PROOF OF THE RAO-BLACKWELL THEOREM

MSE of $T(\underline{X})$:

$MSE(T(\underline{x}), m(\theta))$

$= \mathbb{E}[(T - m(\theta))^2]$ ①

Now $T = \mathbb{E}(V|S)$

$\therefore$ ① becomes

$$\mathbb{E}[(\mathbb{E}(V|S) - m(\theta))^2] = \mathbb{E}[\{\mathbb{E}(V - m(\theta)|S)\}^2] \quad (+)$$

$$Var(V - m(\theta)|S) = \mathbb{E}[(V - m(\theta))^2|S] - \{\mathbb{E}[V - m(\theta)|S]\}^2$$

$$Var(V - m(\theta)|S) \geq 0$$

$$\Rightarrow \mathbb{E}[(V - m(\theta))^2|S] - \{\mathbb{E}[V - m(\theta)|S]\}^2 \geq 0$$

$$\therefore \{\mathbb{E}[V - m(\theta)|S]\}^2 \leq \mathbb{E}[(V - m(\theta))^2|S]$$

$\therefore$ in (+)

$$\mathbb{E}[(T - m(\theta))^2] \leq \mathbb{E}[\mathbb{E}[(V - m(\theta))^2|S]]$$

$$= \mathbb{E}[(V - m(\theta))^2]$$

Hence $\mathbb{E}[(T - m(\theta))^2] \leq \mathbb{E}[(V - m(\theta))^2]$

$\Rightarrow MSE(T) \leq MSE(V)$ ◻.

(i) $X_1, \dots, X_n$ iid $U(0, \theta)$

Likelihood function:

$$L(\theta | \underline{x}) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}\{0 \leq x_i \leq \theta\}$$

$$= \frac{1}{\theta^n} \mathbb{1}\{0 \leq x_1, \dots, x_n \leq \theta\}$$

$$= \frac{1}{\theta^n} \mathbb{1}\{0 \leq \min \underline{x}\} \mathbb{1}\{\max \underline{x} \leq \theta\}$$

$$= g(t, \theta) h(\underline{x})$$

$$g(t, \theta) = \frac{1}{\theta^n} \mathbb{1}\{t \leq \theta\} \qquad t = \max \underline{x}$$

$$h(\underline{x}) = \mathbb{1}\{0 \leq \min \underline{x}\}$$

$\therefore$ By the factoristation criterion
$T(\underline{X}) = \max \underline{X}$ is sufficient for θ.

(ii) Show $U = 2X_1$ is unbiased for θ

Holds since $\mathbb{E}(X_1) = \frac{\theta}{2}$.

(iii) $T = \max \underline{X}$ is sufficient
$U = 2X_1$ is unbiased.

Consider

$$
\begin{aligned}
V &= \mathbb{E}[2X_1 \mid T=t] \\
&= 2\,\mathbb{E}[X_1 \mid T=t] \\
&= 2\Big\{\mathbb{E}[X_1 \mid X_1 = \max \underline{X}]\,\mathbb{P}(X_1 = \max \underline{X}) \\
&\qquad + \mathbb{E}[X_1 \mid X_1 \neq \max \underline{X}]\,\mathbb{P}(X_1 \neq \max \underline{X})\Big\}
\end{aligned}
$$

$X_1 \sim U[0,t]$ because X_1 is not maximum

$$
\begin{aligned}
&= 2\left\{\left(t \times \frac{1}{n}\right) + \frac{t}{2} \times \frac{(n-1)}{n}\right\} \\
&= \frac{(n+1)t}{n}
\end{aligned}
$$

$\therefore\ V = \frac{(n+1)T}{n}$ is an unbiased estimator of θ with $Var(V) \leq Var(U)$.

Suppose that the minimum variance of an unbiased estimator of θ is $v = v(\theta)$.

Let T_1 and T_2 be two <u>distinct</u> unbiased estimators of θ such that

$$Var(T_1) = Var(T_2) = v$$

Define $T = \frac{1}{2}(T_1 + T_2)$

We see $\mathbb{E}(T) = \theta$ i.e. T is unbiased.

$$Var(T) = Var\left(\frac{1}{2}(T_1 + T_2)\right)$$

$$= \frac{1}{4}\{Var(T_1) + Var(T_2) + 2Cov(T_1, T_2)\}$$

Let $Corr(T_1, T_2) = e$

$$\Rightarrow \frac{Cov(T_1, T_2)}{\sqrt{Var(T_1)Var(T_2)}} = e$$

$\Rightarrow \cancel{Corr(T_1, T}$

$$Cov(T_1, T_2) = ev$$

also $|e| \leqslant 1$

Hence

$$Var(T) = \frac{1}{4}\{v + v + 2ev\}$$

$$= \frac{1}{2}v(1+e)$$

Since T is unbiased then $Var(T) \geq v$

$$\Rightarrow \frac{1}{2} v(1+e) \geq v$$

$$1 + e \geq 2$$

$$\Rightarrow e \geq 1$$

$\therefore e = 1$ since $|e| \leq 1$.

$\Rightarrow$ T_1 and T_2 are linearly related

$\therefore T_2 = aT_1 + b$

Since $\mathbb{E}(T_1) = \mathbb{E}(T_2) = \theta$

$\Rightarrow$ $a = 1, \ b = 0$

$\therefore T_1 = T_2$ $\#$ since T_1 and T_2 are distinct!

Hence, T_1 and T_2 are not distinct and the MVUE is unique.

Let $X \sim \text{Bin}(n, \theta)$

Suppose $\mathbb{E}[g(X)] = 0$ for all θ.

$$\Rightarrow \sum_{r=0}^{n} g(r)\, \mathbb{P}(X = r) = 0$$

$$\Rightarrow \sum_{r=0}^{n} g(r) \binom{n}{r} \theta^r (1-\theta)^{n-r} = 0$$

$$\Rightarrow (1-\theta)^n \sum_{r=0}^{n} g(r) \binom{n}{r} \left(\frac{\theta}{1-\theta}\right)^r = 0$$

$$\Rightarrow \sum_{r=0}^{n} a_r z^r = 0 \quad \text{where } a_r = g(r)\binom{n}{r}, \quad z = \frac{\theta}{1-\theta}$$

$\therefore$ $a_r = 0$ for $r = 0, 1, \dots, n$.

$$\Rightarrow g(r)\binom{n}{r} = 0$$

$$\Rightarrow g(r) = 0 \quad \text{for all } r.$$

Hence X is complete for θ.

$X_1, \dots, X_n \quad X_i \sim \text{Bern}(\theta)$

We know $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{R}{n}$ is the MVBUE for θ.

Also know $\mathcal{I}(\theta) = \frac{n}{\theta(1-\theta)}$

Score f'n $U(\theta; \underline{X})$ is

$$U(\theta; \underline{X}) = \mathcal{I}(\theta)(\bar{X} - \theta)$$

$$= \frac{n}{\theta(1-\theta)}(\bar{X} - \theta)$$

We can't obtain a MVBUE of θ^2 because

$$U(\theta; \underline{X}) \neq A(\theta)(T(\underline{X}) - \theta^2)$$

Seek ~~of~~ a MVUE of θ^2.

Guess at $\bar{X}^2$ as an estimator of θ^2

$$\mathbb{E}(\bar{X}^2) = \text{Var}(\bar{X}) + \{\mathbb{E}(\bar{X})\}^2$$

$$= \frac{\theta(1-\theta)}{n} + \theta^2$$

$$= \frac{\theta}{n} - \frac{\theta^2}{n} + \theta^2$$

$$= \frac{\theta}{n} + \frac{(n-1)}{n}\theta^2$$

$$\therefore \mathbb{E}\left(\bar{X}^2 - \frac{\theta}{n}\right) = \frac{(n-1)\,\theta^2}{n}$$

$$\Rightarrow \mathbb{E}\left(\bar{X}^2 - \frac{\bar{X}}{n}\right) = \frac{(n-1)\,\theta^2}{n}$$

$$\therefore \mathbb{E}\left[\frac{n}{n-1}\left(\bar{X}^2 - \frac{\bar{X}}{n}\right)\right] = \theta^2$$

$$\Rightarrow \frac{n}{n-1}\left(\bar{X}^2 - \frac{\bar{X}}{n}\right) \text{ is unbiased for } \theta^2.$$

$$\frac{n}{n-1}\left(\frac{R^2}{n} - \frac{R}{n^2}\right)$$

$$= \frac{R(R-1)}{n(n-1)} \text{ is unbiased for } \theta^2 \quad \sum_{i=1}^{n} X_i = R.$$

$R \sim Bin(n, \theta)$.

* R is complete for θ.
* R is sufficient for θ.

$\therefore$ R is complete sufficient for θ.

Hence $\frac{R(R-1)}{n(n-1)}$ is the unique MVUE of θ^2.

$X_i \sim Exp(\theta)$

Likelihood function

$$L(\theta|\underline{X}) = \prod_{i=1}^{n} \theta e^{-\theta X_i}$$

$$= \theta^n e^{-\theta \sum_i X_i}$$

$\therefore \sum_{i=1}^{n} X_i$ is sufficient.

Also, since $X_1, \ldots, X_n$ are iid $Exp(\theta)$

$$T = \sum_{i=1}^{n} X_i \sim Gamma(n, \theta) \leftarrow$$ a member of the exponential family.

$\therefore$ T is complete, sufficient for θ.

Now $\mathbb{P}(X_1 \leq a) = 1 - e^{-\theta a}$

$$T = X_1 + \sum_{i=2}^{n} X_i$$

$$= X_1 + W \qquad W \sim Gamma(n-1, \theta)$$

Since X_1 and W are independent then

$$f_{X_1,W}(x_1, w) = \theta e^{-\theta x_1} \frac{\theta^{n-1}}{\Gamma(n-1)} w^{n-2} e^{-\theta w}$$

$$f_T(t) = \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t}$$

$X_1 \mid T = t$

$$f_{X_1|T}(x_1|t) = \frac{f_{X_1,W}(x_1, w)}{f_T(t)}$$

$$= \frac{\left\{\cancel{\theta} \cancel{e^{-\theta x_1}} \frac{\cancel{\theta^{n-1}}}{\Gamma(n-1)} w^{n-2} \cancel{e^{-\theta w}}\right\}}{\left\{\frac{\cancel{\theta^n}}{\Gamma(n)} t^{n-1} \cancel{e^{-\theta t}}\right\}}$$

$$= \frac{w^{n-2}}{t^{n-1}} \cdot \frac{\Gamma(n)}{\Gamma(n-1)} \qquad \Gamma(n) = (n-1)!$$

$$= (n-1) \frac{w^{n-2}}{t^{n-1}}$$

$$= (n-1) \frac{(t - x_1)^{n-2}}{t^{n-1}}$$

$$\therefore \mathbb{P}(X_1 \leq a \mid T = t) = \int_0^a f_{X_1|T}(x_1|t)\, dx_1$$

$$= \int_0^a \frac{(n-1)}{t^{n-1}} (t - x)^{n-2}\, dx$$

$$= -\left(\frac{t-x}{t}\right)^{n-1} \Bigg|_0^a$$

$$= 1 - \left(1 - \frac{a}{t}\right)^{n-1}$$