Score function linear form

$$U(\theta;\underline{X}) = A(\theta)(T(\underline{X}) - m(\theta))$$

$$\Rightarrow \frac{\partial \ell(\theta|\underline{X})}{\partial\theta} = A(\theta)(T(\underline{X}) - m(\theta))$$

Take the derivative of the above with respect to θ.

$$\Rightarrow \frac{\partial^2 \ell(\theta|\underline{X})}{\partial\theta^2} = A'(\theta)(T(\underline{X}) - m(\theta)) - A(\theta)m'(\theta)$$

$$\Rightarrow -\frac{\partial^2 \ell(\theta|\underline{X})}{\partial\theta^2} = -A'(\theta)(T(\underline{X}) - m(\theta)) + A(\theta)m'(\theta)$$

Taking expectations of both sides of the above

$$\mathbb{E}\left[-\frac{\partial^2 \ell(\theta|\underline{X})}{\partial\theta^2}\right] = -A'(\theta)\,\mathbb{E}[\underbrace{T(\underline{X}) - m(\theta)}_{0}] + A(\theta)m'(\theta)$$

$$\Rightarrow \mathcal{I}(\theta) = A(\theta)m'(\theta)$$

$$\Rightarrow A(\theta) = \frac{\mathcal{I}(\theta)}{m'(\theta)}$$

Thus : $$U(\theta;\underline{X}) = \frac{\mathcal{I}(\theta)}{m'(\theta)}(T(\underline{X}) - m(\theta))$$

$$\ulcorner \mathrm{CRLB} = \frac{[m'(\theta)]^2}{\mathcal{I}(\theta)} \lrcorner$$

MVBUE: Example 1

$\ulcorner U(\theta; x) = A(\theta)[T(\underline{x}) - m(\theta)]$
$= A^*(\theta)(T(\underline{x}) - \theta) \lrcorner$

$X_1, \ldots, X_n \quad X_i \sim \text{Bern}(\theta)$

$$\mathbb{P}(X_i = x_i) = \theta^{x_i}(1-\theta)^{1-x_i} \qquad x_i \in \{0, 1\}.$$

The likelihood function is

$$L(\theta|\underline{x}) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$
$$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

The log-likelihood is

$$\ell(\theta|\underline{x}) = \left(\sum_{i=1}^{n} x_i\right)\log\theta + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-\theta)$$

Then the score function

$$\frac{\partial \ell(\theta|\underline{x})}{\partial\theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{\left(n - \sum_{i=1}^{n} x_i\right)}{1-\theta}$$

$$= \frac{(1-\cancel{\theta})\sum_{i=1}^{n} x_i - \theta\left(n - \cancel{\sum_{i=1}^{n} x_i}\right)}{\theta(1-\theta)}$$

$$= \frac{1}{\theta(1-\theta)}\left(\sum_{i=1}^{n} x_i - n\theta\right)$$

$$= \frac{n}{\theta(1-\theta)}\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \theta\right)$$

$$= \frac{n}{\theta(1-\theta)}(\bar{x} - \theta)$$

Writing the score function in linear form, we obtain

$$U(\theta; \underline{X}) = \frac{n}{\theta(1-\theta)} (\bar{X} - \theta)$$

$$= A(\theta)(T(\underline{X}) - m(\theta))$$

$\therefore$ $\bar{X}$ is unbiased for θ and attains the Cramér-Rao lower bound. i.e. it's a MVBUE for θ.

we know that $m(\theta) = \theta$

$$\Rightarrow m'(\theta) = 1$$

$$\therefore \text{CRLB} = \frac{1}{\mathcal{I}(\theta)}$$

Here, $A(\theta) = \mathcal{I}(\theta) = \dfrac{n}{\theta(1-\theta)}$

$$\therefore Var(T(\underline{X})) = \frac{\theta(1-\theta)}{n}.$$

$X_1, \dots, X_n$ s.t. $X_i \sim Poi(\lambda)$ | MVBUE Example 2

$$\mathbb{P}(X_i = x_i) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \qquad x_i \in \{0, 1, 2, \dots, \}$$

$$L(\lambda \mid \underline{x}) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$= \frac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}$$

The log-likelihood function is

$$\ell(\lambda \mid \underline{x}) = \left(\sum_{i=1}^{n} x_i\right) \log \lambda - n\lambda - \log\left(\prod_{i=1}^{n} x_i!\right)$$

The score function is

$$\frac{\partial \ell}{\partial \lambda}(\lambda \mid \underline{x}) = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$$

$$= \frac{1}{\lambda}\left(\sum_{i=1}^{n} x_i - n\lambda\right)$$

$$= \frac{n}{\lambda}\left(\frac{\sum_{i=1}^{n} x_i}{n} - \lambda\right)$$

$$= \frac{n}{\lambda}(\bar{x} - \lambda)$$

$\therefore$ The score function is

$$U(\lambda; \underline{X}) = \frac{n}{\lambda}(\bar{X} - \lambda)$$

$\Rightarrow$ $\bar{X}$ is a MVBUE for λ.

and $\mathcal{I}(\lambda) = \dfrac{n}{\lambda}$

$\Rightarrow$ CRLB $= \dfrac{\lambda}{n}$.

i.e. $\operatorname{Var}(\bar{X}) = \dfrac{\lambda}{n}$.

$X_1, \ldots, X_n \quad X_i \sim N(\mu, \sigma^2)$

The likelihood function:

$$L(\mu, \sigma^2 \mid \underline{X}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(X_i - \mu)^2\right\}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2\right\}$$

The log-likelihood is

$$\ell(\mu, \sigma^2 \mid \underline{X}) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$$

The score functions are

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i - \mu)^2$$

The second derivatives are

$$\frac{\partial^2 \ell}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 \ell}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(X_i - \mu)^2$$

$$\frac{\partial^2 \ell}{\partial \mu\, \partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}(X_i - \mu)$$

$$\left[\mathcal{I}(\mu,\sigma^2)\right] = \begin{pmatrix} \mathbb{E}\left(-\frac{\partial^2 \ell}{\partial \mu^2}\right) & \mathbb{E}\left(\frac{-\partial^2 \ell}{\partial \mu \partial \sigma^2}\right) \\ \mathbb{E}\left(-\frac{\partial^2 \ell}{\partial \mu \partial \sigma^2}\right) & \mathbb{E}\left(\frac{\partial^2 \ell}{\partial (\sigma^2)^2}\right) \end{pmatrix}$$

$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial \mu^2}\right] = \frac{n}{\sigma^2}$$

$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial (\sigma^2)^2}\right] = \mathbb{E}\left[\frac{-n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_{i=1}^{n}(X_i-\mu)^2\right]$$

$$= \frac{-n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_{i=1}^{n} \mathbb{E}[(X_i-\mu)^2]$$

$\mathrm{Var}(X_i) = \sigma^2$

$$= \frac{-n}{2\sigma^4} + \frac{n\sigma^2}{\sigma^6}$$

$$= \frac{n}{2\sigma^4}$$

$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial \mu \partial \sigma^2}\right] = \mathbb{E}\left[\frac{1}{\sigma^4}\sum_{i=1}^{n}(X_i - \mu)\right]$$

$$= \frac{1}{\sigma^4}\sum_{i=1}^{n}\mathbb{E}\left(\cancelto{0}{X_i-\mu}\right)$$

$\because$ $\mathbb{E}(X_i) = \mu$

$$= 0$$

Hence, the Fisher information matrix is:

$$\mathcal{I}(\mu,\sigma^2) = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

The inverse of $\mathcal{I}(\mu, \sigma^2)$ is

$$\mathcal{I}^{-1}(\mu, \sigma^2) = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}$$

For estimators of μ, the CRLB for the variance is $\frac{\sigma^2}{n}$.

For estimators σ^2, the CRLB for the variance is $\frac{2\sigma^4}{n}$.

$X \sim \text{Poi}(\theta)$

The pmf of X is

$$\mathbb{P}(X = x) = \frac{\theta^x e^{-\theta}}{x!} \qquad x \in \{0, 1, 2, \dots\}$$

$$= \exp\left(\log(\theta^x) - \theta - \log x!\right)$$

$$= \exp\left(\underbrace{[\log \theta]}_{a(\theta)} \underbrace{x}_{T(x)} \underbrace{-\theta}_{b(\theta)} \underbrace{-\log x!}_{c(x)}\right)$$

$$= \exp\left(a(\theta)T(x) + b(\theta) + c(x)\right)$$

with
$$a(\theta) = \log \theta$$
$$T(x) = x$$
$$b(\theta) = -\theta$$
$$c(x) = -\log x!$$

$\therefore$ X is a member of the exponential family.

$X \sim \text{Beta}(\alpha, \beta)$

The pdf of X is

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$$= \exp\left(\underbrace{\log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)}_{b(\alpha,\beta)} + \underbrace{(\alpha-1)}_{a_1(\alpha,\beta)}\overbrace{\log x}^{T_1(x)} + \underbrace{(\beta-1)}_{a_2(\alpha,\beta)}\overbrace{\log(1-x)}^{T_2(x)}\right)$$

$$= \exp\left(a_1(\alpha,\beta)T_1(x) + a_2(\alpha,\beta)T_2(x) + b(\alpha,\beta) + c(x)\right)$$

with

$$a_1(\alpha,\beta) = \alpha - 1$$

$$a_2(\alpha,\beta) = \beta - 1$$

$$T_1(x) = \log x$$

$$T_2(x) = \log(1-x)$$

$$b(\alpha,\beta) = \log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)$$

$$c(x) = 0.$$

$\therefore$ X is a member of the exponential family.