Junior geometry talk, Oxford

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Abstract

Quiver varieties and their quantizations feature prominently in geometric representation theory. Multiplicative quiver varieties are group-like versions of ordinary quiver varieties whose quantizations involve quantum groups and q-difference operators. In this talk, we will define and give examples of representations of quivers, ordinary quiver varieties, and multiplicative quiver varieties. No previous knowledge of quivers will be assumed. If time permits, we will describe some phenomena that occur when quantizing multiplicative quiver varieties at a root of unity, and work-in-progress with Nicholas Cooney.

1 Introduction

A central theme of this talk is the following:

Theme: quantizations at a root of unity have large centers
and give rise to matrix bundles (Azumaya algebras).

The idea is that you start with a commutative algebra, it has a quantization that depends on a parameter q, and when q is a root of unity, the quantization becomes Azumaya over its center.

I don't expect you to know anything about Azumaya algebras; I'll say more about that later in the talk. For now, you can think of an Azumaya algebra as some sort of generalized matrix bundle.

Nicholas and I are exploring this theme in the case of quantized quiver varieties at a root of unity. This talk will describe some fairly basic things that feature in our research.

Let me give you some examples of this theme:

1. Quantum groups $U_q\mathfrak{g}$. The center is related to the big Bruhat cell Bw_0B. (de Concini, Kac, Procesi)

2. Quantum coordinate algebras $\mathcal{O}_q(G)$. The center is related to the ordinary coordinate algebra $\mathcal{O}(G)$. (de Concini, Lyubashenko, Procesi)

3. Quantum differential operators $\mathcal{D}_q(G/B)$ on the flag variety G/B. The center is related to functions on the cotangent bundle $\mathcal{O}(T^*G/B)$. (Backelin-Kremntizer)

4. Double affine Hecke algebras at a root of unity. The center is related to the Hilbert scheme. (Varagnolo-Vasserot)

5. Hypertoric quantum groups. The center is related to the corresponding multiplicative hypertoric variety. (Cooney, Ganev)
6. (Expected) Quantized multiplicative quiver varieties. The center is related to the ordinary multiplicative quiver variety. (work in progress with Nicholas Cooney)

2 Quiver varieties

2.1 Notation and basics on quiver representations

So what's a quiver? It is a pair consisting of a set of vertices and a set of edges.

Let $Q = (V, E)$ be a finite quiver. Let $\alpha = \alpha(e)$ and $\beta = \beta(e)$ denote the source and target, respectively, of an edge $e \in E$. A **representation** X of Q is a V-graded vector space $\oplus_{v\in V} X_v$ together with a linear map $X_e : X_\alpha \to X_\beta$ for each edge $e \in E$. Note that X_e extends to an endomorphism of $\oplus_{v\in V} X_v$ by $X_e|_{X_v} = \mathrm{Id}_{X_v}$ if $v \neq \alpha(e)$.

A **morphism** of representations $f : X \to Y$ is a morphism $f : \oplus_{v\in V} X_v \to \oplus_{v\in V} Y_v$ of V-graded vector spaces that respects the maps X_e and Y_e for each edge $e \in E$, specifically, the diagram

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "X_e"'] & Y_\alpha \arrow[d, "Y_e"] \\ X_\beta \arrow[r, "f_\beta"] & Y_\beta \end{tikzcd}$$

commutes for every edge e. A morphism $f : X \to Y$ is an isomorphism if it is an isomorphism of V-graded vector spaces. Finally, the **dimension vector** $\dim X \in \mathbb{Z}^V_{\geq 0}$ of a representation X of Q is given by $(\dim X)_v = \dim(X_v)$.

A motivating question is the formation of a moduli space of representations of a quiver Q, in other words, associating to each quiver Q an algebraic space whose points, in some sense, correspond to isomorphism classes of representations of Q. Since the dimension vector is a discrete parameter, this moduli space will be a disjoint union over the set $\mathbb{Z}^V_{\geq 0}$ of possible dimension vectors. Thus we reduce to the consideration of representations of Q with a fixed dimension vector $\mathbf{d} = (d_v) \in \mathbb{Z}^V_{\geq 0}$. Up to a choice of basis, any such representation is the same as an element of the vector space

$$\mathrm{Mat}_{\mathbf{d}}(Q) = \bigoplus_{e\in E} \mathrm{Hom}(\mathbb{C}^{d_\alpha}, \mathbb{C}^{d_\beta}).$$

Observe that there is an action of $G_{\mathbf{d}} = \prod_{i\in V} \mathrm{GL}_{d_i}(\mathbb{C})$ on $\mathrm{Mat}_{\mathbf{d}}(Q)$ given by

$$(g \cdot x)_e = g_\beta \circ x_e \circ g_\alpha^{-1},$$

where $g = (g_i) \in G_{\mathbf{d}}$ and $x = (x_e) \in \mathrm{Mat}_{\mathbf{d}}(Q)$. Moreover, two points of $\mathrm{Mat}_{\mathbf{d}}(Q)$ define isomorphic representations if and only if they lie in the same orbit of $G_{\mathbf{d}}$. Therefore, a moduli space of representations of Q with dimension vector $\mathbf{d}$ will be, roughly speaking, a sort of quotient of $\mathrm{Mat}_{\mathbf{d}}(Q)$ by $G_{\mathbf{d}}$. We turn our attention to a variation of this idea that leads to relevant applications.

2.2 (Additive) quiver variety

The action of $G_{\mathbf{d}}$ on $\mathrm{Mat}_{\mathbf{d}}(Q)$ gives rise to a Hamiltonian action of $G_{\mathbf{d}}$ on $T^*\mathrm{Mat}_{\mathbf{d}}(Q)$. The natural moment map for this action is given by

$$\mu : T^*\mathrm{Mat}_{\mathbf{d}}(Q) \to \mathfrak{g}_{\mathbf{d}}^* \simeq \mathfrak{g}_{\mathbf{d}}$$
$$\mu(X)_i = \sum_{\beta(e)=i} X_e X_{e^\vee} - \sum_{\alpha(e)=i} X_{e^\vee} X_e,$$

where $\mathfrak{g}_{\mathbf{d}} = \mathrm{Lie}(G_{\mathbf{d}}) = \prod_{i\in V} \mathfrak{gl}_{d_i}$ is the Lie algebra of $G_{\mathbf{d}}$.

Fix a character $\chi : \mathrm{GL}_{\mathbf{d}} \to \mathbb{C}^\times$ and consider the Hamiltonian reduction

$$\mathcal{M}_{\mathbf{d}}^\lambda(Q) = \mu^{-1}(0)//_\chi G_{\mathbf{d}} = \mathrm{Proj}(\dots).$$

We will refer to this variety as the ordinary or additive quiver variety of Q

Remark 2.1. Alternatively, take the preimage of a scalar matrix $\lambda = (\lambda_i) \in (\mathbb{C})^V \hookrightarrow \mathfrak{g}_{\mathbf{d}}$, and hence one fixed under the adjoint action of $G_{\mathbf{d}}$ on $\mathfrak{g}_{\mathbf{d}}$.

Remark 2.2. Introduce even more structure (called a framing) to obtain Nakajima quiver varieties (Pavel talked about Nakajima quiver varieities in the AGT seminar on Monday).

Example 2.3. Cyclic quiver, Kleinian singularities.

2.3 Multiplicative quiver varieties

Fix a total order on the set $\bar{E}$ of edges of $\bar{Q}$. Consider the Zariski open set

$$T^*\mathrm{Mat}_{\mathbf{d}}(Q)^\circ = \{X \in T^*\mathrm{Mat}_{\mathbf{d}}(Q) \mid \det(1 + X_e X_{e^\vee}) \neq 0 \text{ for all } e \in E\}$$

of $T^*\mathrm{Mat}_{\mathbf{d}}(Q)$. We equip this set with a slightly different symplectic structure than the one we're used to on the contangent bundle. With this symplectic structure, one defines a group-valued moment map:

$$\tilde{\mu} : T^*\mathrm{Mat}_{\mathbf{d}}(Q)^\circ \to G_{\mathbf{d}}$$
$$\tilde{\mu}(X)_i = \prod_{e\in\bar{E},\alpha(e)=i} (1 + X_{e^\vee} X_e)^{\epsilon(e)} = \prod_{e\in E,\alpha(e)=i} (1 + X_{e^\vee} X_e) \prod_{e\in E,\beta(e)=i} (1 + X_e X_{e^\vee})^{-1},$$

or, more simply,

$$X \mapsto \prod_{e\in\bar{E}} (1 + X_{e^\vee} X_e)^{\epsilon(e)}.$$

Remark 2.4. See AMM for more on group-valued moment maps.

The multiplicative quiver variety is defined as the Hamiltonian reduction

$$\tilde{\mathcal{M}}_{\mathbf{d}}^\xi = \tilde{\mu}^{-1}(1)//_\chi G_{\mathbf{d}} = \mathrm{Proj}(\dots).$$

Remark 2.5. More generally, take the preimage of a scalar matrix $\xi = (\xi_i) \in (\mathbb{C}^\times)^V \hookrightarrow G_{\mathbf{d}}$, and hence one that is fixed under the adjoint action of $G_{\mathbf{d}}$.

Some motivation: first introduced to solve the Deligne-Simpson problem, related to quantum groups and q-difference operators, spherical DAHAs, possibly also to mirror symmetry and baby cases of non-abelian Hodge theory.

3 Quantization of multiplicative quiver varieites

Let me give you the general procedure for quantizing multiplicative quiver varieties, and then we'll focus on a specific example where we can see much ot the structure very explicitly. So if the next five minutes is nonsense to you, you can ignore it and rejoin us later.

- Step 1: Define an algebra $\mathcal{D}_q(e)$ for every edge $\cdot^n \to \cdot^m$ or $\cdot \circlearrowright$ in the quiver. This will be a $U_q(\mathfrak{gl}_n)$-$U_q(\mathfrak{gl}_m)$-bimodule.
  - depends only on the dimension vector at the endpoints
  - We're used to the idea that differential operators quantize cotangent bundles, and indeed that's what we use when considering ordinary (additive) quiver varieties. The quantization of multiplicative quiver varities involve quantum groups and q-difference operators, mostly because we have a Zariski open subset of the cotangent bundle with a funny symplectic structure.
  - Key words: q-difference operators, R-matrix, Heisenberg double.
- Step 2: Construct an algebra of q-difference operators $\mathrm{Mat}_{\mathbf{d}}(Q))$:

  $$\mathcal{D}_q(Q) = \tilde{\bigoplus_e} \mathcal{D}_q(e).$$

  This tensor product takes place in the braided tensor category $U_q(\mathfrak{gl}_{\mathbf{d}})$-mod. So the multiplicative is not componentwise, but takes into account the braiding and the incidency relations between the edges.
- Step 3: Define a quantum moment map $\mu_q : \mathcal{O}_q(GL_{\mathbf{d}}) \to \mathcal{D}_q(\mathrm{Mat}_{\mathbf{d}}Q)$. This quanutm moment map degenerates to the group-valued moment map we had before.
- Step 3: Perform quantum Hamiltonian reduction along μ_q to obtain $\mathcal{M}_q(Q, \mathbf{d})$ to obtained the quantized multiplicative quiver variety:

  $$A_q^\chi = \left(\mathcal{D}_q(Q)/\mathcal{D}_q(Q)(\mu_q(\ker(\epsilon)))\right)^{U_q(\mathfrak{gl}_{\mathbf{d}})}.$$

Expectation: When q is a root of unity, A_q^χ is the global sections of a sheaf $\mathcal{A}$ of algebras on $\tilde{M}_\chi$. Moreover, $\mathcal{A}$ is Azumaya over $\tilde{M}_\chi$.

3.1 The algebra $\mathcal{D}_q(e)$

Rather than say too much more about the general procedure, let's focus on the example of $\mathcal{D}_q(\cdot \to \cdot)$ with dimension vector $(1, 1)$ and what happens at a root of unity. Let $q \in \mathbb{C}^\times$. We're going to consider the following graded algebra

$$\mathcal{D}_q(\cdot \to \cdot) = \mathcal{D}_q(\mathbb{C}) = \mathbb{C}\langle x, \partial, \alpha^{\pm 1}\rangle/(\partial x = q^2 x\partial + (q^2 - 1), \alpha = 1 + x\partial),$$

with $\deg(x) = 1$ and $\deg(\partial) = -1$ and $\deg(\alpha) = 0$. This is a q-version of the ordinary Weyl algebra. It is called an **algebra of q-difference operators** because it has the following natural action on the ring $\mathcal{O}(\mathbb{A}^1) = \mathbb{C}[t]$ of functions on $\mathbb{A}^1$:

$$(x \cdot f)(t) = tf(t) \qquad (\partial \cdot f)(t) = \frac{f(q^2t) - f(t)}{t} \qquad (1 + x\partial) \cdot f(t) = f(q^2t).$$

In words, x acts by translation, ∂ acts by a difference operators, and α acts as an Euler or dilation operator.

Exercise: the q-commutativity property of α:

$$\alpha x = q^2 x\alpha \qquad \text{and} \qquad \alpha\partial = q^{-2}\partial\alpha.$$

Another way to state this property is that α is a 'q-central grading operator'.

Observe that, when $q = 1$, we obtain $\mathcal{D}_1(\mathbb{C}) = \mathbb{C}[x, \partial][(1 + x\partial)^{-1}]$, which is the algebra of functions on the following open subset of $T^*\mathbb{C}$:

$$(T^*\mathbb{C})^\circ := \{(p, w) \in T^*\mathbb{C} \mid 1 + pw \neq 0\}.$$

Thus $\mathcal{D}_q(\mathbb{C})$ can be regarded as a quantization of this open subset.

3.2 The ℓ-center Z_ℓ of $\mathcal{D}_q$

Now we discuss 'quantization at a root of unity'. If q is not a root of unity, then the center $Z(\mathcal{D}_q) = \mathbb{C}$ consists of only scalars.

On the other hand, suppose q is a primitive ℓ-th root of unity, where $\ell > 1$ is odd. Then x^ℓ, ∂^ℓ, and α^ℓ are central in $\mathcal{D}_q$. Define

$$Z_\ell = \langle x^\ell, \partial^\ell \rangle \subseteq \mathcal{D}_q$$

to be the (central) subalgebra generated by x^ℓ and ∂^ℓ.

Fact: $\mathrm{Spec}(Z_\ell) = T^*\mathrm{Mat}_{(1,1)}(\cdot \to \cdot)^\circ = (T^*\mathbb{C})^\circ$ and acquires a symplectic form given by $\omega = \frac{dp \wedge dw}{1+pw}$.

The symplectic form arises from a non-degenerate Poisson structure on Z_ℓ induced from the commutator on $\mathcal{D}_q(\mathbb{C})$. This is the same symplectic form as the one involved in defining the group-valued moment map above.

Question: How is the ℓ-center related to the actual center?

Now we are ready to state the first result:

Theorem 3.1. *Suppose q is a primitive ℓ-th root of unity, where $\ell > 1$ is odd.*

- *The center $Z(\mathcal{D}_q(e))$ of $\mathcal{D}_q(\mathbb{C})$ is precisely the ℓ-center Z_ℓ.*
- *$\mathcal{D}_q(e)$ is Azumaya over Z_ℓ of rank ℓ^2.*

3.3 Azumaya algebra reminders

As promised, I now include some reminders on Azumaya algebras.

Let R be a commuative ring. An R-algebra A is Morita trivial if there is an equivlance of categories A-mod $\simeq R$-mod. An R algebra is Morita invertible if there is an R-algebra B such that $A \otimes_R B$ is Morita trivial.

Definition 3.2. A Morita invertible R-algebra is called an Azumaya algebra over R.

By definition, the set of Azumaya R-algebras up to Morita equivalence forms a group under tensor product. This is known as the Brauer group of R.

We can describe Azumaya algebras in a more geometric way. Let R be a finitely generated algebra over $\mathbb{C}$. Everything I say here can be easily generalized to any commutative ring, but I'll stick with finitely generated algebras over $\mathbb{C}$ for simplicity (and because it is all we'll need). It turns out that an Azumaya algebra over R is just a **matrix bundle**. More precisely,

Definition 3.3. An R-algebra A is *Azumaya* if A is

- finitely generated and projective as an R-module, with nonvanishing rank, and
- for every maximal ideal M of R, $A \otimes_R R/M$ is a matrix algebra over $R/M \simeq \mathbb{C}$.

The first condition says that A is a vector bundle over $\mathrm{Spec}(R)$. The second condition indicates that the fibers of the vector bundle are all matrix algebras.

Remark 3.4. Different authors give different definitions of Azumaya algebras, so I want to note that the second condition is equivalent to the following natural map being an isomorphism:

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$
$$a \otimes b \mapsto [c \mapsto acb].$$

Definition 3.5. An Azumaya algebra A is called *split* if $A \simeq \mathrm{End}_R(P)$ for some projective R-module P.

These definitions can be adapted to define a sheaf of Azumaya algebras (respectively split Azumaya algebras) on a scheme of finite type over $\mathbb{C}$.

Going back to q-difference operators, now we know what it means for $\mathcal{D}_q(\mathbb{C})$ to be Azumaya over Z_ℓ.

Remark 3.6. The algebra $\mathcal{D}_q$ is not a split Azumaya Z_ℓ-algebra since it has no zero-divisors.

3.4 The algebra $\mathcal{D}_q(Q)$

Our next aim is to define a certain algebra $\mathcal{D}_q(Q)$ of q-difference operators on $\mathrm{Mat}_{\mathbf{d}}(Q)$. For this, choose a total order on E and take $\#E$ copies of $\mathcal{D}_q(e)$:

$$\mathcal{D}_q(e_1), \ldots, \mathcal{D}_q(e_n).$$

Idea: Let $\mathcal{D}_q(Q) = \bigotimes_{i=1}^n \mathcal{D}_q(e_i)$ in a (nontrivially) braided tensor category $U_q(\mathfrak{gl}_{\mathbf{d}})$-mod.

Here it is important to recall that if you have a bunch of algebra objects in a braided tensor category, then their tensor product acquires an algebra structure, but it is not the componentwise multiplication that we're used to, but rather the algebra structure takes into account the braiding.

The 'hypertoric' case is known, proven independently by Nicholas and by me.

Theorem 3.7 (Cooney, G.)**.** *Suppose q is a primitive ℓ-th root of unity, where $\ell > 1$ is odd. If $\mathbf{d}_v = 1$ for all vertices, then:*

- *The center $Z(\mathcal{D}_q(Q))$ of $\mathcal{D}_q(Q)$ is precisely the ℓ-center $Z_\ell(Q)$, generated by x_i^ℓ, ∂_i^ℓ, and α_i^ℓ:*
$$Z(\mathcal{D}_q(Q)) = \langle x_i^\ell, \partial_i^\ell, \alpha_i^\ell \rangle.$$
- *The center $Z(\mathcal{D}_q(Q))$ is (isomorphic to) the algebra of functions on*
$$(T^*\mathrm{Mat}_{\mathbf{d}}(Q))^\circ := \{(p,w) \in T^*\mathbb{C}^n \mid 1 + p_i w_i \neq 0 \textit{ for all } i\}.$$
- *$\mathcal{D}_q(Q)$ is Azumaya over Z_ℓ of rank ℓ^{2n}, i.e. the fibers are ismorphic to* $\mathrm{Mat}_{\ell^n}(\mathbb{C})$.

Nicholas and I are working on extending this to higher dimension vectors, starting with the example of $\cdot \to \cdot$ with dimension vector $(1,2)$. Already in this case the techniques used above break down.

Once the Azumaya property is shown, we have some indications that the Azumaya property will descend under Hamiltonian reduction by general principles.

4 Critical points in representation theory

- Affine Lie algebras at critical level (Edward Frenkel)
- Quantum groups at a root of unity
- Rational cherednik algebras at $t = 0$ (Spec of the center is the Calogero-Moser space)
- Lie algebras and Cherednik algebras in positive characteristic