Outline for 'Model categories and dg-categories'

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- The previous lecture (given by Pavel) introduced dg-categories and localized categories. A model structure is an extra structure on a localization pair ($\mathcal{C}$,W) that provides a better way to describe and understand the localized category $\mathcal{C}[W^{-1}]$. In the next lecture (to be given by Lee), we apply model category theory to the category of dg-categories.
- Basic notions
  - Left and right lifting properties for morphisms in a category
  - Model category: The data of three subcategories Fib, Cof, and W satisfying four axioms: 2 of 3, retracts, lifting properties, and functorial factorizations.
  - Cofibrant and fibrant objects; the category M^{cf}; the homotopy category $Ho(M) = M[W^{-1}]$.
- Examples
  - First a lemma: If W and either Fib or Cof is specified, then lifting properties determine the remaining subcategory. In other words, the axioms of a model category are over-determined in some sense.
  - Top – topological spaces. Equivalences are weak homotopy equivalences and fibrations are Serre fibrations. The homotopy category is equivalent to the category of CW complexes with homotopy classes of maps.
  - $C(k)$ – complexes of k-modules, where k is a commutative ring. Equivalences are quasi-isomorphisms. Projective and injective model structures. The homotopy category is the derived category $D(k)$.
  - $C(Sh(X,k))$ – complexes of sheaves of k-modules on a topological space X.
  - $dg-cat$ – the category of dg-categories. Equivalences are quasi-equivalences of categories. The models structure on $C(k)$ is used to define fibrations.
- Homotopy theory for model categories
  - Definitions: cylinder object, (left) homotopy
  - Lemma: Homotopy is an equivalence relation in M^{cf}, so can define $M^{cf}/\sim$.
  - Lemma: If two maps are homotopic, then they are equal in $Ho(M)$. In particular we get a functor $M^{cf}/\sim\, \to Ho(M^{cf})$.
  - Cofbriant and fibrant replacements allow us to define a functor $Ho(M) \to Ho(M^{cf})$.
  - The Whitehead Theorem: The functors $M^{cf}/\sim\, \to Ho(M^{cf}) \leftarrow Ho(M)$ are all equivalences of categories.
- Maps of model categories
  - An adjunction $(F,U) : \mathcal{C} \to \mathcal{D}$ between model categories is called a Quillen adjunction if F preserves cofibrations and trivial cofibrations.

  - Lemma: In this case, F preserves equivalences on cofibrant objects and U preserves equivalences on fibrant objects.
  - Consequently, there are derived functors $(LF, RU) : Ho(\mathcal{C}) \to Ho(\mathcal{D})$ – 'best homotopical approximations' to F and U.
  - If LF is an equivalence, then (F, U) is called a Quillen equivalence.

- $C(k)$-model category structures
  - Loosely, a $C(k)$-model category structure on a model category M is the structure on M of a module category over $C(k)$ such that all internal Homs exist and the action functor has nice properties.
  - Examples: $C(k)$, $C(Sh(X, k))$
  - Enrichment of M over $C(k)$ using internal Homs.

- T-dg-modules for a dg-category T
  - A T-dg-module is defined as a dg-functor $T \to C(k)$.
  - The category $T-\text{Mod}$ of T-dg-modules admits a model structure. The homotopy category $Ho(T-\text{Mod})$ is called the derived category of T and is denoted $D(T)$.
  - A dg-functor $f : T \to T'$ of dg-categories gives rise to a Quillen adjunction $(f_!, f^*) : T-\text{Mod} \to T'-\text{Mod}$.
  - If M is has a $C(k)$-model category structure, then a T-dg-module with coefficients in M is a dg-functor $T \to M$. The category of such modules is denoted M^T and it admits a model structure under mild assumptions on M.

- $\text{Int}(M)$ and Yoneda
  - Let M be a model category with a $C(k)$-model structure. Define $\text{Int}(M)$ as the enrichment of M^{cf} over $C(k)$ using internal Homs.
  - Theorem: $[\text{Int}(M)] \simeq Ho(M)$. In other words, $\text{Int}(M)$ is a dg-enhancement of $Ho(M)$.
  - It turns out that any dg-category can be fully embedded (up to quasi-equivalence) into some dg-category of the form $\text{Int}(M)$.
  - We have a dg-version of the Yoneda embedding that will be useful later:

    $$T \to \text{Int}(T^{op} - \text{Mod})$$

    where x goes to the T^{op}-dg-module $\underline{h}_x : T^{op} \to C(k)$ sending y to $T(y, x)$.