Cohomology of skyscrapers sheaves

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1 Review of sheaf cohomology

Let's give a quick review of sheaf cohomology. Let X be a topological space and R a commutative ring. It is a fact that the category $\mathsf{Shv}(X)$ of sheaves of R-modules has enough injectives. The proof of this is constructive and uses the fact that the category of R-modules has enough injectives. The following Key Facts will allow us to define the sheaf cohomology functor.

1. If F and G are sheaves equipped with injective resolutions $F \to \{I_\bullet\}$ and $G \to \{J_\bullet\}$, then any map $F \to G$ lifts to a chain map $F \to \{I_\bullet\}$ and $G \to \{J_\bullet\}$. The lift is defined uniquely up to chain homotopy equivalence.
2. Injective resolutions are unique up to chain homotopy equivalence.

The map from X to a point gives rise to a pushforward-pullback adjunction. In this case, it takes the form:

$$\mathrm{Hom}_R(M, \Gamma(F)) = \mathrm{Hom}_{\mathsf{Shv}(X)}(\bar{M}, F)$$

where $\bar{M}$ denotes the constant sheaf on X with sections M. In particular, the global sections functor is left exact. The sheaf cohomology of X with coefficients in a sheaf F is defined as the right derived functor of Γ evaluated at F:

$$H^i(X, F) = (R^i\Gamma)(F)$$

To compute the cohomology, choose an injective resolution of F, apply Γ to this resolution, and take the cohomology of the resulting chain complex. The Key Fact (2) above guarantees that this process is well-defined since any additive functor preserves chain homotopy equivalences and since chain homotopy equivalent complexes have isomorphic cohomology. Moreover, Key Fact (1) defines how cohomology handles morphisms.

(Reminder: For a left exact covariant functor or a right exact contravariant functor, take injective resolutions. For a right exact covariant functor or a left exact contravariant functor, take projective resolutions.)

2 Skyscraper sheaves

We illustrate the constructions of the previous section as we prove that cohomology with coefficients in a skyscraper sheaf is always trivial. Let's first say what skyscraper sheaves.

Let X be a topological space, and $x : * \to X$ be the inclusion of a point. Note that the category $\mathsf{Shv}(*)$ of sheaves of R-modules on a point is equivalent to the category of R-modules. For an R-module M, the pushforward sheaf $x_*(M)$ is the so-called skyscraper sheaf of X at x with sections M, and the module of sections defined as

$$x_*(M)(U) = \begin{cases} M & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}.$$

The pullback $x^*(F)$ is the stalk F_x of the sheaf F at x. Therefore, the pullback-pushforward adjunction implies that

$$\mathrm{Hom}_R(F_x, M) = \mathrm{Hom}_{\mathsf{Shv}(X)}(F, x_*(M)).$$

Lemma 1. *If I is an injective R-module then $x_*(I)$ is an injective sheaf.*

Proof. We must show that if $0 \to F \to G$ is an exact sequence of sheaves, then $\mathrm{Hom}(G, x_*(I)) \to \mathrm{Hom}(F, x_*(I)) \to 0$ is exact. Recall (Hartshorne exercise 1.2 in chapter II) that a morphism of sheaves is injective if and only if it is injective on all the stalks. In particular, we have that $0 \to F_x \to G_x$ is an exact sequence of R-modules. Since I is an injective R-module, it follows that $\mathrm{Hom}(G_x, I) \to \mathrm{Hom}(F_x, I) \to 0$ is exact. The adjunction above completes the proof.

A corollary of this lemma is that any product of sheaves of the form $x_*(I_x)$, where I_x is a family of injective R-modules labeled by points in X, is also injective. We remark that an ingredient in the proof that the category $\mathsf{Shv}(X)$ has enough injectives is a product of skyscraper sheaves of injective R-modules.

Proposition 2. *Let M be an R-module. Then the corresponding skyscraper sheaf $x_*(M)$ satisfies $H^i(X, x_*(M)) = 0$ for $i > 0$.*

Proof. Let $M \to \{I_\bullet\}$ be an injective resolution of M as an R-module. Checking stalks shows that

$$x_*(M) \to \{x_*(I_\bullet)\}$$

is an injective resolution of the sheaf $x_*(M)$. Applying global sections returns the original injective resolution $M \to \{I_\bullet\}$. Injective resolutions are exact, hence have no cohomology. The result follows.