THE WONDERFUL COMPACTIFICATION FOR QUANTUM GROUPS

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SECTIONS

- Introduction and motivation
- Definition and basic properties of the wonderful compactification
- The Vinberg semigroup
- Quantum groups, quantum $\mathcal{D}$-modules

Let G be a connected reductive group. Let $G_{\mathrm{adj}} = G/Z(G)$ be the adjoint group.

We're interested in a certain compactification $\overline{G_{\mathrm{adj}}}$ of G_{adj}. Construction to come. Basic properties:

- Smooth projective variety with a $G \times G$ action.
- Open orbit is G_{adj} with left and right multiplication action.
- Closed orbit is $G/B \times G/B$.
- Other orbits are related to partial flag varieties, wonderful compactifications of smaller rank groups, degenerations of G, etc.
- Encodes asymptotics of matrix coefficients

WHY IS IT WONDERFUL?

'Wonderful' is a technical term (a wonderful variety is a type of spherical variety with conditions on the boundary divisors).

Appearances in geometric representation theory:

- character sheaves [Bezrukavnikov-Finkelberg-Ostrik]
- second adjointness for p-adic groups [Bezrukavnikov-Kazhdan, Sakellaridis-Venkatesh]
- constant term & Eisenstien series functors [Drinfeld-Gaitsgory]

Also relevant in the theory of spherical varieties, Poisson geometry, arithmetic geometry.

Question: What is the wonderful compactification for quantum groups? What can we do with it?

- Let G be a connected reductive group over $\mathbb{C}$. Fix a Borel $B \subset G$ and a maximal torus $T \subset B$.
- Let $V = V_\lambda$ be a finite-dimensional irreducible representation of **regular** highest weight λ. Consider

$$\begin{tikzcd} G \arrow[r] \arrow[d] & GL(V) \arrow[r] & End(V) \setminus \{0\} \arrow[d] \\ G_{adj} & & \mathbb{P}(End(V)) \end{tikzcd}$$

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$$\begin{tikzcd} G \arrow[r] \arrow[d] & GL(V) \arrow[r] & End(V) \setminus \{0\} \arrow[d] \\ G_{adj} \arrow[rr, "\psi"] & & \mathbb{P}(End(V)) \end{tikzcd}$$

Lemma. The map ψ is well-defined, injective, and $G \times G$ equivariant.

Definition/Proposition

The **wonderful compactification** $\overline{G_{adj}}$ of G_{adj} is the closure of the image of ψ in $\mathbb{P}(End(V))$. The projective $G \times G$-variety $\overline{G_{adj}}$ is smooth and does not depend on the choice of regular highest weight.

Let $G = SL_2$, and so $G_{adj} = PSL_2$.

Claim: The wonderful compactification is $\overline{PSL_2} = \mathbb{P}^3$.

Proof: The defining representation $\mathbb{C}^2$ is of regular highest weight.

$$\begin{tikzcd} SL_2\mathbb{C} \arrow[r] \arrow[d] & Mat_2\mathbb{C} \setminus \{0\} \arrow[d] \\ PSL_2\mathbb{C} \arrow[r, "\psi"] & \mathbb{P}(Mat_2\mathbb{C}) \simeq \mathbb{P}^3 \end{tikzcd}$$

The image of ψ is dense.

Note: The complement of PSL_2 in $\mathbb{P}^3$ is

$$\{\text{rank 1 matrices}\}/\mathbb{C}^\times = \mathbb{P}^1 \times \mathbb{P}^1$$

The stratification into $G \times G$ orbits is $PSL_2 \coprod (\mathbb{P}^1 \times \mathbb{P}^1)$.

EXAMPLE: G = SL_3 AND HIGHER

WARNING: For $n > 2$, the defining representation of SL_n is not regular, and the wonderful compactification of PSL_n is **not** $\mathbb{P}^{n^2-1}$. (Although this is **a** compactification of PSL_n, there aren't enough $G \times G$ orbits, and the structure isn't as rich.)

Example: For $G = SL_3$, the wonderful compactification is harder to describe than for SL_2. It has four orbits: PSL_3, $SL_3/B \times SL_3/B$, and two that form fibrations (i=1,2)

$$\begin{tikzcd} PSL_2 \arrow[r] & Orb_{\alpha_i} \arrow[d] \\ & \mathbb{P}^2 \times \mathbb{P}^2 \end{tikzcd} \qquad \begin{tikzcd} \overline{PSL_2} = \mathbb{P}^3 \arrow[r] & \overline{Orb_{\alpha_i}} \arrow[d] \\ & \mathbb{P}^2 \times \mathbb{P}^2 \end{tikzcd}.$$

Let Δ be the set of positive simple roots of $\mathrm{Lie}(G)$. For $I \subseteq \Delta$, let P_I and L_I denote the corresponding parabolic subgroup and Levi.

Proposition

The $G \times G$ orbits on $\overline{G_{adj}}$ are in bijection with subsets of Δ. Write Orb_I for the orbit corresponding to $I \subseteq \Delta$. There are fibrations:

$$\begin{tikzcd} L_{adj} \arrow[r] & \mathrm{Orb}_I \arrow[d] \\ & G/P_I \times P_I^- \backslash G \end{tikzcd} \qquad \begin{tikzcd} \overline{L_{adj}} \arrow[r] & \overline{\mathrm{Orb}_I} \arrow[d] \\ & G/P_I \times P_I^- \backslash G \end{tikzcd} .$$

where $\overline{L_{adj}}$ is the wonderful compactification of the adjoint group L_{adj} of L_I, and $\overline{\mathrm{Orb}_I}$ denotes the closure of Orb_I in $\overline{G_{adj}}$.

Extremes: $\mathrm{Orb}_\Delta = G_{adj}$ and $\mathrm{Orb}_\emptyset = G/B \times B^- \backslash G$.

There is a different perspective on the wonderful compactification based on the Vinberg semigroup.

Key tools:

- Peter-Weyl theorem
- Rees algebra of $\mathcal{O}(G)$
- GIT quotients

VINBERG SEMIGROUP

Recall:

$$\left\{ \begin{array}{c} \text{finite-dimensional irreducible} \\ \text{representations of } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{dominant weights} \\ \text{of } G \end{array} \right\} =: \Lambda^+.$$

$$V_\lambda \longleftrightarrow \lambda$$

Peter-Weyl Theorem

There is an isomorphism of $G \times G$ representations

$$\bigoplus_{\lambda \in \Lambda^+} V_\lambda^* \otimes V_\lambda \longrightarrow \mathcal{O}(G)$$

given by matrix coefficients: $f \otimes v \mapsto [g \mapsto \langle f, g \cdot v \rangle]$.

Remark: The multiplication on $\mathcal{O}(G)$ does not respect the decomposition. Reason: $V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus \text{stuff}$.

Let $\Lambda = X^*(T)$ be the weight lattice of G, equipped with the following partial order: $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of non-negative multiples of positive roots.

Lemma. If $V_\nu \subseteq V_\lambda \otimes V_\mu$, then $\nu \leq \lambda + \mu$. Consequently, there is a Λ-filtration on $\mathcal{O}(G)$ given by

$$\mathcal{O}(G)_{\leq \lambda} = \bigoplus_{\mu \leq \lambda} V_\mu^* \otimes V_\mu.$$

Example. If $G = SL_2$, then $\Lambda^+ = \mathbb{Z}_{\geq 0} \subseteq \Lambda = \mathbb{Z}$ and $\alpha = 2$ is the only positive root. Partial order:

$$\cdots \leq -2 \leq 0 \leq 2 \leq 4 \leq \ldots$$

$$\cdots \leq -1 \leq 1 \leq 3 \leq 5 \leq \ldots$$

We have $\mathcal{O}(SL_2) = \mathbb{C}[a, b, c, d]/(ad - bc = 1)$. As a vector space, $\mathcal{O}(SL_2)_{\leq 1}$ is spanned by a, b, c, d.

Definition/Proposition

The Rees algebra of $\mathcal{O}(G)$ is the following subalgebra of $\mathcal{O}(G) \otimes \mathbb{C}[\Lambda]$:

$$\operatorname{Rees}(\mathcal{O}(G)) = \bigoplus_{\lambda \in \Lambda} \mathcal{O}(G)_{\leq \lambda} z^{\lambda}.$$

$\operatorname{Rees}(\mathcal{O}(G))$ is naturally a bialgebra (not Hopf!). Define the Vinberg semigroup as $\mathbb{V}_G = \operatorname{Spec}(\operatorname{Rees}(\mathcal{O}(G)))$.

(Here $\mathbb{C}[\Lambda]$ is the group algebra, so $z^{\lambda+\mu} = z^{\lambda} z^{\mu}$.)

Remark. This is not Vinberg's original definition; see work of Brion.

Example. For $G = SL_2$, we have $\mathbb{V}_{SL_2} = \mathrm{Mat}_2$.

Also: The inclusion $\mathbb{C}[z^{\alpha} \mid \alpha \in \Delta\,] \hookrightarrow \mathcal{O}(\mathbb{V}_G)$ induces a map $\pi : \mathbb{V}_G \to \mathbb{A}^r$. The generic fiber is G; the other fibers are rational degenerations of G.

Since $\mathcal{O}(\mathbb{V}_G) = \mathrm{Rees}(\mathcal{O}(G))$ is graded by $\Lambda = X^*(T)$, there is an action of T on $\mathbb{V}_G$.

Theorem

1. [Martens-Thaddeus] Fix $\lambda \in \Lambda^+$ regular. Then $\overline{G_{adj}} = \mathbb{V}_G /\!\!/_\lambda T$.
2. A Poisson-Lie bracket on $\mathcal{O}(G)$ (e.g. the Sklyanin bracket) on G induces Poisson structures on $\mathbb{V}_G$, $\overline{G_{adj}}$, and each Orb_I.

- For simplicity, assume G is semisimple and simply connected.
- Let $\mathfrak{g} = \mathrm{Lie}(G)$.
- Fix $q \in \mathbb{C}^\times$ and let $U_q(\mathfrak{g})$ be the quantum group.

Example. The quantum group $U_q(\mathfrak{sl}_2)$ is generated by $E, F, K^{\pm 1}$ subject to the relations

$$KE = q^2 EK \qquad KF = q^{-2} FK \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

For q not a root of unity:

$$\left\{ \begin{array}{c} \text{finite-dimensional irreducible} \\ \text{representations of } U_q(\mathfrak{g}) \\ \text{(of type 1)} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{dominant weights} \\ \text{of } G \end{array} \right\} =: \Lambda^+.$$

$$V_\lambda \longleftrightarrow \lambda$$

Definition

The quantized coordinate algebra $\mathcal{O}_q(G)$ is the image of

$$\bigoplus_{\lambda \in \Lambda^+} V_\lambda^* \otimes V_\lambda \longrightarrow U_q(\mathfrak{g})^*.$$

Theorem

The algebra $\mathcal{O}_q(G)$ ...

1. is a Hopf algebra that quantizes $\mathcal{O}(G)$ with the Sklyanin bracket. [de Concini-Lyubashenko]
2. has a Λ-filtration given by $\mathcal{O}_q(G)_{\leq\lambda} = \bigoplus_{\mu \leq \lambda} V_\mu^* \otimes V_\mu$.

Remark. Alternative definitions use FRT algebras and R-matrix quantization.

EXAMPLE

The algebra $\mathcal{O}_q(\mathrm{Mat}_2)$ is generated by elements a, b, c, d with relations

$$ab = qba \qquad cd = qdc$$

$$ac = qca \qquad ad = da + (q - q^{-1})cb \qquad bc = cb \qquad bd = qdb$$

The quantum coordinate algebra of SL_2 is:

$$\mathcal{O}_q(SL_2) = \mathcal{O}_q(\mathrm{Mat}_2)/\langle ad - qbc - 1\rangle.$$

The coproduct and counit are the same as for $\mathcal{O}(SL_2)$. As a vector space, $\mathcal{O}_q(SL_2)_{\leq 1}$ is spanned by a, b, c, d.

Definition

The quantum coordinate ring of the Vinberg semigroup is the Rees algebra $\mathcal{O}_q(\mathbb{V}_G) := \mathrm{Rees}(\mathcal{O}_q(G))$.

Proposition [G.]

1. $\mathcal{O}_q(\mathbb{V}_G)$ is naturally a bialgebra and quantizes $\mathbb{V}_G$.
2. Fix $\lambda \in \Lambda^+$ regular. The graded algebra

$$\bigoplus_{n \geq 0} \mathcal{O}_q(\mathbb{V}_G)_{n\lambda}$$

quantizes the homogeneous coordinate ring of $\overline{G_{adj}}$.

Recall that the Sklyanin bracket induces Poisson structures on $\mathbb{V}_G$, $\overline{G_{adj}}$, and each orbit Orb_I.

Fix $I \subseteq \Delta$.

- Let $\Lambda_I = \mathbb{Z}\{\alpha \mid \alpha \in I\} \subseteq \Lambda$. There is a filtration on $\mathcal{O}_q(G)$ by Λ/Λ_I with associated graded $\mathrm{gr}_I(\mathcal{O}_q(G))$.
- Let $\mathfrak{u}_I^{\pm}$ and $\mathfrak{l}_I$ be the Lie algebras of $R_{unip}(P_I^{\pm})$ and L_I. Regard $\mathfrak{u}_I^{-} \times \mathfrak{l}_I \times \mathfrak{u}_I$ as a subalgebra of $\mathfrak{g} \times \mathfrak{g}$, with $\mathfrak{l}_I$ positioned diagonally.

Theorem [G.]

Fix $I \subseteq \Delta$. There is an isomorphism of graded algebras

$$\mathrm{gr}_I(\mathcal{O}_q(G)) = \mathcal{O}_q(G \times G)^{U_q(\mathfrak{u}_I^{-} \times \mathfrak{l}_I \times \mathfrak{u}_I)},$$

and each of these algebras quantizes Orb_I.

- The right-hand side can be read as 'group mod stabilizer'.

Goal: Understand quantum $\mathcal{D}$-modules on $\overline{\mathrm{G}_{\mathrm{adj}}}$

One approach:

1. The algebra of quantum differential operators on G is the smash product $\mathcal{D}_{\mathrm{q}}(\mathrm{G}) = \mathcal{O}_{\mathrm{q}}(\mathrm{G}) \star \mathrm{U}_{\mathrm{q}}(\mathfrak{g})$.
2. The algebra of quantum differential operators on $\mathbb{V}_{\mathrm{G}}$ is related to the filtration $\mathcal{D}_{\mathrm{q}}(\mathrm{G})_{\leq\lambda} = \mathcal{O}_{\mathrm{q}}(\mathrm{G})_{\leq\lambda} \star \mathrm{U}_{\mathrm{q}}(\mathfrak{g})$.
3. There is expected to be a moment map $\mu_{\mathrm{q}} : \mathrm{U}_{\mathrm{q}}(\mathfrak{t}) \to \mathcal{D}_{\mathrm{q}}(\mathbb{V}_{\mathrm{G}})$.
4. Hamiltonian reduction along μ_{q} gives quantum $\mathcal{D}$-modules on $\overline{\mathrm{G}_{\mathrm{adj}}}$.

Another goal: Understand $\mathcal{D}$-modules on the quantum orbits.

- One way is as quantum $\mathcal{D}$-modules on $G \times G$ that are strongly equivariant for $U_q(\mathfrak{u}_l^- \times \mathfrak{l}_l \times \mathfrak{u}_l)$.
- One would like another way coming from asymptotics of quantum $\mathcal{D}$-modules on the Vinberg semigrroup.
- Functoriality for quantum $\mathcal{D}$-modules is not well understood, but perhaps categories of $\mathcal{D}$-modules on quantum orbits are related by taking associated gradeds.

Application: Conjecturally, one recovers quantum Beilinson-Bernstein localization as asymptotics of matrix coefficients by relating $\mathcal{D}$-modules on the quantum orbits $\mathrm{Orb}_{\Delta} = G_{adj}$ and $\mathrm{Orb}_{\emptyset} = G/B \times B^{-}\backslash G$. [On-going joint work with D. Ben-Zvi and D. Nadler in both the classical and quantum setting].

Thanks!