STAT0017: Selected Topics In Statistics

Topic 2: “Dependence modelling in finance using copulas”

Lecture 1

2019

Material and text books

Accessible Introduction

1. J.-F. Mai and M. Scherer. *Financial Engineering with Copulas Explained.* Financial Engineering Explained. Palgrave Macmillan, 2014
2. R.B. Nelsen. *An Introduction to Copulas.* Springer Series in Statistics. Springer, 2006

Vine Copulas

1. Dorota Kurowicka and Roger Cooke. *Uncertainty Analysis with High Dimensional Dependence Modelling.* John Wiley Sons, Ltd, 2006

Articles

2. Kjersti Aas. Modelling the dependence structure of financial assets: A survey of four copulas. *Norwegian Computing Center*, 2004

Material and text books

Other books

1. F. Durante and C. Sempi. *Principles of Copula Theory.*
   Taylor & Francis, 2015
2. H. Joe. *Dependence Modeling with Copulas.*
   Chapman & Hall/CRC Monographs on Statistics & Applied Probability.
   Taylor & Francis, 2014
3. Harry Joe. *Multivariate models and dependence concepts.*
   Chapman Hall Ltd, London; New York, 1997

Assessment, computer practicals.

1. **Assessment** 2 take-home ICAs
   - Extreme value theory and practice (50%).
     Set: 6/2/2019, Deadline: 21/3/2019
   - Dependence modelling in finance using copulas (50%)
     Set: 6/3/2019, Deadline: 24/4/2019
2. **Computer practicals** in R: 2 practicals.
3. **Office hours**:
   11:00-12.00 Wednesdays
   11:00-12.00 Fridays
   and by appointment outside office hours.
4. **Discussion forum** on Moodle.

Course outline

Lecture 1

- Introduction to copula theory (bivariate case)

Lecture 2

- Simulation (bivariate case)
- Empirical Applications (bivariate case)

Lecture 3, 4

- Vine copulas (multivariate case)

Lecture 5

- Dynamic copulas
- Factor copulas

Value-at-Risk

Let the random variable L denote the loss of a portfolio over the period h.

Definition 2.1 (Value-at-Risk)

$$\mathrm{VaR}_\alpha = \inf\{l \in \mathbb{R} : P(L > l) \leqslant 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geqslant \alpha\}$$

Given some confidence level $\alpha \in (0, 1)$, the Value-at-Risk of a given portfolio is the threshold such that the probability of losing more than this threshold over a time horizon h is equal to $1 - \alpha$.

Identical correlation $\rho = 0.7$, but different dependence structure

Definition 2.2 (Quantile function)

For a CDF $F_X : \mathbb{R} \to [0, 1]$ of a r.v. X, the quantile function of F_X is the function F_X^{-1}:

$$F_X^{-1}(u) = \inf \{x \in \mathbb{R} : F_X(x) \geq u\}$$

where $0 < u \leq 1$, and $F_X^{-1}(0) = \inf \{x \in \mathbb{R} : F_X(x) > 0\}$

Preliminaries

Lemma 2.1 (The Probability Integral Transform)

Let X be a continuous r.v. with CDF $F_X(x)$ which is increasing over the range of X. Then $U \equiv F_X(x) \sim \mathrm{U}(0,1)$.

Proof:

$$\begin{aligned} F_U(u) &= \mathbb{P}(U \leq u) \\ &= \mathbb{P}(F_X(X) \leq u) \\ &= \mathbb{P}(F_X^{-1}(F_X(X)) \leq F_X^{-1}(u)) \\ &= \mathbb{P}(X \leq F_X^{-1}(u)) \\ &= F_X(F_X^{-1}(u)) \\ &= u \end{aligned}$$

where $\quad u \in [0,1]$

Lemma 2.2 (The inverse Probability Integral Transform)

Let $U \sim \mathrm{U}(0,1)$, and F_X be any CDF. Then, $X \equiv F_X^{-1}(U) \sim F_X$

Proof:

$$\begin{aligned}
\mathbb{P}(X \leq x) &= \mathbb{P}(F_X^{-1}(U) \leq x) \\
&= \mathbb{P}(F_X(F_X^{-1}(U)) \leq F_X(x)) \\
&= \mathbb{P}(U \leq F_X(x)) \\
&= F_X(x)
\end{aligned}$$

where $\quad x \in \mathbb{R}$

Definition 2.3

Let S_1 and S_2 be nonempty subsets of $\bar{\mathbb{R}}$. A 2-place real function H is a function whose domain, $Dom\ H = S_1 \times S_2$, is a subset of $\bar{\mathbb{R}}^2$ and whose range, $Ran\ H$, is a subset of $\bar{\mathbb{R}}$.

Definition 2.4

A 2-place real function H is 2-increasing if $V_H([a_1, a_2] \times [b_1, b_2]) \geq 0$ for all rectangles B whose vertices lie in $Dom\ H$.

Note that the statement "H is 2-increasing" neither implies nor is implied by the statement "H is nondecreasing in each argument".

Example

Example 3.8. Let H be the function defined on $[0,1] \times [0,1]$ by $H(x,y) = max(x,y)$. Then H is a nondecreasing function of x and of y. Nevertheless, $V_H([0,1] \times [0,1]) = -1$. Hence, H does not satisfy the 2-increasing property.

Example 3.9. Let H be the function defined on $[0,1] \times [0,1]$ by $H(x,y) = (2x-1)(2y-1)$. Although H is 2-increasing, it is a decreasing function of x for each y in (0,1/2) and a decreasing function of y for each x in (0,1/2).

Lemma 2.3

Let S_1 and S_2 be nonempty subsets of $\bar{\mathbb{R}}$, and let H be a 2-increasing function with domain $S_1 \times S_2$. Let x_1, x_2 be in S_1 with $x_1 \le x_2$, and let y_1, y_2 be in S_2 with $y_1 \le y_2$. Then the function $t \mapsto H(t, y_2) - H(t, y_1)$ is nondecreasing on S_1, and the function $t \mapsto H(x_2, t) - H(x_1, t)$ is nondecreasing on S_2.

Using Lemma 1.1, it can be shown that a 2-increasing function H is nondecreasing in each argument.

Lemma 2.4

Let S_1 and S_2 be nonempty subsets of $\bar{\mathbb{R}}$, and let H be a grounded 2-increasing function with domain $S_1 \times S_2$. Then H is nondecreasing in each argument.

Now suppose that S_1 has a greatest element b_1 and that S_2 has a greatest element b_2. We then say that a function H from $S_1 \times S_2$ into $\mathbb{R}$ has margins, and that the margins of H are the functions F and G given by:

$$\text{Dom}\, F = S_1, \text{ and } F(x) = H\left(x, b_2\right) \text{ for all } x \text{ in } S_1$$

$$\text{Dom}\, G = S_2, \text{ and } G(y) = H\left(b_1, y\right) \text{ for all } y \text{ in } S_2$$

Example 3.12. Let H be the function with domain $[-1,1] \times [0,\infty]$ given by:

$$H(x,y) = \frac{(x+1)(e^y - 1)}{x + 2e^y - 1}$$

Then H is grounded because $H(x,0) = 0$ and $H(-1,y) = 0$.

The margins of H are $F(x)$ and $G(y)$ given by:

$$F(x) = H(x,\infty) = (x+1)/2 \tag{1}$$

$$G(y) = H(1,y) = 1 - e^{-y} \tag{2}$$

Definition 2.5

A d-dimensional copula $C : [0,1]^d \to [0,1]$ is a joint cumulative distribution function (CDF) of a d-dimensional random vector with uniform marginals $U(0,1)$.

1. $C\left(u_1, \ldots, u_j, \ldots, u_d\right) = 0$ if $u_j = 0$ for at least one $j \in \{1, \ldots, d\}$
2. $C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$ for all u_j and $j \in \{1, \ldots, d\}$
3. C is d-increasing, that is, for all $\boldsymbol{a} = (a_1, \ldots, a_d) \in [0,1]^d$ and $\boldsymbol{b} = (b_1, \ldots, b_d) \in [0,1]^d$, where $a_i \leqslant b_i$:

$$V_C([\boldsymbol{a}, \boldsymbol{b}]) = \sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{\sum_{j=1}^{d} i_j} C\left(u_{1i_1}, \ldots, u_{di_d}\right) \geqslant 0$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \ldots, d\}$

Using **Properties 1** and **Properties 3** in Definition 3.13 we can show that $C : [0,1]^2 \to [0,1]$ is nondecreasing in each argument.

Let $a_1,\ a_2$ denote the least elements of $[0,1], [0,1]$, respectively, and set $x_1 = a_1,\ y_1 = a_2$ in Lemma 1.1.

Sklar's Theorem

Theorem 2.1 (Sklar, 1959)

If $X_1, \ldots, X_d$ has joint distribution function $F_{1,\ldots,d}(x_1, \ldots, x_d)$ and marginal distribution functions $F_1(x_1), \ldots, F_d(x_d)$, then there exists appropriate d-dimensional copula C such that, for all $x_1, \ldots, x_d$ in $\bar{\mathbb{R}} = [-\infty, \infty]$:

$$F_{1,\ldots,d}(x_1, \ldots, x_d) = C_{1,\ldots,d}\left(F_1(x_1), \ldots, F_d(x_d)\right)$$

The joint probability density function $f_{1,\ldots,k}(x_1, \ldots, x_d)$ for an absolutely continuous $F_{1,\ldots,d}(x_1, \ldots, x_d)$ with strictly increasing continuous margins $F_1(x_1), \ldots, F_d(x_d)$ is:

$$f_{1,\ldots,d}(x_1, \ldots, x_d) = c_{1,\ldots,d}(F_1(x_1), \ldots, F_d(x_d)) \prod_{i=1}^{d} f_i(x_i)$$

for d-dimensional copula density c.

Example 3.16 (Bivariate Bernoulli distribution)

Let (X_1, X_2) follow a bivariate Bernoulli distribution with $\mathbb{P}(X_1 = x_1, X_2 = x_2) = \frac{1}{4}$, $x_1, x_2 \in \{0, 1\}$. This implies that $\mathbb{P}(X_i = x_i) = \frac{1}{2}$, $x_i \in \{0, 1\}$. *Ran* $F_i = \{0, \frac{1}{2}, 1\}, i \in \{1, 2\}$.

Any copula with $C(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$

For example:

- $C(u_1, u_2) = u_1 u_2$
- $C(u_1, u_2) = \min\left\{u_1, u_2, \frac{\delta(u_1)+\delta(u_2)}{2}\right\}$ where $\delta(u) = u^2$

Invariance principle

Theorem 3.1

Let X_1 and X_2 be continuous random variables with copula $C_{X_1X_2}$. If α and β are strictly, monotonically increasing transformations of X_1 and X_2, respectively, then $C_{\alpha(X_1),\beta(X_2)} = C_{X_1X_2}$.Thus C_{X_1,X_2} is invariant under strictly increasing transformations of X_1 and X_2.

1. $(X_1, \ldots, X_d)$ has copula $C \iff (F_1(X_1), \ldots, F_d(X_d)) \sim C$
2. The copula of gross returns and the copula of log-returns is identical

Example 4.2

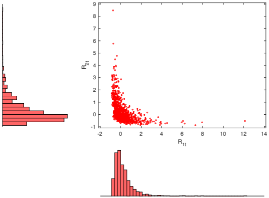
The simple return, R_{it}, on the asset i between dates $t-1$ and t is defined as follows:

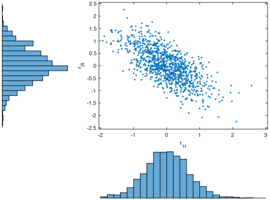
$$R_{it} \equiv \frac{P_{it}}{P_{it-1}} - 1 \tag{3}$$

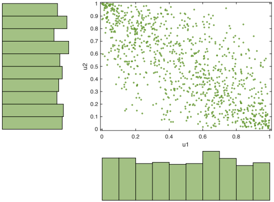
The log-return r_{it} of an asset i is the natural logarithm of its gross return $(1 + R_t)$:

$$r_{it} \equiv \log(1 + R_{it}) = \log \frac{P_{it}}{P_{it-1}} = \log P_{it} - \log P_{it-1} \tag{4}$$

Example 4.2







Examples of copulas

1. Fundamental copulas
2. Implicit copulas
3. Explicit copulas

Fundamental copulas

Independence Copula

- $\Pi(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j$

  since $C\left(F_1\left(x_1\right), \ldots, F_d\left(x_d\right)\right) = F(x_1, \ldots, x_d) = \prod_{j=1}^{d} F_j\left(x_j\right)$

- Therefore, $X_1, \ldots, X_d$ are independent, if and only if, their copula is Π.
- The density is then $c(u_1, \ldots, u_d) = 1, (u_1, \ldots, u_d)' \in [0,1]^d$

Fundamental copulas

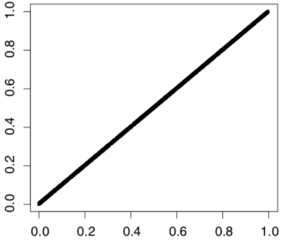
Theorem 3.2 (Fréchet-Hoeffding bounds)

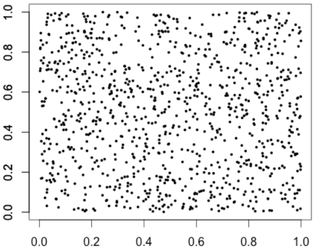
For any d-dimensional copula $C : [0,1]^d \to [0,1]$, and any $\boldsymbol{u} = (u_1, \ldots, u_d) \in [0,1]^d$, the following inequalities hold:

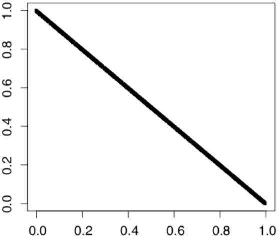
$$W(u_1, \ldots, u_d) \le C(u_1, \ldots, u_d) \le M(u_1, \ldots, u_d)$$

where:

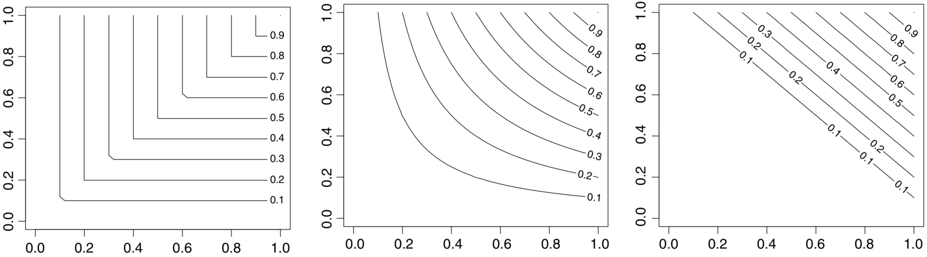
$W(\boldsymbol{u}) = \max\left\{\sum_{j=1}^{d} u_j - d + 1, 0\right\}$ and
$M(\boldsymbol{u}) = \min_{1 \le j \le d}\{u_j\}$





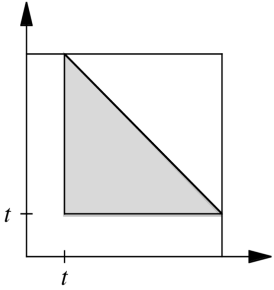


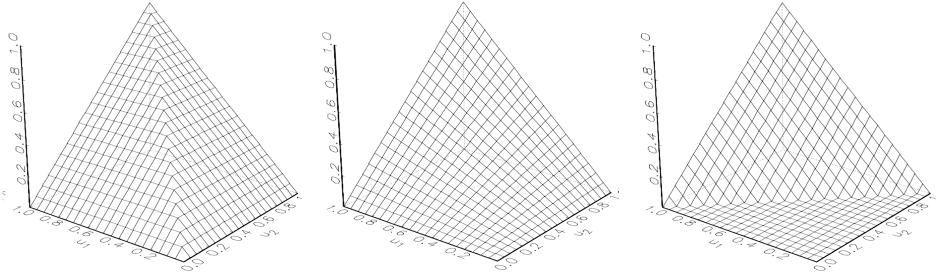
- The comonotonicity copula is the Fréchet upper bound copula.
- The countermonotonicity copula is the two-dimensional Frećhet lower bound copula.



- Note that the points $(t, 1)$ and $(1, t)$ are each members of the level set corresponding to the constant t.
- Hence, the boundary conditions $C(1, t) = t = C(t, 1)$ readily provide the constant for each level set.

- It follows from (4.4) that for a given t in $[0,1]$ the graph of the level set $\{(u_1,u_2) \in [0,1]^2 | C(u_1,u_2) = t\}$ must lie in the shaded triangle whose boundaries are the level sets determined by $M(u_1,u_2) = t$ and $W(u,v) = t$.





- Note that the comonotonicity and countermonotonicity copulas cannot be completely represented in terms of density.
- Hence, the following representation is not feasible:

$$\int_0^u \int_0^v \frac{\partial^2 C(s,t)}{\partial s \partial t} \, ds dt$$

- Here, the value of the double integral is equal to zero almost everywhere, and the copula is said to be *singular*.

Examples of copulas

1. Fundamental copulas
2. Implicit copulas
3. Explicit copulas

Implicit Copulas

Gaussian (Normal) Copula

Normal copula has the linear correlation coefficient ρ as its dependence parameter, although it has no tail dependence.

$$C_G(u_1, u_2|\rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-(r^2 - 2\rho rs + s^2)}{2(1-\rho^2)} \right\} drds$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of a standard normal, and $\rho \in (-1, 1)$.

Implicit Copulas

Student (Student-t) copula

Student-t copula has also the linear correlation coefficient ρ as a measure of dependence. Although Student-t copula has tail dependence, it imposes symmetry in both tails.

$$C_t(u_1, u_2|\rho, \nu) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{r^2 - 2\rho rs + s^2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} drds$$

where ν is the degree-of-freedom parameter, $t_\nu^{-1}(\cdot)$ is the inverse of the standard Student-t cumulative distribution function, and $\rho \in (-1, 1)$.

Method of Inversion: Example

Let's consider the following joint distribution function $F(y_1, y_2)$ with marginal distributions $F_1(y_1)$ and $F_2(y_2)$:

$$F(y_1, y_2) = \exp\left\{-\left[e^{-y_1} + e^{-y_2} - \left(e^{-\theta y_1} + e^{-\theta y_2}\right)^{\frac{-1}{\theta}}\right]\right\} \tag{5}$$

$-\infty < y_1, y_2 < \infty,\ \theta \geq 0$

The marginal distributions can then be found as follows:

$$\lim_{y_2 \to \infty} F(y_1, y_2) = F_1(y_1) = \exp\left(e^{-y_1}\right) \equiv u_1 \tag{6}$$

$$\lim_{y_1 \to \infty} F(y_1, y_2) = F_2(y_2) = \exp\left(e^{-y_2}\right) \equiv u_2 \tag{7}$$

$y_1 = -\log(-\log(u_1))$ and $y_2 = -\log(-\log(u_2))$

Then the corresponding copula is given by:

$$c(u_1, u_2) = u_1 u_2 \exp\left\{\left[(-\log(u_1))^{\theta} + (-\log(u_2))^{\theta}\right]^{-1/\theta}\right\}$$

If the copula $C(u_1, \dots, u_d)$ has a density $c(u_1, \dots, u_d)$, then it can be obtained as follows:

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}$$

If the copula is given in the form of (and the multivariate CDF $F_{\boldsymbol{X}}(\cdot)$ is known):

$$C(u_1, \dots, u_d) = F_{\boldsymbol{X}}\left(F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\right)$$

Then the density can be written as follows:

$$c(u_1, \dots, u_d) = \frac{f_{\boldsymbol{X}}\left(F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\right)}{f_1\left(F_{X_1}^{-1}(u_1)\right) \dots f_d\left(F_{X_d}^{-1}(u_d)\right)}$$

Example 4.10

The density of the Gaussian copula can be written as:

$$c(u_1, \ldots, u_d) = \frac{1}{\sqrt{\det \mathbf{R}}} \exp\left(-\frac{1}{2} \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix}^T \cdot \left(\mathbf{R}^{-1} - \mathbf{I}\right) \cdot \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix} \right)$$

where $\mathbf{I}$ is the identity matrix, and $\mathbf{R}$ is the correlation matrix.

Examples of copulas

1. Fundamental copulas
2. Implicit copulas
3. Explicit copulas

Explicit Copulas

Frank copula
The distribution of Frank copula has the following form:

$$C(u_1, u_2|\theta) = -\theta^{-1} \log \left\{ 1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1} \right\}$$

Frank copula density:

$$\begin{aligned} c(u_1, u_2) &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \\ &= \theta \left(1 - e^{-\theta}\right) e^{-\theta(u_1 + u_2)} \left[\left(1 - e^{-\theta}\right) - \left(1 - e^{-\theta u_1}\right)\left(1 - e^{-\theta u_2}\right)\right]^{-2} \end{aligned}$$

Explicit Copulas

Gumbel copula

The distribution of Gumbel copula has the following form:

$$C(u_1, u_2|\theta) = \exp\left(-\left[(-\log\ u_1)^{\theta} + (-\log\ u_2)^{\theta}\right]^{\frac{1}{\theta}}\right)$$

Gumbel copula density:

$$\begin{aligned} c(u_1, u_1) &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \\ &= C(u_1, u_2)(u_1 u_2)^{-1}\left((-\log u_1)^{\theta} + (-\log u_2)^{\theta}\right)^{-2+2/\theta}(\log u_1 \log u_2)^{\theta-1} \\ &\times \left\{1 + (\theta - 1)\left((-\log u_1)^{\theta} + (-\log u_2)^{\theta}\right)^{-1/\theta}\right\} \end{aligned}$$

Explicit Copulas

Clayton copula

The distribution of Clayton copula has the following form:

$$C(u_1, u_2|\theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}$$

Clayton copula density:

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} = (1+\theta)(u_1 u_2)^{-1-\theta}\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta-2}$$

Next week:

- Construction of Archimedean copulas
- Univariate models
- Simulation from bivariate copulas