assistant

An intriguing proposition. A modification to the fundamental law of motion, where the relationship between force and acceleration is not linear, especially in regimes of very low acceleration. Let us investigate the consequences of this "MOdified Newtonian Dynamics" (MOND) on the motion of a body orbiting a large, homogeneously distributed mass.

The physical system is a spherical mass distribution of total mass M and radius R_b, with a constant density within this radius and zero density outside. A test mass m is in a circular orbit of radius r with velocity $v_c(r)$. The acceleration is purely centripetal, with magnitude $a = v_c^2 / r$.

The proposed law of motion is $\vec{F} = m\mu(a/a_0)\vec{a}$, where the gravitational force $\vec{F}$ is still given by the standard theory of gravitation. The function $\mu(x) = x / (1+x)$ dictates the behavior. We must analyze two distinct spatial regions: inside the mass distribution ($r \le R_b$) and outside of it ($r > R_b$).

Region 1: Inside the Sphere ($r \le R_b$)

First, we must determine the gravitational force acting on the test mass m. For a spherically symmetric body with uniform density, the gravitational force at a radius r inside the sphere is generated only by the mass enclosed within that radius, which we can call $M(r)$.

The density is $\rho = M / (\frac{4}{3}\pi R_b^3)$. The mass enclosed within radius r is $M(r) = \rho \cdot (\frac{4}{3}\pi r^3) = M \frac{r^3}{R_b^3}$.

The gravitational force is therefore: $$F_g = \frac{G M(r) m}{r^2} = \frac{G \left(M \frac{r^3}{R_b^3}\right) m}{r^2} = \frac{GMm}{R_b^3} r$$ The gravitational force increases linearly with the distance from the center.

Now, we equate this gravitational force with the MOND expression for force, $F = m\mu(a/a_0)a$. Substituting $a = v_c^2/r$: $$\frac{GMm}{R_b^3} r = m \mu\left(\frac{v_c^2}{ra_0}\right) \frac{v_c^2}{r}$$ $$\frac{GM}{R_b^3} r^2 = \mu\left(\frac{v_c^2}{ra_0}\right) v_c^2$$

We must now consider the two limiting cases for the acceleration.

Case 1a: High Acceleration ($a \gg a_0$) In this regime, the argument $x = a/a_0$ is large, so $\mu(x) = \frac{x}{1+x} \approx 1$. The law of motion effectively reverts to Newton's second law, $F=ma$. Our equation becomes: $$\frac{GM}{R_b^3} r^2 \approx (1) \cdot v_c^2$$ $$v_c(r) = \sqrt{\frac{GM}{R_b^3}} r$$ In the high-acceleration regime inside the sphere, the orbital velocity increases linearly with radius, which is the standard Newtonian result.

Case 1b: Low Acceleration ($a \ll a_0$) In this regime, the argument $x = a/a_0$ is small, so $\mu(x) = \frac{x}{1+x} \approx x$. The MOND force becomes $F \approx m (a/a_0) a = m a^2/a_0$. Our equation becomes: $$\frac{GM}{R_b^3} r^2 \approx \left(\frac{v_c^2}{ra_0}\right) v_c^2 = \frac{v_c^4}{ra_0}$$ Solving for v_c: $$v_c^4 = a_0 \frac{GM}{R_b^3} r^3$$ $$v_c(r) = \left(\frac{GMa_0}{R_b^3}\right)^{1/4} r^{3/4}$$ In the deep MOND regime inside the sphere, the orbital velocity increases with radius as $r^{3/4}$.

Region 2: Outside the Sphere ($r > R_b$)

Outside a spherically symmetric mass, the gravitational force is as if the entire mass M were concentrated at the center. $$F_g = \frac{GMm}{r^2}$$ Equating this with the MOND force: $$\frac{GMm}{r^2} = m \mu\left(\frac{v_c^2}{ra_0}\right) \frac{v_c^2}{r}$$ $$\frac{GM}{r} = \mu\left(\frac{v_c^2}{ra_0}\right) v_c^2$$

Again, we analyze the two acceleration regimes.

Case 2a: High Acceleration ($a \gg a_0$) Here, $\mu(x) \approx 1$. $$\frac{GM}{r} \approx (1) \cdot v_c^2$$ $$v_c(r) = \sqrt{\frac{GM}{r}}$$ This is the standard Keplerian/Newtonian result for a point mass, where velocity falls off as $1/\sqrt{r}$.

Case 2b: Low Acceleration ($a \ll a_0$) Here, $\mu(x) \approx x$. $$\frac{GM}{r} \approx \left(\frac{v_c^2}{ra_0}\right) v_c^2 = \frac{v_c^4}{ra_0}$$ $$v_c^4 = GMa_0$$ $$v_c(r) = (GMa_0)^{1/4}$$ In the low-acceleration regime far from the mass, the orbital velocity becomes constant, independent of radius. This yields the "flat rotation curve" that is observed in galaxies and is consistent with the Tully-Fischer relation.

Summary of Results

We have determined the behavior of the circular velocity ($v_c(r)$) in four distinct regimes for a body orbiting a homogeneous spherical mass under MOND.

- **For ($r \le R_b$) (Inside the mass):**
  - If acceleration is high (($a \gg a_0$)), we recover the Newtonian result: $$v_c(r) \propto r$$
  - If acceleration is low (($a \ll a_0$)), MOND predicts: $$v_c(r) \propto r^{3/4}$$
- **For ($r > R_b$) (Outside the mass):**
  - If acceleration is high (($a \gg a_0$)), we recover the Keplerian result: $$v_c(r) \propto \frac{1}{\sqrt{r}}$$
  - If acceleration is low (($a \ll a_0$)), MOND predicts a flat rotation curve: $$v_c(r) = (GMa_0)^{1/4} = \text{constant}$$

A complete rotation curve for a galaxy under this model would likely exhibit a combination of these behaviors. Close to the dense center, acceleration would be high, and the velocity would rise linearly (Newtonian). As radius increases, acceleration drops. The velocity curve would transition from the Newtonian (r) dependence towards the MOND ($r^{3/4}$) dependence. After passing the main mass distribution at (R_b), the velocity would begin to decrease, but as acceleration continues to fall into the deep MOND regime (($a \ll a_0$)), the curve would flatten out to the constant velocity predicted by ($v_c = (GMa_0)^{1/4}$), rather than continuing to drop towards zero as in the purely Newtonian case. This theoretical framework provides a mechanism for producing the observed flat rotation curves without invoking unseen matter.