# STAT0017: Selected Topics In Statistics

## Topic 2: “Dependence modelling in finance using copulas”

Lecture 5

2019

**Last week**:

- We considered multivariate models.
- We saw that extending elliptical copulas to higher dimension was straightforward.
- We also saw that it was reasonably simple to generate Archimedean $d$-copulas.
- We considered limitations associated with them:
  - In particular, all the 2-margins of an Archimedean 3-copula were identical.
  - In addition, there was only one parameter which limited the nature of the dependence structure.
- We relaxed the exchangeability property by introducing nested Archimedean copulas.
- We also considered vine copulas.

**Today**:

- Factor Copula Models
- Regime-Switching Copulas

# Material and text books relevant to Lecture 5

**Factor-copulas**


1. H. Joe. *Dependence Modeling with Copulas.*
   Chapman & Hall/CRC Monographs on Statistics & Applied Probability.
   Taylor & Francis, 2014

   See **Chapter 3.10**

2. Alexander J. McNeil, Rdiger Frey, and Paul Embrechts. *Quantitative Risk Management: Concepts, Techniques and Tools.*
   Princeton University Press, 2015

   See **Chapter 3.4.1**

3. Pavel Krupskii and Harry Joe. Factor copula models for multivariate data.
   *Journal of Multivariate Analysis*, 120:85 – 101, 2013.
   ISSN 0047-259X


**Regime-Switching**


4. Chang-Jin Kim and Charles R. Nelson. *State-space models with regime switching: classical and Gibbs-sampling approaches with applications.*
   MIT Press, Cambridge, MA, 1999

5. J.D. Hamilton. *Time Series Analysis.*
   Princeton University Press, 1994
   See **Chapter 22**

# Factor copula model

- Vine copulas have become popular in recent years.
- The number of bivariate copulas (both conditional and unconditional) used in the vine construction is $\frac{d(d-1)}{2}$ for $d$ variables.
- When $d$ is very large (e.g. a large number of asset returns), the conditional independence can be assumed at higher levels of the vine.
- This leads to a truncated vine model, which allows to reduce the total number of parameters.

# The four-dimensional D-vine

The four-dimensional D-vine structure is generally expressed as:

$$
\begin{aligned}
f\left(x_1, x_2, x_3, x_4\right) = & f_1\left(x_1\right) \cdot f_2\left(x_2\right) \cdot f_3\left(x_3\right) \cdot f_4\left(x_4\right) \\
& \cdot c_{12}\left\{F_1\left(x_1\right), F_2\left(x_2\right)\right\} \cdot c_{23}\left\{F_2\left(x_2\right), F_3\left(x_3\right)\right\} \\
& \cdot c_{34}\left\{F_3\left(x_3\right), F_4\left(x_4\right)\right\} \\
& \cdot c_{13|2}\left\{F\left(x_1|x_2\right), F\left(x_3|x_2\right)\right\} \cdot c_{24}|x_3\rangle, F\left(x_4|x_3\right)\} \\
& \cdot c_{14|23}\left\{F\left(x_1|x_2, x_3\right), F\left(x_4|x_2, x_3\right)\right\}
\end{aligned}
$$

# Truncated R-vine copula

Truncated vine copulas have been first considered by Brechmann et al. (2012).

## Definition 3.1 (Truncated R-vine copula)

Let $\boldsymbol{U} = (U_1, \ldots, U_d)'$ be a random vector with uniform margins and let $\ell \in \{0, \ldots, d-1\}$ be level of truncation. Then $\boldsymbol{U}$ is said to be distributed according to the $d$-dimensional $\ell$-truncated R-vine copula $C(\cdot; \mathcal{V}, \mathcal{B}, \boldsymbol{\theta})$, if the $d$-dimensional R-vine copula has the following property:

$$C_{j(e),k(e);D(e)} = \Pi \quad \forall\, e \in E_i,\, i = \ell + 1, \ldots, d-1$$

where $\Pi$ denotes the bivariate independence copula.

# The four-dimensional truncated D-vine

The four-dimensional D-vine structure truncated at level 1 is:

$$
\begin{aligned}
f\left(x_1, x_2, x_3, x_4\right) = & f_1\left(x_1\right) \cdot f_2\left(x_2\right) \cdot f_3\left(x_3\right) \cdot f_4\left(x_4\right) \\
& \cdot c_{12}\left\{F_1\left(x_1\right), F_2\left(x_2\right)\right\} \cdot c_{23}\left\{F_2\left(x_2\right), F_3\left(x_3\right)\right\} \\
& \cdot c_{34}\left\{F_3\left(x_3\right), F_4\left(x_4\right)\right\}
\end{aligned}
$$

# Factor copula model

- The *factor copula* model offers is an alternative approach to truncated vines.
- Let $\mathbf{U} = (U_1, \ldots, U_d)$ be a random vector with $U_i \sim U(0,1)$, $i = 1, \ldots, d$.
- Then the $d$-dimensional copula $C(u_1, \ldots, u_d)$ is the joint cdf of the vector $\mathbf{U}$.
- The $p$ factor copula models are conditional independence models where observed variables $(U_1, \ldots, U_d)$ are conditionally independent given $p$ latent variables $V_1, \ldots, V_p$.

# Factor Models

- **Factor analysis** is one of the techniques of dimension reduction, which is central to multivariate statistical analysis and is widely used in financial risk management (especially in high-dimensions).
- It is a method for modelling the components of an observable $d$-dimensional random vector $\mathbf{X}$, and its covariance structure, in terms of a smaller number of underlying unobservable (latent) common *factors*.
- For example, if the components of $\mathbf{X}$ represent asset returns, then this approach attempts to explain a large part of their variation in terms of a smaller number of stock market index returns.

# Factor copula model

## Theorem 3.1 (Linear factor model)

The random vector $\mathbf{X}$ is said to follow a $p$-factor model if it can be decomposed as follows:

$$\boldsymbol{X} = \boldsymbol{a} + B\boldsymbol{F} + \boldsymbol{\varepsilon}$$

where:

- $\boldsymbol{F} = (F_1, \ldots, F_p)'$ is a random vector of common factors with $p < d$.
- $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d)'$ is a random vector of independent mean zero error terms.
- $B \in \mathbb{R}^{d \times p}$ is a matrix of constant factor loadings
- $\boldsymbol{a} \in \mathbb{R}^d$ is a vector of constants.
- $\mathrm{cov}(\boldsymbol{F}, \boldsymbol{\varepsilon}) = 0$

# Factor copula model

- Let's assume that $V_i$ are independent and identically distributed $U(0,1)$, for $i = 1, \ldots, d$.

## Factor copula model

Let the conditional cdf of $U_j$ given $V_1, \ldots, V_p$ be denoted by $C_{j|V_1,\ldots,V_p}$. Then:

$$C(\boldsymbol{u}) = \int_{[0,1]^p} \prod_{j=1}^{d} C_{j|V_1,\ldots,V_p}(u_j|v_1, \ldots, v_p)\, \mathrm{d}v_1 \ldots \mathrm{d}v_p, \quad \boldsymbol{u} \in [0,1]^d$$

- It is called a *factor copula* model with $C_{j|V_1,\ldots,V_p}$ expressed in terms of a sequence of bivariate copulas that link the observed variables $U_j$ to the latent variables $V_k$.
- Hence, the dependence structure of $\mathbf{U}$ is then defined through conditional distributions $C_{1|V_1,\ldots,V_p}, \cdots, C_{d|V_1,\ldots,V_p}$.

# One-factor copula models

- Consider the case of one factor, i.e. $p = 1$.
- Let's denote the joint cdf of $(U_j, V_1)$ by $C_{j,V_1}$, and its density by $c_{j,V_1}$.
- $C_{j|V_1}$ is just a partial derivative of the copula $C_{j,V_1}$ with respect to the second argument: $C_{j|V_1}(u_j|v_1) = \partial C_{j,V_1}(u_j, v_1)/\partial v_1$

## 1-factor copula model

Let the conditional cdf of $U_j$ given $V_1$ be denoted by $C_{j|V_1}$. Then:

$$C(\boldsymbol{u}) = \int_0^1 \prod_{j=1}^{d} C_{j|V_1}(u_j|v_1)\, \mathrm{d}v_1, \quad \boldsymbol{u} \in [0,1]^d \tag{3.1}$$

# One-factor copula models

- Note that $\frac{\partial}{\partial u} C_{j|V_1}\left(u_j | v_1\right) = \frac{\partial^2}{\partial u_j \partial v_1} C_{j,V_1}\left(u, v_1\right) = c_{j,V_1}\left(u_j, v_1\right)$
- Then the 1-factor copula density is:

$$c\left(u_1, \ldots, u_d\right) = \frac{\partial^d C\left(u_1, \ldots, u_d\right)}{\partial u_1 \cdots \partial u_d} = \int_0^1 \prod_{j=1}^{d} c_{j,V_1}\left(u_j, v_1\right) dv_1$$

- In this model, dependence is defined by $d$ bivariate linking copulas $C_{1,V_1}, \ldots, C_{d,V_1}$.
- When all bivariate linking copulas have lower (upper) tail dependence then all bivariate margins of $\mathbf{U}$ will also be lower (upper) tail dependent respectively.
- Therefore, asymmetric dependence structure can be easily modelled by appropriately selected bivariate linking copulas.

# One-factor copula models

- It will be shown next that when $C_{j,V_1}$ are all bivariate normal copulas, then (3.1) is simply the copula of the multivariate normal distribution with a 1-factor correlation matrix.
- Let $C_{j,V_1}$ be the bivariate normal copula with correlation parameter $\theta_{j1}$, for $j = 1, \dots, d$.
- Let $\Phi$ denote the standard normal cdf, $\phi$ denote the standard normal density function, and let $\Phi_2(\cdot\,; \theta_{j1})$ be bivariate normal cdf with correlation $\theta$.
- Then $C_{j,V_1}(u_j, v_1) = \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v); \theta_{j1}\right)$, and the partial derivative is:

$$C_{j|V_1}\left(u_j|v_1\right) = \Phi\left(\frac{\Phi^{-1}(u_j) - \theta_{j1}\Phi^{-1}(v_1)}{\left(1-\theta_{j1}^2\right)^{1/2}}\right)$$

# One-factor copula models

- Let $u_j = \Phi\left(z_j\right)$ to get a multivariate distribution with $N(0,1)$ margins.

$$
\begin{aligned}
F\left(z_1, \ldots, z_d\right) &:= C\left(\Phi\left(z_1\right), \ldots, \Phi\left(z_d\right)\right) \\
&= \int_0^1 \prod_{j=1}^{d} \Phi\left(\frac{z_j - \theta_{j1}\Phi^{-1}\left(v_1\right)}{\left(1-\theta_{j1}^2\right)^{1/2}}\right) \mathrm{d}v_1 \\
&= \int_{-\infty}^{\infty} \left\{\prod_{j=1}^{d} \Phi\left(\frac{z_j - \theta_{j1} w}{\left(1-\theta_{j1}^2\right)^{1/2}}\right)\right\} \cdot \phi(w)\mathrm{d}w
\end{aligned}
$$

# One-factor copula models

- Therefore, this 1-factor copula model is the multivariate Gaussian model with a 1-factor correlation structure.
- This is because this multivariate cdf comes from the following representation:

$$Z_j = \theta_{j1} W + \sqrt{1 - \theta_{j1}^2}\epsilon_j, \quad j = 1, \dots, d$$

where $W, \epsilon_1, \dots, \epsilon_d$ are i.i.d. $N(0, 1)$ random variables.

- However, this property does not hold for other distributions.
- For example, if $C_{j,V_1}$ is the Student $t$ copula with correlation $\theta_{j1}$, for $j = 1, \dots, d$, and $\nu_i$ degrees of freedom then $c(u_1, \dots, u_d)$ is no longer the multivariate Student $t$ copula density.

- Factor Copula Models
- Regime-Switching Copulas

# Motivation

**Why regime switching?**

- financial markets undergo episodes of rapid growth and dramatic decline of stock prices
- asymmetry in the dependence structure amongst financial returns is well documented
- adequate models able to capture variations in dependence structure are required

# Regime Switching

Consider the copula density $c(u_{1t}, u_{2t}|\boldsymbol{\theta}_k)$ at time $t$.

The regime switching bivariate copula density for $\boldsymbol{u_t} = (u_{1t}, u_{2t})$ can be fully characterized by specifying conditional density as follows:

$$c(\boldsymbol{u_t}|\boldsymbol{\Theta}, S_t) = \sum_{k=1}^{2} 1_{\{k\}}(S_t) \cdot c(\boldsymbol{u_t}|\boldsymbol{\theta}_k)$$

where:

- $\boldsymbol{\Theta}$ is a vector of population parameters which includes parameter vectors $\boldsymbol{\theta}_1$ in *state 1*, and $\boldsymbol{\theta}_2$ in *state 2*;
- $S_t$ is the latent state variable.
- $c(\boldsymbol{u_t}|\boldsymbol{\theta}_k)$ is a bivariate copula.

# Regime Switching

$$c(\boldsymbol{u_t}|\boldsymbol{\Theta}, S_t) = \sum_{k=1}^{2} 1_{\{k\}}(S_t) \cdot c(\boldsymbol{u_t}|\boldsymbol{\theta}_k)$$

- This specification implies that the copula density of a vector $\boldsymbol{u_t}$ at time $t$ depends on a random variable $S_t$ that indicates the current regime.
- In other words, regime determines what the current copula is: once we know what the current regime is, we know the copula for that $t$.

# Regime Switching

- We consider a discrete-time Markov chain in which the state can change at each discrete time point.

- Given a sequence of discrete random variables $S_0, S_1, S_2, \dots$ taking values in some finite or countably infinite set $\Omega$, a sequence $S_n$ is said to be a Markov chain if the following Markov property is satisfied:

$$\mathbb{P}\big(S_{n+1} = j | S_n = i, S_{n-1} = i_{n-1}, \dots, S_0 = i_0\big) \quad = \quad \mathbb{P}\big(S_{n+1} = j | S_n = i\big)$$

# Regime Switching

Let's define one-step transition probabilities for a two-state Markov process as follows:

$$p_{ij} = \mathbb{P}\big(S_{n+1} = j \big| S_n = i\big) \quad \forall\, i, j = 1, 2 \tag{3.2}$$

where $S_t$ denotes the state at date $t$. We can then collect these probabilities to construct the transition probability matrix:

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

where $p_{ij} \geq 0$ for all $i, j = 1, 2$ so that $p_{11} + p_{12} = 1$ and $p_{21} + p_{22} = 1$

# Regime Switching

- The first challenge in estimating Markov regime-switching models is that the value of $S_t$ is unknown because it is unobservable.
- Therefore, in order to estimate regime-switching model when the states are not observed we can consider a decomposition of the joint density of $\boldsymbol{u}_{1:T} = (\boldsymbol{u}_1, ..., \boldsymbol{u}_T)$:

$$\begin{aligned} c(\boldsymbol{u}_{1:T}|\boldsymbol{\theta}) =& c(\boldsymbol{u}_t|\boldsymbol{\theta}) \cdot \prod_{t=2}^{T} c(\boldsymbol{u}_t|\boldsymbol{u}_{1:(t-1)}, \boldsymbol{\theta}) \\ =& \left[ \sum_{k=1}^{2} c(\boldsymbol{u}_t|S_1 = k, \boldsymbol{\theta}_k) P(S_1 = k) \right] \\ & \cdot \prod_{t=2}^{T} \left[ \sum_{k=1}^{2} c(\boldsymbol{u}_t|S_t = k, \boldsymbol{\theta}_k) P(S_t = k|\boldsymbol{u}_{t-1}, \boldsymbol{\theta}) \right] \end{aligned}$$

# Regime Switching

- Because the required probabilities are not observed, we make inferences on the probabilities at time $t$ using the available information up to period $t-1$.

- This constitutes the main idea of the filter outlined in Hamilton (1994) and Kim and Nelson (1999), which we use in order to calculate the filtered probabilities of the process being in each state, based on the availability of the new information.

# Regime Switching

This yields the following algorithm that should be iterated through $t = 1, ..., T$:

**Step 1:** *Prediction of $S_t$*

$$\mathbb{P}(S_t = l | w_{t-1}) = \sum_{k=1}^{2} p_{kl} \mathbb{P}(S_{t-1} = k | w_{t-1})$$

for $l = 1, 2$ and $p_{kl} = \mathbb{P}(S_t = l | S_{t-1} = k)$ is the transition probability between the states $k$ and $l$ as introduced in (3.3).

**Step 2:** *Filtering of $S_t$*

$$\mathbb{P}(S_t = l|w_t) = \frac{c_t(u_1, u_2|S_t = l, w_{t-1})\mathbb{P}(S_t = l|w_{t-1})}{\sum_{k=1}^{2} c_t(u_1, u_2|S_t = k, w_{t-1})\mathbb{P}(S_t = k|w_{t-1})}$$

where $w_t = (w_{t-1}, u_{1t}, u_{2t})'$.