googleai-gemini-2.5-pro

An analysis of the hydrogen atom, as described, requires us to consider the motion of an electron under the influence of a central electrical force from a fixed proton. The system is analogous to a small planet orbiting a massive, stationary star. The primary force governing this interaction is the electrostatic attraction.

A.1: Electron Velocity in a Circular Orbit

To maintain a stable circular orbit of radius (r), the electron must be subject to a constant inward force that provides the necessary centripetal acceleration. This force is the Coulomb attraction between the proton (charge (+e)) and the electron (charge (-e)).

The magnitude of the electrostatic force is given by Coulomb's law: [F_e = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}] For a circular orbit, this force must be equal to the centripetal force required to keep the electron, with mass (m_e), moving at a velocity (v) in a circle of radius (r). The centripetal force is given by Newton's second law: [F_c = m_e a_c = \frac{m_e v^2}{r}] By equating these two forces, we establish the condition for a stable orbit: [\frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} = \frac{m_e v^2}{r}] We can simplify this equation by canceling a factor of (r) on both sides. This leaves us with an expression for (m_e v^2): [m_e v^2 = \frac{e^2}{4\pi\epsilon_0 r}] Solving for the velocity (v), we find: [v = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r}}] This result shows how the electron's speed is fundamentally linked to the strength of the electric interaction and the radius of its orbit.

A.2: Quantized Orbits and the Bohr Radius

A new principle is introduced: the angular momentum of the electron is not continuous but is restricted to discrete, or "quantized," values. The angular momentum (L) for a particle in a circular orbit is the product of its radius and its linear momentum ((p = m_e v)), so (L = r m_e v). This new rule states: [L = r_n m_e v_n = n\hbar] where (n) is a positive integer ((n=1, 2, 3, \dots)) and (v_n) and (r_n) are the velocity and radius corresponding to that integer. From this, we can express the velocity (v_n) as: [v_n = \frac{n\hbar}{m_e r_n}] We now have two distinct expressions for the velocity. By equating our result from A.1 with this new quantum condition, we can determine the allowed radii. [\sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r_n}} = \frac{n\hbar}{m_e r_n}] To solve for (r_n), we first square both sides of the equation: [\frac{e^2}{4\pi\epsilon_0 m_e r_n} = \frac{n^2\hbar^2}{m_e^2 r_n^2}] We can simplify by canceling (m_e) and (r_n) from the denominators: [\frac{e^2}{4\pi\epsilon_0} = \frac{n^2\hbar^2}{m_e r_n}] Rearranging to solve for the radius (r_n), we find: [r_n = n^2 \left(\frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} \right)] This result is remarkable. It shows that the electron cannot orbit at any arbitrary radius, but only at specific distances that scale with the square of the integer (n). This confirms the structure (r_n = n^2 r_1), where the fundamental radius (r_1), called the Bohr radius, is: [r_1 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}] To express this in terms of the fine-structure constant, (\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}), we can rearrange the expression for (r_1) to isolate the term (\frac{e^2}{4\pi\epsilon_0}): [r_1 = \frac{\hbar^2}{m_e} \left(\frac{4\pi\epsilon_0}{e^2} \right)] From the definition of (\alpha), we have (\frac{e^2}{4\pi\epsilon_0} = \alpha \hbar c). Therefore, (\frac{4\pi\epsilon_0}{e^2} = \frac{1}{\alpha \hbar c}). Substituting this into our equation for (r_1): [r_1 = \frac{\hbar^2}{m_e} \left(\frac{1}{\alpha \hbar c} \right) = \frac{\hbar}{m_e c \alpha}] Now, we calculate the numerical value of (r_1): Given: (\hbar \approx 1.054 \times 10^{-34} \text{ J s}), (m_e \approx 9.11 \times 10^{-31} \text{ kg}), (c \approx 3.00 \times 10^8 \text{ m/s}), and (\alpha \approx 7.27 \times 10^{-3}). [r_1 = \frac{1.054 \times 10^{-34}}{(9.11 \times 10^{-31})(3.00 \times 10^8)(7.27 \times 10^{-3})} \approx \frac{1.054 \times 10^{-34}}{1.988 \times 10^{-24}} \approx 0.530 \times 10^{-10} \text{ m}] The Bohr radius is (r_1 = 5.30 \times 10^{-11} \text{ m}).

Next, we find the velocity on the first orbit, (v_1), in terms of (\alpha) and (c). We use the quantization condition for (n=1): (v_1 = \frac{\hbar}{m_e r_1}). Substituting our new expression for (r_1): [v_1 = \frac{\hbar}{m_e \left(\frac{\hbar}{m_e c \alpha} \right)} = \frac{\hbar m_e c \alpha}{m_e \hbar}] The terms (m_e) and (\hbar) cancel, leaving a strikingly simple result: [v_1 = \alpha c]

A.3: Mechanical Energy of the Electron

The total mechanical energy (E) of the electron is the sum of its kinetic energy (K) and its potential energy (U). The kinetic energy is (K = \frac{1}{2}m_e v^2). The electrostatic potential energy of a charge (-e) at a distance (r) from a charge (+e) is (U = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r}). So, the total energy is: [E = K + U = \frac{1}{2}m_e v^2 - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r}] From our force balance in A.1, we found a relationship: (m_e v^2 = \frac{e^2}{4\pi\epsilon_0 r}). We can substitute this into the kinetic energy term: [K = \frac{1}{2} \left(\frac{e^2}{4\pi\epsilon_0 r} \right)] Notice that (K = -\frac{1}{2}U). The total energy becomes: [E = \frac{1}{2} \left(\frac{e^2}{4\pi\epsilon_0 r} \right) - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r}]

Since the radius r is quantized as r_n, the energy must also be quantized. The energy of the n-th level, E_n, is:

$$E_n = -\frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r_n}$$

This is the electron's mechanical energy in terms of e, ϵ_0, and r_n.

Now, we determine the ground state energy E_1 (for $n=1$) in terms of α, m_e, and c.

$$E_1 = -\frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r_1}$$

We can also express E_1 as the negative of the kinetic energy, $E_1 = -K_1 = -\frac{1}{2}m_e v_1^2$. This seems simpler. Using our result $v_1 = \alpha c$:

$$E_1 = -\frac{1}{2} m_e (\alpha c)^2 = -\frac{1}{2} m_e c^2 \alpha^2$$

This expression elegantly connects the ground state energy to the electron's rest mass energy ($m_e c^2$) and the fine-structure constant.

Finally, we compute the numerical value of E_1 in electron-volts (eV). The rest energy of an electron is $m_e c^2 \approx (9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 \approx 8.199 \times 10^{-14} \text{ J}$.

$$E_1 = -\frac{1}{2} (8.199 \times 10^{-14} \text{ J}) (7.27 \times 10^{-3})^2$$

$$E_1 \approx -\frac{1}{2} (8.199 \times 10^{-14}) (5.285 \times 10^{-5}) \approx -2.166 \times 10^{-18} \text{ J}$$

To convert from Joules to eV, we use the conversion $1 \text{ eV} \approx 1.602 \times 10^{-19} \text{ J}$.

$$E_1 (\text{in eV}) = \frac{-2.166 \times 10^{-18} \text{ J}}{1.602 \times 10^{-19} \text{ J/eV}} \approx -13.52 \text{ eV}$$

The ground state energy is approximately -13.5 eV.