googleai-gemini-2.5-pro

To understand the physical nature of this proposed velocity profile, (v_{c,m}(r)), it is essential to examine its behavior in the extreme regimes: when the radius of orbit, (r), is very small compared to the characteristic radius (r_m), and when it is very large.

Simplification of the Velocity Profile

The velocity of an object in a circular orbit is maintained by the gravitational force from the mass enclosed within that orbit. The relationship is given by Newton's laws: [\frac{m v^2}{r} = \frac{G M_{enc}(r) m}{r^2} \implies v^2(r) = \frac{G M_{enc}(r)}{r}] The enclosed mass, (M_m(r)), is found by integrating the given mass density, (\rho_m(r') = \frac{C_m}{r_m^2 + r'^2}), over a sphere of radius (r). [M_m(r) = \int_0^r \rho_m(r') 4\pi r'^2 dr' = 4\pi C_m \int_0^r \frac{r'^2}{r_m^2 + r'^2} dr'] Using the provided integral hint, (\int \frac{x^2}{a^2+x^2}dx = x - a \arctan(x/a)), with (x=r') and (a=r_m), we find: [M_m(r) = 4\pi C_m \left[r' - r_m \arctan\left(\frac{r'}{r_m}\right) \right]*0^r = 4\pi C_m \left(r - r_m \arctan\left(\frac{r}{r_m}\right) \right)] Therefore, the square of the velocity is: [v*{c,m}(r)^2 = \frac{G}{r} M_m(r) = 4\pi G C_m \left(1 - \frac{r_m}{r} \arctan\left(\frac{r}{r_m}\right) \right)] This form must correspond to the one given in the problem, (v_{c,m}(r) = \sqrt{k_1 - \frac{k_2 \arctan(r/r_m)}{r}}), allowing us to proceed.

Case 1: Inner Regions ((r \ll r_m))

In this regime, the ratio (x = r/r_m) is much less than 1. We can use the approximation for the arctangent function provided in the hints, (\arctan(x) \approx x - x^3/3). Let's analyze the term (\frac{r_m}{r} \arctan\left(\frac{r}{r_m}\right)): [\frac{r_m}{r} \arctan\left(\frac{r}{r_m}\right) = \frac{1}{x} \arctan(x) \approx \frac{1}{x} \left(x - \frac{x^3}{3}\right) = 1 - \frac{x^2}{3} = 1 - \frac{r^2}{3r_m^2}] Substituting this back into the expression for (v_{c,m}(r)^2): [v_{c,m}(r)^2 \approx 4\pi G C_m \left(1 - \left(1 - \frac{r^2}{3r_m^2}\right) \right) = 4\pi G C_m \left(\frac{r^2}{3r_m^2} \right)] Taking the square root, we find the velocity profile in the inner regions: [v_{c,m}(r) \approx \left(\sqrt{\frac{4\pi G C_m}{3r_m^2}} \right) r] The velocity increases linearly with the distance from the center. This is the expected behavior for an object moving within a sphere of approximately uniform density, as for (r \ll r_m), the density (\rho_m(r) \approx C_m/r_m^2) is nearly constant.

Case 2: Outer Regions ((r \gg r_m))

In this regime, the ratio (x = r/r_m) is very large. As (x \to \infty), the function (\arctan(x)) approaches a constant value of (\pi/2). Substituting this limit into the velocity expression: [v_{c,m}(r)^2 \approx 4\pi G C_m \left(1 - \frac{r_m}{r} \frac{\pi}{2} \right)] For very large (r), the term (\frac{r_m \pi}{2r}) becomes negligible. Thus, the velocity approaches a constant value: [v_{c,m}(r) \to \sqrt{4\pi G C_m} \quad (\text{as } r \to \infty)] This model predicts a "flat" rotation curve in the outer parts of the galaxy.

Simplification of the Enclosed Mass

Now we examine the enclosed mass (M_m(r)) in the outer region where (r \gg r_m). Our exact expression is: [M_m(r) = 4\pi C_m \left(r - r_m \arctan\left(\frac{r}{r_m}\right) \right)] Using the same approximation, (\arctan(r/r_m) \approx \pi/2): [M_m(r) \approx 4\pi C_m \left(r - r_m \frac{\pi}{2} \right)] While this expression still contains (r_m), for (r \gg r_m), the term linear in (r) dominates the constant offset term (r_m \frac{\pi}{2}). Therefore, the mass simplifies to a form that grows linearly with radius and depends only on (C_m) and (r): [M_m(r) \approx (4\pi C_m) r]

Mass Estimation of NGC 6946

We can now use this theoretical framework to estimate the mass of the galaxy shown in Figure 1. The experimental data points in Fig 1(B) show that the rotation curve for NGC 6946 does indeed flatten out at large radii, consistent with our model for (r \gg r_m).

1. **Determine Model Parameters from Data:** From the graph in Fig 1(B), the constant velocity in the outer region is approximately (v_{c,\infty} \approx 150 \text{ km/s}). We can equate this to our theoretical result: [v_{c,\infty} = \sqrt{4\pi G C_m} \implies 4\pi C_m = \frac{v_{c,\infty}^2}{G}] Using the values (v_{c,\infty} = 1.50 \times 10^5 \text{ m/s}) and (G = 6.674 \times 10^{-11} \text{ N m}^2/\text{kg}^2): [4\pi C_m = \frac{(1.50 \times 10^5 \text{ m/s})^2}{6.674 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}} = 3.371 \times 10^{20} \text{ kg/m}]

2. **Estimate the Galaxy's Radius from the Picture:** The scale bar in Fig 1(A) indicates a length of 18 kpc. The visible extent of the galaxy appears to have a radius, (R_{pic}), that is slightly larger than this bar, perhaps by a factor of 1.25. [R_{pic} \approx 1.25 \times 18 \text{ kpc} = 22.5 \text{ kpc}] We convert this radius to meters: [R_{pic} = 22.5 \text{ kpc} \times (3.09 \times 10^{19} \text{ m/kpc}) = 6.95 \times 10^{20} \text{ m}]

3. **Calculate the Total Mass:** Since this radius is in the outer region of the galaxy, we can use our simplified large-radius mass formula, (M_m(r) \approx (4\pi C_m) r). [M_{galaxy} \approx (3.371 \times 10^{20} \text{ kg/m}) \times (6.95 \times 10^{20} \text{ m}) \approx 2.34 \times 10^{41} \text{ kg}]

4. **Convert to Solar Masses:** It is conventional to express galactic masses in units of solar mass ((1 M_{\odot} = 1.99 \times 10^{30} \text{ kg})). [M_{galaxy} \approx \frac{2.34 \times 10^{41} \text{ kg}}{1.99 \times 10^{30} \text{ kg}/M_{\odot}} \approx 1.18 \times 10^{11} M_{\odot}]

The estimated mass of the galaxy NGC 6946 present in the picture is approximately (1.2 \times 10^{11}) solar masses.