

To determine the relative radial velocity of a galactic object with respect to the Sun, we must first construct a model of their motions. We consider three points in the galactic plane: the galactic center C, the Sun S, and the emitting object E.

Based on the problem's premise, both the Sun and the object E travel in circular orbits around the center C. The Sun's orbit has a radius R_{\odot} and a tangential velocity \vec{v}_{\odot} . The object E has an orbital radius R and a tangential velocity \vec{v}_E . *A key piece of information from our prior investigation (Part C.2) is that for objects outside the galactic bulge, the orbital velocity is constant, independent of the radius. Therefore, the speeds are equal: $|\vec{v}_E| = |\vec{v}_{\odot}| = v_{\odot}$.*

The quantity we seek is the relative radial velocity, $v_{rE/S}$, which is the projection of the relative velocity vector, $\vec{v}_E - \vec{v}_{\odot}$, onto the line of sight from the Sun to the object, represented by the unit vector \hat{u}_{SE} . Mathematically, this is expressed as: $[v_{rE/S} = (\vec{v}_E - \vec{v}_{\odot}) \cdot \hat{u}_{SE} = (\vec{v}_E \cdot \hat{u}_{SE}) - (\vec{v}_{\odot} \cdot \hat{u}_{SE})]$ This means we need to find the component of each object's velocity along the line of sight SE.

Let us analyze the geometry of the situation, which is captured by the triangle SCE. The angle at the Sun, between the line to the galactic center (SC) and the line of sight (SE), is the galactic longitude, l . Let the angle at the object E, between the line CE and the line SE, be α .

The velocity vector \vec{v}_{\odot} is tangential to the Sun's orbit, and thus perpendicular to the radius vector \vec{CS} . The angle between the line of sight SE and the radius \vec{CS} is l . Therefore, the angle between the velocity \vec{v}_{\odot} and the line of sight SE is $90^\circ - l$. The projection of \vec{v}_{\odot} onto the line of sight is: $[\vec{v}_{\odot} \cdot \hat{u}_{SE} = v_{\odot} \cos(90^\circ - l) = v_{\odot} \sin(l)]$

Similarly, the velocity vector \vec{v}_E is tangential to its orbit, and thus perpendicular to its radius vector \vec{CE} . The angle between the line of sight SE and the radius \vec{CE} is α . The angle between the velocity \vec{v}_E and the line of sight is $90^\circ - \alpha$. Its projection is: $[\vec{v}_E \cdot \hat{u}_{SE} = v_E \cos(90^\circ - \alpha) = v_E \sin(\alpha)]$ Since we established $v_E = v_{\odot}$, this becomes $v_{\odot} \sin(\alpha)$.

Substituting these components back into our expression for the relative radial velocity gives: $[v_{rE/S} = v_{\odot} \sin(\alpha) - v_{\odot} \sin(l)]$ To complete this, we must express $\sin(\alpha)$ in terms of the given parameters. We can do this by applying the Law of Sines to the triangle SCE: $[\frac{CE}{\sin(l)} = \frac{CS}{\sin(\alpha)}]$ Substituting the orbital radii, $CE = R$ and $CS = R_{\odot}$: $[\frac{R}{\sin(l)} = \frac{R_{\odot}}{\sin(\alpha)}]$ implies $\sin(\alpha) = \frac{R_{\odot}}{R} \sin(l)$

Now we can replace $\sin(\alpha)$ in our velocity equation: $[v_{rE/S} = v_{\odot} \left(\frac{R_{\odot}}{R} \sin(l) \right) - v_{\odot} \sin(l)]$ Factoring out the common terms yields the final expression for the relative radial velocity: $[v_{rE/S} = v_{\odot} \sin(l) \left(\frac{R_{\odot}}{R} - 1 \right)]$

The second part of the inquiry is to find the object's orbital radius R based on the observable quantities. This requires a simple algebraic rearrangement of the relationship we have just discovered. Starting from our result: $[\frac{v_{rE/S}}{v_{\odot} \sin(l)} = \frac{R_{\odot}}{R} - 1]$ We isolate the term containing R : $[\frac{R_{\odot}}{R} = 1 + \frac{v_{rE/S}}{v_{\odot} \sin(l)}]$ Combining the terms on the right-hand side gives: $[\frac{R_{\odot}}{R} = \frac{v_{\odot} \sin(l) + v_{rE/S}}{v_{\odot} \sin(l)}]$ Finally, inverting the expression to solve for R , we find: $[R = R_{\odot} \left(\frac{v_{\odot} \sin(l)}{v_{\odot} \sin(l) + v_{rE/S}} \right)]$ This remarkable result provides a method to determine the distance of an object from the galactic center using only measurements of its Doppler shift and its position in the sky, a powerful tool for mapping the structure of our own galaxy from within.