9}$$

where $\mathcal{B}'$ is the restriction of $\mathcal{B}$ to $\mathrm{Ran}\, X'$ and, the symmetric algebra is taken inside $\mathrm{Shv}(\mathrm{Ran}\, X)$ using the $\otimes^{\star}$-monoidal structure.

6.2.10. Using the equivalence (6.2.9) and the fact that $C^*_c(\mathrm{Ran}\, X, -)$ commutes with Sym,[14] we get an explicit presentation of the LHS of (6.2.7)

$$C^*_c(\mathrm{Ran}\, X, \mathcal{B}) \simeq \mathrm{Sym}^{>0} C^*_c(X, j_* M) \simeq \mathrm{Sym}^{>0} C^*(X', M), \tag{6.2.11}$$

which appears in the statement of the Atiyah-Bott formula as stated in Theorem 6.1.3.

6.2.12. Now, we are done if we could show that the map in (6.2.7) is an equivalence.

6.3. **Affine Grassmannian and the sheaf $\mathcal{A}$.** Unfortunately, one does not know how to directly prove that (6.2.7) is an equivalence. Instead, [GL14] proceeds with an equivalence of a dual nature, which we will now briefly recall.

6.3.1. The main player in this step is the affine Grassmannian, or more precisely, a factorizable version of the affine Grassmannian. Let G, X as above. The factorizable affine Grassmannian of G, denoted by $\mathrm{Gr}_{\mathrm{Ran}\, X'}$, is the prestack such that for each scheme S,

$$\mathrm{Gr}_{\mathrm{Ran}\, X'}(S) = \{(\mathcal{P}, I, \alpha)\},$$

where

(i) $\mathcal{P}$ is a G-bundle over $S \times X$
(ii) I is a non-empty finite subset of $X'(S)$
(iii) α is a trivialization of $\mathcal{P}$ on the complement of the graph of I.

[14] Note that this is a special case of the fact that $C^*_c(\mathrm{Ran}\, X, -)$ commutes with Chev. And in fact, both are due to the same reasons: that $C^*_c(\mathrm{Ran}\, X, -)$ is continuous and that it's symmetric monoidal.

6.3.2. From the definition, we have the following natural morphism

$$g : \mathrm{Gr}_{\mathrm{Ran}\,X'} \to \mathrm{Ran}\,X',$$

where we forget everything, except for the set I, and similarly another natural morphism

$$u : \mathrm{Gr}_{\mathrm{Ran}\,X'} \to \mathrm{Bun}_G,$$

we we only remember the bundle $\mathcal{P}$.

6.3.3. The map g allows us to define

$$\widetilde{\mathcal{A}}' \simeq \mathrm{Fib}(g_!(\omega_{\mathrm{Gr}_{\mathrm{Ran}\,X'}}) \to \omega_{\mathrm{Ran}\,X'}) \in \mathrm{Shv}(\mathrm{Ran}\,X'),$$

and the map u induces a map at the cohomology level, namely

$$C_*^{\mathrm{red}}(\mathrm{Gr}_{\mathrm{Ran}\,X'}) \to C_*^{\mathrm{red}}(\mathrm{Bun}_G). \tag{6.3.4}$$

Together, we get the following map

$$C_c^*(\mathrm{Ran}\,X', \widetilde{\mathcal{A}}') \to C_*^{\mathrm{red}}(\mathrm{Bun}_G). \tag{6.3.5}$$

6.3.6. Note that since

$$\mathrm{Gr}_{\mathrm{Ran}\,X'} \to \mathrm{Ran}\,X'$$

is pseudo-proper, $\widetilde{\mathcal{A}}'$ is easy to describe. Namely for any finite set of points $\{x_1, x_2, \dots, x_n\} \subset X(k)$, the !-fiber of $\widetilde{\mathcal{A}}'$ at this point is

$$\mathrm{Fib}\left(\bigotimes_{i=1}^n C_*(\mathrm{Gr}_{G_{x_i}}) \to \Lambda\right). \tag{6.3.7}$$

6.3.8. *$\mathcal{A}$ and* Bun_G. The equivalence of a dual nature that we alluded to earlier is given by the following important result (see [GL14, Thm. 3.2.13]).

Theorem 6.3.9. *The map* (6.3.4)*, and hence* (6.3.5)*, is an equivalence.*

6.3.10. Using a variant of the diagonal map

$$\mathrm{Gr} \to \mathrm{Gr} \times \mathrm{Gr},$$

one can equip $\widetilde{\mathcal{A}}'$ with the structure of an object in

$$\mathrm{ComCoAlg}^\star(\mathrm{Ran}\,X').$$

However, note that the sheaf $\widetilde{\mathcal{A}}'$ is not factorizable, since its !-fiber, as described in (6.3.7), is too big, i.e. it's not equivalent to

$$\bigotimes_{i=1}^n C_*^{\mathrm{red}}(\mathrm{Gr}_{G_{x_i}}). \tag{6.3.11}$$

Using a similar reasoning as in the case of $\widetilde{\mathcal{B}}$ and $\mathcal{B}$, we can construct an object $\mathcal{A}' \in \mathrm{coFact}^\star(X')$ with the correct !-fiber as given in (6.3.11), and moreover, $\mathcal{A}'$ has the property that

$$C_c^*(\mathrm{Ran}\,X', \widetilde{\mathcal{A}}') \simeq C_c^*(\mathrm{Ran}\,X', \mathcal{A}').$$

6.3.12. Altogether, we have the following

Proposition 6.3.13. *We have a natural equivalence*

$$C_c^*(\mathrm{Ran}\,X', \mathcal{A}') \simeq C_*^{\mathrm{red}}(\mathrm{Bun}_G).$$

6.4. **Pairing.** We will now describe how the equivalence given by Proposition 6.3.13 helps us show that (6.2.7) is an equivalence.

6.4.1. For any schemes $S, S' \in \mathrm{Sch}$ and any non-empty finite subsets $I \subset X(S)$ and $I' \subset X'(S')$, we have a natural map (which is just a more elaborate variant of (6.2.6))

$$\mathrm{Gr}_{I'} \times S \to \mathrm{Bun}_G \times S' \times S \to S' \times BG_I,$$

which induces a map

$$\mathcal{A}' \boxtimes \mathcal{B} \to \omega_{\mathrm{Ran}X' \times \mathrm{Ran}X},$$

and hence, a pairing (using TakeOut)

$$\mathcal{A}' \boxtimes \mathcal{B} \to \Gamma_{j_{\mathrm{Ran}}!}\omega_{\mathrm{Ran}X'}.$$

6.4.2. Restricting this map to $\mathrm{Ran}X' \times \mathrm{Ran}X'$ gives us the following map

$$\mathcal{A}' \boxtimes \mathcal{B}' \to (\delta_{\mathrm{Ran}X'})_! \omega_{\mathrm{Ran}X'},$$

and hence, using the definition of Verdier duality, a map

$$\mathcal{B}' \to D_{\mathrm{Ran}X'}\mathcal{A}' \tag{6.4.3}$$

between objects in $\mathrm{ComAlg}^\star(\mathrm{Ran}X')$.

6.4.4. It is proved, in fact twice (using different methods), in §17 and §18 of [Gai15], that the restriction of (6.4.3) to the diagonal X' of $\mathrm{Ran}X'$ is an equivalence. Namely, we have

$$\mathcal{B}'|_{X'} \simeq (D_{\mathrm{Ran}X'}\mathcal{A}')|_{X'}. \tag{6.4.5}$$

6.5. **The last steps.** The results that we have just proved in this paper appear in two places in the concluding steps, which are given by Proposition 6.5.1 and 6.5.4. Together, they imply the Atiyah-Bott formula.

Proposition 6.5.1. *$D_{\mathrm{Ran}X'}\mathcal{A}'$ is factorizable, i.e.*

$$D_{\mathrm{Ran}X'}\mathcal{A}' \in \mathrm{Fact}^\star(X') \subset \mathrm{ComAlg}^\star(\mathrm{Ran}X').$$

Proof. It is well-known that for a split semi-simple simply connected group G_0, $C^{\mathrm{red}}_*(\mathrm{Gr}_{G_0}, \Lambda)$ lives in cohomological degrees ≤ -2. Using the fact that

$$\mathrm{Gr}_{\mathrm{Ran}X'} \to \mathrm{Ran}X'$$

is pseudo-proper and that $\mathcal{A}'$ is factorizable, we see that for each non-empty finite set I, $\mathcal{A}'|_{\overset{\circ}{X}'^I}$ lives in (perverse) cohomological degrees $\leq -3|I|$.

Now, by Theorem 3.3.3, we know that there exists an object

$$\mathfrak{a}' \in \mathrm{Lie}^\star(X')^{\leq c_L}$$

such that

$$\mathcal{A}' \simeq \mathrm{Chev}(\mathfrak{a}').$$

Theorem 5.3.1 then implies that

$$D_{\mathrm{Ran}X'}\mathrm{Chev}(\mathfrak{a}') \simeq \mathrm{coChev}(D_{\mathrm{Ran}X'}\mathfrak{a}'),$$

which is known to be factorizable by Theorem 4.1.3 □

Corollary 6.5.2. *The map given in* (6.4.3) *is an equivalence, i.e.*

$$\mathcal{B}' \simeq D_{\mathrm{Ran}X'}\mathcal{A}', \tag{6.5.3}$$

and hence

$$\mathcal{B} \simeq (j_{\mathrm{Ran}})_* \mathrm{coChev}\, D_{X'}\mathfrak{a}' \simeq \mathrm{coChev}\, j_* D_{X'}\mathfrak{a}'.$$

Proof. The first statement is a direct consequence of the proposition above and the equivalence (6.4.5), where as the second statement is the result of Proposition 5.4.1. □

Proposition 6.5.4. *We have the following equivalence induced by Proposition 6.5.1*

$$C^*_c(\mathrm{Ran}X, \mathcal{B}) \simeq C^*_c(\mathrm{Ran}X', \mathcal{A}')^\vee.$$

Proof. We have the following equivalences

$$
\begin{aligned}
&(6.5.5) \qquad & C^*_c(\operatorname{Ran} X, \mathcal{B}) &\simeq C^*_c(\operatorname{Ran} X, \operatorname{coChev} j_* D_{X'} \mathfrak{a}') \\
&(6.5.6) \qquad & &\simeq \operatorname{coChev} C^*_c(X, j_* D_{X'} \mathfrak{a}') \\
& & &\simeq \operatorname{coChev} C^*(X, D_{X'} \mathfrak{a}') \\
& & &\simeq \operatorname{coChev}(C^*_c(X, \mathfrak{a}')^\vee) \\
&(6.5.7) \qquad & &\simeq (\operatorname{Chev}(C^*_c(X', \mathfrak{a}')))^\vee \\
& & &\simeq C^*_c(\operatorname{Ran} X', \operatorname{Chev} \mathfrak{a}')^\vee \\
& & &\simeq C^*_c(\operatorname{Ran} X', \mathcal{A}')^\vee.
\end{aligned}
$$

Here, (6.5.5), (6.5.6) and (6.5.7) are due to Corollary 6.5.2, Theorem 5.1.2 and Theorem 5.3.1 (applied to a point) respectively. □

6.5.8. Finally, as a corollary, we have the Atiyah-Bott formula. Indeed, we have

$$C^{\mathrm{red}}_*(\operatorname{Bun}_G)^\vee \simeq C^*_c(\operatorname{Ran} X', \mathcal{A}')^\vee \simeq C^*_c(\operatorname{Ran} X, \mathcal{B}) \simeq \operatorname{Sym}^{>0} C^*(X', M)$$

where the first, second and third equivalences are due to Proposition 6.3.13, Proposition 6.5.4, and (6.2.11) respectively.

Appendix A. The addFil trick

In this appendix, we will quickly recall, without proof, a useful construction taken from [GR, §IV.2], which allows us to reduce many statements about $\mathcal{P}$-algebras to trivial $\mathcal{P}$-algebras, where $\mathcal{P}$ is an operad in Vect. Throughout this subsection, all categories without any further description will be assumed to be presentable, symmetric monoidal stable infinity over a field k of characteristic 0. Moreover, functors between these categories are assumed to be continuous.

All such categories, along with continuous functors between them, form a category, which we will use

$$\mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{pres,cont}},$$

to denote, or for simplicity

$$\mathrm{DGCat}^{\mathrm{SymMon}}.$$

A.1. **Notations.** For a symmetric monoidal category $\mathcal{C}$, we denote the category of filtered objects in $\mathcal{C}$

$$\mathcal{C}^{\mathrm{Fil}} = \operatorname{Fun}(\mathbb{Z}, \mathcal{C}),$$

the category of functors from $\mathbb{Z}$ to $\mathcal{C}$. Here, $\mathbb{Z}$ is a ordered set, viewed as a category. Similarly, we denote the category of graded objects

$$\mathcal{C}^{\mathrm{gr}} = \operatorname{Fun}(\mathbb{Z}^{\mathrm{set}}, \mathcal{C}),$$

where $\mathbb{Z}^{\mathrm{set}}$ is a the discrete category, whose underlying underlying objects are the integers.[15]

A.2. **Functors.** Now, we will recall several familiar functors between $\mathcal{C}$, $\mathcal{C}^{\mathrm{Fil}}$, and $\mathcal{C}^{\mathrm{gr}}$.

A.2.1. Let

$$V = \cdots \to V_{n-1} \to V_n \to V_{n+1} \to \cdots,$$

be an object in $\mathcal{C}^{\mathrm{Fil}}$. Then, we define

$$\text{ass-gr} : \mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}^{\mathrm{gr}}$$

to be the functor of taking the associated graded object

$$\text{ass-gr}(V)_n = \operatorname{coFib}(V_{n-1} \to V_n),$$

and

$$\mathrm{oblv}_{\mathrm{Fil}} : \mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}$$

[15] In [GR], it's called $\mathbb{Z}^{\mathrm{Spc}}$.

to be the left Kan extension along

$$\mathbb{Z} \to \mathrm{pt}.$$

Namely

$$\mathrm{oblv}_{\mathrm{Fil}}(V) = \operatorname*{colim}_{n\in\mathbb{Z}} V_n.$$

A.2.2. We also use

$$(\mathrm{gr} \to \mathrm{Fil}) : \mathcal{C}^{\mathrm{gr}} \to \mathcal{C}^{\mathrm{Fil}}$$

and

$$\bigoplus : \mathcal{C}^{\mathrm{gr}} \to \mathcal{C}$$

to denote the functor obtained by taking the left Kan extension along

$$\mathbb{Z}^{\mathrm{set}} \to \mathbb{Z},$$

and

$$\mathbb{Z}^{\mathrm{set}} \to \mathrm{pt}$$

respectively.

A.2.3. Note that the categories $\mathcal{C}^{\mathrm{Fil}}$ and $\mathcal{C}^{\mathrm{gr}}$ are equipped with a natural symmetric monoidal structure coming from $\mathcal{C}$, and moreover, the functors ass-gr, $\mathrm{oblv}_{\mathrm{Fil}}$, $\mathrm{gr} \to \mathrm{Fil}$, and $\bigoplus$ are naturally symmetric monoidal.

A.2.4. *Adding a filtration.* Let

$$\mathrm{addFil} : \mathcal{C} \to \mathcal{C}^{\mathrm{Fil}}$$

be the functor defined as follows: for an object V in $\mathcal{C}$,

$$\mathrm{addFil}(V)_n = \begin{cases} V, & \text{when } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

It's easy to see that

$$\bigoplus \circ \text{ass-gr} \circ \mathrm{addFil} \simeq \mathrm{oblv}_{\mathrm{Fil}} \circ \mathrm{addFil} \simeq \mathrm{id}_{\mathcal{C}}.$$

A.3. **Interactions with algebras over an operad.** Let $\mathcal{P}$ be an operad in Vect. Then we have the following pair of functors

$$\mathrm{addFil} : \mathcal{P}\text{-alg}(\mathcal{C}) \to \mathcal{P}\text{-alg}(\mathcal{C}^{\mathrm{Fil}^{>0}}) \qquad \text{and} \qquad \mathrm{oblv}_{\mathrm{Fil}} : \mathcal{P}\text{-alg}(\mathcal{C}^{\mathrm{Fil}^{>0}}) \to \mathcal{P}\text{-alg}(\mathcal{C}).$$

A.3.1. Let

$$F : \mathrm{DGCat}^{\mathrm{SymMon}} \to \mathcal{C}\mathrm{at}_\infty$$

be a functor, where $\mathcal{C}\mathrm{at}_\infty$ is the ∞-category of all ∞-categories. Suppose we have a continuous natural transformation

$$\Phi : \mathcal{P}\text{-alg}(-) \to F(-),$$

i.e. morphisms between two objects in

$$\mathrm{Fun}(\mathrm{DGCat}^{\mathrm{SymMon}}, \mathcal{C}\mathrm{at}_\infty).$$

Then from what we've discussed above, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{P}\text{-alg}(\mathcal{C}) \arrow[r, "\Phi"] & F(\mathcal{C}) \\
\mathcal{P}\text{-alg}(\mathcal{C}^{\mathrm{Fil}}) \arrow[r, "\Phi"] \arrow[u, "\mathrm{oblv}_{\mathrm{Fil}}"] & F(\mathcal{C}^{\mathrm{Fil}}) \arrow[u, "\mathrm{oblv}_{\mathrm{Fil}}"']
\end{tikzcd}$$

which, combined with the fact that

$$\mathrm{oblv}_{\mathrm{Fil}} \circ \mathrm{addFil} \simeq \mathrm{id}_{\mathcal{C}},$$

implies that the following diagram also commutes

$$\begin{tikzcd} \mathcal{P}\text{-alg}(\mathcal{C}) \arrow[r, "\Phi"] \arrow[d, "\mathrm{addFil}"'] & F(\mathcal{C}) \\ \mathcal{P}\text{-alg}(\mathcal{C}^{\mathrm{Fil}}) \arrow[r, "\Phi"] & F(\mathcal{C}^{\mathrm{Fil}}) \arrow[u, "\mathrm{oblv}_{\mathrm{Fil}}"'] \end{tikzcd}$$

A.3.2. Further composing the diagram above with ass-gr and $\bigoplus$ gives us the following commutative diagram

$$\begin{tikzcd} \mathcal{P}\text{-alg}(\mathcal{C}) \arrow[r, "\Phi"] \arrow[d, "\mathrm{addFil}"'] & F(\mathcal{C}) \\ \mathcal{P}\text{-alg}(\mathcal{C}^{\mathrm{Fil}^{>0}}) \arrow[r, "\Phi^{\mathrm{Fil}}"] \arrow[d, "\text{ass-gr}"'] & F(\mathcal{C}^{\mathrm{Fil}^{>0}}) \arrow[u, "\mathrm{oblv}_{\mathrm{Fil}}"'] \arrow[d, "\text{ass-gr}"'] \\ \mathcal{P}\text{-alg}(\mathcal{C}^{\mathrm{gr}^{>0}}) \arrow[r, "\Phi^{\mathrm{gr}}"] \arrow[d, "\oplus"'] & F(\mathcal{C}^{\mathrm{gr}^{>0}}) \arrow[d, "\oplus"'] \\ \mathcal{P}\text{-alg}(\mathcal{C}) \arrow[r, "\Phi"] & F(\mathcal{C}) \end{tikzcd} \tag{A.3.3}$$

We will refer to this as the *fundamental commutative diagram of the* addFil *trick*.

A.3.4. Now, suppose there are two natural transformations

$$\Phi_1, \Phi_2 : \mathcal{P}\text{-alg}(-) \to F(-)$$

equipped with a morphism between them

$$\alpha : \Phi_1 \to \Phi_2.$$

Or more concretely, we have a compatible family of morphisms in $F(\mathcal{C})$

$$\Phi_1(c) \to \Phi_2(c)$$

parametrized by pairs $(\mathcal{C}, c)$ where $c \in \mathcal{C}$ and $\mathcal{C} \in \mathrm{DGCat}^{\mathrm{SymMon}}$, and we want to prove that α is an equivalence.

A.3.5. The top square of the commutative diagram above implies that it suffices to show that

$$\Phi_1^{\mathrm{Fil}} \circ \mathrm{addFil} \to \Phi_2^{\mathrm{Fil}} \circ \mathrm{addFil}$$

is an equivalence. But since ass-gr and $\bigoplus$ are conservative, it suffices to show that

$$\bigoplus \circ \,\text{ass-gr} \circ \Phi_1^{\mathrm{Fil}} \circ \mathrm{addFil} \to \bigoplus \circ \,\text{ass-gr} \circ \Phi_2^{\mathrm{Fil}} \circ \mathrm{addFil}$$

is an equivalence, which, due to the commutativity of the diagrams, is equivalent to

$$\Phi_1 \circ \bigoplus \circ \,\text{ass-gr} \circ \mathrm{addFil} \to \Phi_2 \circ \bigoplus \circ \,\text{ass-gr} \circ \mathrm{addFil}$$

being an equivalence.

A.3.6. The crucial observation of [GR, Prop. IV.2.1.4.6] is the following

Proposition A.3.7. *The functor*

$$\bigoplus \circ \,\text{ass-gr} \circ \mathrm{addFil} : \mathcal{P}\text{-alg}(\mathcal{C}) \to F(\mathcal{C})$$

is canonically equivalent to $\mathrm{triv}_{\mathcal{P}} \circ \mathrm{oblv}_{\mathcal{P}}$*, i.e.*

$$\mathcal{P}\text{-alg}(\mathcal{C}) \xrightarrow{\mathrm{oblv}_{\mathcal{P}}} \mathcal{C} \xrightarrow{\mathrm{triv}_{\mathcal{P}}} \mathcal{P}\text{-alg}(\mathcal{C}).$$

A.3.8. This implies that it suffices to prove that

$$\Phi_1(c) \to \Phi_2(c)$$

is an equivalence only for the case where c is a trivial algebra.

A.4. **A general principle.** More generally, suppose we want to prove a property of $\Phi(c)$ for some $c \in \mathcal{P}\text{-alg}(\mathcal{C})$. Moreover, suppose this property is preserved under under $\mathrm{oblv}_{\mathrm{Fil}}$, and is conservative under $\bigoplus$ and ass-gr. Then, it suffices to prove the case where c has a trivial algebra structure.

Appendix B. Co-filtration and addCoFil

In this appendix, we will collect various notions that are dual to the one in §A. These are used in the body of the paper to give a proof of the addCoFil trick in a special case.

B.1. **Notations.** For a symmetric monoidal category $\mathcal{C}$, we denote the category of co-filtered objects in $\mathcal{C}$

$$\mathcal{C}^{\mathrm{coFil}} = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}).$$

B.2. **Functors.** As in the case of filtration, there are several familiar functors between $\mathcal{C}$, $\mathcal{C}^{\mathrm{coFil}}$, and $\mathcal{C}^{\mathrm{gr}}$.

B.2.1. Let

$$V = \cdots \to V_{n+1} \to V_n \to V_{n-1} \to \cdots,$$

be an object in $\mathcal{C}^{\mathrm{coFil}}$. Then we define

$$\text{ass-gr} : \mathcal{C}^{\mathrm{coFil}} \to \mathcal{C}^{\mathrm{gr}}$$

to be the functor of taking the associated graded object

$$\text{ass-gr}(V)_n = \mathrm{Fib}(V_n \to V_{n-1}),$$

and

$$\mathrm{oblv}_{\mathrm{coFil}} : \mathcal{C}^{\mathrm{coFil}} \to \mathcal{C}$$

to be the right Kan extension along

$$\mathbb{Z}^{\mathrm{op}} \to \mathrm{pt}.$$

Namely

$$\mathrm{oblv}_{\mathrm{coFil}}(V) = \lim_{n \in \mathbb{Z}^{\mathrm{op}}} V_n.$$

B.2.2. Note that the category $\mathcal{C}^{\mathrm{coFil}}$ naturally inherits the monoidal structure coming from $\mathcal{C}$. Moreover, the functor ass-gr is monoidal.

B.2.3. We also use

$$\prod : \mathcal{C}^{\mathrm{gr}} \to \mathcal{C}$$

to denote the right Kan extension along

$$\mathbb{Z}^{\mathrm{set}} \to \mathrm{pt}.$$

Namely

$$\prod((V_n)_{n\in\mathbb{Z}}) = \prod_{n \in \mathbb{Z}} V_n.$$

B.2.4. *Adding a co-filtration.* We will use

$$\mathrm{addCoFil} : \mathcal{C} \to \mathcal{C}^{\mathrm{coFil}}$$

to denote a functor defined as follows: for an object V in $\mathcal{C}$,

$$\mathrm{addCoFil}(V)_n = \begin{cases} V, & \text{when } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

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