Lecture 4: Value-at-Risk

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Value-at-Risk

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- It is used by banks, pensions funds, etc.

- It is also used outside finance in situations where it is necessary to quantify **how much an individual or institution stands to lose** over some given period of time.

Value-at-Risk

- **Value-at-Risk** is probably the most widely used risk measure in financial institutions.
- It is a measure for assessing the risk (i.e. the amount of potential loss) associated with an investment or a portfolio of investments.
- It is used by banks, pensions funds, etc.
- It is also used outside finance in situations where it is necessary to quantify **how much an individual or institution stands to lose** over some given period of time.

Value-at-Risk

- Value-at-Risk is the *market risk* measure prescribed by Basel Accord II and III to ensure that a bank has adequate capital to be able to absorb losses.
- Basel III is a set of international banking regulations established by the Basel Committee on Banking Supervision in order to promote stability in the international financial system.
- It builds on Basel I and Basel II, and was developed in response to the deficiencies in financial regulation revealed by the financial crisis of 2007-08

Value-at-Risk - Introduction

- Suppose a company holds a certain number of financial assets such as stocks, bonds, derivatives, etc. This collection of assets is called a **portfolio**.

- Every day, the value of the portfolio will change, since the price of the individual assets in the portfolio change frequently.

- The company typically wants to know the answer to questions such as "what is the probability of the value of the portfolio dropping by more than $10 million on a given day?". Knowing this is essential to risk management.

Value-at-Risk

Let the random variable L denote the loss of a portfolio over the period h.

Definition 1.1 (Value-at-Risk)

$$\mathrm{VaR}_{\alpha} = \inf\{l \in \mathbb{R} : P(L > l) \leqslant 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geqslant \alpha\}$$

Given some confidence level $\alpha \in (0,1)$, the Value-at-Risk of a given portfolio is the threshold such that the probability of losing more than this threshold over a time horizon h is equal to $1 - \alpha$.

Value at Risk - Definition

- VaR is hence a measure of **risk exposure**
- For example, if $\alpha = 0.99$ and the VaR is $\$1m$, this means that there is a 1% probability of losing more than $\$1m$ in a single day.
- Normally, we will be given the portfolio, and the value α, then asked to find the VaR – i.e. for risk management the company will choose some value of α (e.g. $\alpha = 0.99$) then try to find what sum of money it has a $100(1-\alpha)\%$ chance of losing in a single day.
- Typical values for α are $\alpha = 0.95$ or $\alpha = 0.99$.
- In *market risk* management the time horizon h is usually 1 or 10 days. In *credit risk* and *operational risk* management h is usually 1 year.

Value at Risk - Estimation

- There are many ways of computing the VaR, which have varying degrees of sophistication. We will consider a very simple method.
- In this course, rather than working with monetary loss, we will instead usually focus on **percentage loss**, since this is typically more informative.
- In the simple method, we assume that the percentage change in the portfolio on each day t is a random variable Y_t, **and that the percentage change on each day is independent of the percentage change on all other days**.
- In other words, If V_t denotes the value of the portfolio on day t, then $V_t = Y_t \times V_{t-1}$ where Y_t is a random variable drawn from some distribution $F_Y(\cdot)$. By the above assumption, the Y_t values are **independent and identically distributed**.

Value at Risk - Estimation

- The VaR will be estimated based on historical data. Suppose the company has held the portfolio for $n+1$ days. They hence have n observations $\mathbf{y} = (y_1, \ldots, y_n)$.

- For example, suppose that on the first day the portfolio was worth \$100m, then on the second day it was worth \$99m, then on the third day it was worth \$101m, then on the fourth it was worth \$102m.

Percentage Loss - Empirical Density

- The first 3 values of Y_t are then:

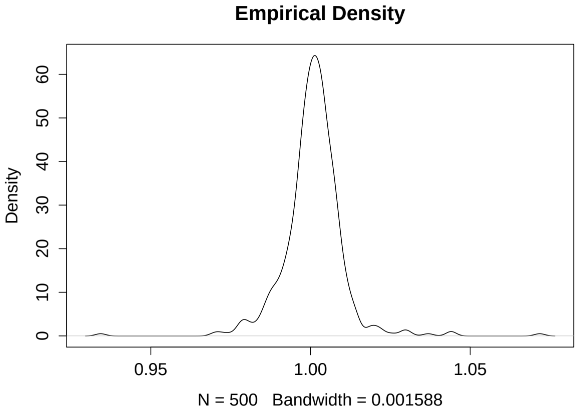
$$Y_1 = \frac{99}{100} = 0.99 \qquad \text{(a 1\% loss)}$$

$$Y_2 = \frac{101}{99} = 1.0202 \qquad \text{(a 2.02\% gain)}$$

$$Y_3 = \frac{102}{101} = 1.0099 \qquad \text{(a 0.99\% gain)}$$

Values smaller than 1 denote a loss.

Percentage Loss - Empirical Density



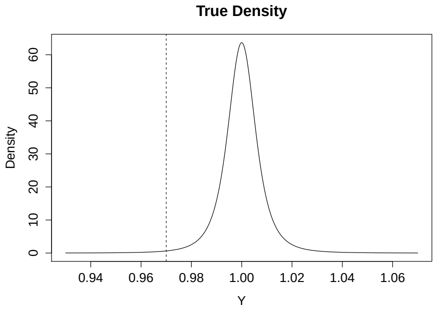
Value at Risk - Computation

- Suppose we know the true distribution $F_Y(\cdot)$ of Y_t. Then we can compute the VaR as follows:
- Find the $100(1-\alpha)^{th}$ quantile of the **true** density of Y_t. Denote this by q.
- For example if $\alpha = 0.99$ then q is equal to the 1^{st} quantile[1], i.e. we find q to satisfy $p(Y_t < q) = 0.01$. In other words, q is the point at which the total area under the curve below that point is equal to 0.01
- Then q is the VaR.

[1] To be precise: 1^{st} 100-quantile or 1^{st} percentile

Percentage Loss - Computation

For example, if the following is the true distribution of Y_t and $\alpha = 0.99$.



Then $q = 0.97$. This means that there is a 1% probability that the value of the portfolio will drop by 3% or more on any given day (since $Y_t = 0.97$ denotes a 3% loss).

Value at Risk - Calculation

- In the **simplest possible model** we might assume that the distribution of each Y_t is Gaussian, $Y_t \sim N(\mu, \sigma^2)$ Then, we would estimate μ and σ, and use this to compute q.
- We use the historical data $\mathbf{y} = (y_1 \ldots, y_n)$ to estimate μ and σ^2.
- First suppose we are frequentists. The maximum likelihood estimates are the usual $\hat{\mu} = 1/n \sum_{t=1}^{n} Y_t$, and $\hat{\sigma}^2 = 1/n \sum_{t=1}^{n} (Y_t - \hat{\mu})^2$.
- For a given α, a simple estimate of q would be the value satisfying:

$$\int_{-\infty}^{q} p(y_t | \hat{\mu}, \hat{\sigma}^2) dy_t = (1 - \alpha)$$

Value at Risk - Simple Frequentist Example

- For example, suppose the empirical mean of the historical Y_t was 1.01, with standard deviation 0.01, i.e. $Y_i \sim N(1.01, 0.01)$.

- This means that on average the portfolio gains 1% in value each day, but the returns have a standard deviation of 1%.

- If $\alpha = 0.99$ then q can be easily computed using R software:

  ```
  qnorm(0.01, 1.01, 0.01) = 0.9867
  ```

- So there is a 1% chance of the portfolio dropping by 1.33% or more in a given day (Note: $1 - 0.9867 = 0.0133$).

Value at Risk - Bayesian Approach

Several problems with this simple frequentist approach:

1. It does not take uncertainty about μ and σ^2 into account - we used point estimates (MLE). But these estimates will not be accurate - we could potentially underestimate risk because we are not taking this uncertainty into account.

2. It does not incorporate prior information about the future portfolio returns – using historical data is important, but we may also have beliefs about the future which aren't reflected in previous history.

3. The usual issues about the difficulty of communicating frequentist statements to non-statisticians, and how they get misinterpreted.

We will hence explore a Bayesian approach to VaR analysis instead. In this case we need to perform Bayesian inference for the parameters μ, σ^2 of the Gaussian distribution.

Value at Risk - Bayesian Approach

- Suppose we are Bayesians. We start with a prior distribution $p(\mu, \sigma^2)$ on the unknown parameters of the Gaussian distribution governing the percentage daily change Y_t.

- This prior is chosen to reflect our beliefs about the future portfolio returns. Remember: μ_t is the average return on a given day $E[Y_t]$, and σ^2 is the variance $Var[Y_t]$.

- We then update this to get the posterior $p(\mu, \sigma^2 | y_1, \ldots, y_n)$ given the historical data.

- This posterior then captures all our knowledge about the distribution of Y_t based on both our prior knowledge, and the historical data. We can then obtain q based on this.

Value at Risk - Calculation

- So, the first question is: how do we go about computing the posterior $p(\mu, \sigma^2|\mathbf{y})$?

- We saw how to do this in Lecture 1 when the likelihood $p(y|\theta)$ was Binomial, and the prior $p(\theta)$ was a Beta distribution.

- We now need to choose a suitable prior $p(\mu, \sigma^2)$ and update this given the Gaussian likelihood $p(y|\mu, \sigma^2)$.

Bayesian Inference for the Gaussian Distribution

- We now discuss how to perform Bayesian inference for parameters of the Gaussian distribution which describes the daily percentage changes in the portfolio value.

- Let's start with the simplest case. Suppose that the variance σ^2 is **known**. In this case we only need to estimate μ (which, remember, denotes the average daily percentage change in value).

- It can be shown that the conjugate prior in this case is also Gaussian: $p(\mu) = N(\mu_0, \sigma_0^2)$ where μ_0 and σ_0^2 control the shape of the prior, and hence reflect prior beliefs about μ.

- In other words, we represent our prior beliefs about the average change in the portfolio value μ by a Gaussian distribution, with parameters μ_0 and σ_0^2.

Bayesian Inference for the Gaussian Distribution

- So in summary, we have that $Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$. We know σ^2 (for now) and need to estimate μ.
- The prior on μ is $p(\mu) = N(\mu_0, \sigma_0^2)$.
- It is **vital** to understand that σ^2 and σ_0^2 here denote two very different quantities:
  - σ^2 is the variance of Y_t (i.e. the variance of the percentage returns).
  - σ_0^2 is a parameter of the prior distribution for μ and represents the uncertainty in our prior beliefs about μ.
- Similarly, μ and μ_0 are different quantities:
  - μ_0 is the mean of the prior distribution for μ and, hence represents our prior beliefs about μ.

Gaussian Distribution With Known Variance

Let's consider Bayesian estimation of the mean of a univariate Gaussian distribution whose variance is assumed to be known.

1. **Likelihood**

$$\begin{aligned} p(\mathbf{y}|\mu,\sigma^2) &= \prod_{i=1}^{n} p(y_i|\mu,\sigma^2) \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right) \end{aligned} \tag{1}$$

Gaussian Distribution With Known Variance

$$
\begin{aligned}
p(\mathbf{y}|\mu,\sigma^2) &= \overbrace{\frac{1}{(2\pi)^{\frac{n}{2}}}}^{\text{constant}} \frac{1}{(\sigma^2)^{\frac{n}{2}}} \exp\left(-\frac{1}{2\sigma^2}\{ns^2 + n(\bar{y}-\mu)^2\}\right) \\
&\propto \underbrace{\frac{1}{(\sigma^2)^{\frac{n}{2}}} \exp\left(-\frac{ns^2}{2\sigma^2}\right)} \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) \qquad (2)
\end{aligned}
$$

where $s^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$

When σ^2 is known, it can be treated as constant, and the above expression can be written as:

$$
\begin{aligned}
p(\mathbf{y}|\mu) &\propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) \\
&\propto \mathcal{N}(\bar{y}|\mu, \frac{\sigma^2}{n})
\end{aligned}
$$

Gaussian Distribution With Known Variance

2 Prior

Since the likelihood has the form:

$$p(\mathbf{y}|\mu) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right)$$

The *conjugate prior* will have the following form:

$$p(\mu) \propto \exp\left(-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right)$$

That is, $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$:

$$p(\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2}}\exp\left(-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right)$$

Gaussian Distribution With Known Variance

3 Posterior Distribution

$$p(\mu|\mathbf{y},\sigma^2) \propto p(\mu) \times p(\mathbf{y}|\mu,\sigma^2)$$

$$\propto \exp\left(-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right) \times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right)$$

$$= \exp\left(-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right)$$

Gaussian Distribution With Known Variance

$$= \exp\left(-\frac{1}{2\sigma_0^2}(\mu^2 - 2\mu\mu_0 + \mu_0^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2) \right)$$

$$= \exp\left(-\frac{\mu^2}{2\sigma_0^2} + \frac{\mu\mu_0}{\sigma_0^2} - \frac{\mu_0^2}{2\sigma_0^2} - \frac{\sum_{i=1}^n y_i^2}{2\sigma^2} + \frac{\mu\sum_{i=1}^n y_i}{\sigma^2} - \frac{n\mu^2}{2\sigma^2} \right)$$

$$= \exp\left(-\frac{\mu^2}{2}\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right) + \mu\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_n^{i=1} y_i}{\sigma^2}\right) - \left(\frac{\mu_0^2}{2\sigma_0^2} + \frac{\sum_{i=1}^n y_i^2}{2\sigma^2}\right)\right)$$

Gaussian Distribution With Known Variance

Let's define new parameters σ_n^2 and μ_n:

$$\sigma_n^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}} \tag{3}$$

$$\mu_n = \sigma_n^2 \left(\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2} \right) \tag{4}$$

Gaussian Distribution With Known Variance

Then:

$$p(\mu|\mathbf{y}, \sigma^2) \propto \exp\left(-\frac{1}{2\sigma_n^2}(\mu^2 - 2\mu\mu_n + \mu_n^2)\right)$$

$$= \exp\left(-\frac{1}{2\sigma_n^2}(\mu - \mu_n)^2\right) \tag{5}$$

Example

- Suppose a company has held a stock portfolio for 10 days.
- The observed returns on these days are:

  $$\mathbf{y} = (0.997, 1.034, 1.012, 1.042, 1.017, 0.994, 1.040, 1.037, 1.022, 0.994)$$

- The company is interested in learning about the distribution of the returns, for the purpose of computing the VaR and controlling risk.

Example

- It is assumed that the percentage returns follow a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$.

- Based on previous experience, the company knows that the true standard deviation of the returns is equal to $\sigma = 0.02$ (so $\sigma^2 = 0.0004$).

- The company does not know the true mean μ, so it has to be estimated.

- Based on its knowledge of other stock portfolios, it decides that its prior beliefs are best represented by a prior $p(\mu) \sim N(1, 0.01^2)$ – i.e. it believes the average percentage daily change is 1 (no change), and the standard deviation 0.01 measures the uncertainty in this prior belief.

Example (continued)

- Substituting information provided in the example into the equations (3) and (4) gives $\mu_n = \mu_{10} = 1.0135$ and $\sigma_n^2 = \sigma_{10}^2 = 0.0053^2$, so the posterior is:

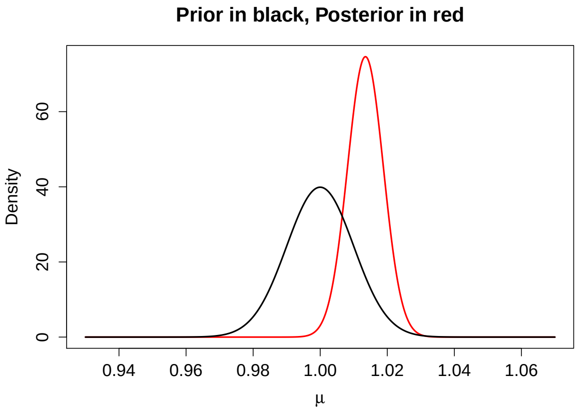
$$p(\mu|\mathbf{y}, \sigma^2) = \mathcal{N}(1.0135, 0.0053^2)$$

- The empirical mean of company's gross returns:

$$\bar{y} = \frac{1}{10}\sum_{t=1}^{10} = 1.0189$$

Example (continued)

The company's posterior beliefs about μ:



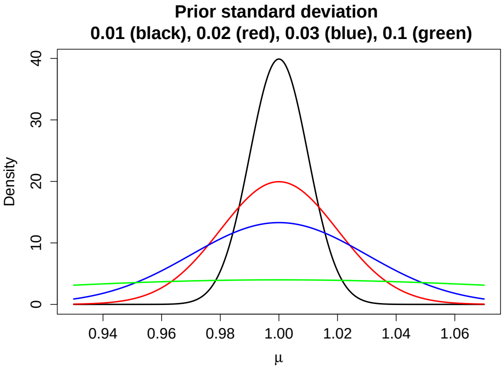
Note that the posterior mean has shifted towards the empirical mean of the data ($\bar{y} = 1.0189$).

Non-informative Priors

- In practice we may not have strong prior beliefs about μ.
- It could also be that we do not want our analysis to be influenced by our prior beliefs (e.g. $Beta(1,1)$).
- In this case we can choose a prior that has an extremely large uncertainty and doesn't impose much prior information on the likelihood.
- In the current case we can represent complete uncertainty about μ by making the prior variance parameter σ_0^2 very large.
- This parameter reflects how uncertain we are about μ before observing the data.

Non-informative Prior

Consider what happens when the prior on μ is $N(1, \sigma_0^2)$, and the variance σ_0^2 increases.



The prior is getting more and more disperse, corresponding to less and less information.

Non-informative Priors

- As σ^2 becomes larger and larger, the prior $p(\mu) = N(\mu_0, \sigma_0^2)$ becomes less and less informative.
- Ideally to represent complete ignorance we would take $\sigma_0^2 = \infty$ (or more formally, take the limit as σ_0^2 tends to infinity).
- This leads to something called an improper prior – the prior is no longer a valid probability distribution.
- Does this matter? In this case no. The posterior works out to be well-defined.

Non-informative Priors

Recall we had:

$$\sigma_n^2 = \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}$$

$$\mu_n = \frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{t=1}^n Y_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}$$

As we let $\sigma_0^2 \to \infty$, the choice of μ_0 no longer matters, and the above expressions can be written as follows:

$$\sigma_n^2 = \left(\frac{n}{\sigma^2}\right)^{-1} = \frac{\sigma^2}{n}$$

$$\mu_n = \frac{\left(\frac{\sum_{t=1}^n Y_i}{\sigma^2}\right)}{\left(\frac{n}{\sigma^2}\right)} = \frac{\sum_{t=1}^n Y_i}{n}$$

Non-informative Priors

- So if we have no strong prior beliefs about μ and use an uninformative prior, the posterior is simply:

$$p(\mu|\mathbf{y},\sigma^2) = N\left(\frac{\sum_{t=1}^{n} Y_i}{n}, \frac{\sigma^2}{n}\right)$$

- This makes intuitive sense – without any prior knowledge, our posterior mean is simply the empirical mean, and the posterior variance is the empirical variance.

- So we can still do Bayesian inference even when we have no strong prior beliefs about the parameter μ - just take prior that has a very high variance.

Back to the Example

- Suppose the company in the previous example had no strong beliefs about μ and they used this non-informative prior. Plugging in the numbers, their posterior would now be:

$$p(\mu|\mathbf{y}, \sigma^2) = N(1.0189, 0.0063^2)$$

  (recall that 1.0189 was the empirical mean)

- Previously, the posterior was $N(1.0135, 0.0053^2)$. This has been pulled away from the empirical mean, due to the prior belief that μ was around 1.

- Note that the posterior variance when using the non-informative prior is larger than when using the informative prior.

Computing the VaR

- Let $\tilde{Y}$ denote the simple gross return on an arbitrary day, d, in the future, which has the same distribution as the gross returns in the historical sample, i.e $\tilde{Y} \sim N(\mu, \sigma^2)$.
- We don't know μ, but we have its posterior distribution $p(\mu|y_1, \ldots, y_n)$.
- We now want to ask questions like "what is the probability of the portfolio losing more than 3% of its value on this day?".
- In the simple frequentist example, we did not incorporate any uncertainty about μ when we computed VaR - we simply used a point estimate.

Computing the VaR

- To answer "what is the probability of the portfolio losing more than 3% of its value on an arbitrary day?", we need to compute $p(\tilde{Y} < 0.97)$.

- Similarly, to answer "what is the probability of the portfolio losing more than 5% of its value on an arbitrary day?", we need $p(\tilde{Y} < 0.95)$.

Posterior predictive distribution

- Let Y be a random variable with a known distribution $p(y|\theta)$, where θ is unknown.
- Before the data $\mathbf{y}$ is observed, the marginal distribution of the unknown but observable y is as follows:

$$p(y) = \int p(y,\theta)d\theta = \int p(\theta)p(y|\theta)d\theta$$

- Once the data $\mathbf{y}$ have been observed, we can predict an unknown $\tilde{y}$ from the same process:

$$\begin{aligned} p(\tilde{y}|\mathbf{y}) &= \int p(\tilde{y},\theta|\mathbf{y})d\theta \\ &= \int p(\tilde{y}|\theta,\mathbf{y})p(\theta|\mathbf{y})d\theta \\ &= \int p(\tilde{y}|\theta)p(\theta|\mathbf{y})d\theta \end{aligned}$$

Computing the VaR

- We are essentially trying to predict $p(\tilde{Y} < z)$ based on the historical data $\mathbf{y} = (y_1, \ldots, y_n)$.
- We do this by using the historical data to find the posterior distribution for the unknown μ, and then predict based on this.
- In other words, we are interested in the distribution of $\tilde{Y}$ based on incorporating information from the historical data. We write this distribution as $p(\tilde{y}|\mathbf{y})$.

Computing the VaR

- So, we have:

$$p(\tilde{y}|\mu) = N(\mu, \sigma^2)(\text{where } \sigma^2 \text{ is known})$$

$$p(\mu|\mathbf{y}, \sigma^2) = N(\mu_n, \sigma_n^2) \text{ (posterior based on historic data } \mathbf{y})$$

- By the theorem of total probability we have:

$$p(\tilde{y}|\mathbf{y}) = \int p(\tilde{y}|\mu)p(\mu|\mathbf{y})d\mu$$

- This is the fundamental equation of Bayesian prediction.

Computing the VaR

- It can be shown that in our conjugate Gaussian case:

$$p(\tilde{y}|\mathbf{y}) = N(\mu_n, \sigma_n^2 + \sigma^2)$$

- In order to find:

$$p(\tilde{Y} < z|\mathbf{y})$$

  we need:

$$\int_{-\infty}^{z} p(\tilde{y}|\mathbf{y}) d\tilde{y}$$

- which is just a Gaussian integral which can be easily computed.

Example

- To return to our previous example, recall that the company (using the informative prior) had a posterior distribution

$$p(\mu|\mathbf{y}) = N(1.0135, 0.0053^2)$$

so $\mu_n = 1.0135$ and $\sigma_n^2 = 0.0053^2$. The variance σ^2 was known to be 0.0004.

- Hence have:

$$\tilde{y}|\mathbf{y} \sim N(1.0135, 0.0053^2 + 0.0004) = N(1.0135, 0.000428)$$

Example

- If the company wants to know the probability of the portfolio dropping in value by more than 3% on a given day, we compute the following integral:

$$\int_{-\infty}^{0.97} \frac{1}{\sqrt{2 \times 0.000428\pi}} e^{\left(\frac{(\tilde{y}-1.0135)^2}{2\times 0.000428}\right)} d\tilde{y} = 0.01774811$$

- Thus, there is a 1.8% probability of the portfolio dropping in value by more than 3% in one day.

Note on Bayesian Inference With Unknown Variance

- In the previous analysis we assumed that the gross returns Y_t had a $N(\mu, \sigma^2)$ distribution where σ^2 was known.
- In practice σ^2 will usually be unknown. We must hence estimate both μ and σ^2.
- For a Bayesian analysis, we need a prior $p(\mu, \sigma^2)$ on both parameters. Ideally, to make it easier to specify the prior, we perhaps want to treat the parameters as being independent and put a separate prior on each: $p(\mu, \sigma^2) = p(\mu)p(\sigma^2)$.
- Unfortunately, this leads to a prior distribution that is not conjugate! We do not yet have the tools to work with non-conjugate priors, so we will leave this for now and revisit it later in the course.

Next week:

- Estimating the probability of extreme events occurring.
- We will look at techniques from the area of extreme value statistics.