2.5 hr written exam

Answer ALL questions. Section A carries 40% of the total marks and Section B carries 60%. The relative weights attached to each question are as follows: A1 (6), A2 (9), A3 (7), A4 (9), A5 (9), B1 (30), B2 (30). The numbers in square brackets s indicate the relative weight attached to each part of the question.

- Unless otherwise indicated, in all questions $\{\epsilon_t\}$ denotes a sequence of uncorrelated zero-mean random variables with constant variance σ^2, i.e. $\{\epsilon_t\} \sim WN(0, \sigma^2)$, where $WN(0, \sigma^2)$ denotes white noise.
- The 97.5^{th} percentile point of the standard Normal distribution is 1.96.
- The transpose of a vector or matrix $\mathbf{A}$ is denoted by $\mathbf{A}^\top$.
- For a process $\{Y_t\}$, we define $\nabla Y_t = Y_t - Y_{t-1}$ and $\nabla_s Y_t = Y_t - Y_{t-s}$.

Section A

A1 (a) Give an analytical expression for each of the following models using the backshift operator (for instance, model AR(1) should be written as $(1 - \phi B)X_t = \epsilon_t$):

(i) ARMA(1,1).

(ii) ARIMA(1,1,2).

(iii) SARIMA$(0, 0, 1) \times (2, 1, 0)_5$. [3]

(b) Re-express an ARMA$(1, 1)$ model with parameters $\phi = 1/2$ and $\theta = -1/4$ as an MA(∞) one. Give the coefficient of ϵ_{t-4}. [3]

(a) (i)

(ii)

(iii)

(b) The coefficient of ϵ_{t-4} is equal to $\frac{1}{16} + \frac{1}{32} = 3/32$.

A2 Consider the processes $Y_t = \phi Y_{t-4} + \epsilon_t$ and $Z_t = \epsilon_t - \theta\epsilon_{t-8}$, with $|\phi| < 1$ and $|\theta| < 1$. You are also given the process

$$X_t = Y_t + (1 - \phi B^8)^{-1} Z_t$$

(a) Which seasonal ARMA models do the $\{Y_t\}$ and $\{Z_t\}$ processes correspond to? [2]

(b) Find which seasonal ARMA model does the $\{X_t\}$ process correspond to. [4]

(c) Describe briefly the shape of the ACF and the PACF plots for the $\{Z_t\}$ process. [3]

(a)

(b) The model is $\text{SARMA}(0,0)\times(3,3)_4$.

(c)

A3 (a) Consider the following stationary AR(1) process:

$$Y_t = \phi Y_{t-1} + \epsilon_t$$

Let $\gamma(k)$ be its autocovariance function. The variance of the white noise is σ^2. You are reminded that for an AR(1) model we have $\gamma(0) = \sigma^2/(1-\phi^2)$ and $\gamma(k) = \phi\gamma(k-1)$, $k \geq 1$.
The following sample autocovariances have been calculated, from given observations of an AR(1) process:

Lag, k	0	1	3
$c(k)$	2	1.5	-2

Use these values to produce estimates of parameters ϕ and σ^2 via the method of moments. [3]

(b) Suppose that $y_1, \ldots, y_T$ is an observed time series of length T. Explain briefly how you would fit a $\text{SARMA}(0,1)\times(1,0)_4$ model to the observed time series by least squares. All involved quantities should be carefully defined in your answer. [4]

(a) $\hat{\phi} = c(1)/c(0) = 0.75$; and $\hat{\sigma}^2 = c(0)(1-\hat{\phi}^2) = 7/8$.

(b)

A4 (a) A stationary time series of length 1,600 produced the sample autocorrelations: $r(1) = 0.4$, $r(2) = 0.5$, $r(3) = 0.04$, $r(4) = 0.0$. We are also given that the variance of the white noise is $\sigma^2 = 3$.

(i) Starting by the Yule-Walker equations, find the sample partial autocorrelation for lag $k = 2$. [3]

(ii) You are also given that $r(5) = 0.04$ and $\hat{\phi}_{33} = -0.34$. Suggest an ARMA model based on all provided sample ACF and PACF values. Explain briefly your answer. [3]

(b) You are given the MA(1) model:

$$X_t = \mu + \epsilon_t - \epsilon_{t-1}$$

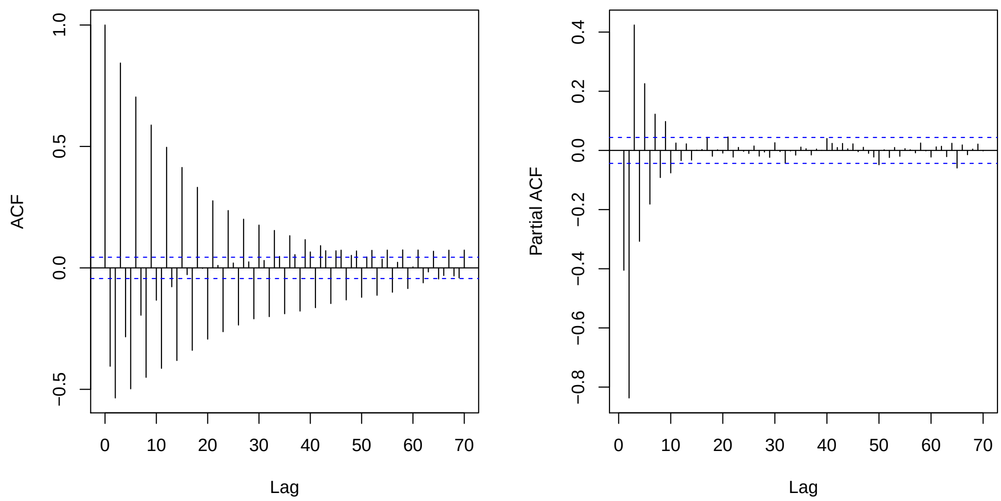
with white noise variance $\sigma^2 = 1$. Let $\overline{X}$ be the sample mean of 100 data points from this model. Find the variance of $\overline{X}$. [3]

(a) (i) $\phi_{22} = (r(2) - r(1)^2)/(1 - r(1)^2)$. Thus, $\phi_{22} \approx 0.405$.

(ii)

(b) $\mathrm{Var}(\overline{X}) = (100 - 2 \cdot 99)/100^2 = 0.0002$.

A5 An analyst wants to fit an ARMA model to a time series of 2,000 observations. The observations provided the sample autocorrelations (left plot) and partial autocorrelations (right plot) shown in the following plots:



(a) Based on these plots, do you think there is an autoregressive part in the model? Is there a moving-average part? Explain your answers. [3]

(b) Do you think that there are complex roots in the characteristic polynomial of the AR-part? Explain your answer. Which values do the dashed horizontal lines shown in the plots correspond to? [3]

(c) Suggest a suitable parsimonious model for this time series. Explain briefly your answer. [3]

(a)

(b)

(c) Maybe an ARMA$(2, 1)$ would be ok.

Section B

B1 A business analyst considered data from the monthy sales (in thousands of pounds) of a telecommunications company. After exploring the available data, she decided to model the sales as a SARMA$(1,0) \times (0,1)_{12}$ process of the following form:

$$X_t - 200 = \tfrac{1}{4}(X_{t-1} - 200) + \epsilon_t - \tfrac{1}{2}\epsilon_{t-12}$$

where $\epsilon_t \sim WN(0, \sigma^2)$ with $\sigma^2 = 8$. The table below shows some of the available monthly observations, together with corresponding estimated residuals:

Month / Year	Sales	Residuals (e_t)
Jan 2016	190	-8
Feb 2016	194	-4
Mar 2016	198	4
⋮	⋮	⋮
Dec 2016	204	8
Jan 2017	206	

(a) Show that the missing estimated residual for January of 2017 is equal to 1. [3]

(b) Produce forecasts for the sales in February & March of 2017. [4]

(c) Show analytically that:

(i) $\text{Cov}(X_t, \epsilon_{t-1}) = \frac{1}{4}\sigma^2$. [3]

(ii) $\text{Cov}(X_t, \epsilon_{t-11}) = \frac{1}{4^{10}}\text{Cov}(X_t, \epsilon_{t-1})$. [4]

(d) Using the results in (c) or otherwise, find (approximately) the stationary variance $\text{Var}(X_t)$. *Hint: As* $\frac{1}{4^{10}}$ *is a very small number, you are advised to replace it with* 0 *to simplify your calculations.* [4]

(e) If we were to forecast further and further into the future, what value will the estimate of future sales converge to? What value will the variance of the forecast error converge to? Explain briefly your answers without making any calculations. [3]

(f) Derive in detail the variance for the error of the estimate for March 2017 and calculate a 95% prediction interval. [5]

(g) Assume that in some other scenario, one claims that the $\{\epsilon_t\}$ process appearing in the definition of the model is not white noise, but it is in fact such that $\nabla_{12}^2 \epsilon_t$ is indeed white noise. Determine which model would the $\{X_t\}$ process correspond to in such a case. [4]

(a)

(b)

$$\hat{X}_{t+1} = 203.5$$

$$\hat{X}_{t+2} = 198.875$$

(c) (i) $\mathrm{Cov}(X_t, \epsilon_{t-1}) = \sigma^2/4$.

(ii) $\mathrm{Cov}(X_t, \epsilon_{t-11}) = \frac{1}{4^{10}}\mathrm{Cov}(X_t, \epsilon_{t-1})$.

(d) $(15/16)\mathrm{Var}(X_t) = (5/4)\sigma^2$, i.e. $\mathrm{Var}(X_t) = (4/3)\sigma^2$.

(e)

(f) The variance of the error is $(1^2 + \frac{1}{16})\sigma^2 = \frac{17}{16} \cdot 8 = 8.5$.
The confidence interval is: $198.875 \pm 1.96 \cdot \sqrt{8.5} = [193.16, 204.59]$.

(g) X_t would be $\mathrm{SARIMA}(1,0,0) \times (0,2,1)_{12}$.

B2 The temperature setting in an industrial furnace is preset to some determined level. However, the actual temperature in the furnace changes randomly with time. In particular, at time t (measured in minutes) the temperature (in degrees Celsius) is X_t, which satisfies

$$X_t = 0.2X_{t-1} + \epsilon_t$$

where ϵ_t is white noise with variance 2.

Physical constraints entail that the temperature X_t cannot be measured directly. However an estimate of the temperature can be obtained by measuring, on an appropriate scale, the intensity of light emitted by the furnace at a certain wavelength. Let $\{Y_t\}$ be the sequence of light intensities: physical arguments can be used to show that

$$Y_t = X_t + \delta_t + 0.1\delta_{t-1}$$

where $\{\delta_t\}$ is white noise with variance 1, uncorrelated with $\{\epsilon_t\}$.

(a) What is the covariance between X_t and Y_t? [4]

(b) Let $\mathbf{S}_t$ be a vector defined as

$$\mathbf{S}_t = \begin{pmatrix} X_t \\ Y_t \\ 0.1\delta_t \end{pmatrix} .$$

Show that $\mathbf{S}_t$ satisfies

$$\mathbf{S}_t = \mathbf{C}\,\mathbf{S}_{t-1} + \mathbf{H}_t \ ,$$

for some 3×3 matrix $\mathbf{C}$, where $\mathbf{H}_t = (\epsilon_t,\, \epsilon_t + \delta_t,\, 0.1\delta_t)^\top$. Write down the matrix $\mathbf{C}$ which is required in this representation, and also write down the variance-covariance matrix $\mathbf{V}$ of $\mathbf{H}_t$. [5]

(c) Although the light intensity is a useful proxy measure for X_t, it cannot be measured perfectly. In fact, the observed light intensity Z_t is given by

$$Z_t = Y_t + \eta_t$$

where $\{\eta_t\}$ is a white noise sequence with variance 1, uncorrelated with both $\{\epsilon_t\}$ and $\{\delta_t\}$.

What is the vector $\mathbf{B}$ required to write Z_t as $Z_t = \mathbf{B}^\top\mathbf{S}_t + \eta_t$? [2]

(d) The Kalman Filter algorithm determines that, if $\hat{\mathbf{S}}_{t|t-\ell}$ is the best estimator of $\mathbf{S}_t$ based on the observed Z's up to time $t-\ell$ and $\mathbf{P}_{t|t-\ell}$ is the variance-covariance matrix of the associated estimation error, then $\hat{\mathbf{S}}_{t|(t-1)} = \mathbf{C}\,\hat{\mathbf{S}}_{t-1|(t-1)}$ and $\mathbf{P}_{t|(t-1)} = \mathbf{C}\mathbf{P}_{t-1|(t-1)}\mathbf{C}^\top + \mathbf{V}$. The Kalman Filter also determines that

$$\begin{aligned} \hat{\mathbf{S}}_{t|t} &= \hat{\mathbf{S}}_{t|(t-1)} + \mathbf{P}_{t|(t-1)}\,\mathbf{B}\,(Z_t - \mathbf{B}^\top\,\hat{\mathbf{S}}_{t|(t-1)})/f_t \\ \text{and } \mathbf{P}_{t|t} &= \mathbf{P}_{t|(t-1)} - \mathbf{P}_{t|(t-1)}\,\mathbf{B}\mathbf{B}^\top\mathbf{P}_{t|(t-1)}/f_t \ , \\ \text{where } f_t &= \mathbf{B}^\top\,\mathbf{P}_{t|(t-1)}\,\mathbf{B} + \mathrm{Var}(\eta_t) \ . \end{aligned}$$

(i) Based on the above formulae, produce an estimate of the state vector $\mathbf{S}_1$. [4]

(ii) The associated error variance for the estimate in (i) is found to be

$$\mathbf{P}_{1|0} = \begin{pmatrix} 0.9 & 0.3 & 0.0 \\ 0.3 & 0.9 & 0.2 \\ 0.0 & 0.2 & 0.4 \end{pmatrix}$$

At time 1, the observed light intensity is $Z_1 = 0.75$ units. Obtain an updated estimate of the state vector given this information.

Assuming that the temperature fluctuations are normally distributed, give a 95% confidence interval for the temperature in the furnace at time $t = 1$. [7]

(e) (i) Find out which ARMA(p, q) model does the $\{Z_t\}$ process correspond to. [4]

(ii) For the ARMA(p, q) model you have identified, calculate all non-zero auto-covariances of the MA-part. [4]

(a)

$$\text{Var}(X_t) = 1/0.96$$

(b)

$$\mathbf{S}_t = \begin{pmatrix} 0.2 & 0 & 0 \\ 0.2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{S}_{t-1} + \mathbf{H}_t \ ,$$

Also,

$$\mathbf{V} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 0.1 \\ 0 & 0.1 & 0.01 \end{pmatrix}$$

(c)

(d) (i)

$$\hat{\mathbf{S}}_{1|0} = \mathbf{0}$$

[4]

(ii)

$$\hat{\mathbf{S}}_{1|1} = \begin{pmatrix} 0.118 \\ 0.355 \\ 0.079 \end{pmatrix}$$

$$\mathbf{P}_{1|1} == \begin{pmatrix} 0.853 & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$$

The 95% confidence interval for X_1 is

$$0.118 \pm 1.96\sqrt{0.853} = [-1.692, 1.928]$$

(e) (i) Z_t is an ARMA$(1,2)$ process.

(ii) For the MA-part of Z_t:

$$\gamma(0) = 4.0504$$
$$\gamma(1) = -0.298$$
$$\gamma(2) = -0.02$$

END OF PAPER