assistant

I am presented with an empirical observation, a graph suggesting a relationship between the total mass of a spiral galaxy, M_{tot}, and its velocity, $v_{c,\infty}$, in its outer regions. The relationship appears linear on a log-log plot, which suggests a power law of the form $M_{tot} \propto v_{c,\infty}^{\gamma}$. My task is to determine if the theoretical mass distribution model proposed earlier, $\rho_m(r) = C_m / (r_m^2 + r^2)$, can account for this observed law, and if so, to determine the exponent γ.

First, I must establish the theoretical connection between the total mass and the asymptotic velocity based on this model. The velocity of an object in a circular orbit at radius r is determined by the gravitational force from the mass $M(r)$ enclosed within that radius. The centripetal force required for the orbit is provided by gravity: $$\frac{m v_c(r)^2}{r} = \frac{G m M(r)}{r^2}$$ This gives a direct relation for the velocity: $$v_c(r)^2 = \frac{G M(r)}{r}$$

The enclosed mass $M(r)$ is found by integrating the given mass density $\rho_m(x)$ over the volume of a sphere of radius r: $$M(r) = \int_0^r 4\pi x^2 \rho_m(x) \,dx = \int_0^r 4\pi x^2 \frac{C_m}{r_m^2 + x^2} \,dx$$ Using the integral hint provided in the problem, $\int_0^r \frac{x^2}{a^2 + x^2} dx = r - a \arctan(r/a)$, with $a = r_m$, I find the enclosed mass: $$M(r) = 4\pi C_m \left[x - r_m \arctan\left(\frac{x}{r_m}\right)\right]_0^r = 4\pi C_m \left(r - r_m \arctan\left(\frac{r}{r_m}\right)\right)$$

The velocity in the outer regions, $v_{c,\infty}$, is the velocity as r becomes very large. Let's examine the behavior of $v_c(r)^2$: $$v_c(r)^2 = \frac{G}{r} \left[4\pi C_m \left(r - r_m \arctan\left(\frac{r}{r_m}\right)\right) \right] = 4\pi G C_m \left(1 - \frac{r_m}{r} \arctan\left(\frac{r}{r_m}\right)\right)$$ As $r \to \infty$, the term $\frac{r_m}{r}$ goes to zero, causing the second term in the parenthesis to vanish. This leaves a constant velocity: $$v_{c,\infty}^2 = 4\pi G C_m$$ This confirms the model produces the flat rotation curves observed. This equation provides a relationship between the model parameter C_m and the observable $v_{c,\infty}$.

Now, I can express the total mass of the galaxy, M_{tot}, which is the mass enclosed within its total radius, R. $$M_{tot} = M(R) = 4\pi C_m \left(R - r_m \arctan\left(\frac{R}{r_m}\right)\right)$$

I have two expressions, one for $v_{c,\infty}^2$ and one for M_{tot}, both involving the parameter C_m. I can eliminate C_m to find a direct relationship between mass and velocity. From the velocity expression, I can write $C_m = \frac{v_{c,\infty}^2}{4\pi G}$. Substituting this into the equation for total mass gives: $$M_{tot} = 4\pi \left(\frac{v_{c,\infty}^2}{4\pi G}\right) \left(R - r_m \arctan\left(\frac{R}{r_m}\right)\right)$$ $$M_{tot} = \left[\frac{1}{G} \left(R - r_m \arctan\left(\frac{R}{r_m}\right)\right) \right] v_{c,\infty}^2$$

This equation has the desired form $M_{tot} = \eta v_{c,\infty}^{\gamma}$. By comparing the two, I can identify the theoretical exponent and the coefficient. The exponent predicted by this model is: $$\gamma = 2$$ The coefficient is $\eta = \frac{1}{G} (R - r_m \arctan(R/r_m))$. The problem states to assume the radius R of a galaxy does not depend on its mass. If we also assume r_m is a characteristic scale that is similar for all spiral galaxies, then η can be treated as a constant.

Now, I must compare this theoretical prediction with the empirical data presented in the Tully-Fischer relation plot. The plot shows $\log_{10}(M_{tot})$ versus $\log_{10}(v_{c,\infty})$. A power law $M_{tot} = \eta v_{c,\infty}^{\gamma_{TF}}$ becomes a linear equation upon taking the logarithm: $$\log(M_{tot}) = \log(\eta) + \gamma_{TF} \log(v_{c,\infty})$$ On this plot, the exponent γ_{TF} is simply the slope of the line. I will estimate the slope from the green line shown in Figure 4 (right).

Let's select two points on the line to calculate the slope, $\gamma_{TF} = \frac{\Delta y}{\Delta x} = \frac{\Delta(\log M)}{\Delta(\log v)}$.

- Point 1: $x_1 = \log(v) = 2.0$. The corresponding y-value is $y_1 = \log(M) = 10.0$.
- Point 2: $x_2 = \log(v) = 2.5$. The corresponding y-value is $y_2 = \log(M) = 12.0$.

The slope is: $$\gamma_{TF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12.0 - 10.0}{2.5 - 2.0} = \frac{2.0}{0.5} = 4.0$$

The empirical data from observations of many galaxies suggests an exponent of $\gamma_{TF} = 4$. The theoretical model, however, predicts an exponent of $\gamma = 2$. There is a clear discrepancy. The model, while capable of producing flat rotation curves for an individual galaxy, does not correctly capture the observed scaling law that relates the mass and velocity across different galaxies. The observed total mass scales as the fourth power of the asymptotic velocity, a much steeper relationship than the second power predicted by the model.