STAT0017: Selected Topics In Statistics

Topic 2: “Dependence modelling in finance using copulas”

Lecture 3

2019

Last week:

- We considered Archimedean copulas in more detail.
- In particular, we looked at the construction of Archimedean copulas.
- We established that Archimedean copulas satisfy all 3 properties.
- We looked at several ways of simulating observations from this class of copulas.
- We also considered simulation from Gaussian and t copulas.

Today we are going to consider:

- measures of dependence
- univariate models

Survival Copulas

- In some applications, it might be of interest to determine the probability that a patient, device, or other object of interest will survive beyond any given specified time t.
- This probability is given by the survival function (or reliability function) $\overline{F}(x) = P(X > x) = 1 - F(x)$, where $F(\cdot)$ denotes the distribution function of X.

Let (X_1, X_2) be a pair of random variables with joint distribution function $F(\cdot, \cdot)$. Then the "joint survival function" is given by:

$$\overline{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

with the marginal survival functions $\overline{F_1}(x_1)$ and $\overline{F_2}(x_2)$, respectively.

Survival Copulas

- Can the Sklar's theorem be extended to joint survival functions?

Let the copula of X_1 and X_2 is C. Then the joint survival function is given as follows:

$$\begin{aligned}\overline{F}(x_1, x_2) &= 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2) \\ &= \overline{F_1}(x_1) + \overline{F_2}(x_2) - 1 + C(F_1(x_1), F_2(x_2)) \\ &= \overline{F_1}(x_1) + \overline{F_2}(x_2) - 1 + C(1 - \overline{F_1}(x_1), 1 - \overline{F_2}(x_2))\end{aligned}$$

Survival Copulas

- Let's define a function $\widehat{C}$ from $\mathbb{I}^2$ into $\mathbb{I}$ by :

$$\widehat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2) \tag{3.2}$$

Then the joint survival function can be expressed as follows:

$$\overline{F}(x_1, x_2) = \widehat{C}(\overline{F_1}(x_1), \overline{F_2}(x_2))$$

The function $\widehat{C}$ is then referred to as the "survival copula" of X_1 and X_2.

- **Question:** Is $\widehat{C}$ a copula?

Example 3.5 (Gumbel's bivariate exponential distribution)

Consider the copula $C(u_1, u_2)$ for Gumbel's bivariate exponential distribution:

$$C(u_1, u_2) = u_1 + u_2 - 1 + (1 - u_1)(1 - u_2)\, e^{-\theta \ln(1-u_1)\ln(1-u_2)}$$

where $\theta \in [0, 1]$

Using (3.2):

$$\widehat{C}(u_1, u_2) = u_1 u_2\, e^{-\theta \ln u_1 \ln u_2}$$

Survival function of copulas

Let $C : [0,1]^d \to [0,1]$ be a d-dimensional copula of a d-dimensional random vector with uniform marginals $U(0,1)$. Then the survival or tail functions, which we denote by $\overline{C}$ is as follows:

$$\overline{C}\,(u_1, \ldots, u_d) = P\,(U_1 > u_1, \ldots, U_d > u_d)$$

Let $d = 2$, then:

$$\overline{C}(u_1, u_2) = P[U_1 > u_1, U_2 > u_2] = 1 - u_1 - u_2 + C(u_1, u_2) = \widehat{C}(1 - u_1, 1 - u_2)$$

Note the difference between the survival copula $\widehat{C}$ and the joint survival function $\overline{C}$ for two uniform random variables $Uniform(0,1)$ with joint distribution function C.

- **Question:** Is $\overline{C}$ a copula?

Coefficients of Tail Dependence

Definition 3.1 (Upper tail dependence)

Let X_1 and X_2 be random variables with marginal distribution functions $F_1(x_1)$ and $F_2(x_2)$ respectively. The coefficient of upper tail dependence of X_1 and X_2 is:

$$\lambda_{\mathrm{U}} := \lambda_{\mathrm{U}}(X_1, X_2) = \lim_{q \to 1^-} P\left(X_2 > F_2^{-1}(q) | X_1 > F_1^{-1}(q)\right) \tag{3.3}$$

given a limit $\lambda_{\mathrm{U}} \in [0, 1]$ exists.

- If $\lambda_{\mathrm{U}} \in (0, 1]$, then X_1 and X_2 are said to exhibit upper tail dependence.
- if $\lambda_{\mathrm{U}} = 0$, then they are asymptotically independent in the upper tail.

Coefficients of Tail Dependence

Definition 3.2 (Lower tail dependence)

Let X_1 and X_2 be random variables with marginal distribution functions $F_1(x_1)$ and $F_2(x_2)$ respectively. The coefficient of upper tail dependence of X_1 and X_2 is:

$$\lambda_L := \lambda_L\left(X_1, X_2\right) = \lim_{q \to 0^+} P\left(X_2 \leqslant F_2^{-1}(q) | X_1 \leqslant F_1^{-1}(q)\right) \tag{3.4}$$

given a limit $\lambda_\mathrm{L} \in [0, 1]$ exists.

- If $\lambda_\mathrm{L} \in (0, 1]$, then X_1 and X_2 are said to exhibit lower tail dependence.
- if $\lambda_\mathrm{L} = 0$, then they are asymptotically independent in the lower tail.

Coefficients of Tail Dependence

For lower tail dependence, and using $C(u_1, u_2) = F\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right)$, we obtain:

Theorem 3.1

$$\lambda_L = \lim_{q \to 0^+} P\left(X_2 \leqslant F_2^{-1}(q) | X_1 \leqslant F_1^{-1}(q)\right) \tag{3.5}$$

$$= \lim_{q \to 0^+} \frac{P\left(X_2 \leqslant F_2^{-1}(q), X_1 \leqslant F_1^{-1}(q)\right)}{P\left(X_1 \leqslant F_1^{-1}(q)\right)} \tag{3.6}$$

$$= \lim_{q \to 0^+} \frac{C(q, q)}{q} \tag{3.7}$$

given a limit $\lambda_{\mathrm{L}} \in [0, 1]$ exists.

- If $\lambda_{\mathrm{U}} \in (0, 1]$, then the copula C has upper tail dependence.
- if $\lambda_{\mathrm{U}} = 0$, then the copula C has no upper tail dependence.

Coefficients of Tail Dependence

If $F_1(x_1)$ and $F_2(x_2)$ are continuous marginal distribution functions, then it can be shown that these parameters λ_{U} and λ_{L} can be expressed in terms of the unique copula C.

Theorem 3.2

$$\begin{aligned}
\lambda_{\mathrm{U}} &= \lim_{q\to 1^-} P\left(X_2 > F_2^{-1}(q) | X_1 > F_1^{-1}(q)\right) && (3.8)\\
&= \lim_{q\to 1^-} P\left(F_2(X_2) > q | F_1(X_1) > q\right) && (3.9)\\
&= \lim_{q\to 1^-} \frac{\overline{C}(q,q)}{1-q} = \lim_{q\to 1^-} \frac{1-2q+C(q,q)}{1-q} && (3.10)\\
&= 2 - \lim_{q\to 1^-} \frac{1-C(q,q)}{1-q} && (3.11)
\end{aligned}$$

given a limit $\lambda_{\mathrm{U}} \in [0,1]$ exists. $\overline{C}$ is the joint survival function for two uniform $U(0,1)$ random variables whose joint cumulative distribution function (CDF) is the copula C.

Example 3.13

The coefficient of tail dependence for the Gaussian copula:

$$
\begin{aligned}
\lambda_{\mathrm{U}} = \lambda_{\mathrm{L}} =& 2 \lim_{x \to -\infty} P\left(X_2 \leqslant x \middle| X_1 = x\right) \\
=& 2 \lim_{x \to -\infty} \Phi(x\sqrt{1-\rho}/\sqrt{1+\rho}) \\
=& 0 \qquad\qquad \text{provided} \quad \rho < 1
\end{aligned}
$$

The coefficient of tail dependence for the t copula:

$$
\lambda_{\mathrm{U}} = \lambda_{\mathrm{L}} = 2\ t_{v+1}\left(-\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}}\right) \qquad \text{provided} \quad \rho > -1
$$

Archimedean Copulas

Corollary 3.1

Let C be an Archimedean copula with generator φ. Then:

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - \varphi^{[-1]}(2\varphi(t))}{1 - t} = 2 - \lim_{s \to 0^+} \frac{1 - \varphi^{[-1]}(2s)}{1 - \varphi^{[-1]}(s)} \tag{3.12}$$

and

$$\lambda_L = \lim_{t \to 0^+} \frac{\varphi^{[-1]}(2\varphi(t))}{t} = \lim_{s \to \infty} \frac{\varphi^{[-1]}(2s)}{\varphi^{[-1]}(s)} \tag{3.13}$$

Today we are going to consider:

- measures of dependence
- univariate models

Univariate models

- We have seen that in probabilistic terms, a copula is a joint cumulative distribution function (CDF) of a d-dimensional random vector with uniform marginals $U(0,1)$.
- This means that before we can fit copula to financial asset returns, we need to apply appropriate transformations.
- Suppose we want to model the dependence structure of a random vector $(X_1, \ldots, X_d)'$ with marginal distribution functions $F_1(x_1), \ldots, F_d(x_d)$.
- Then $U \equiv F_i(X_i) \sim \mathrm{U}(0,1)$ for $i = 1, \ldots, d$.
- We saw that the implication of a copula being invariant under strictly increasing transformations is that if $(X_1, \ldots, X_d)'$ has copula C, then $(F_1(X_1), \ldots, F_d(X_d))'$ also has copula C.

- Time series of financial asset prices often display unit-root behaviour.
- Hence, log-return series are often analysed (which are stationary).

Let $P_t,\ t = 0, \ldots, T$ represent a time series of financial asset prices. Then log-returns r_t on P_t is:

$$r_t = \ln(P_t) - \ln(P_{t-1}) = \log\left(\frac{P_t}{P_{t-1}}\right) = \log\left(1 + \frac{P_t - P_{t-1}}{P_{t-1}}\right)$$

Note: Taylor-expansion of r_t shows that $r_t \approx \frac{P_t - P_{t-1}}{P_{t-1}}$

Time series of FTSE100

Figure 1 : FTSE100 (price)

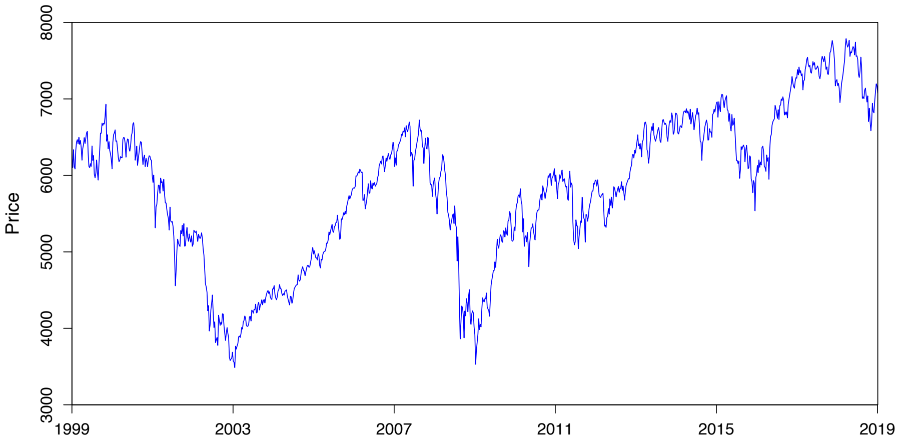
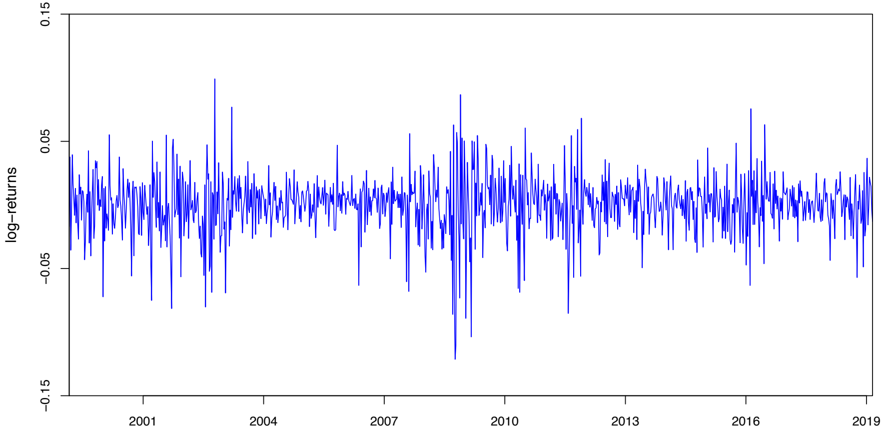


Figure 2 : FTSE100 (log-returns)



Models for volatility

- From Figure 2 it is evident that the variance is not constant.
- Another evident feature is *volatility clustering*.
- *Volatility clustering* refers to the tendency of large changes in financial asset prices to be followed by large changes, and small changes to be followed by small changes (Mandelbrot, 1963).
- Hence, it is sensible to consider a model which is able to take these features into account.
- This provides a motivation for the ARCH class of models.

Models for volatility

Autoregressive conditionally heteroscedastic (ARCH) models

- Modelling and forecasting stock market volatility has attracted growing attention by academics and practitioners during the past two decades.
- Volatility is frequently used as a measure of the risk of financial assets.
- It measured by the standard deviation or variance of financial asset returns.

ARCH model

Definition 3.3

The $ARCH(p)$ model of Engle (1982) is defined by:

$$x_{it} = \mu + u_{it} \quad \text{where} \quad u_{it} = h_{it}^{\frac{1}{2}} \epsilon_{it}$$

$$h_{it} = \omega_i + \sum_{p=1}^{P} \alpha_{ip} u_{it-p}^2$$

where h_{it} is the conditional variance given past information, and ϵ_{it} are $i.i.d.$ random variables, $\omega_i,\ \alpha_{ip} > 0$ assures that the conditional variance h_{it} is positive.

Limitations of ARCH(p) models

ARCH(p) models are rarely used in practice since they have limitations associated with them:

- There is no clear guidance on the choice of the number of lags m of the squared residual.
- The number of lags of the squared error required to capture all of the dependence could be very large, and hence the number of parameters to be estimated. In this case, the resulting model would not be parsimonious.
- Non-negativity constraints might be violated due to the large number of parameters in the model. This is because the more parameters there are in the model, the more likely it is that there will be at least one negative estimated value.

GARCH model

To address limitations associated with ARCH model, Bollerslev (1986) and Taylor (1986) independently developed the GARCH model.

Definition 3.4

The $GARCH(p, q)$ model of Bollerslev (1986) is defined by:

$$x_{it} = \mu + u_{it} \quad \text{where} \quad u_{it} = h_{it}^{\frac{1}{2}}\epsilon_{it}$$

$$h_{it} = \omega_i + \sum_{p=1}^{P} \alpha_{ip} u_{it-p}^2 + \sum_{q=1}^{Q} \beta_{iq} h_{it-q}$$

where h_{it} is the conditional variance given past information, and ϵ_{it} are $i.i.d.$ random variables, ω_i, β_{iq}, $\alpha_{ip} > 0$ assures that the conditional variance h_{it} is positive and $\sum_{p=1}^{P} \alpha_{ip} + \sum_{q=1}^{Q} \beta_{iq} < 1$ ensures stationarity.

- The GARCH model can be shown to be an ARMA model for the conditional variance.
- For example, consider the GARCH(1, 1) model:

$$h_{it} = \omega_i + \alpha_{i1} u_{it-1}^2 + \beta_{i1} h_{it-1}$$

- Then it can be shown to be an ARMA(1,1) process for the squared errors.

$$u_{it}^2 = \omega_i + (\alpha_{i1} + \beta_{i1}) u_{it-1}^2 + v_t - \beta_{i1} v_{t-1}$$

- Why GARCH models are more popular than ARCH models?
- GARCH model is a more parsimonious model of the conditional variance than a high-order ARCH model.
- For instance, GARCH(1,1) model can be written as an infinite order ARCH model:

$$\begin{aligned} h_{it} &= \omega_i + \alpha_{i1} u_{it-1}^2 + +\alpha_{i2} u_{it-2}^2 + +\alpha_{i3} u_{it-3}^2 + \dots \\ &= \omega_i + \sum_{p=1}^{\infty} \alpha_{ip} u_{it-p}^2 \end{aligned}$$

- It is important to note that although the conditional variance of u_{it}^2 is changing, the unconditional variance is constant.
- Again, consider the GARCH(1, 1) model.
- Then the unconditional variance of u_{it} is constant and defined as follows:

$$\text{var}\left(u_{it}\right) = \frac{\omega_i}{1 - \left(\alpha_{i1} + \beta_{i1}\right)}$$

- From this definition it is evident why the restriction $\alpha_{i1}+\beta_{i1} < 1$ is required.
- For $\alpha_{i1}+\beta_{i1} \geq 1$, the unconditional variance of u_{it} is not defined.
- When $\alpha_{i1} + \beta_{i1} = 1$, the model is termed IGARCH, i.e. integrated GARCH, which means a "unit root in variance".

TGARCH model

- An extension of GARCH model has been proposed in order to account for the *leverage effects*.
- *Leverage effects* refers to the tendency of volatility to react differently to a positive change in price than to a negative change.

Definition 3.5

The threshold $GARCH(p, q)$ model is defined by:

$$x_{it} = \mu + u_{it} \quad \text{where} \quad u_{it} = h_{it}^{\frac{1}{2}} \epsilon_{it}$$

$$h_{it} = \omega_i + \sum_{p=1}^{P} \alpha_{ip} u_{it-p}^2 + \sum_{o=1}^{O} \gamma_{io} u_{it-o}^2 I_{[u_{it-o}<0]} + \sum_{q=1}^{Q} \beta_{iq} h_{it-q}$$

This volatility model is also known as the GJR model, named after the authors Glosten, Jagannathan and Runkle (1993).

where h_{it} is the conditional variance given past information, and ϵ_{it} are $i.i.d.$ random variables, $\omega_i, \; \beta_{iq}, \; \gamma_{io}, \; \alpha_{ip} > 0$ assures that the conditional variance h_{it} is positive.

Univariate (marginal) model building

Univariate model building consists of the following steps:

1. Test for serial dependence in log-returns and, if necessary, build an ARMA model to remove any linear dependence.
2. Test for the presence of ARCH effects and, if they are statistically significant, specify a volatility model.
3. Estimate parameters of the conditional mean and volatility models.
4. Check if the fitted model conforms to the specifications of a stationary process, and refine it if necessary.

- For most asset return series a simple AR model might be needed.
- When daily returns are considered, the mean equation might require indicator variables in order to account for possible differences in the behaviour of asset returns over particular days of the week.
- There could be cases when the serial correlations are weak or non-existent.
- In such cases, building a mean equation involves removing the sample mean of asset returns form the time series.

Testing for ARCH Effect

- Let $u_{it} = x_{it} - \mu$ be the residuals from the mean equation.
- Then the series of squared residuals u_{it}^2 is used to test for the ARCH effects (i.e. conditional heteroscedasticity).
- There are two available tests:
  1. The Ljung-Box test
  2. Lagrange multiplier (LM) test

Testing for ARCH Effect

2. **The Ljung-Box test**

   The test statistic is:

$$Q(m) = T(T+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{T-k} \sim \chi_m^2$$

   where:

   T is the sample size

   $\hat{\rho}_k$ is the sample autocorrelation at lag k

   m is the number of lags that are being tested.

Testing for ARCH Effect

2. **The Ljung-Box test**

   The test statistic is:

$$Q(m) = T(T+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{T-k} \sim \chi_m^2$$

- The hypothesis to be tested:

  H_0 : All of the first m autocorrelation coefficients are jointly zero

  H_1 : The data are not independently distributed

Testing for ARCH Effect

2. **Lagrange multiplier (LM) test**

1 Run the following linear regression and extract residuals $\hat{u}_{it}$:

$$x_{it} = \mu + u_{it}$$

2 Generate squared residuals $\hat{u}_{it}^2$, and run the following regression on m own lags:

$$\hat{u}_{it}^2 = \alpha_0 + \alpha_1 \hat{u}_{it-1}^2 + \alpha_2 \hat{u}_{it-2}^2 + \cdots + \alpha_q \hat{u}_{it-m}^2 + v_{it}$$

where v_{it} is an error term

3 Obtain R^2 from the above regression.

Testing for ARCH Effect

2. **Lagrange multiplier (LM) test**

4 The test statistic is then constructed as $T * R^2$, where T is the number of observations and R^2 is the coefficient of multiple correlation from previous regression.

5 The test statistic $T * R^2 \sim \chi^2_{(m)}$

6 The hypothesis to be tested:

$$H_0 : \alpha_1 = 0 \text{ and } \alpha_2 = 0 \text{ and } \alpha_3 = 0 \text{ and } \dots \text{ and } \alpha_m = 0$$
$$H_1 : \alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0 \text{ or } \alpha_3 \neq 0 \text{ or } \dots \text{ or } \alpha_m \neq 0$$

Next week:

- Vine Copulas