STAT0017: Selected Topics In Statistics

Topic 2: “Dependence modelling in finance using copulas”

Lecture 2

2019

Last week:

- We defined what is a copula is.
  - * What is the probabilistic definition of a **copula**?
- We considered properties that a function should satisfy in order to be a copula.
  - * Can you name any of these properties?
- Copulas (or copulae) allow us to understand and study dependence independently of the margins.
- We introduced fundamental copulas, implicit copulas, and briefly explicit copulas.

Today:

- We are going to consider explicit copulas more explicitly.
- In particular, we will consider Archimedian copulas.
- We will also see how can we simulate observations from copulas.

Material and text books relevant to Lecture 2

References

1. Edward W. Frees and Emiliano A. Valdez. Understanding relationships using copulas.
   North American Actuarial Journal, 2(1):1–25, 1998
2. R.B. Nelsen. *An Introduction to Copulas.*
   Springer Series in Statistics. Springer, 2006
   See **Chapter 4**
3. Alexander J. McNeil, Rdiger Frey, and Paul Embrechts. *Quantitative Risk Management: Concepts, Techniques and Tools.*
   Princeton University Press, 2015
   See **Chapter 5.4**

Definition 1.1

Let φ be a continuous, strictly decreasing function from $\mathbb{I} = [0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$. The *pseudo-inverse* of φ is the function $\varphi^{[-1]}$ with $\operatorname{Dom}\varphi^{[-1]} = [0,\infty]$ and $\operatorname{Ran}\varphi^{[-1]} = \mathbb{I}$ given by:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & 0 \leq t \leq \varphi(0) \\ 0 & \varphi(0) \leq t \leq \infty \end{cases}$$

Note that $\varphi^{[-1]}$ is continuous and nonincreasing on $[0,\infty]$, and strictly decreasing on $[0,\varphi(0)]$.

Archimedean Copulas

Furthermore, $\varphi\left(\varphi^{[-1]}(u)\right) = u$ on $\mathbb{I}$ and:

$$\varphi\left(\varphi^{[-1]}(t)\right) = \begin{cases} t & 0 \le t \le \varphi(0) \\ \varphi(0) & \varphi(0) \le t \le \infty \end{cases}$$
$$= \min(t, \varphi(0))$$

Note that if $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{-1}$.

Archimedean Copulas

Lemma 1.1

Let φ be a continuous, strictly decreasing function from $\mathbb{I}$ to $[0, \infty]$ such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the *pseudo-inverse* of φ in Definition 1.1. Let C be the function from $\mathbb{I}^d$ to $\mathbb{I}$ given by:

$$C(u_1, \ldots, u_d) = \varphi^{[-1]}(\varphi(u_1) + \cdots + \varphi(u_d)) \tag{1.1}$$

Then C satisfies the boundary conditions for a copula:

1. $C(u_1, \ldots, u_j, \ldots, u_d) = 0$ if $u_j = 0$ for at least one $j \in \{1, \ldots, d\}$
2. $C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$ for all u_j and $j \in \{1, \ldots, d\}$

Archimedean Copulas

Definition 1.2

Let φ be a continuous, strictly decreasing function from $\mathbb{I}$ to $[0, \infty]$ such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ defined in Definition 1.1. Then the function C from $\mathbb{I}^2$ to $\mathbb{I}$ given by (1.1) is a copula if and only if φ is convex.

- Copulas of the form given by (1.1) are called *Archimedean* copulas.
- The function φ is called a *generator* of the copula.
- If $\varphi(0) = \infty$, we say that φ is a *strict* generator.
- Hence, $\varphi^{[-1]} = \varphi^{-1}$ and $C(u_1, \ldots, u_d) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_d))$ is said to be a *strict* Archimedean copula.

Example

Example 1.5. Let $\varphi(t) = -\ln t$ for t in $[0, 1]$. Because $\varphi(0) = \infty$, φ is strict. Hence, $\varphi^{[-1]}(t) = \varphi^{-1}(t) = \exp(-t)$.

Generate copula C using (1.1):

$$C(u, v) = \exp(-[(-\ln u) + (-\ln v)]) = uv = \Pi(u, v)$$

Question: Is $\Pi(u, v)$ a strict Archimedean copula?

Example

Example 1.6. Let $\varphi(t) = 1 - t$ for t in $[0, 1]$. Then, $\varphi^{[-1]}(t) = 1 - t$ for t in $[0, 1]$ and 0 for $t > 1$, i.e. $\varphi^{[-1]}(t) = \max(1 - t, 0)$.

Again, generate copula C using (1.1):

$$C(u, v) = \max(u + v - 1, 0) = W(u, v)$$

Question: Is $W(u, v)$ a strict Archimedean copula?

Copula	Generator $\phi(t)$	Parameter range	Strict	Lower	Upper
Gumbel	$(-\ln t)^{\theta}$	$\theta \geqslant 1$	Yes	Π	M
Clayton	$\frac{1}{\theta}\left(t^{-\theta}-1\right)$	$\theta \geqslant -1$	$\theta \geqslant 0$	W	M
Frank	$-\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$	$\theta \in \mathbb{R}$	Yes	W	M

Table 1: Table summarizing generator functions for widely used copulas, parameter space and limiting cases.

5 minute break

Check Your Understanding

- What is the probabilistic interpretation of the grounded property of a copula?

Bivariate Copula Simulation

Sklar's theorem allows modelling the dependency structure separately from the marginals.

To simulate bivariate data with a particular copula dependence structure, the following steps are performed:

1. Simulate uniform random variables $\{U_1, \ldots, U_n\}$ with the given copula dependency structure.
2. Generate sample $\{X_1, \ldots, X_n\} = \left\{F_{X_1}^{-1}(U_1), \ldots, F_{X_n}^{-1}(U_n)\right\}$.

Note that a random variable X_i can have any desired distribution by choosing arbitrary marginal quantile function $F_{X_i}^{-1}$.

The Gaussian copula

It can be noted here that Step 2 is straightforward. However, it is Step 1 that requires some extra work.

The Gaussian copula is a distribution over the unit cube $[0,1]^d$.

$$C_G(u_1, \ldots, u_d|\rho) = \Phi_G\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)|\rho\right)$$

This copula is constructed from a standard multivariate normal distribution over $\mathbb{R}^d$ by using the Probability Integral Transform (PIT):

$$\{U_1, \ldots, U_d\} = \{\Phi(Y_1), \ldots, \Phi(Y_d)\}$$

where Φ is the cumulative distribution function of a standard normal.

Bivariate Gaussian copula

Let's focus on the bivariate case, $d = 2$.

Let Y_1 and Y_2 be independent and standard $\mathcal{N}(0, 1)$ random variables such that:

$$\boldsymbol{Y} = [\,Y_1,\, Y_2] \sim N\left([0,0], \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

$$\{\,U_1,\, U_2\} = \{\Phi\,(\,Y_1)\,,\Phi\,(\,Y_2)\}$$

$\star$ When $\rho = 0$, Y_1 and Y_2 are said to be independent!

Bivariate Gaussian copula

Question: How can we induce statistical dependence between these two random variables, Y_1 and Y_2?

First, let's consider a general case:

$$X_1 \sim \mathcal{N}\left(\mu_{X_1}, \sigma^2_{X_1}\right)$$
$$X_2 \sim \mathcal{N}\left(\mu_{X_2}, \sigma^2_{X_2}\right)$$

Bivariate Gaussian copula

Let $Z_1, Z_2 \sim \mathcal{N}(0,1)$ such that:

$$f(z_1, z_2) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(z_1^2 + z_2^2\right)\right\} \tag{1.2}$$

Next, let's transform these standard normal random variables Z_1, Z_2 to random variables, X_1 and X_2, that follow a normal distribution with arbitrary parameters $\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}, \sigma_{X_2}, \rho$.

$$X_1 = \sigma_{X_1} Z_1 + \mu_{X_1} \tag{1.3}$$

$$X_2 = \sigma_{X_2}\left(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2\right) + \mu_{X_2} \tag{1.4}$$

Bivariate Gaussian copula

Next, let's examine the marginal distributions of X_1 and X_2.

$$\mathbb{E}(X_1) = \sigma_{X_1}\mathbb{E}(Z_1) + \mu_{X_1} = \mu_{X_1} \tag{1.5}$$
$$\mathbb{E}(X_2) = \sigma_{X_2}\left(\rho\mathbb{E}(Z_1) + \sqrt{1-\rho^2}\mathbb{E}(Z_2)\right) + \mu_{X_2} = \mu_{X_2} \tag{1.6}$$

$$Var(X_1) = \sigma^2_{X_1} Var(Z_1) + 0 = \sigma^2_{X_1} \tag{1.7}$$
$$Var(X_2) = \sigma^2_{X_2}\rho^2 Var(Z_1) + \sigma^2_{X_2}(1-\rho^2) Var(Z_2) + 0 = \sigma^2_{X_2} \tag{1.8}$$

Hence: $X_1 \sim \mathcal{N}\left(\mu_{X_1}, \sigma^2_{X_1}\right)$ and $X_2 \sim \mathcal{N}\left(\mu_{X_2}, \sigma^2_{X_2}\right)$

Bivariate Gaussian copula

It can also be shown that the correlation between X_1 and X_2 is ρ.

$$\begin{aligned}
\mathrm{Cov}(X_1, X_2) &= \mathbb{E}\left\{(X_1 - \mathbb{E}(X_1))(X_2 - \mathbb{E}(X_2))\right\} \\
&= \mathbb{E}\left\{(\sigma_{X_1} Z_1 + \mu_{X_1} - \mu_{X_1})\left(\sigma_{X_2}\left[\rho Z_1 + \sqrt{1-\rho^2} Z_2\right] + \mu_{X_2} - \mu_{X_2}\right)\right\} \\
&= \mathbb{E}\left\{(\sigma_{X_1} Z_1)\left(\sigma_{X_2}\left[\rho Z_1 + \sqrt{1-\rho^2} Z_2\right]\right)\right\} \\
&= \sigma_{X_1}\sigma_{X_2}\mathbb{E}\left\{\rho Z_1^2 + \sqrt{1-\rho^2} Z_1 Z_2\right\} \\
&= \sigma_{X_1}\sigma_{X_2}\rho\mathbb{E}\left\{Z_1^2\right\} + 0 \\
&= \sigma_{X_1}\sigma_{X_2}\rho
\end{aligned}$$

Hence:

$$\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} = \rho \tag{1.9}$$

Bivariate Gaussian copula

Now we can easily obtain $Y_1, Y_2 \sim \mathcal{N}(0,1)$ with the dependence determined by ρ as follows:

$$\mu_{Y_1} = 0, \quad \mu_{Y_2} = 0$$
$$\sigma_{Y_1} = 1 \quad \sigma_{Y_2} = 1$$

$$Y_1 = \sigma_{Y_1} Z_1 + \mu_{Y_1} = Z_1 \tag{1.10}$$
$$Y_2 = \sigma_{Y_2}\left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_{Y_2} = \rho Z_1 + \sqrt{1-\rho^2} Z_2 \tag{1.11}$$

$$\boldsymbol{Y} = [\, Y_1,\, Y_2] \sim N\left([0,0], \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

Bivariate Gaussian copula

We have seen that when $\rho = 0$, Y_1 and Y_2 are said to be independent!

Question: What is the copula of (Y_1, Y_2)?
How can you generate observations from that copula?

Multivariate Gaussian copula

We can define the multivariate Normal distribution $\boldsymbol{X} \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for any dimension d by the following stochastic representation:

$$\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{L}\boldsymbol{Z}$$

where:

- $\boldsymbol{\mu} \in \mathbb{R}^d$ is the mean vector.
- $\boldsymbol{L}$ is the lower triangular matrix known as the *Cholesky factor* of the Cholesky decomposition.
- $\boldsymbol{Z} = (Z_1, \ldots, Z_d)'$ is a d-dimensional random vector of independent standard normal random variables Z_i, $i \in \{1, \ldots, d\}$

Multivariate Gaussian copula

The simulation from the multivariate Gaussian copula can be performed easily using the following steps:

1. Compute the $d \times d$ Cholesky *factor* $\boldsymbol{L}$ of $\boldsymbol{\Sigma}$
2. Simulate a sample $\boldsymbol{Z}$ of size $d \times 1$ from $N(0, 1)$
3. Create a $d \times 1$ vector $\boldsymbol{U} = \Phi(\boldsymbol{L}\boldsymbol{Z})$
4. Repeat steps $2 - 3$ n times

where Φ is the cumulative distribution function of a standard normal.

Cholesky decomposition (factorization)

Every symmetric, positive definite matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$ can be factored as:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$$

where $\boldsymbol{L}$ is lower triangular matrix with positive diagonal elements.

- $\boldsymbol{L}$ is called the *Cholesky factor* of $\boldsymbol{A}$
- it can be interpreted as a generalized "square root" of a positive definite matrix $\boldsymbol{A}$

Cholesky decomposition (factorization)

Example:

Let's find the Cholesky decomposition of $\mathbf{\Sigma}$ for the general covariance matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

We need to solve for a, b, c:

$$a^2 = \sigma_X^2 \quad ab = \rho\sigma_X\sigma_Y \quad b^2 + c^2 = \sigma_Y^2$$

$$a = \sigma_X$$

$$b = \rho\sigma_X\sigma_Y/a = \rho\sigma_Y$$

$$c = \sqrt{\sigma_Y^2 - b^2} = \sigma_Y\left(1 - \rho^2\right)^{1/2}$$

Cholesky decomposition (factorization)

This is how we obtained the required transformation in equations (1.3) and (1.4).

$$
\begin{aligned}
\boldsymbol{X} &= \boldsymbol{\mu} + \boldsymbol{L}\boldsymbol{Z} \\
\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \begin{pmatrix} \mu_{X_1} \\ \mu_{X_2} \end{pmatrix} + \begin{pmatrix} \sigma_{X_1} & 0 \\ \rho\sigma_{X_2} & \sigma_{X_2}\left(1-\rho^2\right)^{1/2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \\
&= \begin{pmatrix} \mu_{X_1} \\ \mu_{X_2} \end{pmatrix} + \begin{pmatrix} \sigma_{X_1} Z_1 \\ \rho\sigma_{X_2} Z_1 + \sigma_{X_2}\left(1-\rho^2\right)^{1/2} Z_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
X_1 &= \mu_{X_1} + \sigma_{X_1} Z_1 \\
X_2 &= \mu_{X_2} + \sigma_{X_2}\left[\rho Z_1 + \left(1-\rho^2\right)^{1/2} Z_2\right]
\end{aligned}
$$

Student-t copula

The Student-t copula is a distribution over the unit hypercube $[0,1]^d$.

$$C_t(u_1,\ldots,u_d|\rho,\nu) = t^d\left(t_\nu^{-1}(u_1),\ldots,t_\nu^{-1}(u_d)|\rho,\nu\right)$$

In a similar fashion, this copula can be constructed from a multivariate t distribution over $\mathbb{R}^d$ by using the Probability Integral Transform (PIT):

$$\{U_1,\ldots,U_d\} = \{t_\nu(Y_1),\ldots,t_\nu(Y_d)\}$$

where $t_\nu(\cdot)$ is the standard Student-t cumulative distribution function with ν degrees of freedom

Student-t copula

The multivariate t distribution with ν degrees of freedom $\boldsymbol{X} \sim Student^d(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ can be defined by the following stochastic representation:

$$\boldsymbol{X} = \boldsymbol{\mu} + \sqrt{\frac{\nu}{S}}\boldsymbol{L}\boldsymbol{Z}$$

where:

- $\boldsymbol{\mu} \in \mathbb{R}^d$ is the mean vector.
- S denotes a random variable following a chi-squared distribution with ν degree of freedom, $S \sim \chi^2_{(v)}$.
- $\boldsymbol{L}$ is the lower triangular matrix known as the *Cholesky factor* of the Cholesky decomposition.
- $\boldsymbol{Z} = (Z_1, \dots, Z_d)'$ is a d-dimensional random vector of independent standard normal random variables Z_i, $i \in \{1, \dots, d\}$

Student-t copula

Next we simulate from the Student-t copula:

1. Compute the 2×2 Cholesky *factor* $\boldsymbol{L}$ of $\boldsymbol{\Sigma}$
2. Simulate a sample $\boldsymbol{Z}$ of size 2×1 from $N(0,1)$
3. Simulate a sample S of size 1 from $\chi^2_{(\nu)}$
4. $\star$ Compute a 2×1 vector $\boldsymbol{Y} = \sqrt{\frac{\nu}{S}} \boldsymbol{L} \boldsymbol{Z}$
5. Create a 2×1 vector $\boldsymbol{U} = t_\nu(\boldsymbol{Y})$
6. Repeat steps $2 - 5$ n times

$t_\nu(\cdot)$ is the standard Student-t cumulative distribution function with ν degrees of freedom

5 minute break

Check Your Understanding

- How would you simulate from fundamental copulas:

$$W(u,v) = \max(u+v-1,0)$$
$$\Pi(u,v) = uv$$
$$M(u,v) = \min(u,v)$$

Archimedean copulas

- Previous approach can only be implemented when the functional form of the joint distribution is known.
- Usually this is not the case.
- However, we might have some information about the dependence structure, which can be described using a particular copula.
- In this situation, we simulate directly observations from the multivariate uniform distribution over the d-dimensional unit hypercube:

$$\{U_1, \ldots, U_d\}$$

Archimedean copulas

- The Archimedean class of copulas is particularly popular.
- We have already seen several members of this class

A very important function that we will make use of is the 1-dimensional conditional CDF, which is defined as:

Conditional CDF

$$
\begin{aligned}
G_{d|1\ldots d-1}\left(u_d \middle| u_1,\ldots,u_{d-1}\right) &= \mathbb{P}\left(U_d \leq u_d \middle| U_1 = u_1,\ldots,U_{d-1} = u_{d-1}\right) \\
&= \frac{\frac{\partial^{d-1}}{\partial u_1 \ldots \partial u_{d-1}} C_{1\ldots d}\left(u_1,\ldots,u_d\right)}{\frac{\partial^{d-1}}{\partial u_1 \ldots \partial u_{d-1}} C_{1\ldots d-1}\left(u_1,\ldots,u_{d-1}\right)}
\end{aligned}
$$

Archimedean copulas

- Let's focus on the bivariate case, $d = 2$.

Consider the conditional distribution of U_1 given the occurrence of the value $U_2 = u_2$:

$$
\begin{aligned}
c_{1|2}(u_1, u_2) =& \mathbb{P}\{U_1 \leq u_1 | U_2 = u_2\} \\
=& \lim_{\Delta u_2 \to 0} \frac{C(u_1, u_2 + \Delta u_2) - C(u_1, u_2)}{\Delta u_2} \\
=& \frac{\partial}{\partial u_2} \mathbf{C}(u_1, u_2)
\end{aligned}
\tag{1.12}
$$

because $\frac{\partial}{\partial u_1} \mathbf{C}(u_1, 1) = 1$

This also implies that the conditional CDF may be derived directly from the copula itself:

$$F_{1|2}\left(x_1|x_2\right) = \frac{\partial F_{12}\left(x_1, x_2\right)}{\partial x_2} / f_2\left(x_2\right) = \frac{\partial}{\partial u_2} C\left(F_{X_1}\left(x_1\right), F_{X_2}\left(x_2\right)\right)$$

where $u_2 \equiv F_{X_2}(x_2)$

The method of conditional distributions

The simulation of uniform variates for a given copula C can be accomplished with this following general algorithm:

1. Generate 2 independent uniform random variables v_1 and v_2 from $U(0,1)$
2. Set $u_1 = v_1$
3. Set $u_2 = G_{2|1}^{-1}(v_2|u_1)$, where $G_{2|1}(u_2|u_1) = \frac{\partial}{\partial u_1} C_{12}(u_1, u_2)$

Clayton

- **Clayton copula**

1. Generate 2 independent uniform random variables v_1 and v_2 from $U(0,1)$
2. Set $u_1 = v_1$
3. Set $u_2 = \left(v_1^{-\theta}\left(v_2^{-\theta/(\theta+1)} - 1\right) + 1\right)^{-1/\theta}$

The desired pair is then (u_1, u_2)

Frank

- **Frank copula**

1. Generate 2 independent uniform random variables v_1 and v_2 from $U(0,1)$
2. Set $u_1 = v_1$
3. Set $u_2 = -\frac{1}{\theta}\log\left(1 + \frac{v_2\left(1-e^{-\theta}\right)}{v_2\left(e^{-\theta v_1}-1\right)-e^{-\theta v_1}}\right)$

Gumbel

- **Gumbel copula**

1. Generate 2 independent uniform random variables v_1 and v_2 from $U(0,1)$
2. Set $u_1 = v_1$
3. $v_2 = \exp\left\{-\left[(-\log u_1)^\theta + (-\log u_2)^\theta\right]^{1/\theta}\right\} \cdot \left(1 + \left[\frac{\log u_2}{\log u_1}\right]^\theta\right)^{\frac{1}{\theta}-1} \cdot u_1^{-1}$

For the Gumbel copula, the conditional distribution $G_{2|1}(u_2|u_1)$ is not directly invertible, but can be solved numerically for u_2.

The method of conditional distributions

- There is an alternative equivalent algorithm based on the Archimedean construction.
- Generate random variables $X_1, \ldots, X_d$ having known distribution $F_{\boldsymbol{X}}(x_1, \ldots, x_d) = C\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right)$, where the copula function is:

$$C\left(u_1, \ldots, u_d\right) = \varphi^{-1}\left(\varphi\left(u_1\right) + \cdots + \varphi\left(u_d\right)\right)$$

- The idea is to simulate the full distribution of $(X_1, \ldots, X_d)$ by recursive simulation of the conditional distribution of X_k given $X_1, \ldots, X_{k-1}$ for $k = 2, \ldots, d$.

The method of conditional distributions

The joint probability density function of $X_1, \ldots, X_k$ is:

$$f_k(x_1, \ldots, x_k) = \frac{\partial^k}{\partial x_1 \ldots \partial x_k} \varphi^{-1} \{\varphi[F_1(x_1)] + \ldots + \varphi[F_k(x_k)]\}$$

$$= \varphi^{-1(k)} \{\varphi[F_1(x_1)] + \ldots + \varphi[F_k(x_k)]\} \prod_{i=1}^{k} \varphi^{(1)}[F_i(x_i)] F_i^{(1)}(x_i)$$

where the superscript notation (j) means the j-th mixed partial derivative.

The method of conditional distributions

Hence, we can express the conditional distribution of X_k given $X_1, \ldots, X_{k-1}$ as follows:

$$
\begin{aligned}
f_k\left(x_k | x_1, \ldots, x_{k-1}\right) =& \frac{f_k\left(x_1, \ldots, x_k\right)}{f_{k-1}\left(x_1, \ldots, x_{k-1}\right)} \\
=& \varphi^{(1)}\left[F_k\left(x_k\right)\right] F^{(1)}\left(x_k\right) \frac{\varphi^{-1(k)}\left\{\varphi\left[F_1\left(x_1\right)\right] + \cdots + \varphi\left[F_k\left(x_k\right)\right]\right\}}{\varphi^{-1(k-1)}\left\{\varphi\left[F_1\left(x_1\right)\right] + \ldots + \varphi\left[F_{k-1}\left(x_{k-1}\right)\right]\right\}}
\end{aligned}
$$

The method of conditional distributions

Next, we can obtain the conditional distribution function of X_k given $X_1, \ldots, X_{k-1}$ as follows:

$$F_k\left(x_k | x_1, \ldots, x_{k-1}\right) = \int_{-\infty}^{x_k} f_k(x | x_1, \ldots, x_{k-1})\, dx$$

$$= \frac{\varphi^{-1(k-1)}\left\{\varphi\left[F_1\left(x_1\right)\right] + \cdots + \varphi\left[F_k\left(x_k\right)\right]\right\}}{\varphi^{-1(k-1)}\left\{\varphi\left[F_1\left(x_1\right)\right] + \ldots + \varphi\left[F_{k-1}\left(x_{k-1}\right)\right]\right\}}$$

$$= \frac{\varphi^{-1(k-1)}\left\{c_{k-1} + \varphi\left[F_k\left(x_k\right)\right]\right\}}{\varphi^{-1(k-1)}\left(c_{k-1}\right)}$$

where $c_k = \varphi\left[F_1\left(x_1\right)\right] + \cdots + \varphi\left[F_k\left(x_k\right)\right]$

The method of conditional distributions

Algorithm

1. Generate d independent uniformly distributed random variables $U_1, ..., U_d \sim Uniform(0,1)$.

2. Set $X_1 = F_1^{-1}(U_1)$ and $c_0 = 0$.

3. For $k = 2, \ldots, p$, recursively calculate:

$$F_k(X_k | x_1, \ldots, x_{k-1}) \equiv U_{k|1,\ldots,k-1} = \frac{\varphi^{-1(k-1)}\{c_{k-1} + \varphi[F_k(x_k)]\}}{\varphi^{-1(k-1)}(c_{k-1})}$$

   and define X_k to be the solution of the equation:

$$U_{k|1,\ldots,k-1} \cdot \varphi^{-1(k-1)}(c_{k-1}) = \varphi^{-1(k-1)}\{c_{k-1} + \varphi[F_k(x_k)]\}$$

The method of conditional distributions

- The method of conditional distributions becomes quite expensive for $d > 2$.
- However, for certain Archimedean copulas fast algorithm exists when the Laplace transform of some positive random variable results in the inverse generator function $\varphi^{-1}(s)$.
- This is the case of the following copulas:
    - Clayton
    - Frank
    - Gumbel

Simulation of LT-Archimedean copulas

Algorithm

1. Generate a random variable V with distribution function $F_V(\cdot)$ such that the Laplace-Stieltjes transform $\tau(s)$ of $F_V(\cdot)$, is the inverse of the generator $\varphi(t)$ of the required copula, i.e. $\tau(s) = \varphi^{-1}(s)$.

2. Simulate d independent uniformly distributed random variables $U_1, ..., U_d \sim Uniform(0,1)$.

3. Obtain uniform random variables with the required dependence $\boldsymbol{U} = \left(\tau\left(-\frac{\ln U_1}{V}\right), \ldots, \tau\left(-\frac{\ln U_d}{V}\right)\right)'$

Clayton

- **Clayton copula**

1. Generate a gamma variable $V \sim Gamma(\frac{1}{\theta}, 1)$ with $\theta > 0$.
2. Simulate d independent uniformly distributed random variables $U_1, ..., U_d \sim Uniform(0, 1)$.
3. Obtain $U = \left((1 + -\frac{\ln U_d}{V})^{\frac{1}{\theta}}, \ldots, (1 + -\frac{\ln U_d}{V})^{\frac{1}{\theta}} \right)'$

The distribution function of V has Laplace transform $\tau(s) = (1 + s)^{\frac{1}{\theta}}$.

Frank

- **Frank copula**

1. Generate a discrete random variable V with probability mass function $p(k) = P(V = k) = \frac{(1-\exp(-\theta))^k}{k\theta}$ for $k = 1, 2, \ldots$ and $\theta > 0$.

2. Simulate d independent uniformly distributed random variables $U_1, ..., U_d \sim Uniform(0, 1)$.

3. Obtain $\boldsymbol{U} = \left(\frac{\ln\left[1+e^{-\frac{\ln U_1}{V}}(e^{\alpha}-1)\right]}{\alpha}, \ldots, \frac{\ln\left[1+e^{-\frac{\ln U_d}{V}}(e^{\alpha}-1)\right]}{\alpha} \right)'$

The distribution function of V has Laplace transform $\tau(s) = \frac{\ln[1+e^{s}(e^{\alpha}-1)]}{\alpha}$.

Gumbel

- **Gumbel copula**

1. Generate a positive stable variable $V \sim \mathrm{St}(\frac{1}{\theta}, 1, \gamma, 0)$, where $\gamma = (cos(\pi/(2\theta)))^{\theta}$ and $\theta > 1$

2. Simulate d independent uniformly distributed random variables $U_1, ..., U_d \sim Uniform(0, 1)$.

3. Obtain $\boldsymbol{U} = \left(\exp\left[-\left(-\frac{\ln U_1}{V}\right)^{1/\alpha}\right], \dots, \exp\left[-\left(-\frac{\ln U_d}{V}\right)^{1/\alpha}\right]\right)'$

The distribution function of V has Laplace transform $\tau(s) = \exp\left(-s^{1/\alpha}\right)$.

Next week:

- Univariate Models
- Vine Copulas