assistant

An object moving in a circle, such as a star in a galaxy, must be subject to a constant inward force that prevents it from flying off in a straight line. This is the essence of circular motion. The acceleration required to maintain a circular path of radius (r) at a constant speed (v_c) is directed towards the center of the circle, and its magnitude is given by ($a_c = v_c^2 / r$).

According to the fundamental law of motion, a force is required to produce this acceleration. For an object of mass (m), this centripetal force must be ($F_c = m a_c = m v_c^2 / r$).

In the scenario of a galaxy, this inward force is gravity. The problem provides a description of the gravitational field not as a direct force, but as a potential, ($\varphi(r)$), which represents the potential energy per unit mass. Therefore, the gravitational potential energy of our star of mass (m) at a distance (r) from the center is ($U(r) = m \varphi(r)$).

A central concept connecting force and potential energy is that the force is the negative gradient of the potential energy. In this spherically symmetric case, the force acts along the radial direction, and its magnitude is given by the derivative with respect to (r): [$F_g = -\frac{dU}{dr}$] Substituting the expression for ($U(r)$), we find the magnitude of the gravitational force on the star: [$F_g = -\frac{d}{dr}(m \varphi(r))$] Since the mass ($m$) of the star is constant, we can write: [$F_g = -m \frac{d\varphi}{dr}$] The negative sign indicates that the force is attractive, pulling the star towards the region of lower potential energy (the center). The magnitude of this force is therefore ($|F_g| = m \frac{d\varphi}{dr}$), assuming the potential increases with radius.

For a stable circular orbit to exist, the gravitational force must provide precisely the required centripetal force. We can therefore equate the magnitudes of these two forces: [$F_c = |F_g|$] [$\frac{m v_c^2}{r} = m \frac{d\varphi}{dr}$] A remarkable feature immediately appears: the mass (m) of the orbiting object cancels from both sides of the equation. This implies that the orbital velocity at a given radius depends only on the structure of the gravitational field itself, not on the mass of the object moving within it.

Solving for the circular velocity, (v_c), we rearrange the equation: [$v_c^2 = r \frac{d\varphi}{dr}$] Taking the square root yields the final expression for the velocity: [$v_c = \sqrt{r \frac{d\varphi}{dr}}$]

This result connects a directly observable quantity, the circular velocity (v_c), to a fundamental but unobservable property of the galaxy, the gradient of its gravitational potential.

To verify this theoretical relationship, one could design an experiment. The goal would be to measure the rotation curve, ($v_c(r)$), of a galaxy and use it to infer the underlying potential.

Experimental Procedure:

1. **Equipment:** A telescope equipped with a spectrometer.
2. **Method:**
   - Observe a spiral galaxy that we see edge-on or nearly edge-on.
   - Measure the distance from the galactic center, (r), for various bright regions (e.g., nebulae or star clusters).
   - Using the spectrometer, measure the Doppler shift of a known spectral emission line (such as the 21cm hydrogen line) from each of these regions.
   - This Doppler shift gives the line-of-sight velocity of the material. After correcting for the galaxy's overall motion away from us, this can be interpreted as the circular velocity, (v_c), at that radius (r).
   - Collect data for many points across the galaxy to build a plot of (v_c) as a function of (r). This plot is the galaxy's rotation curve.
3. **Analysis:** With the experimental data for ($v_c(r)$), we can use our derived formula, rearranged as ($\frac{d\varphi}{dr} = \frac{v_c(r)^2}{r}$), to calculate the value of the gravitational potential gradient at any radius. By integrating this result numerically, ($\varphi(r) = \int_0^r \frac{v_c(r')^2}{r'} dr'$), we could map the entire gravitational potential of the galaxy. This, in turn, allows us to deduce the distribution of mass required to create such a potential.

This formula is an idealization. It assumes perfectly circular orbits and a spherically symmetric potential. Real galaxies are flattened, and stars can have slightly elliptical orbits. Nonetheless, this simple model provides a powerful first step in linking the observable dynamics of a galaxy to its unseen mass structure.