googleai-gemini-2.5-pro

Here we investigate the physics of popping a champagne bottle, focusing on the conditions inside the bottle, the phenomena upon opening, and the subsequent motion of the cork.

Part C.1: Pressure Inside the Bottle

We are presented with a closed system containing a fixed total amount of carbon dioxide, $(n_T = 0.2)$ mol, distributed between the liquid champagne $((V_L = 750)$ mL$)$ and the gaseous headspace $((V_G = 25)$ mL$)$. To determine the pressure (P_i) in the headspace, we must consider the equilibrium between these two phases.

The total number of moles is the sum of the moles in the liquid, (n_L), and the moles in the gas, (n_G): $[n_T = n_L + n_G]$

We can express the number of moles in each phase in terms of the pressure (P_i).

1. In the liquid, the concentration of dissolved CO_2, (c_e), is governed by Henry's Law, $(c_e = k_H P_i)$. The number of moles is this concentration multiplied by the liquid volume: $(n_L = c_e V_L = k_H P_i V_L)$.
2. In the gas phase, we assume the CO_2 behaves as an ideal gas. The ideal gas law, $(PV=nRT)$, gives the number of moles in the headspace as $(n_G = \frac{P_i V_G}{R T_0})$, where (T_0) is the absolute temperature and (R) is the ideal gas constant.

Substituting these expressions into our conservation equation for the number of moles gives: $[n_T = k_H P_i V_L + \frac{P_i V_G}{R T_0}]$

We can now solve for the pressure (P_i) by factoring it out: $[P_i = \frac{n_T}{k_H V_L + \frac{V_G}{R T_0}}]$

Now, we can calculate the pressure for the two given temperatures, using $(n_T = 0.2)$ mol, $(V_L = 7.5 \times 10^{-4})$ m³, $(V_G = 2.5 \times 10^{-5})$ m³, and $(R = 8.314)$ J·mol⁻¹·K⁻¹.

Case 1: Temperature $(T_0 = 6^\circ)$C = 279.15 K Given $(k_H(6^\circ\text{C}) = 5.4 \times 10^{-4})$ mol·m⁻³·Pa⁻¹. The term for dissolved moles is $(k_H V_L = (5.4 \times 10^{-4})(7.5 \times 10^{-4}) = 4.05 \times 10^{-7})$ mol·Pa⁻¹. The term for gaseous moles is $(\frac{V_G}{R T_0} = \frac{2.5 \times 10^{-5}}{8.314 \times 279.15} = 1.077 \times 10^{-8})$ mol·Pa⁻¹. $[P_i = \frac{0.2}{4.05 \times 10^{-7} + 1.077 \times 10^{-8}} = \frac{0.2}{4.158 \times 10^{-7}} \approx 4.81 \times 10^5 \text{ Pa}]$ So, at (6°)C, the pressure is $(P_i \approx 4.81)$ bar.

Case 2: Temperature $(T_0 = 20^\circ)$C = 293.15 K Given $(k_H(20^\circ\text{C}) = 3.3 \times 10^{-4})$ mol·m⁻³·Pa⁻¹. The term for dissolved moles is $(k_H V_L = (3.3 \times 10^{-4})(7.5 \times 10^{-4}) = 2.475 \times 10^{-7})$ mol·Pa⁻¹. The term for gaseous moles is $(\frac{V_G}{R T_0} = \frac{2.5 \times 10^{-5}}{8.314 \times 293.15} = 1.026 \times 10^{-8})$ mol·Pa⁻¹. $[P_i = \frac{0.2}{2.475 \times 10^{-7} + 1.026 \times 10^{-8}} = \frac{0.2}{2.578 \times 10^{-7}} \approx 7.76 \times 10^5 \text{ Pa}]$ So, at (20°)C, the pressure is $(P_i \approx 7.76)$ bar.

Part C.2: Fog Formation

Upon opening the bottle, the high-pressure CO_2 gas in the headspace expands rapidly against the atmosphere. This process is described as adiabatic and reversible. We can determine the final temperature (T_f) of the gas after it expands to the atmospheric pressure $(P_f = P_0 = 1)$ bar.

For a reversible adiabatic expansion of an ideal gas, the relationship between pressure and temperature is given by $(P^{1-\gamma} T^\gamma = \text{constant})$. Therefore: $[P_i^{1-\gamma} T_0^\gamma = P_f^{1-\gamma} T_f^\gamma]$ Solving for the final temperature (T_f): $[T_f = T_0 \left(\frac{P_f}{P_i}\right)^{\frac{\gamma-1}{\gamma}}]$ Using the given adiabatic coefficient $(\gamma = 1.3)$, the exponent is $((\gamma-1)/\gamma = 0.3/1.3 \approx 0.2308)$. We will use the pressure values provided for this section.

Case 1: Initial Temperature $(T_0 = 6^\circ)$C = 279.15 K With $(P_i = 4.69)$ bar and $(P_f = 1)$ bar: $[T_f = 279.15 \text{ K} \times \left(\frac{1}{4.69}\right)^{0.2308} \approx 279.15 \times 0.700 \approx 195.4 \text{ K}]$

Case 2: Initial Temperature $(T_0 = 20^\circ)$C = 293.15 K With $(P_i = 7.45)$ bar and $(P_f = 1)$ bar: $[T_f = 293.15 \text{ K} \times \left(\frac{1}{7.45}\right)^{0.2308} \approx 293.15 \times 0.629 \approx 184.4 \text{ K}]$

Now, we must determine if these final temperatures are low enough to cause the CO_2 to solidify, forming a "blue fog". This occurs if the final temperature (T_f) is below the sublimation temperature of CO_2 at the final pressure of 1 bar. We find this sublimation temperature, (T_{sub}), from the given vapor pressure equation by setting $P_{\text{sat}}^{\text{CO}2} = P_0 = 1$ bar. $\left[\log_{10}\left(\frac{P_0}{P_0}\right) = \log_{10}(1) = 0 = A - \frac{B}{T_{\text{sub}}+C}\right]$ $\left[T_{\text{sub}} = \frac{B}{A} - C = \frac{1.30 \times 10^3}{6.81} - (-3.49) = 190.9 + 3.49 = 194.39 \text{ K}\right]$

We can now compare our final temperatures to this sublimation point:

- At ($T_0 = 6^\circ$C, the final temperature is ($T_f \approx 195.4$) K. Since ($T_f > T_{\text{sub}}$), the CO_2 gas does not solidify. The problem states that in this case, a grey-white fog appears from water vapor condensation. This corresponds to **statement (1)**.
- At ($T_0 = 20^\circ$C, the final temperature is ($T_f \approx 184.4$) K. Since ($T_f < T_{\text{sub}}$), the CO_2 gas will partially turn into solid crystals, creating a blue fog. This corresponds to **statement (4)**.

Thus, the true statements are (1) and (4).

Part C.3: Maximum Height of the Cork

To find the maximum height the cork reaches, we must first determine its kinetic energy as it leaves the bottleneck. We can do this using the work-energy theorem. The net work done on the cork during its travel through the bottleneck (($l_0 = 0.025$) m) is equal to its final kinetic energy (K_e). $[K_e = W_{\text{net}} = W_{\text{pressure}} + W_{\text{friction}} + W_{\text{gravity}}]$

The work is done by three forces: the pressure from the expanding gas, the friction with the bottleneck, and gravity.

1. **Work by Pressure ((W_{pressure})):** The net work from pressure is the work done by the expanding internal gas, (W_{gas}), minus the work done against the external atmospheric pressure, (W_{atm}). For the adiabatic expansion from volume (V_G) to ($V_f = V_G + S l_0$), where ($S = \pi (d/2)^2$) is the cork's cross-sectional area: $[W_{\text{gas}} = \frac{P_f V_f - P_i V_i}{1-\gamma}]$ The work done against the atmosphere is ($W_{\text{atm}} = -P_0 (V_f - V_G)$).
2. **Work by Friction ((W_{friction})):** The friction force is ($F_f = \alpha A$), where (A) is the contact area ($A(x) = \pi d (l_0 - x)$). The problem states that initially, the pressure force slightly overcomes friction. We can use this to find the friction parameter (\alpha), or more directly, the initial friction force: $[F_{f, \text{initial}} = (P_i - P_0)S]$ Since the friction force decreases linearly from this initial value to zero as the cork travels a distance (l_0), the work done *by* friction (a negative value) is the average force times the distance: $[W_{\text{friction}} = - \frac{F_{f, \text{initial}} + 0}{2} \times l_0 = -\frac{1}{2}(P_i - P_0)S l_0]$
3. **Work by Gravity ((W_{gravity})):** Gravity does negative work as the cork moves up: ($W_{\text{gravity}} = -mg l_0$).

Calculations: Using ($T_0 = 6^\circ$C, so ($P_i = 4.69 \times 10^5$) Pa.

- Cork area: ($S = \pi (0.018/2)^2 = 2.545 \times 10^{-4}$) m².
- Volume change: ($\Delta V = S l_0 = (2.545 \times 10^{-4})(0.025) = 6.36 \times 10^{-6}$) m³.
- Initial volume: ($V_i = V_G = 2.5 \times 10^{-5}$) m³.
- Final volume: ($V_f = V_i + \Delta V = 3.136 \times 10^{-5}$) m³.
- Final pressure: ($P_f = P_i (V_i/V_f)^\gamma = (4.69 \times 10^5)(\frac{2.5}{3.136})^{1.3} \approx 3.48 \times 10^5$) Pa.

Now we calculate the work terms:

- ($W_{\text{gas}} = \frac{(3.48 \times 10^5)(3.136 \times 10^{-5}) - (4.69 \times 10^5)(2.5 \times 10^{-5})}{1 - 1.3} = \frac{10.91 - 11.725}{-0.3} \approx 2.72$) J.
- ($W_{\text{atm}} = -(10^5)(6.36 \times 10^{-6}) = -0.64$) J.
- Total ($W_{\text{pressure}} = 2.72 - 0.64 = 2.08$) J.
- ($W_{\text{friction}} = -\frac{1}{2}(4.69 \times 10^5 - 10^5)(2.545 \times 10^{-4})(0.025) = -1.18$) J.
- ($W_{\text{gravity}} = -(0.01 \text{ kg})(9.81 \text{ m/s}^2)(0.025 \text{ m}) = -0.00245$) J.

The kinetic energy at exit is: $[K_e = 2.08 - 1.18 - 0.00245 = 0.898 \text{ J}]$

Once the cork leaves the bottle, it behaves as a projectile. By conservation of energy, its initial kinetic energy is converted into gravitational potential energy at its maximum height (H_c). $[K_e = mgH_c]$ $[H_c = \frac{K_e}{mg} = \frac{0.898}{0.01 \times 9.81} = \frac{0.898}{0.0981} \approx 9.15 \text{ m}]$

The maximum height reached by the cork is approximately 9.15 meters.