assistant

It has been proposed that Newton's second law, $F=ma$, might not be the complete picture and could require modification in regimes of extremely low acceleration, such as those found in the outer regions of galaxies. A specific proposal is the theory of MOdified Newtonian Dynamics (MOND), which posits that the relationship between force and acceleration is given by $\vec{F} = m\mu\left(\frac{a}{a_0}\right)\vec{a}$, where a_0 is a new fundamental constant of nature with units of acceleration.

Let us investigate the consequences of this proposal in the simplest non-trivial scenario: a single body of mass m, such as a star, in a stable circular orbit of radius r around a large central mass M. The gravitational force is still given by Newton's law of universal gravitation, $\vec{F}_g = -\frac{GMm}{r^2}\hat{r}$. For a circular orbit, the acceleration is purely centripetal, with magnitude $a = v_c^2/r$.

The problem specifies we should explore the limit where the acceleration is very small compared to the MOND constant, i.e., $a \ll a_0$. The proposed function is $\mu(x) = x / (1 + x)$. In the limit where $x = a/a_0 \ll 1$, the denominator $(1+x)$ approaches 1. Therefore, the function simplifies considerably: $\mu\left(\frac{a}{a_0}\right) \approx \frac{a}{a_0}$ In this low-acceleration regime, the MOND force law becomes: $F \approx m\left(\frac{a}{a_0}\right)a = \frac{ma^2}{a_0}$ This is a profound departure from Newtonian dynamics, suggesting force is proportional to acceleration squared.

Let's apply this to our orbiting star. Equating the gravitational force with this modified dynamical law gives: $\frac{GMm}{r^2} = \frac{ma^2}{a_0}$ Now, we substitute the expression for centripetal acceleration, $a = v_{c,\infty}^2/r$, where we use $v_{c,\infty}$ to denote the constant velocity in these outer regions: $\frac{GMm}{r^2} = \frac{m}{a_0}\left(\frac{v_{c,\infty}^2}{r}\right)^2 = \frac{m v_{c,\infty}^4}{a_0 r^2}$ A remarkable thing happens: the dependence on the radius r on both sides of the equation cancels out entirely. $GM = \frac{v_{c,\infty}^4}{a_0}$ Rearranging this equation to express the total mass M in terms of the orbital velocity gives: $M = \left(\frac{1}{Ga_0}\right)v_{c,\infty}^4$ This is a direct theoretical prediction for the relationship between the total mass of a galaxy and the flat part of its rotation curve. Comparing this to the empirical Tully-Fischer relation form, $M \propto v^\gamma$, we find that the MOND theory predicts an exponent of **$\gamma = 4$**.

This theoretical result can be tested. We can use the empirical data from the Tully-Fischer relation to determine the value of this proposed new constant, a_0. Rearranging our derived equation: $a_0 = \frac{v_{c,\infty}^4}{GM}$ From the Tully-Fischer plot in Figure 4, we can select a representative point that lies on the best-fit line. A convenient point is where $\log_{10}(v_{c,\infty} / 1\,\text{km/s}) = 2.3$, which corresponds to $\log_{10}(M_{tot} / M_\odot) \approx 11$. Let's extract these values:

- $v_{c,\infty} = 10^{2.3} \,\text{km/s} \approx 199.5 \,\text{km/s} \approx 2.0 \times 10^5 \,\text{m/s}$. This is consistent with the velocity for NGC 6946 from Figure 1.
- $M_{tot} = 10^{11} M_\odot = 10^{11} \times (1.99 \times 10^{30} \,\text{kg}) = 1.99 \times 10^{41} \,\text{kg}$.

Now, we can calculate a_0: $a_0 = \frac{(2.0 \times 10^5 \,\text{m/s})^4}{(6.67 \times 10^{-11} \,\text{N m}^2/\text{kg}^2)(1.99 \times 10^{41} \,\text{kg})} = \frac{1.6 \times 10^{21} \,\text{m}^4/\text{s}^4}{1.327 \times 10^{31} \,\text{m}^3/\text{s}^2} \approx 1.2 \times 10^{-10} \,\text{m/s}^2$ This value, derived from astronomical observations, is remarkably consistent with the value of $a_0 \approx 10^{-10} \,\text{m/s}^2$ suggested by the theory's proponents.

Finally, we must check if the theory is being applied in the correct regime. The derivation was contingent on the assumption that $a \ll a_0$. Let's calculate the actual acceleration of a star in the outer part of a galaxy like NGC 6946 using the standard Newtonian definition of centripetal acceleration. From question D.2, we found the acceleration in the outer regions is $a_m \approx 5 \times 10^{-11} \,\text{m/s}^2$. Let's compare this to our calculated value of a_0: $\frac{a_m}{a_0} = \frac{5 \times 10^{-11} \,\text{m/s}^2}{1.2 \times 10^{-10} \,\text{m/s}^2} \approx 0.42$ The ratio is less than 1, which confirms that the acceleration is indeed in the low-acceleration regime where MOND is expected to become relevant. The condition $a \ll a_0$ is the idealized limit, but our result confirms we are operating in the correct domain ($a \lesssim a_0$) and not in the high-acceleration Newtonian regime ($a \gg a_0$). Thus, the MOND framework appears to be self-consistent with the observational data it seeks to explain.