groq-kimi-k2

B.1 Volume corresponding to (m_{add})

To determine the physical meaning of (m_{add}), we must first establish the force (F) required to hold the tube in equilibrium. We consider the system to be the tube itself plus the column of mercury contained within it. The total downward gravitational force on this system is ($(m_{\text{tb}} + m_{\text{merc}})g$), where (m_{tb}) is the mass of the tube and (m_{merc}) is the mass of the mercury inside.

This downward weight is counteracted by two upward forces: the externally applied force (F) and the force exerted by the atmospheric pressure of the surrounding mercury bath on the open base of the tube. If the opening (the bottom of the narrow tube, with area (S_t)) is at a depth (z_{bottom}) below the bath surface (($z = 0$)), the pressure at that depth is ($P_{\text{bath}} = P_a - \rho g z_{\text{bottom}}$). However, a more direct approach is to recognize that the atmospheric pressure (P_a) on the bath's surface provides an upward lift on the base of the mercury column. This lift is equivalent to ($P_a S_t$).

Therefore, the force balance equation is:

$$F + P_a S_t = (m_{\text{tb}} + m_{\text{merc}})g$$

Solving for (F), we find the required pulling force:

$$F = (m_{\text{tb}} + m_{\text{merc}})g - P_a S_t$$

The problem defines the additional mass, (m_{add}), through the equation ($F = (m_{\text{tb}} + m_{\text{add}})g$). By comparing our derived expression for (F) with the given one, we can isolate (m_{add}):

$$(m_{\text{tb}} + m_{\text{add}})g = (m_{\text{tb}} + m_{\text{merc}})g - P_a S_t$$

$$m_{\text{add}}g = m_{\text{merc}}g - P_a S_t$$

$$m_{\text{add}} = m_{\text{merc}} - \frac{P_a S_t}{g}$$

This result shows that (m_{add}) is not the mass of a simple, contiguous volume of mercury. It represents the total mass of mercury in the tube, (m_{merc}), reduced by a mass equivalent to the atmospheric lift, ($P_a S_t / g$).

The question asks to "color the area corresponding to the volume of liquid mercury that is responsible for the term (m_{add})". Given that (m_{add}) tracks (m_{merc}) with a constant offset (for a fixed (P_a)), the physical volume whose mass governs the behavior of (m_{add}) is the entire volume of mercury within the tube, (V_{merc}). The term (m_{add}) is the *effective* added mass, but the *responsible* physical volume is that which contains all the mercury.

Therefore, the area to be colored on the diagram is the entire shaded region representing the mercury column within both the narrow tube and the bulb.

B.2 Sketch of (m_{add}) versus (h_t)

We will analyze the evolution of (m_{add}) as the tube is lifted, which means as (h_t) (the altitude of the junction) increases from ($h_t = -H_b$) to ($h_t = H_t$). Our derived relation is ($m_{\text{add}} = m_{\text{merc}} - P_a S_t / g$). For this part, ($P_a = P_0$) is constant, so the behavior of ($m_{\text{add}}(h_t)$) is identical to that of ($m_{\text{merc}}(h_t)$), merely shifted by a constant value. We must analyze ($m_{\text{merc}}(h_t) = \rho V_{\text{merc}}(h_t)$).

First, we calculate the barometric height for mercury under the given standard atmospheric pressure ($P_0 = 1.000 \times 10^5 , \text{Pa}$):

$$H_{\text{baro}} = \frac{P_0}{\rho g} = \frac{1.000 \times 10^5 , \text{Pa}}{(13.5 \times 10^3 , \text{kg/m}^3)(9.8 , \text{m/s}^2)} \approx 0.756 , \text{m} = 75.6 , \text{cm}$$

The behavior of the mercury column has three distinct regimes:

Regime 1: Tube completely full

A vacuum (the Torricellian vacuum) can only form if the top of the tube's inner volume is higher than H_{baro}. The top of the tube is at altitude $h_{\text{top}} = h_t + H_b$.

The tube remains full as long as $h_t + H_b < H_{\text{baro}}$, i.e.

$$h_t < H_{\text{baro}} - H_b = 75.6\,\text{cm} - 20\,\text{cm} = 55.6\,\text{cm}$$

In this regime ($-20\,\text{cm} \le h_t < 55.6\,\text{cm}$), the volume of mercury is the total internal volume of the tube, $V_{\text{merc}} = S_t H_t + S_b H_b$, which is constant. Thus, m_{add} is constant.
The slope of the graph $\mathrm{d}m_{\text{add}}/\mathrm{d}h_t$ is **0**.

Regime 2: Vacuum forms, mercury level in the bulb

When $h_t > 55.6\,\text{cm}$, a vacuum forms and the mercury level z_e stabilizes at

$$z_e = H_{\text{baro}} = 75.6\,\text{cm}$$

As long as the junction h_t is below this level ($h_t < H_{\text{baro}}$), the mercury surface will be in the upper bulb section. This regime holds for

$$55.6\,\text{cm} < h_t < 75.6\,\text{cm}$$

The volume of mercury is

$$V_{\text{merc}}(h_t) = S_t H_t + S_b(H_{\text{baro}} - h_t)$$

The mass m_{add} is

$$m_{\text{add}}(h_t) = \rho\left[S_t H_t + S_b(H_{\text{baro}} - h_t)\right] - \frac{P_0 S_t}{g}$$

which is linear in h_t. The slope is

$$\frac{\mathrm{d}m_{\text{add}}}{\mathrm{d}h_t} = -\rho S_b$$

$$\text{Slope} = -(13.5 \times 10^3\,\text{kg/m}^3)(200 \times 10^{-4}\,\text{m}^2) = \mathbf{-270\,\textbf{kg/m}}$$

Regime 3: Vacuum forms, mercury level in the narrow tube

When the junction is lifted above the barometric height ($h_t > H_{\text{baro}} = 75.6\,\text{cm}$), the fixed mercury level $z_e = H_{\text{baro}}$ is now in the lower, narrow part of the tube. The bulb is empty of mercury.

This regime holds for

$$75.6\,\text{cm} < h_t \le 80\,\text{cm}$$

The volume of mercury is

$$V_{\text{merc}}(h_t) = S_t(H_{\text{baro}} - z_{\text{bottom}}) = S_t(H_{\text{baro}} - (h_t - H_t))$$

The mass m_{add} is

$$m_{\text{add}}(h_t) = \rho S_t(H_{\text{baro}} + H_t - h_t) - \frac{P_0 S_t}{g}$$

which is also linear in h_t. The slope is

$$\frac{\mathrm{d}m_{\text{add}}}{\mathrm{d}h_t} = -\rho S_t$$

$$\text{Slope} = -(13.5 \times 10^3 \, \text{kg/m}^3)(5 \times 10^{-4} \, \text{m}^2) = \mathbf{-6.75 \, \textbf{kg/m}}$$

Sketch and Angular Points

The graph of $m_{\text{add}}(h_t)$ consists of three linear segments:

- An initial horizontal line from $h_t = -20 \, \text{cm}$ to the first angular point.
- A line with a steep negative slope from the first to the second angular point.
- A line with a shallow negative slope from the second angular point to $h_t = 80 \, \text{cm}$.

The angular points (kinks) occur at the transitions between regimes:

- First angular point:
  $$h_{t,1} = H_{\text{baro}} - H_b = 75.6 - 20 = \mathbf{55.6 \, \textbf{cm}}$$

- Second angular point:
  $$h_{t,2} = H_{\text{baro}} = \mathbf{75.6 \, \textbf{cm}}$$

B.3 Amplitude of m_{add} variation (Δm_{add})

Here, the tube's position h_t is fixed, while the atmospheric pressure $P_a(t) = P_0 + P_1(t)$ varies. The amplitude of the pressure variation is $A = 5 \times 10^2 \, \text{Pa}$. We are told the mercury surface z_e remains in the bulb. This means we are always in the physical situation of Regime 2 from the previous analysis.

The expression for m_{add} as a function of P_a is:

$$m_{\text{add}}(t) = m_{\text{merc}}(t) - \frac{P_a(t) S_t}{g}$$

In this regime, the mercury height is given by the barometric law:

$$z_e(t) = \frac{P_a(t)}{\rho g}$$

The volume of mercury is

$$V_{\text{merc}}(t) = S_t H_t + S_b(z_e(t) - h_t) = S_t H_t + S_b\left(\frac{P_a(t)}{\rho g} - h_t\right)$$

Substituting this into the equation for m_{add}:

$$m_{\text{add}}(t) = \rho\left[S_t H_t + S_b\left(\frac{P_a(t)}{\rho g} - h_t\right)\right] - \frac{P_a(t) S_t}{g}$$

$$m_{\text{add}}(t) = \rho S_t H_t + \frac{S_b P_a(t)}{g} - \rho S_b h_t - \frac{S_t P_a(t)}{g}$$

Group constant and variable terms:

$$m_{\text{add}}(t) = (\rho S_t H_t - \rho S_b h_t) + \frac{P_a(t)}{g}(S_b - S_t)$$

The entire variation in m_{add} comes from the variation in $P_a(t)$. The relationship is linear. The amplitude of the variation of m_{add}, which we denote by Δm_{add}, is directly proportional to the amplitude of the pressure variation, A.

Let $P_a(t) = P_0 + P_1(t)$, where $P_1(t)$ has amplitude A.
The variable part of $m_{\text{add}}(t)$ is $\dfrac{P_1(t)}{g}(S_b - S_t)$.
Therefore,

$$\Delta m_{\text{add}} = \frac{A}{g}(S_b - S_t)$$

Now, we substitute the numerical values:

- $A = 5 \times 10^2 \, \text{Pa}$
- $g = 9.8 \, \text{m/s}^2$
- $S_b = 200 \, \text{cm}^2 = 2.00 \times 10^{-2} \, \text{m}^2$

- $S_t = 5 \, \text{cm}^2 = 5 \times 10^{-4} \, \text{m}^2$

$$\Delta m_{\text{add}} = \frac{5 \times 10^2}{9.8}(2.00 \times 10^{-2} - 5 \times 10^{-4}) = \frac{500}{9.8}(0.02 - 0.0005) = \frac{500}{9.8}(0.0195) \approx 51.02 \times 0.0195 \approx 0.9949 \, \text{kg}$$

Hence

$$\Delta m_{\text{add}} \approx \mathbf{0.995 \, \textbf{kg}}$$