STAT0017: Selected Topics In Statistics

Topic 2: “Dependence modelling in finance using copulas”

Lecture 4

2019

Last week:

- We considered one kind of dependence measure: the coefficient of tail dependence.
- We also considered survival copula and survival function of a copula C.
- We looked at time series models that take into account ARCH/-GARCH effects present in financial asset returns.
- We also looked at two test for detecting serial dependence and ARCH/GARCH effects.

Today:

- We are going to consider multivariate copulas.
- We will briefly discuss Multivariate Archimedean Copulas and comment on their limitations.
- We will then introduce a more flexible class called Vine copulas.

Material and text books relevant to Lecture 4

References

1. Dorota Kurowicka and Roger Cooke. *Uncertainty Analysis with High Dimensional Dependence Modelling.*
   John Wiley Sons, Ltd, 2006
2. Kjersti Aas. Modelling the dependence structure of financial assets: A survey of four copulas.
   Norwegian Computing Center, 2004
3. R.B. Nelsen. *An Introduction to Copulas.*
   Springer Series in Statistics. Springer, 2006
   See **Chapter 4.6**

Multivariate Archimedean Copulas

- We have considered construction of bivariate Archimedean copulas.
- In financial applications it is often of interest to model the dependence structure of more than 2 variables.
- We have seen that elliptical copulas can be easily extended to $d > 2$ dimensions.

Multivariate elliptical Copula

Multivariate Gaussian Copula

$$C(u_1 \dots u_d) = \Phi^d_{\mathbf{R}}\left(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)\right)$$
$$= \frac{1}{\sqrt{(2\pi)^d |\mathbf{R}|}} \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} \exp\left(-\frac{1}{2}\mathbf{s}^T \mathbf{R}^{-1} \mathbf{s}\right) ds_1 \dots ds_d$$

where $\Phi^d_{\mathbf{R}}$ is the joint distribution function of $\mathbf{Y}$ with standard normal components, $\mathbf{R}$ is the linear correlation matrix.

Multivariate Student's t-copula

$$C(u_1 \dots u_d) = t^d_{v,\mathbf{R}}\left(t_v^{-1}(u_1), \dots, t_v^{-1}(u_d)\right)$$
$$= \frac{\Gamma\left(\frac{v+d}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{(\pi v)^d |\mathbf{R}|}} \int_{-\infty}^{t_v^{-1}(u_1)} \dots \int_{-\infty}^{t_v^{-1}(u_d)} \left(1 + \frac{\mathbf{s}^T \mathbf{R}^{-1} \mathbf{s}}{v}\right)^{-\frac{v+d}{2}} ds_1 \dots ds_d$$

where $t^d_{v,\mathbf{R}}$ is the joint distribution function of $\mathbf{Y}$ with degrees of freedom ν, $\mathbf{R}$ is the linear correlation matrix.

Multivariate Archimedean Copulas

The Archimedean copula can be extended to the multivariate case:

$$C(u_1, \ldots, u_d) = \varphi^{[-1]}(\varphi(u_1) + \cdots + \varphi(u_d)) \quad u_1, \ldots, u_d \in [0,1] \quad (3.1)$$

- The functions $C^d(u_1, \ldots, u_d)$ in (3.1) are the serial iterates of the Archimedean 2-copula generated by φ.
- That is, if $C^2(u_1, u_2) = C(u_1, u_2) = \varphi^{[-1]}(\varphi(u_1) + \varphi(u_2))$, then for $d \geq 3$:

$$C^d(u_1, \ldots, u_d) = C\left(C^{d-1}(u_1, \ldots, u_{d-1}), u_d\right)$$

Multivariate Archimedean Copulas

Example 3.1

Using $\varphi(t) = -\log(t)$ and $\varphi^{-1}(t) = \exp(-t)$ in (3.1) generates Π^d for $d > 2$:

$$\Pi^d(u_1, \dots, u_d) = \exp(-[(-\ln u_1) + \cdots + (-\ln u_n)]) = u_1 \cdots u_d$$

- However, this technique of composing Archimedean copulas generally fails.

Example 3.2

Using $\varphi(t) = 1 - t$ in (3.1) generates W^d, which fails to be a copula for any $d > 2$.

$$\varphi^{[-1]}(t) = \left\{ \begin{array}{cc} 1 - t & \text{for } t \in \mathbb{I} \\ 0 & \text{for } t > 1 \end{array} \right\} = \max\{1 - t, 0\}$$

We saw that **Definition 1.2** in Lecture 2 outlines the properties of φ (continuous, strictly decreasing and convex, with $\varphi(1) = 0$) needed for C^d in (3.1) to be a copula for $d = 2$:

Definition 1.2 (Lecture 2)

Let φ be a continuous, strictly decreasing function from $\mathbb{I}$ to $[0, \infty]$ such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ defined in Definition 1.1. Then the function C from $\mathbb{I}^2$ to $\mathbb{I}$ given by (3.1) is a copula if and only if φ is convex.

This begs the question:

What additional properties of φ and $\varphi^{[-1]}$ are required to guarantee that C^d in (3.1) is a copula for $n \geq 3$?

Multivariate Archimedean Copulas

Definition 3.1

A function $g(t)$ is *completely monotonic* on an interval J if it is continuous there and has derivatives of all orders that alternate in sign. That is, it satisfies the following condition:

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0$$

for all t in the interior of J and $k = 0, 1, 2, \ldots$

As a consequence, if $g(t)$ is completely monotonic on $[0, \infty)$ and $g(c) = 0$ for some (finite) $c > 0$, then g must be essentially zero on $[0, \infty)$.

Multivariate Archimedean Copulas

- Thus, if the pseudo-inverse $\varphi^{[-1]}$ of an Archimedean generator φ is completely monotonic, it must be positive on $[0, \infty)$.
- That is, φ is strict and $\varphi^{[-1]} = \varphi^{-1}$.

Definition 3.2

Let φ be a continuous strictly decreasing function from $\mathbb{I}$ to $[0, \infty]$ such that $\varphi(0) = \infty$ and $\varphi(1) = 0$, and let φ^{-1} denote the inverse of φ. If C^d is the function from $\mathbb{I}^d$ to $\mathbb{I}$ given by (4.6.1), then C^d is a d-copula for all $d \geq 2$ if and only if φ^{-1} is completely monotonic on $[0, \infty)$.

Example (Gumbel-Hougaard copula)

Example 3.3

Let $\varphi(t) = (-\ln t)^\theta$, $\theta \geq 1$, which can be used to generate the bivariate Gumbel-Hougaard copula. Here $\varphi^{-1} = \exp(-t^{\frac{1}{\theta}})$, which is completely monotonic. Therefore, the Gumbel-Hougaard family of 2-copulas can be generalized to a family of d-copulas for $\theta \geq 1$ and any $d \geq 2$:

$$C(u_1, \ldots, u_d) = \left(u_1^{-\theta} + u_2^{-\theta} + \cdots + u_d^{-\theta} - d + 1\right)^{-1/\theta}$$

Example (Clayton copula)

Example 3.4

Let $\varphi(t) = t^{-\theta} - 1$ for $\theta > 0$. Then $\varphi^{-1} = (1+t)^{-\theta}$, which is a completely monotonic function on $[0, \infty)$. Therefore, the family of bivariate Clayton copulas can be generalized to a family of d-copulas for $\theta > 0$ and any $d \geq 2$:

$$C(u_1, \ldots, u_d) = \left(u_1^{-\theta} + u_2^{-\theta} + \cdots + u_d^{-\theta} - d + 1\right)^{-1/\theta}$$

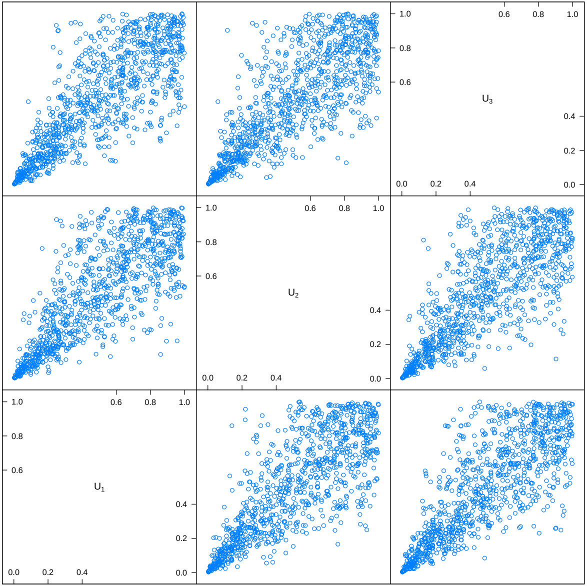
Limitations of this approach

- We have seen that it is reasonably simple to generate Archimedean d-copulas.
- However, there are limitations associated with them.
- In general all the k-margins of an Archimedean d-copula are identical.
- In addition, there are usually only one or two parameters which limits the nature of the dependence structure they can capture.

Limitations of this approach

- In the case of Archimedean copulas C^d is a symmetric function in its arguments.
- Due to this property, Archimedean copulas are sometimes called *exchangeable*.
- The implications of exchangeability property is that the $y = x$ axis is the axis of reflection symmetry for the underlying distribution.
- Another consequence of exchangeability is that, given a d-variate Archimedean copula and $m \in \{2, \ldots, d-1\}$, any two m-variate margins from this copula describe the same m-variate distribution.
- For example, for $d = 3$ and $m = 2$, and assuming the joint distribution of (U_1, U_2, U_3) is a d-variate Archimedean copula, the joint distribution of (U_1, U_2) is equal to the joint distribution of (U_1, U_3) or (U_2, U_3).

Archimedean copula d=3

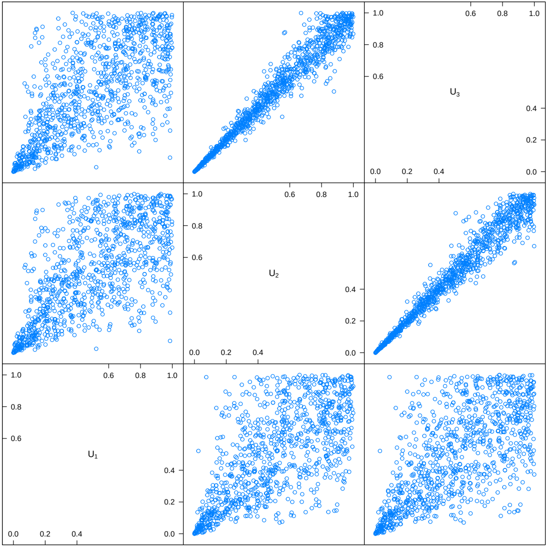


Nested Archimedean copula

- Nested Archimedean copulas relax the exchangeability property.
- They can be obtained by plugging in Archimedean copulas into each other.
- Let's consider the $d = 3$ case, then the bivariate Archimedean copula C_{12} can be plugged into a bivariate Archimedean copula C_{123}:

$$C_{123}\left(C_{12}\left(u_1, u_2\right), u_3\right) = \psi_{123}^{-1}\left(\psi_{123}\left(\psi_{12}^{-1}\left(\psi_{12}\left(u_1\right) + \psi_{12}\left(u_2\right)\right)\right) + \psi_{123}\left(u_3\right)\right)$$

Nested Archimedean copula d=3



- Vine copulas

Regular Vine Copulas

Sklar's theorem allows modelling the dependency structure separately from the marginals.

Definition 3.3

A set of linked trees $\mathcal{V} = \{T_1, \ldots, T_{d-1}\}$ is a regular vine on n elements if:

1. T_1 is a tree with nodes $N_1 = \{1, \ldots, d\}$ and a set of edges E_1.
2. T_i is a tree with nodes $N_i = E_{i-1}$ and edge set E_i for $i = 2, \ldots, d-1$.
3. If $a = \{a_1, a_2\}$ and $b = \{b_1, b_2\}$ are two nodes in N_i connected by an edge, then exactly one element of a equals one element of b for $i = 2, \ldots, d-1$.

Pair-copula construction (PCC)

A useful standard result from statistics says that the joint density can be decomposed by conditioning as follows:

$$f_{1,\dots,d}(x_1, \dots, x_d) = f_1(x_1) \cdot f_{2|1}(x_2|x_1) \cdots f_{d|1,\dots,d-1}(x_d|x_1, \dots, x_{d-1})$$

Pair-copula construction (PCC)

Combining two previous results, we can represent 3-dimensional joint density as follows:

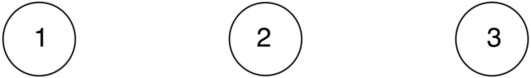
$$
\begin{aligned}
f_{1,2,3}(x_1, x_2, x_3) \;=\; & \underbrace{f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3)}_{\text{marginal densities}} \\
& \times \underbrace{c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3))}_{\text{unconditional pair copula densities}} \\
& \times \underbrace{c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))}_{\text{conditional pair copula density}}
\end{aligned}
$$

The decomposition is **not unique**, 3 possible PCC constructions.

Pair-copula construction (PCC)

In order to organize them, Bedford and Cooke (2001) introduced graphical models called **regular vines (R-vines)**.

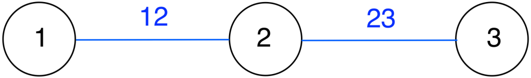
$$
\begin{aligned}
f_{1,2,3}(x_1, x_2, x_3) &= f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \\
&\quad \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \\
&\quad \cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))
\end{aligned}
$$



Pair-copula construction (PCC)

In order to organize them, Bedford and Cooke (2001) introduced graphical models called **regular vines (R-vines)**.

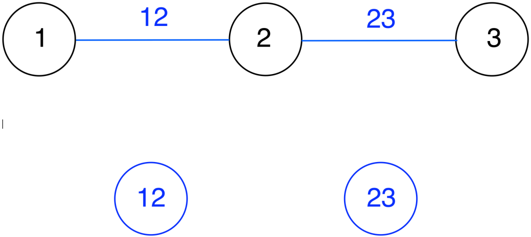
$$
\begin{aligned}
f_{1,2,3}(x_1, x_2, x_3) &= f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \\
&\quad \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \\
&\quad \cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))
\end{aligned}
$$



Pair-copula construction (PCC)

In order to organize them, Bedford and Cooke (2001) introduced graphical models called **regular vines (R-vines)**.

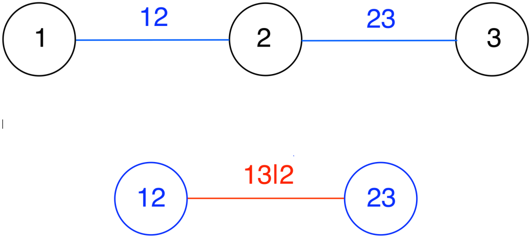
$$
\begin{aligned}
f_{1,2,3}(x_1, x_2, x_3) \quad = \quad & f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \\
& \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \\
& \cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))
\end{aligned}
$$



Pair-copula construction (PCC)

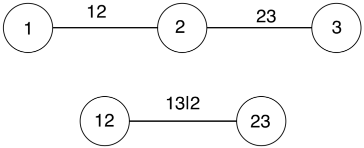
In order to organize them, Bedford and Cooke (2001) introduced graphical models called **regular vines (R-vines)**.

$$
\begin{aligned}
f_{1,2,3}(x_1, x_2, x_3) \quad = \quad & f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \\
& \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \\
& \cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))
\end{aligned}
$$



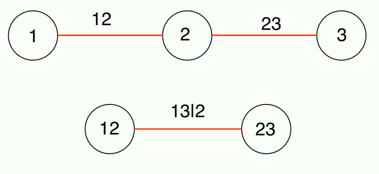
Model selection

Model = vine structure + copula families + copula parameters



Model selection

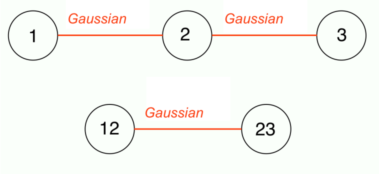
Model = vine structure + copula families + copula parameters



Gaussian
Student's
Survival Gumbel
Clayton
Frank
Joe
BB1
BB6
BB7
BB8

Model selection

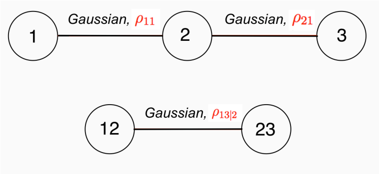
Model = vine structure + copula families + copula parameters

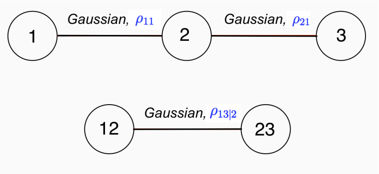


Gaussian
Student's
Survival Gumbel
Clayton
Frank
Joe
BB1
BB6
BB7
BB8

Model selection

Model = vine structure + copula families + copula parameters





Model selection

The joint log-likelihood of the model given the observed data $\mathbf{x} = (\mathbf{x_1}, ..., \mathbf{x_T})'$ is as follows

$$
\begin{aligned}
L(\mathscr{V}, \mathrm{B}, \boldsymbol{\theta}, \boldsymbol{\alpha}) &= \sum_{t=1}^{T} \log f_{1,2,3}(\boldsymbol{x_t} | \mathscr{V}, \mathrm{B}, \boldsymbol{\theta}, \boldsymbol{\alpha}) \\
&= \sum_{t=1}^{T} \sum_{i=1}^{3} \log f_i(x_{it} | \alpha_i) + \\
&\quad + \sum_{t=1}^{T} \log c_{1,2,3}(F_1(x_1), F_2(x_2), F_3(x_3)) | \mathscr{V}, \mathrm{B}, \boldsymbol{\theta}, \alpha_i)
\end{aligned}
$$

Definition 3.4

A regular vine copula $C = ((\mathcal{V}), B(\mathcal{V}), \boldsymbol{\theta}(B(\mathcal{V})))$ in d dimensions is a multivariate distribution function such that for a random vector $\boldsymbol{U} = (U_1, \ldots, U_d)' \sim C$ with uniform margins:

1. $\mathcal{V}$ is a regular vine on n elements.
2. $B(\mathcal{V}) = \{C_{i(e),j(e)|D(e)} | e \in E_m, m = 1, \ldots, d-1\}$ is a set of $d(d-1)/2$ copula families that identify the conditional distributions of $U_{i(e)}, U_{j(e)} | \boldsymbol{U}_{D(e)}$.
3. $\boldsymbol{\theta}(B(\mathcal{V})) = \{\boldsymbol{\theta}_{i(e),j(e)|D(e)} | e \in E_m, m = 1, \ldots, d-1\}$ is the set of parameter vectors associated with the copulas in $B(\mathcal{V})$.

Vine copula density

The probability density function $f_{1:d}$ of $\mathbf{x} = (x_1, \ldots, x_d)' \in \mathbb{R}^d$ of a d-dimensional regular vine distribution $F_{1:d}$ is as follows:

$$f_{1:d}(\boldsymbol{x}|\mathcal{V}, B, \boldsymbol{\theta}) = \left[\prod_{m=1}^{d-1} \prod_{e \in E_m} C_{i(e),j(e)|D(e)} \left(F_{i(e)|D(e)}, F_{j(e)|D(e)} | \boldsymbol{\theta}_{i(e),j(e)|D(e)}\right)\right] \times \left[\prod_{k=1}^{d} f_k\left(x_k\right)\right] \tag{3.2}$$

where $F_{i(e)|D(e)} := F_{i(e)|D(e)}\left(x_{i(e)}|\boldsymbol{x}_{D(e)}\right)$ and $F_{j(e)|D(e)} := F_{j(e)|D(e)}\left(x_{j(e)}|\boldsymbol{x}_{D(e)}\right)$. These conditional distribution functions are determined as follows:

$$\begin{aligned} F_{i(e)|D(e)}\left(x_{i(e)}|\boldsymbol{x}_{D(e)}\right) &= F_{\mathscr{C}_{e,a}|D_e}\left(x_{\mathscr{C}_{e,a}}|\boldsymbol{x}_{D_e}\right) \\ &= \frac{\partial C_{\mathscr{C}_{a,a_1},\mathscr{C}_{a,a_2}}\left(F_{\mathscr{C}_{a,a_1}|D_a}\left(x_{\mathscr{C}_{a,a_1}}|\boldsymbol{x}_{D_a}\right), F_{\mathscr{C}_{a,a_2}|D_a}\left(x_{\mathscr{C}_{a,a_2}}|\boldsymbol{x}_{D_a}\right)\right)}{\partial F_{\mathscr{C}_{a,a_2}|D_a}\left(x_{\mathscr{C}_{a,a_2}}|\boldsymbol{x}_{D_a}\right)} \end{aligned}$$

Next Week:

- We are going to consider time-varying dependence structure.
- We will also consider factor copulas
- Student presentation