## Exercises 5

1. Show that the $L_1$ estimator for the regression coefficients in a general linear model (Section 2.3 (i)) is an M–estimator by

   (a) stating its objective function $\rho_{L_1}$ (convince yourself that the conditions in 2.3.1 (i) are fulfilled);

   (b) deriving its $\psi$-function $\psi_{L_1}$.

   (c) Show that the $L_1$ estimator is the maximum likelihood estimator of the regression parameter vector (based on a sample $y_1, \ldots, y_N$ from a general linear model) when the error distribution is double exponential with mean zero. Recall that the density of a double exponential distribution with mean $\xi$ is given by

   $$f(x) = \frac{\lambda}{2} \exp\left\{-\lambda|x-\xi|\right\}, \qquad x \in \mathbb{R}.$$

   Note: you do not need to derive a closed formula for the estimator!

2. In Section 2.3.1(iv) it is stated that the equations on which M–estimators are based are similar to weighted least squares where the weights are

   $$w(u) = \frac{\psi(u)}{u},$$

   where $u$ is a standardised residual. Derive these weights for

   (a) the $L_1$ estimator and

   (b) the Bisquare objective function.

   Sketch the two functions and discuss (in words only) their implications for the estimation procedure.

3. Attached you find an `R` output with information on three regression analyses of the Turnip data from last week (cf. Exercises 3). In all three cases Model 2 (including an $x_2^2$–term) is used (refer to Exercises 3 for the LS-fit).

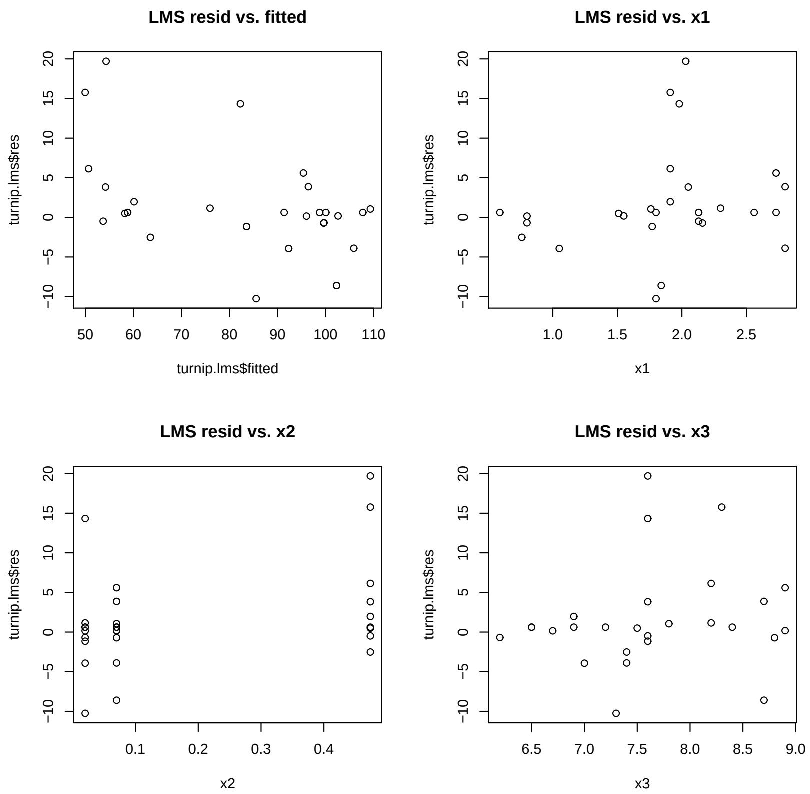
   (a) Compare the results of the three methods of estimation (least squares, LMS and MM).

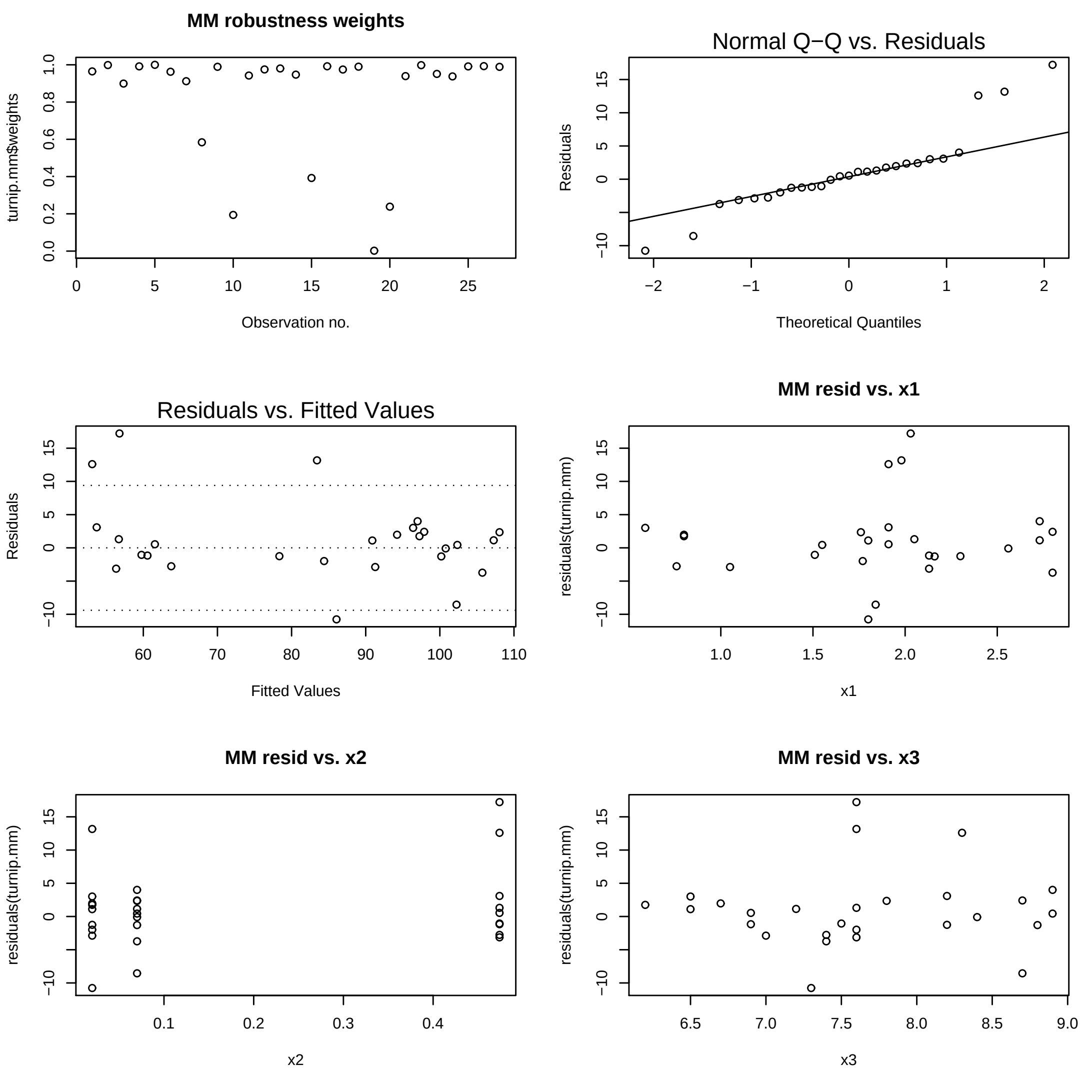
   (b) Compare the weights given by the MM-estimator with the observations that appear to be outliers in the least squares analysis.

   (c) Compare the residual plots of the robust methods with the ones obtained by least squares (for Model 2 only, cf. Exercises 3).

```
## Robust regression of turnip data - Model 2 (with x2^2 term)
#

## LMS-estimator
#
> turnip.lms <- lqs(y~x1+x2+x3+I(x2^2), method="lms")
# Coefficients
> turnip.lms$coef
 (Intercept)           x1           x2           x3      I(x2^2)
  136.761961    -6.115702   678.416120    -7.283058 -1496.680490
# MAD scale estimator for residual standard dev.
> mad(turnip.lms$residuals)
[1] 2.008249

## MM-estimator
#
> turnip.mm <- lmrob(y~x1+x2+x3+I(x2^2))
> summary(turnip.mm)

Call:
lmrob(formula = y ~ x1 + x2 + x3 + I(x2^2))

Weighted Residuals:
     Min       1Q   Median       3Q      Max
-10.7607  -1.6381   0.5403   2.3836  17.2128

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   126.472     11.163  11.330 1.19e-10 ***
x1             -4.546      1.568  -2.900  0.00831 **
x2            620.051    112.034   5.534 1.46e-05 ***
x3             -6.038      1.793  -3.368  0.00278 **
I(x2^2)     -1372.950    222.622  -6.167 3.31e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Robust residual standard error: 3.758
Convergence in 18 IRWLS iterations

Robustness weights:
 observation 19 is an outlier with |weight| <= 0.0019 ( < 0.0037);
 one weight is ~= 1. The remaining 25 ones are summarized as
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
 0.1940  0.9377  0.9646  0.8691  0.9899  0.9988
```

```
# Residuals for LS, LMS and MM-estimator
> cbind(residuals(turnip.lm2),turnip.lms$res,residuals(turnip.mm))
          [,1]        [,2]         [,3]
1    3.90473931   1.0541322   2.35241937
2    1.12596610   0.1811983   0.44002513
3    3.29922069   5.5977273   4.00431262
4    2.15543576   0.6165289   1.13899074
5   -0.08607526   0.6165289  -0.08772078
6    1.82977107   3.8692149   2.41484813
7   -3.00371153  -3.8987603  -3.73510389
8   -8.00270725  -8.6018595  -8.54931796
9   -1.22263889  -0.7165289  -1.29075346
10  11.52486123  14.3287190  13.16915525
11   4.11617058   0.6165289   3.00793934
12   2.62842638   0.1574380   1.97028560
13   2.91554846  -0.6840909   1.75107328
14  -2.82205564  -3.9287190  -2.88168543
15 -11.88895521 -10.2570248 -10.76065328
16   0.79044012   0.6165289   1.10860702
17  -3.38224341  -1.1555785  -1.98550608
18  -3.88219250   1.1555785  -1.25306794
19  13.95654785  19.7030992  17.21276244
20   8.77051714  15.7673554  12.59413892
21  -0.63205844   6.1390496   3.09029646
22  -1.86554103   1.9710744   0.54034444
23  -4.32490276  -2.5204545  -2.76835054
24  -6.50673566  -0.4853306  -3.13263693
25  -3.82476475   0.6165289  -1.15953417
26  -3.59695349   0.4946281  -1.05500332
27  -1.97610886   3.8254132   1.30368257
```


LMS resid vs. fitted
turnip.lms$res
turnip.lms$fitted
LMS resid vs. x1
turnip.lms$res
x1
LMS resid vs. x2
turnip.lms$res
x2
LMS resid vs. x3
turnip.lms$res
x3


MM robustness weights
turnip.mm$weights
Observation no.
Normal Q−Q vs. Residuals
Residuals
Theoretical Quantiles
Residuals vs. Fitted Values
Residuals
Fitted Values
MM resid vs. x1
residuals(turnip.mm)
x1
MM resid vs. x2
residuals(turnip.mm)
x2
MM resid vs. x3
residuals(turnip.mm)
x3