From Lecture 4

The mles are $\hat{\mu} = \bar{x}$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Log-likelihood

$$\ell(\mu, \sigma^2 | \underline{x}) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n}x_i - n\mu\right)$$

$$\frac{\partial \ell}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\underbrace{\sigma^4}_{=(\sigma^2)^2}}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial^2 \ell}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 \ell}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial^2 \ell}{\partial \mu \partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}(x_i - \mu)$$

For Fisher information matrix:

$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial \mu^2}\right] = \frac{n}{\sigma^2}$$

$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial\mu\,\partial\sigma^2}\right] = \mathbb{E}\left[\frac{1}{\sigma^4}\sum_{i=1}^{n}(X_i-\mu)\right]$$

$$= \frac{1}{\sigma^4}\left[\sum_{i=1}^{n}\mathbb{E}(X_i) - n\mu\right]$$

$$= 0. \qquad \text{(independence)}$$

$$\mathbb{E}\left[-\frac{\partial^2 \ell}{\partial(\sigma^2)^2}\right] = -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_{i=1}^{n}\mathbb{E}\left[(X_i-\mu)^2\right]$$

$$= -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}n\sigma^2$$

$$= \frac{n}{2\sigma^4}$$

$\therefore$ The Fisher information matrix is

$$\mathcal{I}(\mu,\sigma^2) = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

$$\Rightarrow \mathcal{I}^{-1}(\mu,\sigma^2) = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}.$$

$\therefore$ The asymptotic distribution of $\begin{pmatrix} \hat{\mu} \\ \hat{\sigma}^2 \end{pmatrix}$ is

$$\begin{pmatrix} \hat{\mu} \\ \hat{\sigma}^2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}, \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^2}{n} \end{pmatrix} \right)$$

We take expectation wrt the posterior $\theta | \underline{x}$ $(\pi(\theta|\underline{x}))$.

$$\mathbb{E}_{\theta|\underline{x}}[(\theta - t)^2] = \mathbb{E}_{\theta|x}[\theta^2 - 2t\theta + t^2]$$

$$= \mathbb{E}_{\theta|x}(\theta^2) - 2t\,\mathbb{E}_{\theta|x}(\theta) + t^2 \quad (+)$$

Recall

$$\text{Var}_{\theta|x}(\theta) = \mathbb{E}_{\theta|x}(\theta^2) - \{\mathbb{E}_{\theta|x}(\theta)\}^2$$

$$\Rightarrow \mathbb{E}_{\theta|x}(\theta^2) = \text{Var}_{\theta|x}(\theta) + \{\mathbb{E}_{\theta|x}(\theta)\}^2$$

$\therefore$ In $(+)$

$$\mathbb{E}_{\theta|x}[(\theta - t)^2] = \text{Var}_{\theta|x}(\theta) + \{\mathbb{E}_{\theta|x}(\theta)\}^2 - 2t\,\mathbb{E}_{\theta|x}(\theta) + t^2$$

$$= \text{Var}_{\theta|x}(\theta) + \{\mathbb{E}_{\theta|x}(\theta) - t\}^2$$

Minimised where $\mathbb{E}_{\theta|x}(\theta) - t = 0$

$$\therefore \mathbb{E}_{\theta|x}(\theta) = t$$

$$\Rightarrow \theta^* = \mathbb{E}_{\theta|x}(\theta).$$

Let X = No. of people who are aware of the campaign.

$X \sim Bin(300, p)$

$p \in (0, 1)$ observed $x = 123$.

Company : prior mean : 0.35

range : 0.25 to 0.45

$\therefore$ Set $2 \times sd = 0.1$

$\Rightarrow sd = 0.05$

$\Rightarrow$ Prior variance $= 0.05^2$

$= 0.0025$.

Let, a priori, $p \sim Beta(a, b)$

$\therefore$ Prior mean $\dfrac{a}{a+b} = 0.35$ ①

$$\frac{ab}{(a+b)^2(a+b+1)} = 0.0025 \quad ②$$

$a = 31.5$

$b = 58.5$

$\therefore$ Prior is s.t.

$$\pi(p) \propto p^{30.5}(1-p)^{57.5}$$

Likelihood function

$$L(p \mid x=123) \propto p^{123}(1-p)^{177}$$

$\therefore$ ~~Posterior~~ Posterior :

$$\pi(p \mid \underline{x}) \propto \pi(p) \times L(p \mid x=123)$$

$$= p^{153.5}(1-p)^{234.5}$$

$\therefore$ posterior is

$$p \mid x=123 \sim \text{Beta}(154.5, 235.5)$$

Posterior mean : $\dfrac{154.5}{(154.5+235.5)} = 0.397$.

The likelihood function is

$$L(\mu|x,\sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

The prior is

$$\mu \sim N(\theta, \tau^2)$$

$$\pi(\mu) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{\frac{-1}{2\tau^2}(\mu-\theta)^2\right\}$$

$$\propto \exp\left\{-\frac{1}{2\tau^2}(\mu-\theta)^2\right\}$$

Posterior $\propto$ Prior x Likelihood

$$\therefore \pi(\mu|\underline{x}) \propto \exp\left\{-\frac{1}{2\tau^2}(\mu-\theta)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$= \exp\left\{-\frac{1}{2\tau^2}(\mu^2 - 2\theta\mu + \theta^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} x_i^2 - 2\mu\sum_{i=1}^{n}x_i + n\mu^2\right)\right\}$$

A B

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)\mu^2 - 2\left(\frac{\theta}{\tau^2}+\frac{\sum x_i}{\sigma^2}\right)\mu\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)\left[\mu^2 - \frac{2\left(\frac{\theta}{\tau^2}+\frac{\sum x_i}{\sigma^2}\right)\mu}{\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)}\right]\right\}$$

$$\pi(\mu|x) \propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)\left[\mu - \frac{\left(\frac{\theta}{\tau^2}+\frac{\sum x_i}{\sigma^2}\right)}{\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)}\right]^2\right\}$$

This is $\propto$ to pdf of a Normal distribution with mean $\dfrac{\left(\frac{\theta}{\tau^2}+\frac{\sum x_i}{\sigma^2}\right)}{\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)}$

variance $\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-1}$

Normal Sampling : Example 2

Given $\mu \sim N(25, 10)$

$X_i \sim N(\mu, 4) \quad i = 1, \ldots, 9$

$\bar{x} = 20$

Take the last example and set

$\theta = 25$

$\tau^2 = 10$

$\sigma^2 = 4$

$$\sum_{i=1}^{n} x_i = n \times \bar{x} = 9 \times 20 = 180.$$

Then, the posterior is normal with mean

$$\frac{\left(\frac{\theta}{\tau^2} + \frac{\sum x_i}{\sigma^2}\right)}{\left(\frac{1}{\tau^2} + \frac{n}{\sigma^2}\right)} = \frac{\left(\frac{25}{10} + \frac{180}{4}\right)}{\left(\frac{1}{10} + \frac{9}{4}\right)} \quad \cancel{= \frac{950}{47}}$$

$$= \frac{\left(\frac{1900}{40}\right)}{\left(\frac{94}{40}\right)} = \frac{1900}{94} = \frac{950}{47}$$

Variance : $\left(\frac{1}{\tau^2} + \frac{n}{\sigma^2}\right)^{-1} = \left(\frac{1}{10} + \frac{9}{4}\right)^{-1}$

$$= \frac{40}{94} = \frac{20}{47}$$

$\therefore$ The posterior is

$$\mu \mid \underline{x} \sim N\left(\frac{950}{47}, \frac{20}{47}\right)$$

Note: Prior variance has now reduced (from 10 to $\frac{20}{47} \approx 0.43$)

Posterior mean $= \frac{950}{47} = 20.2$

This is much closer to the sample mean than to the prior mean.