googleai-gemini-2.5-pro

I am asked to determine the wavelength of a photon emitted from a specific transition within the hydrogen atom.

The problem describes a "hyperfine structure" transition, which releases a photon. The energy of any emitted particle of light, or photon, must be equal to the energy difference, (\Delta E), between the initial and final states of the atom. The relationship between a photon's energy and its wavelength, (\lambda), is one of the foundational principles of light, given by (E_{photon} = \frac{hc}{\lambda}), where (h) is Planck's constant and (c) is the speed of light. Therefore, if I can determine the energy of this hyperfine transition, (\Delta E_{HF}), I can find its wavelength.

The problem provides a path to find this energy. It states that the hyperfine energy difference, (\Delta E_{HF}), is related to the "fine structure" energy difference, (\Delta E_F), by the approximation (\Delta E_{HF} \approx 3.72 \frac{m_e}{m_p} \Delta E_F). My first task, then, is to determine (\Delta E_F).

The fine structure energy difference (\Delta E_F) is described as the energy difference between the electron's magnetic moment, (\vec{M}_s), being parallel versus anti-parallel to the magnetic field (\vec{B}_1) it experiences. The potential energy of a magnetic moment in a magnetic field is given by (U = -\vec{M}_s \cdot \vec{B}_1).

- When parallel, the energy is (U_{parallel} = -M_s B_1).
- When anti-parallel, the energy is (U_{anti-parallel} = +M_s B_1). The energy difference is thus (\Delta E_F = U_{anti-parallel} - U_{parallel} = 2M_s B_1).

To proceed, I need expressions for (M_s) and (B_1). The problem gives the magnitude of the electron's magnetic moment as (M_s \approx \frac{e}{m_e}\hbar). The magnetic field (\vec{B}_1) is produced by the proton orbiting the electron (in the electron's reference frame) at the Bohr radius, (r_1). A moving charge constitutes a current and generates a magnetic field. The magnitude of this field from a single charge (e) moving at velocity (v_1) in a circle of radius (r_1) is given by the law of Biot and Savart for a point charge: (B_1 = \frac{\mu_0}{4\pi} \frac{e v_1}{r_1^2}).

Combining these, I can write the fine structure energy difference as: [\Delta E_F = 2 \left(\frac{e\hbar}{m_e}\right) \left(\frac{\mu_0}{4\pi} \frac{e v_1}{r_1^2}\right) = \frac{2 e^2 \hbar \mu_0 v_1}{4\pi m_e r_1^2}] This expression must now be simplified and related to the quantities (\alpha) and the ground state energy (E_1), as requested in the first part of question A.5. From the preceding questions, I know that the velocity in the first orbit is (v_1 = \alpha c), and the quantization of angular momentum ((m_e v_1 r_1 = \hbar)) gives the first Bohr radius as (r_1 = \frac{\hbar}{m_e v_1} = \frac{\hbar}{m_e \alpha c}).

Substituting these into the expression for (\Delta E_F): [\Delta E_F = \frac{2 e^2 \hbar \mu_0 (\alpha c)}{4\pi m_e} \left(\frac{1}{r_1^2}\right) = \frac{2 e^2 \hbar \mu_0 \alpha c}{4\pi m_e} \left(\frac{m_e \alpha c}{\hbar}\right)^2 = \frac{2 e^2 \hbar \mu_0 \alpha c}{4\pi m_e} \frac{m_e^2 \alpha^2 c^2}{\hbar^2} = \frac{e^2 \mu_0 m_e \alpha^3 c^3}{2\pi \hbar}] This expression seems correct, but it is complex. I am asked to relate it to (\alpha) and (E_1). Let me try to relate the terms using the definition of (\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}) and the relation (c^2 = 1/(\epsilon_0 \mu_0)). From the definition of (\alpha), I have (e^2 = 4\pi\epsilon_0 \hbar c \alpha). Substituting this and (\mu_0 = 1/(\epsilon_0 c^2)): [\Delta E_F = \frac{(4\pi\epsilon_0 \hbar c \alpha) \left(\frac{1}{\epsilon_0 c^2}\right) m_e \alpha^3 c^3}{2\pi \hbar} = \frac{4\pi \hbar \alpha m_e \alpha^3 c^2}{2\pi \hbar} = 2 m_e c^2 \alpha^4] Now, I must relate this to (E_1). The ground state energy (E_1) is the sum of kinetic and potential energies. I have previously found it can be expressed as (E_1 = -\frac{1}{2} m_e v_1^2 = -\frac{1}{2} m_e (\alpha c)^2 = -\frac{1}{2} m_e c^2 \alpha^2). From this, I can see that (m_e c^2 \alpha^2 = -2E_1). Substituting this into my expression for (\Delta E_F): [\Delta E_F = 2 (m_e c^2 \alpha^2) \alpha^2 = 2(-2E_1)\alpha^2 = -4E_1\alpha^2] Since (E_1) is a negative energy for a bound state, (\Delta E_F) is positive, which is physically sensible for an energy splitting. This fulfills the first part of A.5.

Now I can find the hyperfine energy splitting: [\Delta E_{HF} \approx 3.72 \frac{m_e}{m_p} \Delta E_F = 3.72 \frac{m_e}{m_p} (-4E_1\alpha^2)] The energy of the emitted photon is (E_{photon} = \Delta E_{HF}). The wavelength is therefore: [\lambda_{HF} = \frac{hc}{E_{photon}} = \frac{hc}{\Delta E_{HF}} = \frac{hc}{3.72 \frac{m_e}{m_p} (-4E_1\alpha^2)}] This is the symbolic expression for the wavelength.

To find the numerical value, I will calculate the value of each term.

1. **Ground State Energy (E_1):** (E_1 = -\frac{1}{2} m_e c^2 \alpha^2). The rest energy of an electron is (m_e c^2 \approx 0.511 \times 10^6) eV. The constant (\alpha) is given as (\approx 7.27 \times 10^{-3}). [E_1 \approx -\frac{1}{2} (0.511 \times 10^6 \text{ eV}) (7.27 \times 10^{-3})^2 \approx -13.5 \text{ eV}]

2. **Hyperfine Energy Splitting (\Delta E_{HF}):** The mass ratio is (\frac{m_e}{m_p} \approx \frac{9.11 \times 10^{-31} \text{ kg}}{1.67 \times 10^{-27} \text{ kg}} \approx 5.45 \times 10^{-4}). [\Delta E_{HF} \approx 3.72 (5.45 \times 10^{-4}) (-4) (-13.5 \text{ eV}) (7.27 \times 10^{-3})^2] [\Delta E_{HF} \approx (2.027 \times 10^{-3}) (54 \text{ eV}) (5.285 \times 10^{-5}) \approx 5.78 \times 10^{-6} \text{ eV}]
3. **Wavelength (\lambda_{HF}):** Using the convenient relation (hc \approx 1240) eV·nm: [\lambda_{HF} = \frac{hc}{\Delta E_{HF}} \approx \frac{1240 \text{ eV}\cdot\text{nm}}{5.78 \times 10^{-6} \text{ eV}} \approx 2.145 \times 10^8 \text{ nm}] Converting this to meters: (2.145 \times 10^8 \text{ nm} = 0.2145) m.

The question asks for the numerical value with two significant digits.

Final Answer: The expression for the wavelength is (\lambda_{HF} = \frac{hc}{3.72 \frac{m_e}{m_p} (-4E_1\alpha^2)}). The numerical value is (\lambda_{HF} \approx 0.21 \text{ m}), or 21 cm.