googleai-gemini-2.5-pro

To determine the velocity of an object in a circular orbit within this galaxy, we must begin from the first principle governing such motion: the gravitational force provides the necessary centripetal force.

Let us consider a small test mass (m) moving in a circular orbit of radius (r) around the galactic center. For this orbit to be stable, the gravitational force (F_g) pulling the object toward the center must be equal to the centripetal force (F_c) required to keep it in its circular path.

A key insight, derived from the law of universal gravitation for a spherically symmetric body, is that the gravitational force on our test mass at radius (r) is determined solely by the mass enclosed within that radius. We shall call this enclosed mass ($M_m(r)$). The mass distributed in shells outside the radius (r) exerts no net force on the object.

This allows us to write the force balance as: $$F_g = F_c$$ $$\frac{G M_m(r) m}{r^2} = \frac{m v_{c,m}(r)^2}{r}$$

By canceling the test mass (m) and simplifying, we can express the orbital velocity we seek, ($v_{c,m}(r)$), in terms of the enclosed mass: $$v_{c,m}(r) = \sqrt{\frac{G M_m(r)}{r}}$$

Our primary task now shifts to calculating this enclosed mass, ($M_m(r)$). The problem provides the volumetric mass density as a function of radius, given by Equation (1): $$\rho_m(x) = \frac{C_m}{r_m^2 + x^2}$$ where (x) is the radial distance from the center.

To find the total mass within a sphere of radius (r), we must integrate this density over the volume of the sphere. We can do this by summing the mass of infinitesimally thin spherical shells, each at a radius (x) with a thickness (dx). The volume of such a shell is ($dV = 4\pi x^2 dx$).

The mass (dM) of one shell is therefore: $$dM = \rho_m(x) , dV = \left(\frac{C_m}{r_m^2 + x^2}\right) (4\pi x^2 dx)$$

Integrating this expression from the center ($(x=0)$) out to the radius of the orbit ($(x=r)$) yields the total enclosed mass ($M_m(r)$): $$M_m(r) = \int_0^r dM = \int_0^r 4\pi C_m \frac{x^2}{r_m^2 + x^2} dx$$

We can factor out the constants: $$M_m(r) = 4\pi C_m \int_0^r \frac{x^2}{r_m^2 + x^2} dx$$

The problem provides a helpful mathematical tool to solve this specific integral: ($\int_0^r \frac{x^2}{a^2 + x^2} dx = r - a \arctan(r/a)$). By identifying ($a = r_m$), we can directly evaluate our integral: $$\int_0^r \frac{x^2}{r_m^2 + x^2} dx = r - r_m \arctan\left(\frac{r}{r_m}\right)$$

Substituting this result back into our expression for the enclosed mass gives: $$M_m(r) = 4\pi C_m \left[r - r_m \arctan\left(\frac{r}{r_m}\right) \right]$$

Finally, we place this complete expression for the enclosed mass back into our equation for the orbital velocity: $$v_{c,m}(r) = \sqrt{\frac{G}{r} \left(4\pi C_m \left[r - r_m \arctan\left(\frac{r}{r_m}\right) \right] \right)}$$

Rearranging this to better reveal its structure, we get: $$v_{c,m}(r) = \sqrt{(4\pi G C_m) \left(\frac{r - r_m \arctan(r/r_m)}{r} \right)}$$

This is the velocity profile corresponding to the given mass density. The problem asks to show it can be written in the form ($v_{c,m}(r) = \sqrt{k_1 \frac{r - k_2 \arctan(r/k_2)}{r}}$). (Note: Physical reasoning requires the term in the numerator to be positive, which confirms this structure, as ($x > \arctan(x)$) for ($x>0$)).

By comparing our derived expression to the target form, we can identify the constants (k_1) and (k_2) by direct inspection: $$k_1 = 4\pi G C_m$$ $$k_2 = r_m$$

Thus, we have shown that the velocity profile takes the required form and have expressed the constants (k_1) and (k_2) in terms of the fundamental parameters (C_m), (r_m), and (G).