* Lecture 8: pp. 18-21

$$\left.\begin{aligned} Y_{10} &= Y_9 + \varepsilon_{10} \\ Y_9 &= Y_8 + \varepsilon_9 \\ &\vdots \\ Y_3 &= Y_2 + \varepsilon_3 \\ Y_2 &= \underbrace{Y_1}_{=0} + \varepsilon_2 \end{aligned}\right\}$$

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

Thus $Y_{10} = \sum_{t=2}^{10} \varepsilon_t$

1. $$\begin{aligned} Var(Y_{10}) &= Var\left(\sum_{t=2}^{10} \varepsilon_t\right) \\ &= \sum_{t=2}^{10} var(\varepsilon_t) \\ &= \sum_{t=2}^{10} \sigma^2 = \underbrace{\sigma^2 + \dots + \sigma^2}_{9 \text{ times}} \\ &= 9\sigma^2 \end{aligned}$$

2. $$\mathbb{E}\{Y_{10}\} = \mathbb{E}\left\{\sum_{t=2}^{10} \varepsilon_t\right\} = \sum_{t=2}^{10} \mathbb{E}\{\varepsilon_t\} = 0 + \dots + 0 = 0$$

$$Y_{10} \sim N(0, 9\sigma^2)$$

Conditional distribution

1. $\mathbb{E}\{Y_{10} \mid Y_1 \dots Y_9\} = \mathbb{E}\{Y_9 \mid Y_1 \dots Y_9\} + \mathbb{E}\{\varepsilon_{10} \mid Y_1 \dots Y_9\}$

$= Y_9 + \mathbb{E}\{\varepsilon_{10}\}$ by independence of ε_{10} and $Y_1 \dots Y_9$

$= Y_9 + 0$

$= Y_9$

2. $\mathrm{Var}(Y_{10} \mid Y_1 \dots Y_9) = \mathrm{Var}(Y_9 \mid Y_1 \dots Y_9) + \mathrm{Var}(\varepsilon_{10} \mid Y_1 \dots Y_9)$

$= 0 + \mathrm{Var}(\varepsilon_{10})$ by independence

$= \sigma^2$

Thus $Y_{10} \mid Y_1 \dots Y_9 \sim N(Y_9, \sigma^2)$

* Lecture 8: p.33

To understand how the model works, a definition of the conditional variance of a r.v. u_t is required.

$$\sigma_t^2 = Var(u_t \mid u_{t-1}, u_{t-2}, \dots)$$
$$= \mathbb{E}\{(u_t - \mathbb{E}(u_t))^2 \mid u_{t-1}, u_{t-2}, \dots\}$$
$$= \mathbb{E}\{u_t^2 \mid u_{t-1}, u_{t-2}, \dots\}$$
$$= \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \dots + \alpha_p u_{t-p}^2$$

* GARCH(1,1) as an ARMA(1,1)

Note that u_t^2 and σ_t^2 at time t are not the same:

$$u_t^2 - \sigma_t^2 = V_t$$

or

$$\sigma_t^2 = u_t^2 - V_t$$

substitute in
$$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$
$$u_t^2 - V_t = \omega + \alpha_1 u_{t-1}^2 + \beta_1 (u_{t-1}^2 - V_{t-1})$$
$$u_t^2 = \omega + (\alpha_1 + \beta_1) u_{t-1}^2 + V_t - \beta_1 V_{t-1}$$

* GARCH(1,1) as ARCH(∞) p. 47

$$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

$$\sigma_{t-1}^2 = \omega + \alpha_1 u_{t-2}^2 + \beta_1 \sigma_{t-2}^2$$

$$\sigma_{t-2}^2 = \omega + \alpha_1 u_{t-3}^2 + \beta_1 \sigma_{t-3}^2$$

Substituting

$$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \beta_1\left(\omega + \alpha_1 u_{t-2}^2 + \beta_1\left(\omega + \alpha_1 u_{t-3}^2 + \beta_1 \sigma_{t-3}^2\right)\right)$$

$$= \omega + \alpha_1 u_{t-1}^2 + \beta_1 \omega + \beta_1 \alpha_1 u_{t-2}^2 + \beta_1^2 \omega + \beta_1^2 \alpha_1 u_{t-3}^2 + \beta_1^3 \sigma_{t-3}^2$$

After an infinite amount of successive substitutions

$$\sigma_t^2 = \underbrace{\omega(1 + \beta_1 + \beta_1^2 + \ldots)} + \alpha_1 u_{t-1}^2 (1 + L\beta_1 + L^2 \beta_1^2 + \ldots) + \underbrace{\beta_1^\infty}_{\beta_1^\infty \to 0} \sigma_0^2$$

Hence

$$\sigma_t^2 = \gamma + \underbrace{\alpha_1}_{\text{let } \alpha = \gamma_1} u_{t-1}^2 (1 + L\beta_1 + L^2 \beta^2 + \ldots)$$

$$= \gamma + \gamma_1 u_{t-1}^2 + \underbrace{\gamma_2}_{= \alpha_1 \beta_1} u_{t-2}^2 + \gamma_3 u_{t-3}^2 + \ldots$$

Thus, GARCH(1,1) containing only 3 parameters in the conditional variance equation is a very parsimonious model that allows ARCH(∞)