googleai-gemini-2.5-pro

We are asked to reconcile two different descriptions of a spiral galaxy. One is a model for the mass density, given by the expression (\rho_m(r) = \frac{C_m}{r_m^2 + r^2}). The other is a model for the gravitational potential in the galactic plane ((z=0)), given by (\phi_G(r, 0) = \phi_0 \ln(r/r_0)). It seems unlikely that these two descriptions are equivalent for all radii (r), but perhaps they converge under certain limiting conditions. Let us investigate this.

The fundamental link between the mass distribution and the gravitational potential (\phi) is the gravitational force it produces. For a test mass (m) in a circular orbit, the force can be expressed in two ways. First, it is the gradient of the potential energy, which for radial dependence is (F_g = m \frac{d\phi}{dr}). Second, according to Newton's law of gravitation and his shell theorem, the force is determined by the total mass (M_{enc}(r)) enclosed within the radius (r), such that (F_g = \frac{G m M_{enc}(r)}{r^2}).

By equating these two expressions for the force, we find a direct relationship between the potential's derivative and the enclosed mass: [\frac{d\phi}{dr} = \frac{G M_{enc}(r)}{r^2}] Our first task must be to calculate this enclosed mass (M_{enc}(r)) from the given density profile (\rho_m(r)). We can do this by integrating the density over the volume of a sphere of radius (r): [M_{enc}(r) = \int_0^r \rho_m(r') , 4\pi (r')^2 , dr' = \int_0^r \frac{C_m}{r_m^2 + (r')^2} , 4\pi (r')^2 , dr'] [M_{enc}(r) = 4\pi C_m \int_0^r \frac{(r')^2}{r_m^2 + (r')^2} , dr'] This integral can be solved. By rewriting the numerator as ((r')^2 + r_m^2 - r_m^2), or by using the integral identity provided in the problem statement, we find: [M_{enc}(r) = 4\pi C_m \left[r' - r_m \arctan\left(\frac{r'}{r_m}\right) \right]_0^r = 4\pi C_m \left(r - r_m \arctan\left(\frac{r}{r_m}\right) \right)] Now we can substitute this result back into our expression for the potential's derivative: [\frac{d\phi}{dr} = \frac{G}{r^2} \left[4\pi C_m \left(r - r_m \arctan\left(\frac{r}{r_m}\right) \right) \right] = 4\pi G C_m \left(\frac{1}{r} - \frac{r_m}{r^2} \arctan\left(\frac{r}{r_m}\right) \right)] This is the potential gradient derived from the mass model. We must now compare it to the potential gradient from the (\phi_G) model. For (\phi_G(r, 0) = \phi_0 \ln(r/r_0)), the derivative is simply: [\frac{d\phi_G}{dr} = \frac{\phi_0}{r}] We are looking for a regime where our derived (\frac{d\phi}{dr}) approximates this simple (\frac{1}{r}) form. Let's examine the two suggested regimes.

Case 1: The inner galaxy, (r \ll r_m). Here, the argument of the arctangent function, (x = r/r_m), is very small. We can use the approximation (\arctan(x) \approx x). [\frac{d\phi}{dr} \approx 4\pi G C_m \left(\frac{1}{r} - \frac{r_m}{r^2} \left(\frac{r}{r_m}\right) \right) = 4\pi G C_m \left(\frac{1}{r} - \frac{1}{r} \right) = 0] This approximation is too coarse. Let's use a more precise one, (\arctan(x) \approx x - x^3/3). [\frac{d\phi}{dr} \approx 4\pi G C_m \left(\frac{1}{r} - \frac{r_m}{r^2} \left[\frac{r}{r_m} - \frac{1}{3}\left(\frac{r}{r_m}\right)^3 \right] \right) = 4\pi G C_m \left(\frac{1}{r} - \frac{1}{r} + \frac{r}{3r_m^2} \right) = \frac{4\pi G C_m}{3r_m^2} r] In this regime, the potential gradient is proportional to (r), which does not match the required (\frac{1}{r}) form.

Case 2: The outer galaxy, (r \gg r_m). Here, the argument (x = r/r_m) is very large. In this limit, the arctangent function approaches a constant value: (\arctan(x) \to \pi/2). [\frac{d\phi}{dr} \approx 4\pi G C_m \left(\frac{1}{r} - \frac{r_m}{r^2} \frac{\pi}{2} \right)] For very large (r), the term proportional to (1/r^2) becomes negligible compared to the term proportional to (1/r). We are left with: [\frac{d\phi}{dr} \approx \frac{4\pi G C_m}{r}] This expression has precisely the form (\frac{\phi_0}{r}). Therefore, the two models are reconciled in the regime (r \gg r_m). By comparing the two expressions for the potential gradient, we can identify (\phi_0): [\phi_0 = 4\pi G C_m] The problem states that under this condition, the circular velocity (v_c(r)) no longer depends on (r). Let us verify this. The centripetal force required to keep a mass (m) in a circular orbit of radius (r) at velocity (v_c) is (F_c = \frac{m v_c^2}{r}). This must be provided by the gravitational force, (F_g = m \frac{d\phi}{dr}). [\frac{m v_c^2}{r} = m \frac{d\phi}{dr} \implies v_c^2 = r \frac{d\phi}{dr}] Substituting our result for the potential gradient in the (r \gg r_m) regime: [v_c^2 = r \left(\frac{\phi_0}{r} \right) = \phi_0] This is a remarkable conclusion. The orbital velocity becomes constant, independent of the radius, at large distances from the center. This matches the "flat rotation curves" observed in many galaxies. The velocity is given by: [v_c = \sqrt{\phi_0}]