MSRI GRADUATE SEMINAR: THE GEOMETRIC SATAKE CORRESPONDENCE

ANA CARAIANI, IORDAN GANEV, AND ARNO KRET

1. Overview

The goal of this seminar is to understand Timo Richarz's proof of the geometric Satake correspondence in [Ri]. Let G be a conected, reductive group over a separably closed field F. The loop group of G is the group functor on F-algebras given by

$$G((t)) : R \mapsto G(R((t))).$$

The positive loop group is the group functor on F-algebras given by

$$G[[t]] : R \mapsto G(R[[t]]).$$

The *affine Grassmannian* is the fpqc quotient

$$\mathrm{Gr}_G := G((t))/G[[t]].$$

It is an ind-projective ind-scheme. The Satake category is the category $P_{G[[t]]}(\mathrm{Gr}_G)$ of $G[[t]]$-equivariant l-adic perverse sheaves on Gr_G (here, l is different from the characteristic of F). This is equipped with a convolution product $\star$ and with a fiber functor ω given by taking global cohomology. Our goal is to understand the proof of the following theorem.

Theorem 1.0.1. [Ri]

(1) *The pair* $(P_{G[[t]]}(\mathrm{Gr}_G), \star)$ *admits a unique symmetric monoidal structure such that the fiber functor* ω *is symmetric monoidal.*
(2) *The fiber functor* ω *is a faithful, exact tensor functor and it induces, via the Tannakian formalism, an equivalence of tensor categories*

$$(P_{G[[t]]}(\mathrm{Gr}_G), \star) \xrightarrow{\sim} (\mathrm{Rep}_{\bar{\mathbb{Q}}_l} \hat{G}, \otimes),$$

where $\hat{G}$ *is the Langlands dual group of* G *over* $\bar{\mathbb{Q}}_l$.

2. Program

2.1. **Lecture 1: The classical (arithmetic) Satake isomorphism.** Speaker: Macky Makisumi. State the classical Satake isomorphism, define the Satake transform, sketch the proof that it is an isomorphism, give an explicit example in the case of GL_n. References: [Gr, He].

2.2. **Lecture 2: Motivation and overview of the proof of the geometric Satake correspondence.** Speaker: Sam Raskin. Explain why one should be able to upgrade the Satake isomorphism to an equivalence of categories, talk about the function-sheaf dictionary, give an overview of the proof.

2.3. **Lecture 3: The affine Grassmannian for GL_n.** Speaker: Yotam Hendel. Talk briefly about sheaves as functors of points and fpqc descent. Define the affine Grassmannian Gr_{GL_n} for GL_n. Define local and global vector bundles and local and global versions of the affine Grassmannian: $\mathrm{Gr}^{\mathrm{loc}}_{GL_n}$ and $\mathrm{Gr}^{\mathrm{glob}}_{GL_n}$. Explain gluing, i.e. the Beauville-Laszlo theorem. Prove the sequence of isomorphisms

$$\mathrm{Gr}_{GL_n} \simeq \mathrm{Gr}^{\mathrm{loc}}_{GL_n} \simeq \mathrm{Gr}^{\mathrm{glob}}_{GL_n}.$$

References: [BL].

2.4. **Lecture 4: The affine Grassmannian Gr_G.** Speaker: Justin Hilburn. Let G be any connected reductive group. Define G-bundles. Give the example of GL_n-bundles and explain why they are the same as vector bundles. Also give SL_n and Sp_{2n} as examples. Explain the Tannakian formalism. Define the affine Grassmannian Gr_G for G, local and global versions. Sketch how to prove that it is an ind-projective ind-scheme of ind-finite type.

2.5. **Lecture 5: Perverse sheaves.** Speaker: Quoc Ho. Define perverse sheaves and equivariant perverse sheaves. Talk about the middle extension and intersection cohomology. References: [BBD, Ac, Ar, dCM, HTT, Cl].

2.6. **Lecture 6: Perverse sheaves and nearby cycles.** Speaker: Quoc Ho. Talks about Braden's theorem. Define nearby and vanishing cycles. Explain why nearby cycles preserve perversity. References: [Br, BBD, DG]

2.7. **Lecture 7: Convolution and fusion, part 1.** Speaker: Akhil Matthew. Define convolution, talk about the Beilinson-Drinfeld Grassmannian, draw the global convolution diagram. References: Section 2 of [Ri], [Ga].

2.8. **Lecture 8: Convolution and fusion, part 2.** Speaker: Iordan Ganev. Sketch the proof that convolution is fusion. References: Section 2 of [Ri].

2.9. **Lecture 9: The Tannakian structure.** Speaker: Jacob Matherne. Describe the stratification of Gr_G in terms of orbits. Describe the simple objects in the category of equivariant perverse sheaves on the affine Grassmannian for G in terms of intersection cohomology. Prove that the Satake category is Tannakian. References: Section 3 of [Ri].

2.10. **Lecture 10: Recovering $\hat{G}$.** Speaker: Andreas Mihatsch. Sketch how to identify the group of tensor automorphisms of the fiber functor (given by taking global cohomology) with the reductive group $\hat{G}$. Briefly explain how to reconstruct the root datum of a split reductive group from the Grothendieck semiring of its algebraic representations. References: Section 4 and Appendix B of [Ri].

References

[Ac] P. Achar, "Lecture notes on perverse sheaves," `https://www.math.lsu.edu/~pramod`

[Ar] A. Arabia: "Faisceaux pervers sur les variétés algbriques complexes. Correspondance de Springer (d'après Borho-MacPherson)", `http://webusers.imj-prg.fr/~alberto.arabia`

[BBD] A. Beilinson, J. Bernstein, and P. Deligne:"Faisceaux pervers"

[BL] A. Beauville and Y. Laszlo: "Une lemme de descente"

[Br] T. Braden,"Hyperbolic localization of intersection cohomology"

[Cl] D. Clausen: "The Springer correspondence"

[dCM] M. A. de Cataldo, L. Migliorini. "The Decomposition Theorem, Perverse Sheaves, and the Topology of Algebraic Maps." (Bulletin of the AMS)

[DG] V. Drinfeld and D. Gaitsgory: "On a theorem of Braden"
[Ga] D. Gaitsgory: "Construction of central elements in the affine Hecke algebra via nearby cycles"
[Gr] B. Gross: "The Satake Transfrom"
[He] F. Herzig: "A Satake isomorphism in characteristic p"
[HTT] Hotta, Takeuchi, Tanisaki: "D-modules, perverse sheaves and representation theory", Birkhaeuser.
[MV] Mirković, Vilonen: "Geometric Langlands duality and representations of algebraic groups over commutative rings"
[Ri] T. Richarz: "A new approach to the geometric Satake equivalence"