UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**STAT3004**
ASSESSMENT PATTERN	:	**STAT3004A**
MODULE NAME	:	**Decision and Risk**
DATE	:	**24 May 2016**
TIME	:	**2:30 pm**
TIME ALLOWED	:	**2 hours 30 mins**

This paper is suitable for candidates who attended classes for this module in the following academic year(s):

2015/16

2016 Examination Paper for STAT3004

Answer ALL questions. Section A carries 40% of the total marks and Section B carries 60%. The relative weights attached to each question are as follows: A1 (20), A2 (10), A3 (10), B1 (30), B2 (30). The numbers in square brackets indicate the relative weights attached to each part question.

Formula Sheet

- If X has a Gamma(α, β) distribution then:

$$p(X|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} X^{\alpha-1} e^{-\beta X}, \quad X > 0$$

where $\Gamma(\alpha)$ is the Gamma function. The relationship between the parameters α, β and the mean/variance of X are:

$$E[X] = \frac{\alpha}{\beta}, \quad Var[X] = \frac{\alpha}{\beta^2}$$

$$\alpha = \frac{E[X]^2}{Var[X]}, \quad \beta = \frac{E[X]}{Var[X]}$$

- If X has a Normal distribution $N(\mu, \sigma^2)$ then:

$$p(X|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X-\mu)^2}{2\sigma^2}}, \quad -\infty < X < \infty$$

- If X has an Inverse-Gamma distribution $IG(\alpha, \beta)$ then:

$$p(X|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} X^{-\alpha-1} e^{-\beta/X}, \quad X > 0$$

- if X has a Generalized Pareto Distribution with threshold u and parameters K and σ, then the corresponding cumulative distribution function is:

$$p(X \leq D|u,k,\sigma) = 1 - \left(1 - k\frac{D-u}{\sigma}\right)^{(1/k)}, \quad X > u$$

The method of moments estimates of k and σ based on a sample $X_1, \ldots, X_n$ where all X_i are greater than u are:

$$\hat{\sigma} = \bar{X}'(\bar{X}'^2/s^2 + 1)/2$$
$$\hat{k} = (\bar{X}'^2/s^2 - 1)/2$$

where $\bar{X}'$ and s^2 denote the sample mean and variance:

Turn Over

$$\bar{X}' = \frac{1}{n}\sum_{i=1}^{n} X_i', \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i' - \bar{X}')^2$$

for the transformed observations $X_i' = X_i - u$.

- The Trapezoid Rule approximates definite integrals by:

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$$

Section A

A1 A city has recently experienced high levels of flooding, and would like to quantify the risk of future floods occurring. Assume that the inter-event times between floods are independent and identically distributed, and follow an Exponential(λ) distribution. The most recent 5 inter-event times (measured in years) are $Y_1 = 3.2$, $Y_2 = 5.2$, $Y_3 = 1.1$, $Y_4 = 4.2$, $Y_5 = 3.6$. Let $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4, Y_5)$ denote this available historical data.

a) Assuming a Gamma(α, β) prior for λ, show that the posterior distribution for λ also has a Gamma distribution. For the data above, give the parameters of this posterior distribution in terms of α and β. [5]

b) Using a noninformative Gamma(0,0) prior, write down the expected value of λ given the above data [2]

c) Suppose that the most recent flood occurred today. We would like to predict the time until the next flood. Let $\tilde{Y}$ denote how many years pass until the next flood. Show that the predictive distribution $p(\tilde{Y}|\mathbf{Y})$ is:

$$p(\tilde{Y}|\mathbf{Y}) = \frac{\tilde{\beta}^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha})} \frac{\Gamma(\tilde{\alpha}+1)}{(\tilde{\beta}+\tilde{Y})^{\tilde{\alpha}+1}}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the parameters of the posterior distribution for λ based on the above data and prior. [5]

d) Write down the integral for computing the probability that the next flood will occur in less than 5 years, using a Gamma$(0,0)$ prior and the above data. You do not need to evaluate the integral. [4]

e) Use the Trapezoid rule to approximate the integral from d), and hence find (approximately) the probability that the next flood will occur in less than 5 years. Remember that $\Gamma(n) = (n-1)! = 1 \times 2 \times \ldots \times n-1$ when n is an integer. [4]

A2 a) Explain what is meant by the 99% Value-at-Risk of a financial portfolio. [2]

b) Write down the general form of the GARCH(1,1) model and explain what common feature of financial data such a model is designed to capture . [2]

c) The log returns of a particular stock have a $N(0, \sigma^2)$ distribution, where the variance σ^2 obeys the following ARCH(2) model:

$$\sigma_t^2 = \beta_0 + \beta_1 Y_{t-1}^2 + \beta_2 Y_{t-2}^2$$

Suppose the log returns on the last 2 days have been $Y_1 = 0.4$, $Y_2 = -0.1$. Write down an integral expression to compute the 99% Value-at-Risk on day 3, assuming that the ARCH parameters have known values $\beta_0 = 0.5$, $\beta_1 = 0.2$ and $\beta_2 = 0.1$. You do not need to evaluate the integral. [3]

d) Now suppose that although β_0 and β_1 are known and equal to the above values, β_2 is unknown. Bayesian inference has been performed based on some earlier historical data, and the resulting **posterior** distribution for β_2 is $\beta_2 \sim N(0.1, 0.2^2)$. Write down an integral expression to compute the 99% Value-at-Risk on day 3 based on this posterior distribution. You do not need to evaluate any parts of the integral. [3]

A3 Two coins are placed in a box. One is unbiased, the other has a 70% probability of landing heads. You pick one at random, toss it 5 times, and then need to decide whether the coin you picked is unbiased. The losses, associated with i) mistakenly claiming that the unbiased coin is biased, and ii) mistakenly claiming that the biased coin is unbiased, are equal. If the coin is correctly classified as biased/unbiased then there is zero loss. Let action a_0 be claiming the coin is unbiased, and action a_1 be claiming the coin is biased.

a) Assuming you observe the sequence of tosses Y=HHHTH (where H=Head and T=Tail), write down the posterior risk ratio $R(a_0|Y)/R(a_1|Y)$, and hence show that action a_1 is optimal in the sense of minimising the risk. [5]

b) Let c_0 denote the cost of mistakenly classifying a biased coin as being unbiased. Also, let kc_0 denote the cost of mistakenly classifying an unbiased coin as being biased. For what values of k would it optimal to classify the coin as unbiased, given the above sequence of tosses? [5]

Section B

B1 A recently formed group of political dissidents have been committing attacks against the government. The number of attacks each year for each of the last six years is:

Year	Number of Attacks
2009	1
2010	0
2011	2
2012	4
2013	3
2014	3

a) Suppose that the numbers of attacks each year are independent and identically distributed draws from a Poisson distribution with unknown parameter λ. A security expert specializing in terrorism tells you that she believes that that there will be 2 attacks a year on average. She is not completely sure about this, so she would like her prior to have a variance of 2. Find the corresponding values of α and β for the conjugate Gamma(α, β) prior that matches this mean and variance (you may want to use the formula sheet). [3]

b) Using the above prior and data, show that the predictive distribution for predicting how many attacks occur on an arbitrary year in the future $\tilde{Y}$ is:

$$p(\tilde{Y}|\mathbf{Y}) = \frac{1}{\tilde{Y}!}\frac{7^{15}}{\Gamma(15)}\frac{\Gamma(15+\tilde{Y})}{8^{15+\tilde{Y}}}, \quad \tilde{Y} = 0, 1, 2, \ldots$$

where $\mathbf{Y}$ denotes the 6 observations in the above table. [6]

c) Based on the above predictive distribution, compute the probability of there being more than 2 attacks in a given year. Remember that $\Gamma(n) = (n-1)!$ when n is an integer. [6]

d) The security expert suspects that there may have been a structural change in how the behavior of the group operated at the end of the year 2011. If this is the case, then it would imply that a change point has occurred in 2011, so that the observations in years 2012-2014 come from a different distribution from those in the years 2009-2011. Define the two models:

$$M_0 : \text{There has been no change point in the behaviour of the group}$$
$$M_1 : \text{There was a change point at the end of 2011}$$

Assuming that both models are equally likely, and using the same Gamma prior from part a), compute $p(M_i|\mathbf{Y})$ for both models and hence decide whether the data supports the expert's belief. Note that you should use the prior from part a) in Model M_0, and use the same prior for both segments in Model M_1. Do not consider any other possible change point locations except the one specified in Model M_1. [15]

B2 The daily log-returns of a particular stock over the last 6 years are as follows:

$$Y_1 = -0.21, Y_2 = -0.30, Y_3 = -0.42, Y_4 = -0.11, Y_5 = 0.12, Y_6 = 0.02$$

Assume that the log-returns are independent and identically distributed draws from a Normal distribution with known mean 0 and unknown variance σ^2

a) The Inverse-Gamma(α, β) distribution (see formula sheet) is the conjugate prior for σ^2. Show that the posterior distribution given the above data is IG$(\alpha + 3, \beta + 0.17)$. You do not need to evaluate any integrals or normalising constants. [5]

b) Show that the posterior predictive distribution $p(\tilde{Y}|\mathbf{Y})$ for a future log-return value $\tilde{Y}$ using the above data $\mathbf{Y} = (Y_1, \ldots, Y_6)$ and an IG(α, β) prior, has the form:

$$\frac{1}{\sqrt{2\pi}} \frac{\tilde{\beta}^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha})} \frac{\Gamma(\tilde{\alpha} + 1/2)}{(\tilde{\beta} + \tilde{Y}^2/2)^{\tilde{\alpha}+1/2}}$$

and find values for $\tilde{\alpha}$ and $\tilde{\beta}$ in terms of α, β and $Y_1, \ldots, Y_6$. [5]

c) Using this predictive distribution and an IG(0,0) prior, use the Trapezoid rule to find the probability that the log-return of the stock on the next day is between -0.5 and -1.0. You may use the fact that:

$$\frac{1}{\sqrt{2\pi}} \frac{0.17^3}{\Gamma(3)} \Gamma(3.5) = 0.0033$$

[5]

d) State the Pickands-Balkema-de Haan theorem and explain why it may be of use to risk analysis [5]

e) Find the corresponding probability of tomorrow's log-return being between -0.5 and -1.0 using a Generalised Pareto Distribution (GPD) to approximate the distribution of the log-returns below the threshold $u = -0.20$. Use the method of moments to find point estimates for the GPD parameters (see formula sheet). [7]

f) Discuss the strengths and weaknesses of modelling the data by a Normal distribution versus a GPD in this context. [3]

END OF PAPER