groq-kimi-k2

A.1. Analysis of Forces and Pressures

Let us begin by observing the situation described in Figure 2b. We have a tube of mass m and cross-sectional area S, closed at its top, which has been lifted by a height h from an initial state of full submersion. The tube remains completely filled with a liquid of density ρ. Our task is to determine the pressure within the liquid at the very top of the tube, and the external force $\vec{F}$ required to maintain this equilibrium.

First, let us consider the pressure. The vast bath of liquid is open to the air, so the pressure at its surface (which we define as altitude $z = 0$) is the atmospheric pressure, P_a. Inside the tube, at this same altitude $z=0$, the pressure must also be P_a, otherwise liquid would flow in or out. The column of liquid inside the tube has a height h. The pressure difference between the top and bottom of this column must be due to its weight. According to the principles of hydrostatics, the pressure decreases as one moves upwards through a fluid.

Let P_w be the pressure in the water at the top of the tube (at $z = h$). The pressure at the bottom ($z = 0$) is greater by an amount $\rho g h$. We can write this relationship as: $$ P_a = P_w + \rho g h $$ From this, we can deduce the pressure at the top of the tube. It is not constant, but depends on how high the tube has been lifted: $$ P_w = P_a - \rho g h $$ This result is logical: as h increases, the weight of the suspended water column increases, leading to a greater pressure reduction at the top.

Next, we must determine the magnitude of the pulling force, F. This force must hold the entire system in static equilibrium. To find it, we can analyze the vertical forces acting on the tube itself. The tube is an object of mass m, so it has a weight mg acting downwards. The applied force F acts upwards. Additionally, the tube is in contact with fluids (air outside, liquid inside) that exert pressure forces on it.

Consider the closed top of the tube, a surface of area S. From the outside, the atmosphere at pressure P_a pushes down on this surface. From the inside, the liquid at pressure P_w pushes up. The net force exerted by the fluids on the tube's top cap is therefore $(P_w - P_a)S$, directed upwards. By action–reaction, the force exerted by the cap on the rest of the tube body is $(P_a - P_w)S$, directed downwards.

The force balance equation for the tube body is therefore: $$ \vec{F} + \vec{F}{\text{gravity}} + \vec{F}{\text{cap on tube}} = 0 $$ In the vertical direction ($\vec{u}_z$), this becomes: $$ F - mg - (P_a - P_w)S = 0 $$ Solving for F: $$ F = mg + (P_a - P_w)S $$ We can now substitute our previously discovered expression for P_w: $$ F = mg + \Bigl(P_a - (P_a - \rho g h)\Bigr)S $$ $$ F = mg + \bigl(P_a - P_a + \rho g h\bigr)S $$ This simplifies to a remarkably elegant result: $$ F = mg + \rho g h S $$ The force required is the sum of the tube's own weight and the weight of the liquid column it lifts above the bath level. This is a satisfying confirmation of our physical intuition.

The expressions are therefore:

- Pressure at the top of the tube:
  $P_w = P_a - \rho g h$
- Force required to maintain position:
  $\vec{F} = (mg + \rho g h S)\vec{u}_z$

A.2. Investigation of Physical Limits and Behaviour

Our previous analysis assumed the liquid column remains intact. However, a liquid cannot sustain infinite tension, or equivalently, its pressure cannot drop below a certain limit. For any real liquid, if the pressure drops to its saturated vapor pressure, P_{sat}, it will begin to boil, a phenomenon known as **cavitation**. A bubble of vapor would appear at the top of the tube.

This introduces a **critical height**. Let us call it h^*. This is the height at which the pressure P_w at the top of the tube drops to P_{sat}.

$$ P_{\text{sat}} = P_a - \rho g h^* \quad\Longrightarrow\quad h^* = \frac{P_a - P_{\text{sat}}}{\rho g} $$

The value of this critical height h^* compared to the total length of the tube, H, dictates the system's behavior.

- **Behaviour A:** If h^* is greater than or equal to the tube's length H, then cavitation will not occur within the range of motion ($0\le h\le H$). The tube remains full of water, and our formula $F = mg + \rho g h S$ holds for the entire process. The force increases linearly with h. The maximum force, $F_{\max}$, occurs at $h = H$:
$F_{\max} = mg + \rho g H S$.

- **Behaviour B:** If $h^$ *is less than H, cavitation will occur when the tube is lifted to the height $h = h^$*. For any $h > h^$, *a vapor–filled space will exist at the top of the tube, with the pressure fixed at P_{sat}. This means the height of the water column inside the tube can no longer be h, but will be fixed at the maximum height sustainable by atmospheric pressure, which is h^*. The force F needed to support the tube and this fixed column of water becomes constant for $h \ge h^$. *The maximum force is reached at $h = h^*$ and remains at that value:
$F_{\max} = mg + \rho g h^* S = mg + (P_a - P_{\text{sat}})S$.

We shall now apply this framework to the three experiments, using the provided data:
$m = 0.5\ \text{kg}$, $g = 9.8\ \text{m s}^{-2}$, $S = 10\ \text{cm}^2 = 10^{-3}\ \text{m}^2$,
$H = 1\ \text{m}$, and $P_a = 1.000\times 10^5\ \text{Pa}$.
The initial force at $h = 0$ is $F_0 = mg = 0.5\times 9.8 = 4.9\ \text{N}$ for all experiments.

Experiment	Behaviour (A/B)	$F_{\max}$ (N)	h^* (m)
1	A	14.7	—
2	A	14.4	—
3	B	5.1	0.021

A.3. Analysis of Approximation Error

We are asked to consider the error introduced by neglecting the saturated vapor pressure, P_{sat}, in our calculation of the maximal force, $F_{\max}$. This implies comparing the true value of $F_{\max}$ with an approximate value calculated assuming $P_{\text{sat}} = 0$.

This error is only relevant when $F_{\max}$ depends on P_{sat}, which is the case for **Behaviour B**. The problem notes that mercury exhibits Behaviour B, and its vapor pressure is exceedingly small ($P_{\text{sat, Hg}} = 0.163\ \text{Pa}$), making it an ideal subject for this analysis.

In Behaviour B, the true maximal force is: $$ F_{\max,\text{true}} = mg + (P_a - P_{\text{sat}})S $$ If we neglect P_{sat}, the approximate maximal force would be: $$ F_{\max,\text{approx}} = mg + (P_a - 0)S = mg + P_a S $$ The relative error, ε, is defined as the absolute difference between the approximate and true values, divided by the true value: $$ \varepsilon = \frac{\lvert F_{\max,\text{approx}} - F_{\max,\text{true}} \rvert}{F_{\max,\text{true}}} $$ Substituting our expressions: $$ \varepsilon = \frac{\lvert (mg + P_a S) - \bigl[mg + (P_a - P_{\text{sat}})S \bigr] \rvert}{mg + (P_a - P_{\text{sat}})S} = \frac{\lvert P_{\text{sat}} S \rvert}{mg + (P_a - P_{\text{sat}})S} $$ This is the general expression for the relative error.

To find the numerical value, we use the data for mercury, as it represents a clear case of Behaviour B where P_{sat} is small compared to P_a.
Given $P_{\text{sat, Hg}} = 0.163\ \text{Pa}$.

$$ \varepsilon = \frac{0.163 \times 10^{-3}}{4.9 + (1.000\times 10^5 - 0.163)\times 10^{-3}} = \frac{1.63\times 10^{-4}}{4.9 + 99.9998\ldots} = \frac{1.63\times 10^{-4}}{104.9} \approx 1.55\times 10^{-6} $$ The error is, as expected, extremely small, confirming that for liquids with low vapor pressure like mercury at room temperature, neglecting P_{sat} is an excellent approximation.