googleai-gemini-2.5-pro

The problem asks for the energy difference, (\Delta E_F), associated with the fine structure. The text states this arises from the interaction of the electron's magnetic moment, (\vec{M_s}), with the magnetic field, (\vec{B_1}), produced by the proton's apparent motion. The two states are defined by the magnetic moment being parallel and anti-parallel to the field.

The potential energy of a magnetic dipole (\vec{M}) in a magnetic field (\vec{B}) is a fundamental concept, given by (U = -\vec{M} \cdot \vec{B}). For the two cases described:

- When (\vec{M_s}) is parallel to (\vec{B_1}), the angle is 0, so (U_{\parallel} = -M_s B_1).
- When (\vec{M_s}) is anti-parallel to (\vec{B_1}), the angle is (\pi), so (U_{\nparallel} = -M_s B_1 \cos(\pi) = +M_s B_1).

The energy difference (\Delta E_F) is the separation between these two energy levels: [\Delta E_F = U_{\nparallel} - U_{\parallel} = (M_s B_1) - (-M_s B_1) = 2 M_s B_1]

To proceed, I must find expressions for (M_s) and (B_1). The problem provides the magnitude of the electron's magnetic moment: [M_s = \frac{e}{m_e}\hbar]

Next, I need the magnetic field (B_1). This field is generated at the electron's position by the orbiting proton. From the electron's reference frame, the proton (charge (+e)) moves in a circular orbit of radius (r_1) with velocity (v_1). A moving charge constitutes a current. The time for one orbit (the period) is (T = \frac{2\pi r_1}{v_1}). The equivalent electric current (I) is the charge passing a point per unit time: [I = \frac{e}{T} = \frac{e v_1}{2\pi r_1}]

The magnetic field at the center of a circular current loop is given by (B = \frac{\mu_0 I}{2r}). Applying this to our situation: [B_1 = \frac{\mu_0}{2r_1} \left(\frac{e v_1}{2\pi r_1} \right) = \frac{\mu_0 e v_1}{4\pi r_1^2}] This confirms the result from A.4.

Now I can substitute my expressions for (M_s) and (B_1) into the equation for (\Delta E_F): [\Delta E_F = 2 \left(\frac{e\hbar}{m_e} \right) \left(\frac{\mu_0 e v_1}{4\pi r_1^2} \right) = \frac{2 \mu_0 e^2 \hbar v_1}{4\pi m_e r_1^2} = \frac{\mu_0 e^2 \hbar v_1}{2\pi m_e r_1^2}]

The goal is to express this in terms of the fine-structure constant, (\alpha), and the ground state energy, (E_1). I must use the relations provided or derived in previous parts. From A.2, we have (v_1 = \alpha c). Let's substitute this into the expression for (\Delta E_F): [\Delta E_F = \frac{\mu_0 e^2 \hbar (\alpha c)}{2\pi m_e r_1^2}]

This expression still contains (e^2), (r_1), and (\mu_0). Let's use the definition of (\alpha) and other fundamental relations to simplify this. The definition given is (\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}). This allows me to write (e^2 = 4\pi\epsilon_0 \hbar c \alpha). I also know the speed of light is related to the vacuum permittivity and permeability: (c^2 = \frac{1}{\epsilon_0 \mu_0}), which means (\mu_0 = \frac{1}{\epsilon_0 c^2}).

Substituting these for (\mu_0) and (e^2): [\Delta E_F = \frac{(\frac{1}{\epsilon_0 c^2}) (4\pi\epsilon_0 \hbar c \alpha) \hbar (\alpha c)}{2\pi m_e r_1^2}] Many terms cancel out. (\epsilon_0), (\pi), and one factor of (c) in the numerator and denominator cancel. [\Delta E_F = \frac{(\frac{1}{c}) (4 \hbar \alpha) \hbar (\alpha c)}{2 m_e r_1^2} = \frac{4 \hbar^2 \alpha^2}{2 m_e r_1^2} = \frac{2 \hbar^2 \alpha^2}{m_e r_1^2}]

This is much simpler. Now I need to eliminate (r_1) in favor of (E_1). From A.3, the total mechanical energy is (E_n = -\frac{e^2}{8\pi\epsilon_0 r_n}). For the ground state, (n=1): [E_1 = -\frac{e^2}{8\pi\epsilon_0 r_1}] We can rewrite this as (r_1 = -\frac{e^2}{8\pi\epsilon_0 E_1}). Substituting (e^2 = 4\pi\epsilon_0 \hbar c \alpha): [r_1 = -\frac{4\pi\epsilon_0 \hbar c \alpha}{8\pi\epsilon_0 E_1} = -\frac{\hbar c \alpha}{2E_1}]

Now, substitute this expression for (r_1) into our equation for (\Delta E_F): [\Delta E_F = \frac{2 \hbar^2 \alpha^2}{m_e \left(-\frac{\hbar c \alpha}{2E_1}\right)^2} = \frac{2 \hbar^2 \alpha^2}{m_e \left(\frac{\hbar^2 c^2 \alpha^2}{4E_1^2}\right)}] The terms (\hbar^2) and (\alpha^2) cancel completely. [\Delta E_F = \frac{2}{m_e \left(\frac{c^2}{4E_1^2}\right)} = \frac{8 E_1^2}{m_e c^2}]

This is a valid expression, but it is not in the form "a function of (\alpha) and (E_1)". I seem to have eliminated (\alpha). Let me re-examine the relationship between (E_1) and (\alpha). The total energy can also be expressed via the Virial theorem for a (1/r) potential, which states (E = -K), where (K) is the kinetic energy. [E_1 = -K_1 = -\frac{1}{2}m_e v_1^2] Using (v_1 = \alpha c), we get: [E_1 = -\frac{1}{2}m_e (\alpha c)^2 = -\frac{1}{2}m_e \alpha^2 c^2] This gives a direct relation between (E_1), (m_e), and (\alpha). I can rearrange this to find an expression for (m_e c^2): [m_e c^2 = -\frac{2E_1}{\alpha^2}]

Now I can substitute this into my previous result for (\Delta E_F): [\Delta E_F = \frac{8 E_1^2}{m_e c^2} = \frac{8 E_1^2}{\left(-\frac{2E_1}{\alpha^2}\right)}] One factor of (E_1) cancels. [\Delta E_F = \frac{8 E_1 \alpha^2}{-2} = -4 \alpha^2 E_1]

This result is elegant and directly relates the fine structure energy splitting to the ground state energy and the fine-structure constant. Since (E_1) is a negative binding energy and (\alpha^2) is positive, the overall sign is positive, as an energy difference must be.

Final Answer for A.5, part one:

The energy difference of the fine structure is given by (\Delta E_F = 2 M_s B_1). Using the given magnitude for the electron's magnetic moment, (M_s = \frac{e\hbar}{m_e}), and the magnetic field from the orbiting proton, (B_1 = \frac{\mu_0 e v_1}{4\pi r_1^2}), we have: [\Delta E_F = \frac{\mu_0 e^2 \hbar v_1}{2\pi m_e r_1^2}] The ground state energy is (E_1 = -\frac{1}{2}m_e v_1^2), and for the ground state orbit, the velocity is (v_1 = \alpha c). This gives a direct relationship: (E_1 = -\frac{1}{2}m_e \alpha^2 c^2). By substituting expressions for the constants and orbital parameters in terms of (\alpha), (E_1), (m_e), and (c), we find the relation. A more direct path is to relate (\Delta E_F) and (E_1) through their dependence on (m_e c^2). Starting from our intermediate result (\Delta E_F = \frac{8 E_1^2}{m_e c^2}) and the relation (m_e c^2 = -\frac{2E_1}{\alpha^2}), we substitute for (m_e c^2): [\Delta E_F = \frac{8 E_1^2}{-2E_1/\alpha^2} = -4\alpha^2 E_1]

Expression for (\Delta E_F): [\Delta E_F = -4\alpha^2 E_1]