googleai-gemini-2.5-pro

Here is the solution for question C.1.

The motion of a point mass m in a potential is governed by the force, which can be found from the gradient of the potential energy (U = m\phi_c). We are interested in the motion along the z-axis, so we need to find the force component (F_z). This force is related to the potential by: [F_z = -\frac{\partial U}{\partial z} = -m \frac{\partial \phi_c}{\partial z}]

The given potential is (\phi_c(r, z) = \phi_0 \ln\left(\frac{r}{r_0}\right) \exp\left(-\frac{z^2}{z_0^2}\right)). Since we are considering motion where r is constant, the term (\phi_0 \ln(r/r_0)) acts as a constant pre-factor. Let's compute the partial derivative with respect to z: [\frac{\partial \phi_c}{\partial z} = \phi_0 \ln\left(\frac{r}{r_0}\right) \frac{\partial}{\partial z} \left[\exp\left(-\frac{z^2}{z_0^2}\right) \right]] Using the chain rule, we get: [\frac{\partial \phi_c}{\partial z} = \phi_0 \ln\left(\frac{r}{r_0}\right) \exp\left(-\frac{z^2}{z_0^2}\right) \left(-\frac{2z}{z_0^2} \right) = -\frac{2z}{z_0^2} \phi_0 \ln\left(\frac{r}{r_0}\right) \exp\left(-\frac{z^2}{z_0^2}\right)]

Now, we can write the force (F_z): [F_z = -m \left(-\frac{2z}{z_0^2} \phi_0 \ln\left(\frac{r}{r_0}\right) \exp\left(-\frac{z^2}{z_0^2}\right) \right) = \frac{2mz}{z_0^2} \phi_0 \ln\left(\frac{r}{r_0}\right) \exp\left(-\frac{z^2}{z_0^2}\right)] Applying Newton's second law, (F_z = m\ddot{z}), we arrive at the equation of motion: [m\ddot{z} = \frac{2mz}{z_0^2} \phi_0 \ln\left(\frac{r}{r_0}\right) \exp\left(-\frac{z^2}{z_0^2}\right)] Dividing by m, the equation of motion for the vertical displacement z is: [\ddot{z} - \left(\frac{2 \phi_0}{z_0^2} \ln\left(\frac{r}{r_0}\right) \right) z \exp\left(-\frac{z^2}{z_0^2}\right) = 0]

To investigate the stability of the galactic plane ((z=0)) and find the frequency of small oscillations, we examine the behavior of the force for small z (i.e., (|z| \ll z_0)). In this limit, we can use the Taylor approximation (\exp(x) \approx 1) for small (x). Here, (x = -z^2/z_0^2), so (\exp(-z^2/z_0^2) \approx 1).

The equation of motion simplifies to a linear differential equation: [\ddot{z} - \left(\frac{2 \phi_0}{z_0^2} \ln\left(\frac{r}{r_0}\right) \right) z \approx 0] This equation is of the form (\ddot{z} + \omega_0^2 z = 0), which describes simple harmonic motion, provided that the coefficient of z is negative. This requires us to define (\omega_0^2) as: [\omega_0^2 = - \frac{2 \phi_0}{z_0^2} \ln\left(\frac{r}{r_0}\right)] For the equilibrium at (z=0) to be stable, the motion must be oscillatory, which means (\omega_0^2) must be positive. The problem states (\phi_0 > 0) and (z_0^2 > 0). Therefore, the sign of (\omega_0^2) depends on the term (-\ln(r/r_0)).

The condition for stability ((\omega_0^2 > 0)) is (-\ln(r/r_0) > 0), which implies (\ln(r/r_0) < 0). This is true if and only if (r/r_0 < 1), or (r < r_0). This confirms that for (r < r_0), the galactic plane (z=0) is a stable equilibrium position.

The angular frequency of these small oscillations is found by taking the square root: [\omega_0 = \sqrt{-\frac{2 \phi_0}{z_0^2} \ln\left(\frac{r}{r_0}\right)}]