LECTURE 10
10/12/18

Given $X \sim Exp(\lambda)$

$$f_X(x) = \lambda e^{-\lambda x} \quad x > 0.$$
$$= g(t, \lambda)\, h(x)$$

$$g(t, \lambda) = \lambda e^{-\lambda t}$$
$$h(x) = 1$$

$\therefore$ $t = x$ is sufficient for λ.

X is sufficient.

CDF of X: $F_X(x) = 1 - e^{-\lambda x}$

$\therefore$ $F_X(X) \sim U(0,1)$

$$\mathbb{P}(a \leqslant F_X(x) \leqslant b) = 1 - \alpha$$

$$\mathbb{P}\left(\frac{\alpha}{2} \leqslant F_X(x) \leqslant 1 - \frac{\alpha}{2}\right) = 1 - \alpha$$

$$\mathbb{P}\left(\frac{\alpha}{2} \leqslant 1 - e^{-\lambda X} \leqslant 1 - \frac{\alpha}{2}\right) = 1 - \alpha$$

$$\mathbb{P}\left(\frac{\alpha}{2} - 1 \leqslant -e^{-\lambda X} \leqslant -\frac{\alpha}{2}\right) = 1 - \alpha$$

$$\mathbb{P}\left(1 - \frac{\alpha}{2} \geqslant e^{-\lambda X} \geqslant \frac{\alpha}{2}\right) = 1 - \alpha$$

$$\mathbb{P}\left(\log\left(1 - \frac{\alpha}{2}\right) \geqslant -\lambda X \geqslant \log\left(\frac{\alpha}{2}\right)\right) = 1 - \alpha$$

$$\Rightarrow \mathbb{P}\left(-\frac{1}{X}\log\left(1-\frac{\alpha}{2}\right) \leq \lambda \leq -\frac{1}{X}\log\left(\frac{\alpha}{2}\right)\right) = 1-\alpha$$

$$\mathbb{P}\left(\left\{-\frac{1}{X}\log\left(1-\frac{\alpha}{2}\right) \leq \lambda\right\} \cap \left\{-\frac{1}{X}\log\left(\frac{\alpha}{2}\right) \geq \lambda\right\}\right) = 1-\alpha$$

$\therefore \left[-\frac{1}{X}\log\left(1-\frac{\alpha}{2}\right), -\frac{1}{X}\log\left(\frac{\alpha}{2}\right)\right]$ is a $100(1-\alpha)\%$ confidence interval for λ.

Let Y = No. of defective components in a sample of 110.

$$Y \sim \text{Bin}(110, \theta).$$

We wish to test

H_0: Sample came from factory A $\Rightarrow \theta = 0.2$

vs.

H_1: Sample came from factory B. $\Rightarrow \theta = 0.3$

Set of possible decisions is:

$D = \{\text{Reject } H_0, \text{ Retain } H_0\}$

$\therefore$ The loss function is

$$L(\theta, \text{Retain } H_0) = \begin{cases} 0 & \theta = 0.2 \\ b & \theta = 0.3 \end{cases}$$

$$L(\theta, \text{Reject } H_0) = \begin{cases} a & \theta = 0.2 \\ 0 & \theta = 0.3 \end{cases}$$

A priori

$\pi(\theta = 0.2) = 0.5$ and $\pi(\theta = 0.3) = 0.5$

$$\mathbb{P}(Y=y\mid\theta) = \binom{110}{y}\theta^{y}(1-\theta)^{110-y}$$

$$\left.\pi(\theta\mid Y=y) = \frac{\pi(\theta)\,\mathbb{P}(Y=y\mid\theta)}{\mathbb{P}(Y=y)}\right\}\ \text{BAYES THEOREM}$$

$$\cancel{\pi}\mathbb{P}(Y=y) = \cancel{\pi(\theta)}\,\mathbb{P}(Y=y\mid\theta)\pi(\theta) + \mathbb{P}(Y=y\mid\theta)\pi(\theta)$$

$$= \binom{110}{28}0.2^{28}(0.8)^{82}\times 0.5$$

$$+ \binom{110}{28}0.3^{28}(0.7)^{82}\times 0.5$$

$$= 0.08306.$$

$$\pi(\theta)\mathbb{P}(Y=28\mid\theta=0.2) = 0.03$$

$$\pi(\theta)\mathbb{P}(Y=28\mid\theta=0.3) = 0.049$$

$$\mathbb{P}(\theta=0.2\mid Y=28) = 0.4004$$

$$\mathbb{P}(\theta=0.3\mid Y=28) = 0.5996$$

$$\mathbb{E}[L(\theta,\text{Retain } H_0)\mid y] = 0\times\mathbb{P}(\theta=0.2\mid y) + b\times\mathbb{P}(\theta=0.3\mid y)$$

$$= 0.5996b$$

$$\mathbb{E}[L(\theta, \text{Reject } H_0)] = a \times \mathbb{P}(\theta = 0.2 \mid y) + 0 \times \mathbb{P}(\theta = 0.3 \mid y)$$

$$= 0.4004a$$

We reject H_0 if

$$0.4004a \underset{\leq}{\cancel{>}} 0.5996b.$$

$X_1, \ldots, X_n \overset{iid}{\sim} Exp(\theta)$

Likelihood function:

$$L(\theta | \underline{X}) = \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$= \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

$$= \theta^n e^{-\theta t} \quad \text{where } t = \sum_{i=1}^{n} x_i$$

Prior distribution:

$\theta \sim Gamma(\alpha, \beta)$.

$\mathbb{E}(\theta) = 2 \qquad Var(\theta) = 1 \qquad$ a priori

$$\Rightarrow \frac{\alpha}{\beta} = 2 \qquad \frac{\alpha}{\beta^2} = 1$$

$$\Rightarrow \alpha = 2\beta \qquad \alpha = \beta^2$$

$$\Rightarrow \beta^2 = 2\beta$$

$$\beta(\beta - 2) = 0$$

$$\beta = 2$$

$$\alpha = 4.$$

Prior pdf:

$$\pi(\theta) = \frac{2^4}{\Gamma(4)} \theta^{4-1} e^{-2\theta}$$

Using Bayes' theorem:

$$\pi(\theta|\underline{x}) \propto \pi(\theta) \times L(\theta|\underline{x})$$

$$= \left\{\frac{2^4}{\Gamma(4)} \theta^{4-1} e^{-2\theta}\right\} \times \theta^n e^{-\theta t}$$

$$\propto \theta^{(n+4)-1} e^{-(t+2)\theta}$$

Posterior is proportional to a Gamma$(n+4, t+2)$ pdf.

$\therefore \theta|\underline{x} \sim \text{Gamma}(n+4, t+2)$

(b) Exact posterior credible interval.

$$\mathbb{P}(a < \theta < b \,|\, \underline{x}) = 1-\alpha$$

$$\Rightarrow \int_a^b \pi(\theta|\underline{x})\, d\theta = 1-\alpha$$

$$\int_a^b \frac{(t+2)^{n+4}}{\Gamma(n+4)} \theta^{(n+4)-1} e^{-(t+2)\theta}\, d\theta = 1-\alpha$$

In general, since n is large.

$\theta|t \sim N(\mathbb{E}(\theta|\underline{x}), Var(\theta|\underline{x}))$ approximately.

$$\theta|t \sim N\left(\frac{n+4}{t+2}, \frac{n+4}{(t+2)^2}\right) = N\left(\frac{54}{24.2}, \frac{54}{(24.2)^2}\right)$$

$$\mathbb{P}\left(-1.96 < \frac{\theta - \frac{54}{24.2}}{\sqrt{\frac{54}{24.2^2}}} < 1.96 \,\Big|\, t\right) = 0.95$$

$$\mathbb{P}\left(\frac{54}{24.2} - 1.96\sqrt{\frac{54}{24.2^2}} < \theta < \frac{54}{24.2} + 1.96\sqrt{\frac{54}{24.2^2}}\right) = 0.95$$

$= (1.636, 2.826)$ is a 95% posterior credible interval for θ.

$X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$ σ^2 known.

(a) $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ distribution of mle $\bar{X}$.

$\therefore$ 95% confidence interval for μ is

$$\left[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right].$$

(b) In a Bayesian setting, prior for μ is

$\mu \sim N(\psi, \phi^2)$

$$\pi(\mu) = \frac{1}{\sqrt{2\pi\phi^2}} \exp\left\{-\frac{1}{2\phi^2}(\mu - \psi)^2\right\}$$

Likelihood function

$$L(\mu | \underline{x}, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}$$

Posterior:

BAYES THEOREM

$$\begin{aligned}\pi(\mu | \underline{x}) &\propto \pi(\mu) L(\mu | \underline{x}) \\ &= (2\pi\phi^2)^{-1/2}(2\pi\sigma^2)^{-n/2} \\ &\quad \times \exp\left\{-\frac{1}{2\phi^2}(\mu - \psi)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\} \\ &\propto \exp\left\{-\frac{1}{2\phi^2}(\mu - \psi)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}\end{aligned}$$

$$\pi(\mu|\underline{x}) \propto \exp\left\{ -\frac{1}{2\phi^2}(\mu^2 - 2\psi\mu + \psi^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} x_i^2 - 2\mu\sum_{i=1}^{n} x_i + n\mu^2\right)\right\}$$

$$\propto \exp\left\{ -\frac{1}{2\phi^2}(\mu^2 - 2\psi\mu) - \frac{1}{2\sigma^2}\left(n\mu^2 - 2\mu\sum_i x_i\right)\right\}$$

$$= \exp\left\{ -\left(\frac{1}{2\phi^2} + \frac{n}{2\sigma^2}\right)\left(\mu^2 - 2\frac{\left(\frac{\psi}{2\phi^2} + \frac{\sum_{i=1}^{n} x_i}{2\sigma^2}\right)}{\frac{1}{2\phi^2} + \frac{n}{2\sigma^2}}\mu\right)\right\}$$

$$= \exp\left\{ -\frac{(\sigma^2 + n\phi^2)}{2\phi^2\sigma^2}\left(\mu^2 - 2\left(\frac{\psi\sigma^2 + \phi^2\sum x_i}{\sigma^2 + n\phi^2}\right)\mu\right)\right\}$$

Is proportional to a normal pdf with mean

$$\frac{\psi\sigma^2 + \phi^2\sum x_i}{\sigma^2 + n\phi^2} = \mu_1$$

and variance $\dfrac{\phi^2\sigma^2}{\sigma^2 + n\phi^2} = \sigma_1^2$

Posterior $\mu|\underline{x} \sim N(\mu_1, \sigma_1^2)$

Credible interval:

$$\mathbb{P}\left(-1.96 < \frac{\mu-\mu_1}{\sigma_1} < 1.96\right) = 0.95$$

central
$\therefore$ 95% central credible interval is:

$$(\mu_1 - 1.96\sigma_1, \mu_1 + 1.96\sigma_1)$$

$$\frac{\psi\sigma^2 + \phi^2 \Sigma x_i}{\sigma^2 + n\phi^2} = \frac{\psi\frac{\sigma^2}{n} + \phi^2\overline{x}}{\sigma^2/n + \phi^2} = \mu_1$$

variance

$$\frac{\phi^2\sigma^2}{\sigma^2+n\phi^2} = \frac{\phi^2\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}+\phi^2} = \sigma_1^2$$

$$\lim_{n\to\infty} \mu_1 = \overline{x}$$

$$\lim_{n\to\infty} \sigma_1^2 = \frac{\sigma^2}{n}$$

as $n \to \infty$ $\mu | \underline{x} \sim N\left(\overline{x}, \frac{\sigma^2}{n}\right)$