$H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Reject H_0 if

$$\frac{\sup_{\mu, \sigma^2} L(\mu, \sigma^2 \mid \underline{x}, H_1)}{\sup_{\sigma^2} L(\sigma^2 \mid \mu_0, \underline{x}, H_0)} > k$$

Under H_1:

$$L(\mu, \sigma^2 \mid \underline{X}) = (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \right\}$$

Log-likelihood function:

$$\ell(\mu, \sigma^2 \mid \underline{X}) = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2$$

Get mles:

$$\left. \frac{\partial \ell}{\partial \mu} \right|_{\hat{\mu}, \hat{\sigma}^2} = 0 \quad \Rightarrow \quad \hat{\mu} = \bar{X}$$

$$\left. \frac{\partial \ell}{\partial \sigma^2} \right|_{\hat{\mu}, \hat{\sigma}^2} = 0 \quad \Rightarrow \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\mu})^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\sup_{\mu,\sigma^2} L(\mu,\sigma^2 \mid \underline{X}, H_1) = (2\pi\hat{\sigma}^2)^{-n/2} \exp\left\{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n}(X_i-\hat{\mu})^2\right\}$$

$$= (2\pi\hat{\sigma}^2)^{-n/2} \exp\left\{-\frac{n}{2}\right\}$$

Under H_0:

$$L(\sigma^2 \mid \mu_0, \underline{X}, H_0) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_0)^2\right\}$$

$$\ell(\sigma^2 \mid \mu_0, \underline{X}, H_0) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_0)^2$$

$$\left.\frac{\partial \ell}{\partial \sigma^2}\right|_{\tilde{\sigma}^2} = 0 \implies \tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2$$

$$\sup_{\sigma^2} L(\sigma^2 \mid \mu_0, \underline{X}, H_0) = (2\pi\tilde{\sigma}^2)^{-n/2} \exp\left\{-\frac{1}{2\tilde{\sigma}^2}\sum_{i=1}^{n}(X_i-\mu_0)^2\right\}$$

$$= (2\pi\tilde{\sigma}^2)^{-n/2} \exp\left\{-\frac{n}{2}\right\}$$

The likelihood ratio is:

$$\frac{(2\pi\hat{\sigma}^2)^{-n/2} e^{-n/2}}{(2\pi\tilde{\sigma}^2)^{-n/2} e^{-n/2}}$$

$$= \left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right)^{n/2}$$

$\therefore$ Reject H_0 if

$$\left[\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]^{n/2} > k$$

$$\left[\frac{\sum_{i=1}^{n}(X_i-\bar{X}+\bar{X}-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]^{n/2} > k$$

$$\left[\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2+n(\bar{X}-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]^{n/2} > k$$

$$\left[1+\frac{n(\bar{X}-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]^{n/2} > k$$

$\ulcorner \bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$

$$\left[1 + f(\underline{X})\right]^{\frac{n}{2}}$$

As $f(\underline{X}) \longrightarrow \infty$

$$\left[1 + f(X)\right]^{\frac{n}{2}} \longrightarrow \infty.$$

$$\left[1 + \frac{n(\bar{X} - \mu_0)^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]^{n/2}$$ is a monotonic increasing function of

$$\frac{n(\bar{X} - \mu_0)^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

$$\frac{n(\bar{X} - \mu_0)^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} > c \iff \left[1 + \frac{n(\bar{X} - \mu_0)}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]^{\frac{n}{2}} > k$$

for some c.

$$\frac{(\bar{X} - \mu_0)}{\sigma/\sqrt{n}} \sim N(0,1) \text{ under } H_0$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$\Rightarrow$ If $Z \sim N(0,1)$

$U \sim \chi^2_\nu$

Where Z and U are independent

$$\frac{Z}{\sqrt{\frac{U}{\nu}}} \sim t_{n-1}$$

$$\frac{\left[\frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}\right]}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} \sim t_{n-1}$$

$$\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \sim t_{n-1}$$

$\therefore$ Likelihood ratio is

$$\left[1 + \frac{T^2}{n-1}\right]^{n/2}$$

$\therefore$ We reject H_0 if
$|T|$ is large.

$\therefore$ Test is
Reject H_0 if

$$|T| > k'$$

where
$$\mathbb{P}(|T| > k' | H_0) = \alpha.$$

$T \sim t_{n-1}$ under H_0.

Critical region

$$C = \{ \underline{x} : |T| > k' \}$$
$$= \left\{ \underline{x} : \left| \frac{\sqrt{n}(\bar{x} - \mu_0)}{s} \right| > k' \right\}$$

$$s = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2,$$

Given $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

We know that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$\therefore \dfrac{(n-1)S^2}{\sigma^2}$ is a pivotal quantity.

$$\mathbb{P}\left(\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right) < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1}\left(\frac{\alpha}{2}\right)\right) = 1-\alpha$$

$$\mathbb{P}\left(\frac{1}{\chi^2_{n-1}(1-\frac{\alpha}{2})} > \frac{\sigma^2}{(n-1)S^2} > \frac{1}{\chi^2_{n-1}(\frac{\alpha}{2})}\right) = 1-\alpha$$

$$\mathbb{P}\left(\frac{(n-1)S^2}{\chi^2_{n-1}(1-\frac{\alpha}{2})} > \sigma^2 > \frac{(n-1)S^2}{\chi^2_{n-1}(\frac{\alpha}{2})}\right) = 1-\alpha$$

$$= \mathbb{P}\left(\left\{\frac{(n-1)S^2}{\chi^2_{n-1}(1-\frac{\alpha}{2})} > \sigma^2\right\} \cap \left\{\frac{(n-1)S^2}{\chi^2_{n-1}(\frac{\alpha}{2})} < \sigma^2\right\}\right) = 1-\alpha$$

$\therefore$ $100(1-\alpha)\%$ confidence interval for σ^2 is:

$$\left[\frac{(n-1)S^2}{\chi^2_{n-1}(\frac{\alpha}{2})}, \frac{(n-1)S^2}{\chi^2_{n-1}(1-\frac{\alpha}{2})}\right]$$

$X \sim Bin(n, \theta)$

$$L(\theta|X) = \mathbb{P}(X = x;\theta)$$
$$= \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

The log-likelihood function is

$$\ell(\theta|\underline{X}) = \log\binom{n}{x} + X\log\theta + (n-x)\log(1-\theta)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{X}{\theta} - \frac{(n-X)}{1-\theta}$$

MLE solve $\left.\frac{\partial \ell}{\partial \theta}\right|_{\hat{\theta}} = 0 \Rightarrow \hat{\theta} = \frac{X}{n}.$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{X}{\theta^2} - \frac{(n-X)}{(1-\theta)^2}$$

$$\therefore \mathcal{I}(\theta) = \mathbb{E}\left[-\frac{\partial^2 \ell}{\partial \theta^2}\right] = \frac{n}{\theta(1-\theta)}$$

$100(1-\alpha)\%$ confidence interval for θ is

$$: \left[\hat{\theta} - \frac{z_{\frac{\alpha}{2}}}{\sqrt{\mathcal{I}(\hat{\theta})}}, \hat{\theta} + \frac{z_{\frac{\alpha}{2}}}{\sqrt{\mathcal{I}(\hat{\theta})}}\right]$$

$$\Rightarrow \left[\hat{\theta} - Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} \, , \, \hat{\theta} + Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} \right]$$