①

$X_1, \ldots, X_n$ are a random sample with $\mathbb{E}(X_i) = \mu$; $Var(X_i) = \sigma^2$.

We know $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ is unbiased for μ.

We might guess, therefore, that $\bar{X}^2$ is unbiased for μ^2.

$$\mathbb{E}(\bar{X}^2) = \mathbb{E}\left(\frac{1}{n^2} \sum_i \sum_j X_i X_j\right)$$

$$= \frac{1}{n^2} \sum_i \sum_j \mathbb{E}(X_i X_j)$$

$$= \frac{1}{n^2} \left\{ \sum_i \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i X_j) \right\}$$

$$= \frac{1}{n^2} \left\{ \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i)\,\mathbb{E}(X_j) \right\}$$

$$\ulcorner\; Var(X_i) = \mathbb{E}(X_i^2) - \{\mathbb{E}(X_i)\}^2$$

$$\Rightarrow \sigma^2 = \mathbb{E}(X_i^2) - \mu^2$$

$$\Rightarrow \mathbb{E}(X_i^2) = \sigma^2 + \mu^2 \;\lrcorner$$

$$= \frac{1}{n^{\cancel{2}}} \left\{ \cancel{n}(\sigma^2 + \mu^2) + \cancel{n}(n-1)\mu^2 \right\}$$

$$= \mu^2 + \frac{\sigma^2}{n}.$$

$\therefore \bar{X}^2$ is biased for μ^2.

Let $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ is unbiased for σ^2

$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n} X_i^2 - n \bar{X}^2 \right\}$$

$$\mathbb{E}[S^2] = \frac{1}{n-1} \left\{ \sum_{i=1}^{n} \mathbb{E}(X_i^2) - n \mathbb{E}(\bar{X}^2) \right\}$$

$$= \sigma^2$$

Hence $\bar{X}^2 - \frac{S^2}{n}$ is unbiased for μ^2.

(2) $X_1, \dots, X_n$ iid $\mathrm{Exp}\left(\frac{1}{\mu}\right)$

The likelihood function is

$$L(\mu|\underline{x}) = \prod_{i=1}^{n} \left(\frac{1}{\mu}\right) e^{-\frac{x_i}{\mu}}$$

$$= \mu^{-n} \exp\left(-\frac{1}{\mu} \sum_{i=1}^{n} x_i\right)$$

The log-likelihood is

$$\ell(\mu|\underline{x}) = -n\log\mu - \frac{1}{\mu}\sum_{i=1}^{n} x_i$$

To find the mle we solve

$$\left.\frac{\partial \ell}{\partial \mu}\right|_{\mu=\hat{\mu}} = 0$$

$$\frac{\partial \ell}{\partial \mu} = -\frac{n}{\mu} + \frac{1}{\mu^2}\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \quad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}.$$

The maximum likelihood estimator is

$$\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\mathbb{E}(\hat{\mu}) = \mathbb{E}\left(\frac{1}{n}\sum_i X_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i)$$

$$= \mu.$$

$\Rightarrow$ $\hat{\mu}$ is unbiased.

We'll examine $Var(\hat{\mu})$

$$Var(\hat{\mu}) = Var\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n}X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i) \quad \text{since } X_1, \dots, X_n \text{ are iid}$$

$$= \frac{1}{n^2} n \mu^2$$

$$= \frac{\mu^2}{n}$$

$$\lim_{n\to\infty} Var(\hat{\mu}) = \lim_{n\to\infty}\frac{\mu^2}{n} = 0$$

$\therefore$ $\overline{X}$ is consistent for μ.

③ $U = \frac{1}{2}X + \frac{1}{2}Y$; $V = \frac{1}{3}X + \frac{2}{3}Y$

$\mathbb{E}(U) = \theta$ and $\mathbb{E}(V) = \theta$ (both unbiased).

$$\begin{aligned}
Var(U) &= Var\left(\frac{1}{2}X + \frac{1}{2}Y\right) \\
&= \frac{1}{4}Var(X) + \frac{1}{4}Var(Y) \\
&= \frac{1}{4}(4+9) = \frac{13}{4} = 3.25.
\end{aligned}$$

$$\begin{aligned}
Var(V) &= Var\left(\frac{1}{3}X + \frac{2}{3}Y\right) \\
&= \frac{1}{9}Var(X) + \frac{4}{9}Var(Y) \\
&= \frac{4}{9} + 4 = \frac{40}{9} = 4.44
\end{aligned}$$

We'd prefer U to V because U has smaller variance.

The relative efficiency of V compared to U is

$$\begin{aligned}
Eff(V, U) &= 100 \times \left(\frac{Var(U)}{Var(V)}\right) \\
&= 100 \times \left(\frac{3.25}{4.44}\right) \approx 73.2\%.
\end{aligned}$$

④ Want to prove $\mathbb{E}[U(\theta; \underline{X})] = 0$.

$$\mathbb{E}\left[U(\theta;\underline{X})\right] = \mathbb{E}\left[\frac{\partial}{\partial\theta}\ell(\theta;\underline{X})\right]$$

$$= \int_{\mathcal{X}} \left\{\frac{\partial}{\partial\theta}\ell(\theta|\underline{X})\right\} f(\underline{x};\theta)\, d\underline{x}$$

sample space of $X_1, \dots, X_n$.

joint pdf of $X_1, \dots, X_n$.

$$= \int_{\mathcal{X}} \left\{\frac{\partial}{\partial\theta}\log L(\theta|\underline{x})\right\} f(\underline{x};\theta)\, d\underline{x}$$

$$= \int_{\mathcal{X}} \left\{\frac{\partial}{\partial\theta}\log f(\underline{x};\theta)\right\} f(\underline{x};\theta)\, d\underline{x}$$

$$= \int_{\mathcal{X}} \frac{\left\{\frac{\partial}{\partial\theta} f(\underline{x};\theta)\right\}}{\cancel{f(\underline{x};\theta)}} \cancel{f(\underline{x};\theta)}\, d\underline{x}$$

$$= \int_{\mathcal{X}} \frac{\partial}{\partial\theta} f(\underline{x};\theta)\, d\underline{x}$$

$$= \frac{\partial}{\partial\theta}\left\{\int_{\mathcal{X}} f(\underline{x};\theta)\, d\underline{x}\right\}$$

$$= \frac{\partial}{\partial\theta}(1)$$

$$= 0$$

$\blacksquare$

⑤ We know that

$$\mathbb{E}[U(\theta;\underline{X})] = 0$$

$$\Rightarrow \int_x \left\{ \frac{\partial}{\partial\theta} \log f(\underline{x};\theta) \right\} f(\underline{x};\theta)\, d\underline{x} = 0$$

Differentiation of the above with respect to θ yields

$$\int_x \left\{ \frac{\partial^2 \log f(\underline{x};\theta)}{\partial\theta^2} \right\} f(\underline{x};\theta)\, d\underline{x}$$

Product rule

$$+ \int_x \left\{ \frac{\partial}{\partial\theta} \log f(\underline{x};\theta) \right\} \frac{\partial}{\partial\theta} f(\underline{x};\theta)\, d\underline{x} = 0$$

$$\Rightarrow \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2} \ell(\theta|\underline{x}) \right]$$

$$+ \int_x \left\{ \frac{\partial}{\partial\theta} \log f(\underline{x};\theta) \right\} \frac{\partial}{\partial\theta} f(\underline{x};\theta)\, d\underline{x} = 0 \qquad (*)$$

But

$$\frac{\partial}{\partial\theta} \log f(\underline{x};\theta) = \frac{\frac{\partial}{\partial\theta} f(\underline{x};\theta)}{f(\underline{x};\theta)}$$

$$\Rightarrow \frac{\partial}{\partial\theta} f(\underline{x};\theta) = \left\{ \frac{\partial}{\partial\theta} \log f(\underline{x};\theta) \right\} f(\underline{x};\theta)$$

Substitution of the above into $(*)$ yields

$$\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ell(\theta|\underline{x})\right]$$

$$+\int_x \left\{\frac{\partial}{\partial\theta}\log f(\underline{x};\theta)\right\}\left\{\frac{\partial}{\partial\theta}\log f(\underline{x};\theta)\right\} f(\underline{x};\theta)\,d\underline{x} = 0$$

$$\Rightarrow \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ell(\theta|\underline{x})\right] + \mathbb{E}\left[\left\{\frac{\partial}{\partial\theta}\ell(\theta|\underline{x})\right\}^2\right] = 0$$

$$\Rightarrow \mathbb{E}\left[\left\{\frac{\partial}{\partial\theta}\ell(\theta|\underline{x})\right\}^2\right] = \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ell(\underline{\theta}|\underline{x})\right]$$

$$\therefore\ \mathcal{I}(\theta) = \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ell(\theta|\underline{x})\right]$$