Section A

A1 (a) If the posterior distribution $p(\theta|\mathbf{y})$ is in the same family as the prior probability distribution $p(\theta)$, the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior for the likelihood function $p(\mathbf{y}|\theta)$.

(b) In this case, the θ-dependent part of the likelihood is $\theta^Y(1-\theta)^{N-Y}$, which matches the θ-dependent part of the Beta distribution, $\theta^{\alpha-1}(1-\theta)^{\beta-1}$.

(c) We choose the prior parameters to match the empirical mean and variance of survey data on proportions in other communities.

(d) The posterior distribution is $Beta(658.88, 354.27)$.

(e)

$$p(\tilde{Y}|Y) = \binom{M}{\tilde{Y}} \frac{B(\tilde{\alpha}+\tilde{Y}, \tilde{\beta}+M-\tilde{Y})}{B(\tilde{\alpha}, \tilde{\beta})}$$

where $\tilde{\alpha} = 658.88$, and $\tilde{\beta} = 354.27$

A2 (a) Using Bayes' Theorem:

$$p(\theta = 0|Y) = \frac{p(Y|\theta = 0) \cdot p(\theta = 0)}{p(Y)} = \frac{0.071}{0.071 + 0.111 + 0.02} = 0.350$$

$$p(\theta = 1|Y) = \frac{p(Y|\theta = 1) \cdot p(\theta = 1)}{p(Y)} = \frac{0.111}{0.071 + 0.111 + 0.02} = 0.551$$

$$p(\theta = 2|Y) = \frac{p(Y|\theta = 2) \cdot p(\theta = 2)}{p(Y)} = \frac{0.02}{0.071 + 0.111 + 0.02} = 0.098$$

(b) $p(\pi^*, a_0|Y) = 2.343$
$p(\pi^*, a_1|Y) = 1.444$
$p(\pi^*, a_2|Y) = 4.654$
Optimal decision is to choose action a_1 (or classifying the village's risk status as $\theta = 1$).

(c) See Lecture 2 pp. 41-47. Decision theory provides a principled way to summarise the posterior by a particular point estimate.

You would need to list different loss functions considered in Lecture 2, how penalties are being imposed, and the corresponding point estimates.

A3 (a) Suppose $Y_t \sim N(0, \sigma_t^2)$, then GARCH(1,1) model for the conditional variance is $\sigma_t^2 = \beta_0 + \beta_1 Y_{t-1}^2 + \beta_2 \sigma_{t-1}^2$

(b) Numerical techniques could be used such as Gaussian quadrature or MCMC.

Section B

B1 See revision session

(a)

$$p(\tilde{Y}|Y) = \frac{B(4, 23 + \tilde{Y})}{B(3, 24)}$$

(b) BIC for Geometric model is -9.97, and for Poisson is -8.89. Hence Poisson gives a better fit to this data.

(c) See Lecture 4 p.48 and Lecture 7 p.18. The BIC approximation to the marginal likelihood may not be accurate due to the small number of observations.

(d) The PBH theorem states that for any distribution $p(Y)$ that satisfies certain regularity conditions, we have that $p(Y \leq D|Y > u)$ is asymptotically a Generalised Pareto Distribution as $u \to \infty$.

(e) $\hat{k} = 1.75$, $\hat{\sigma} = 4.125$

$$p(\tilde{Y} \geq 11.5) = 0.37$$

(f) Any reasonable answer.

B2 (a) See Lecture 5 pp. 14-16 for similar derivation. Note there was a typo in this question. The marginal likelihood of the observations up to and including the change point is as follows:

$$p(Y_1, \ldots, Y_\tau|\tau) = \int p(Y_1, \ldots, Y_\tau|\sigma^2)p(\sigma^2)d\sigma^2 = (2\pi)^{-\tau/2} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\tilde{\alpha})}{\tilde{\beta}^{\tilde{\alpha}}}$$

where $\tilde{\alpha} = \alpha + \tau/2$ and $\tilde{\beta} = \beta + \sum_{i=1}^{\tau} \frac{Y_i^2}{2}$. Substitute $\alpha = 1$ and $\beta = 1$ to get the required expression.

(b)

$$p(Y_{\tau+1},\ldots,Y_n|\tau) = (2\pi)^{-(4-\tau)/2}\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\tilde{\alpha})}{\tilde{\beta}^{\tilde{\alpha}}}$$

where $\tilde{\alpha} = \alpha + (4-\tau)/2$ and $\tilde{\beta} = \beta + \sum_{i=\tau+1}^{4}\frac{Y_i^2}{2}$. Substitute $\alpha = 1$ and $\beta = 1$ to get the required expression.

(c) $\frac{p(M_0|Y)}{p(M_1|Y)} = 1.2 \times \frac{\Gamma(2)\Gamma(2)}{\Gamma(2.5)\Gamma(1.5)}$
If this expression evaluates to 1.02, then that would mean that the model M_0 gives a better fit to the data.

(d) For a threshold p, Value-at-risk is a number (or percentage) X such that the probability of a portfolio value dropping by more than X over specified time period is equal to p.

(e) Any reasonable answer. Potential practical challenges: time-varying volatility; change point possibility, etc.

End of Paper