assistant

Part B. Acoustic emission of a bursting bubble

B.1 Expression for the film retraction speed, v_f

I am observing a bubble at the surface of the liquid. A hole appears and rapidly expands, driven by some intrinsic property of the liquid. This property is surface tension, σ, which represents an energy per unit area. As the hole grows, the total surface area of the film decreases, releasing this surface energy. The problem states that half of this released energy is converted into the kinetic energy of the liquid that accumulates in the rim of the hole. My goal is to find the speed, v_f, at which the rim retracts.

Let's consider the physical quantities involved. The speed v_f must depend on the driving energy source, the surface tension σ (in units of energy/area, or M/T^2), and the inertia of the material being moved, which depends on the liquid density ρ_ℓ (M/L^3) and the film thickness h (L). A dimensional argument suggests how these might combine to form a velocity (L/T). The ratio σ/ρ_ℓ has units of L^3/T^2. To get L^2/T^2 (velocity squared), I must divide by a length. The natural length scale here is the film thickness h. Thus, I anticipate that v_f will be proportional to $\sqrt{\sigma / (\rho_\ell h)}$.

To find the precise relationship, I will formulate an energy balance. In a small time interval dt, the radius of the hole increases from r to $r + v_f dt$.

The area of the film that vanishes during this interval is an annulus, $dA = \pi(r + v_f dt)^2 - \pi r^2 \approx 2\pi r v_f dt$. Since the film has two surfaces (one in contact with the air, one with the CO_2 in the bubble), the total reduction in surface area is $2 dA$. The surface energy released is therefore $|dE_{surface}| = \sigma \cdot (2 \cdot dA) = 4\pi\sigma r v_f dt$.

According to the problem, half of this energy becomes kinetic energy: $dE_k = (1/2) \cdot |dE_{surface}| = 2\pi\sigma r v_f dt$.

This energy is transferred to the mass of the liquid in the annulus, dM, which is collected by the rim. The volume of this annulus is $dA \cdot h$, so its mass is $dM = \rho_\ell \cdot dA \cdot h = \rho_\ell \cdot (2\pi r v_f dt) \cdot h$. The kinetic energy gained is that of this newly accelerated mass, $dE_k = (1/2) dM v_f^2$.

Substituting dM: $dE_k = (1/2) \cdot (2\pi\rho_\ell r h v_f dt) \cdot v_f^2 = \pi\rho_\ell r h v_f^3 dt$.

Now, I can equate the two expressions for dE_k: $\pi\rho_\ell r h v_f^3 dt = 2\pi\sigma r v_f dt$

Dividing both sides by π, ρ_ℓ, r, v_f, and dt (assuming $v_f \neq 0$), I find: $\rho_\ell h v_f^2 = 2\sigma$

Solving for v_f, I arrive at the expression for the film retraction speed: [v_f = \sqrt{\frac{2\sigma}{\rho_\ell h}}]

B.2 Frequency of a Helmholtz resonator, f_0

The problem proposes modeling the sound emission using a Helmholtz resonator. This system consists of a gas-filled cavity of volume V_0 connected to the outside by a narrow neck of area S. A "piston" of gas of mass m_p resides in this neck. If this piston is displaced by a small distance z into the cavity, it compresses the gas. The increased pressure creates a restoring force, pushing the piston back. This is the archetype of a simple harmonic oscillator. My task is to find its natural frequency of oscillation.

The restoring force F is $F = -\Delta P \cdot S$, where ΔP is the pressure change inside the cavity. The oscillation is rapid, so we can assume the compression and expansion are adiabatic, governed by the relation $P V^\gamma = \text{constant}$.

Let the initial state be (P_0, V_0). When the piston moves inward by z, the new volume is $V = V_0 - S z$, and the new pressure is P. From the adiabatic relation, $P = P_0(V_0/V)^\gamma = P_0(V_0 / (V_0 - S z))^\gamma = P_0(1 - Sz/V_0)^{-\gamma}$.

For a small displacement, $Sz/V_0 \ll 1$. I can use the binomial approximation $(1+x)^\alpha \approx 1+\alpha x$: $P \approx P_0(1 + (-\gamma)(-Sz/V_0)) = P_0(1 + \gamma Sz/V_0)$

The pressure change is $\Delta P = P - P_0 = \gamma P_0 Sz/V_0$. The restoring force on the piston is $F = -\Delta P \cdot S = -(\gamma P_0 S^2 z/V_0)$.

This is precisely the form of Hooke's Law, $F = -kz$, where the effective spring constant of the gas is $k = \gamma P_0 S^2/V_0$. The equation of motion for the piston is $m_p \cdot d^2z/dt^2 = F = -kz$. This describes simple harmonic motion with an angular frequency ω = sqrt(k/mp).

The oscillation frequency f_0 is $\omega/(2\pi)$: $f_0 = (1/(2\pi)) * \text{sqrt}(k/mp) = (1/(2\pi)) * \text{sqrt}((\gamma P_0 S^2/V_0)/mp)$

This can be written as: [f_0 = \frac{S}{2\pi} \sqrt{\frac{\gamma P_0}{m_p V_0}}]

B.3 Bubble radius a and film thickness h

I am now equipped with two physical models and experimental data. I can combine them to determine the geometric properties of the bursting bubble—its radius a and the thickness h of the film cap. I have two unknowns, so I need two independent equations.

The first equation comes from the maximum sound frequency, f_max = 40 kHz. This maximum occurs when the hole has reached its final radius, r = rc. At this point, I can apply the Helmholtz frequency formula with the following substitutions:

- Cavity volume: $V_0 = (4/3)\pi a^3$ (the bubble volume)
- Neck area: $S = \pi rc^2$
- Piston mass: $mp = (8/3)\varrho g*rc^3$

Substituting these into the frequency equation: $f\_max = (\pi rc^2 / (2\pi)) * \text{sqrt}(\gamma P_0 / (((8/3)\varrho g*rc^3) * ((4/3)\pi a^3)))$ $f\_max = (rc^2 / 2) * \text{sqrt}(\gamma P_0 / ((32/9)\pi\varrho g*rc^3a^3))$ Squaring both sides to simplify the algebra: $f\_max^2 = (rc^4 / 4) * (\gamma P_0 / ((32/9)\pi\varrho g*rc^3a^3)) = (rc^4 / 4) * (9\gamma P_0 / (32\pi\varrho g*rc^3a^3))$ $f\_max^2 = (9\gamma P_0 * rc) / (128\pi\varrho g*a^3)$

This equation relates rc and a. The problem provides another relation: $rc = 2*\text{sqrt}(3)*a^2*\text{sqrt}(\varrho\ell*g_0/\sigma)$. I can substitute this into my frequency result to obtain an equation solely in terms of a.

$f\_max^2 = (9\gamma P_0 / (128\pi\varrho g*a^3)) * (2*\text{sqrt}(3)*a^2*\text{sqrt}(\varrho\ell*g_0/\sigma))$ $f\_max^2 = (18*\text{sqrt}(3)*\gamma P_0) / (128\pi\varrho g*a) * \text{sqrt}(\varrho\ell*g_0/\sigma)$

Now, I can solve for the bubble radius a: $a = (18*\text{sqrt}(3)*\gamma P_0) / (128\pi\varrho g*f\_max^2) * \text{sqrt}(\varrho\ell*g_0/\sigma)$

Let's insert the numerical values:

- $\gamma = 1.3$
- $P_0 = 1.0 \times 10^5$ Pa
- $\varrho g = 1.8$ kg·m^{-3}
- $f\_max = 4.0 \times 10^4$ Hz
- $\sigma = 47 \times 10^{-3}$ J·m^{-2}
- $\varrho\ell = 1.0 \times 10^3$ kg·m^{-3}
- $g_0 \approx 9.81$ m·s^{-2}

First, the term under the square root: $\text{sqrt}((1.0e3 * 9.81) / 47e-3) = \text{sqrt}(2.087e5) \approx 456.8$ m^{-1}. $a = (18 * 1.732 * 1.3 * 1.0e5) / (128 * \pi * 1.8 * (4.0e4)^2) * 456.8$ $a = (4.053 \times 10^6) / (723.8 * 1.6 \times 10^9) * 456.8$ $a = (4.053 \times 10^6) / (1.158 \times 10^{12}) * 456.8 = (3.50 \times 10^{-6}) * 456.8 \approx 1.60 \times 10^{-3}$ m So, the bubble radius is $a \approx 1.6$ mm.

The second equation comes from the bursting time, $tb = 3 \times 10^{-2}$ ms $= 3 \times 10^{-5}$ s. The hole opens at a constant speed vf over this time to reach its final radius rc. Therefore, rc = vf * tb. Using my result from B.1: $rc = \text{sqrt}(2\sigma / (\varrho\ell h)) * tb$

I can solve this for the film thickness h: $h = 2\sigma*tb^2 / (\varrho\ell*rc^2)$. First, I must calculate rc using the value of a I just found: $rc = 2*\text{sqrt}(3) * a^2 * \text{sqrt}(\varrho\ell*g_0/\sigma) = 2*\text{sqrt}(3) * (1.60 \times 10^{-3})^2 * (456.8)$ $rc = 3.464 * (2.56 \times 10^{-6}) * 456.8 \approx 4.05 \times 10^{-3}$ m.

Now I can find h: $h = (2 * 47 \times 10^{-3} * (3 \times 10^{-5})^2) / (1.0 \times 10^3 * (4.05 \times 10^{-3})^2)$ $h = (94 \times 10^{-3} * 9 \times 10^{-10}) / (1.0 \times 10^3 * 16.4 \times 10^{-6})$ $h = (8.46 \times 10^{-11}) / (1.64 \times 10^{-2}) \approx 5.16 \times 10^{-9}$ m. The thickness of the film is $h \approx 5.2$ nm.

The final results are:

- Bubble radius: **$a \approx 1.6$ mm**
- Film thickness: **$h \approx 5.2$ nm**