Our likelihood ratio test has rejection region

$$C = \left\{ \underline{x} : \frac{h_1(\underline{x})}{h_0(\underline{x})} > k \right\}$$

Define $\phi_C(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \in C \\ 0 & \text{if } \underline{x} \notin C \end{cases}$

Let D be the critical region for any other test of the same hypotheses of size α.

Define $\phi_D(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \in D \\ 0 & \text{if } \underline{x} \notin D \end{cases}$

If $\phi_C(\underline{x}) = 1 \Rightarrow \dfrac{h_1(\underline{x})}{h_0(\underline{x})} > k$

$$\Rightarrow h_1(\underline{x}) - k h_0(\underline{x}) > 0$$

If $\phi_C(\underline{x}) = 0 \Rightarrow h_1(\underline{x}) - k h_0(\underline{x}) \leq 0$

Consider

$$(\phi_C(\underline{x}) - \phi_D(\underline{x}))(h_1(\underline{x}) - k h_0(\underline{x})) \geq 0 \quad \forall \underline{x}.$$

(always the product of two terms with the same sign)

$h_1(\underline{x}) = f_1(\underline{x}) \leftarrow$ joint density of $\underline{x}$ under H_1

$h_0(\underline{x}) = f_0(\underline{x}) \leftarrow$ joint density of $\underline{X}$ under H_0.

$$\int_{\mathcal{X}} (\phi_c(\underline{x}) - \phi_D(\underline{x}))(h_1(\underline{x}) - k h_0(\underline{x}))\, d\underline{x} \geq 0$$

$$\int_{\mathcal{X}} \phi_c(\underline{x}) h_1(\underline{x})\, d\underline{x} - \int_{\mathcal{X}} \phi_D(\underline{x}) h_1(\underline{x})\, d\underline{x}$$

$$- k \left(\int_{\mathcal{X}} \phi_c(\underline{x}) h_0(\underline{x})\, d\underline{x} - \int_{\mathcal{X}} \phi_D(\underline{x}) h_0(\underline{x})\, d\underline{x} \right) \geq 0$$

$$\int_{\mathcal{X}} \phi_c(\underline{x}) h_1(\underline{x})\, d\underline{x} - \int_{\mathcal{X}} \phi_D(\underline{x}) h_1(\underline{x})\, d\underline{x}$$

$$- k[\cancel{\alpha - \alpha}^{\,0}] \geq 0$$

$$\underbrace{\int_{\mathcal{X}} \phi_c(\underline{x}) h_1(\underline{x})\, d\underline{x}} - \int_{\mathcal{X}} \phi_D(\underline{x}) h_1(\underline{x})\, d\underline{x} \geq 0$$

$$\mathbb{P}(\underline{X} \in C \mid H_1) - \mathbb{P}(\underline{X} \in D \mid H_1) \geq 0$$

Power of test with crit. region C

Power of test with crit. region D

$\therefore$ Most powerful test of size α has critical region C.

$H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$ $(\mu_1 > \mu_0)$

Let $X_1, \ldots, X_n$ $X_i \sim N(\mu, \sigma^2)$

Under H_0

$X_i \sim N(\mu_0, \sigma^2)$

Likelihood f'n

$$h(\mu_0 | \underline{X}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-1}{2\sigma^2}(X_i - \mu_0)^2\right\}$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_0)^2\right\}$$

Under H_1:

$X_i \sim N(\mu_1, \sigma^2)$

$$h(\mu_1 | \underline{X}) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_1)^2\right\}$$

The most powerful test of size α of H_0 vs. H_1 has rejection region given by:

$$\frac{h_1(\mu_1 | \underline{X})}{h_0(\mu_0 | \underline{X})} > k$$

$$\Rightarrow \frac{\cancel{(2\pi\sigma^2)^{-n/2}} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_1)^2\right\}}{\cancel{(2\pi\sigma^2)^{-n/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_0)^2\right\}} > k$$

$$\Rightarrow \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_1)^2 + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_0)^2\right\} > k$$

$$\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(\cancel{X_i^2} - 2\mu_1 X_i + \mu_1) - \sum_i(\cancel{X_i^2} - 2\mu_0 X_i + \mu_0^2)\right]\right\} > k$$

$$\exp\left\{-\frac{1}{2\sigma^2}\left[-2[\mu_1+\mu_0]\sum_{i=1}^{n}X_i + n(\mu_\phi^2-\mu_0^2)\right]\right\} > k \quad (+)$$

This is a monotonic increasing function $\sum_{i=1}^{n} X_i$.

$$\therefore (+) \Rightarrow \sum_{i=1}^{n} X_i > c$$

$$\Rightarrow \bar{X} > c'$$

Under H_0:

$$X_i \sim N(\mu_0, \sigma^2)$$

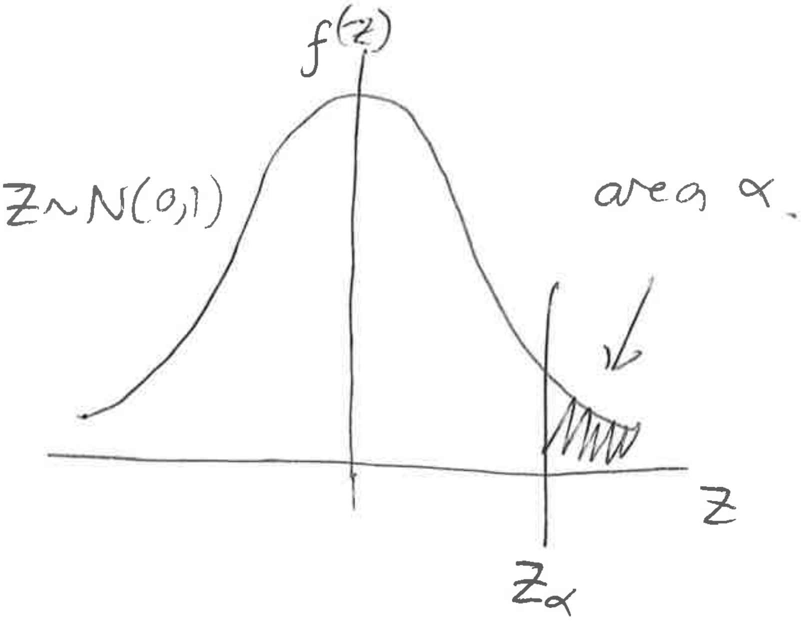
$$\Rightarrow \bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

$$\therefore \mathbb{P}(\bar{X} > c' \mid H_0) = \alpha$$

$$\Rightarrow \mathbb{P}\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > \frac{c'-\mu_0}{\sigma/\sqrt{n}}\right) = \alpha$$

$$\frac{c' - \mu_0}{\sigma/\sqrt{n}} = z_\alpha$$

$$c' = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$



$\therefore$ Reject H_0 if

$$\bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}.$$

$$C = \left\{ \underline{x} : \bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$$

(b)

Test has size α ✓

Is the test most powerful for <u>all</u> $\mu_1 > \mu_0$?

Test is uniformly most powerful as C does <u>not</u> depend on μ_1.

$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

$X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$$L(\theta | \underline{X}) = \theta^{\Sigma x_i}(1-\theta)^{n - \Sigma x_i}$$

Log-likelihood:

$$\ell(\theta | \underline{X}) = (\Sigma X_i) \log \theta + (n - \Sigma X_i) \log(1-\theta)$$

MLE: $\hat{\theta} = \dfrac{\sum_{i=1}^{n} X_i}{n}$.

Test statistic. Let $R = \sum_{i=1}^{n} X_i$ (no. of successes)

$$X^2 = 2[\ell(\hat{\theta} | \underline{X}) - \ell(\theta_0 | \underline{X})]$$

$$= 2\left\{R \log\left(\frac{R}{n}\right) + (n-R)\log\left(1 - \frac{R}{n}\right) - R\log\theta_0 - (n-R)\log(1-\theta_0)\right\}$$

$$= 2\left\{R\left[\log\left(\frac{R}{n}\right) - \log\theta_0\right] + (n-R)\left[\log\left(1-\frac{R}{n}\right) - \log(1-\theta_0)\right]\right\}$$

Under H_0: $X^2 \sim \chi^2_1$

$\therefore$ For a test of size α, reject H_0 if $X^2 > \chi^2_1(1-\alpha)$

(b) $r = 18$ voters who will vote for the mayor

$n = 80$

$H_0: \theta = 0.3$ vs. $H_1: \theta \neq 0.3$

$\therefore \theta_0 = 0.3$

Test statistic: 2.2645

$\chi^2_1(0.05) = 3.84$

$\therefore X^2 = 2.2645 < 3.84$

$\Rightarrow$ Don't reject H_0.

$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

$$\ell(\theta|\underline{X}) = R\log\theta + (n-R)\log(1-\theta)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{R}{\theta} - \frac{(n-R)}{(1-\theta)}$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{R}{\theta^2} + \frac{(n-R)}{(1-\theta)^2}$$

$$\mathcal{I}(\theta) = \mathbb{E}\left[-\frac{\partial^2 \ell}{\partial \theta^2}\right] \qquad \mathbb{E}[R] = n\theta.$$

$$= \cancel{\theta(1-\theta)} \frac{n}{\theta(1-\theta)}$$

For a Wald test of size α reject H_0 if

$$\mathcal{I}(\theta_0)(\hat{\theta} - \theta_0)^2 > \chi^2_1(1-\alpha)$$

$$W = \frac{n}{\theta_0(1-\theta_0)}\left(\frac{R}{n} - \theta_0\right)^2 > \chi^2_1(1-\alpha).$$

Election example $R = 18$

$n = 80$

$\theta_0 = 0.3$

$W = 2.14857 < 3.84$

Retain H_0.

$H_0 : \theta = \theta_0$

vs.

$H_1 : \theta \neq \theta_0$

$X_1, \dots, X_n \overset{iid}{\sim} Bern(\theta)$

$$U(\theta; X) = \frac{\partial \ell}{\partial \theta}$$

$$= \frac{R}{\theta} - \frac{(n-R)}{1-\theta}$$

$$\mathcal{I}(\theta) = \frac{n}{\theta(1-\theta)}$$

$\therefore$ Score test statistic:

$$X^2 = \frac{[U(\theta_0; \underline{X})]^2}{I(\theta_0)}$$

$$= \frac{\theta_0(1-\theta_0)}{n}\left[\frac{R}{\theta_0} - \frac{(n-R)}{1-\theta_0}\right]^2 \sim \chi^2_1$$

under H_0.

(b) $r = 18$

$n = 80$

$\theta_0 = 0.3$

$X^2 = 2.14857 < 3.84 \implies$ Retain H_0.