# 2.5 hr written exam

Answer ALL questions. Section A carries 40% of the total marks and Section B carries 60%. The relative weights attached to each question are as follows: A1 (8), A2 (9), A3 (9), A4 (6), A5 (8), B1 (30), B2 (30). The numbers in square brackets indicate the relative weight attached to each part of the question.

- Unless otherwise indicated, in all questions $\{\epsilon_t\}$ denotes a sequence of uncorrelated zero-mean random variables with constant variance $\sigma^2$, i.e. $\{\epsilon_t\} \sim WN(0, \sigma^2)$, where $WN(0, \sigma^2)$ denotes white noise.
- The $97.5^{th}$ percentile point of the standard Normal distribution is 1.96.
- The transpose of a vector or matrix $\mathbf{A}$ is denoted by $\mathbf{A}^\top$.
- Wold's formula gives the autocovariance function of the stationary process $X_t = \sum_{j=0}^{\infty} b_j \epsilon_{t-j}$ as $\gamma(k) = \sigma^2 \sum_{j=0}^{\infty} b_j b_{j+k}$.
- If $|\lambda| < 1$, then $\sum_{j=0}^{\infty} \lambda^j = \frac{1}{1-\lambda}$.

TURN OVER

# Section A

**A1** (a) Give an analytical mathematical expression for each of the following models using the backshift operator (for example, model AR(1) should be written as $(1-\phi B)X_t = \epsilon_t$):

i) $\mathrm{ARMA}(1,2)$ [1]

ii) $\mathrm{SARMA}(0,1)\times(1,0)_4$ [2]

iii) $\mathrm{SARIMA}(0,1,1)\times(1,1,1)_6$ [2]

(b) We are given the following models:

i) $X_t = -\frac{1}{16}X_{t-4} + \epsilon_t + \epsilon_{t-2}$

ii) $X_t = \frac{1}{2} + X_{t-1} + \epsilon_t - \frac{3}{2}\,\epsilon_{t-4}$

For each of i), ii), determine whether the model is stationary or not, and whether it is invertible or not. Explain your answers. [3]

(a)

(b) i) Stationary; not invertible.

ii) Not stationary; not invertible.

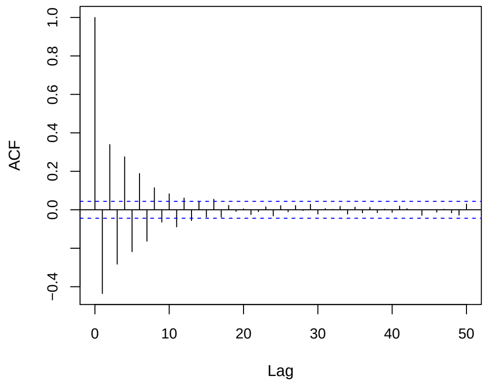
**A2** You are given the following stationary ARMA(1,1) model:

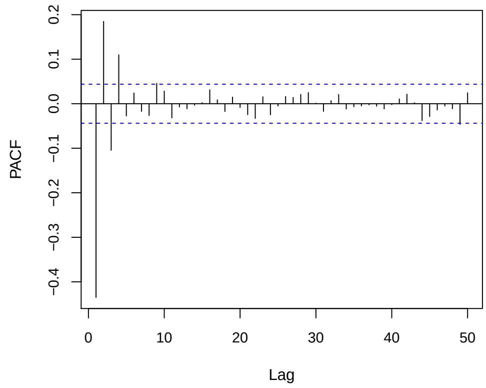
$$X_t = \phi X_{t-1} + \epsilon_t - \theta\epsilon_{t-1}\ , \quad |\phi| < 1\ ,$$

(a) Re-express this model as a $\mathrm{MA}(\infty)$ one. [3]

(b) Using your answer in (a) and Wold's formula or otherwise, calculate $\gamma(4)$. What is the exact value of $\gamma(4)$ when $\theta = \phi$? [3]

(c) You are given that $\phi > 0$. Are the following ACF and PACF plots consistent with this ARMA model? Explain your answer. [3]




(a)

(b) When $\phi = \theta$, we have that $\gamma(4) = 0$.

(c) No, they are not.

**A3** Consider the following MA(2)-model:

$$Y_t = \epsilon_t - \theta\epsilon_{t-1} + \tfrac{1}{2}\,\epsilon_{t-2}$$

Data from this model have provided the following estimates of the autocorrelation function:

| Lag, $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $r(k)$ | 0.5 | 0.45 | 0.00 | -0.04 | 0.01 |

and the estimate of the autocovariance $\hat{\gamma}(1) = 0.75$.

(a) Use the above sample information to estimate $\theta$ with the *method of moments*. Note that the model is required to be invertible. [3]

(b) Find now an estimate of the variance of the white noise $\sigma^2$, again with the method of moments. [3]

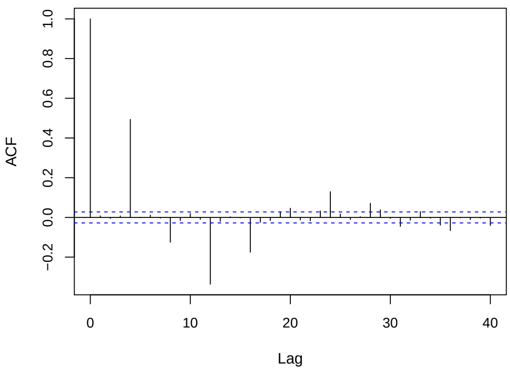
(c) Find the lag 1 and 2 sample partial autocorrelations. [3]

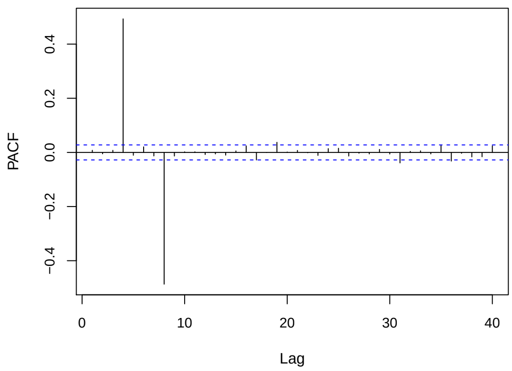
(a) $\hat{\theta} = -\frac{1}{2}$

(b) $\hat{\sigma}^2 = 1$

(c) $\hat{\phi}_{11} = 0.5; \quad \hat{\phi}_{22} = 0.27$

**A4** A statistician wants to fit an ARMA model to a given time series. The empirical autocorrelation and partial autocorrelation functions of the time series are as shown in the following plots:




a) Based on the above figures, suggest an ARMA model for this time series. Explain why both plots are consistent with the model you have suggested. [4]

(b) Looking at these plots, is there any evidence for existence of complex roots in any of the characteristic polynomials of the model? [2]

(a) SAR$(2)_4$

(b) Yes.

**A5** (a) Consider the following stationary AR(1) model:

$$Y_t = 1 + \phi Y_{t-1} + \epsilon_t \ , \quad |\phi| < 1$$

Find the variance of $Y_t$. [2]

(b) Consider an arbitrary stationary process $(X_t)$, with mean $\mu_X$, variance $\sigma_X^2$ and autocorrelation function $\rho_X(k)$, $k \geq 0$. Given the observations $X_1, X_2, \ldots, X_T$, the sample mean $\overline{X} = \sum_{i=1}^{T} X_i/T$ is obtained as an estimate of $\mu_X$.

i) Show that the variance of the estimator $\overline{X}$ is equal to:

$$\frac{\sigma_X^2}{T} \cdot \Big(1 + 2\sum_{k=1}^{T-1}(1 - \tfrac{k}{T})\rho_X(k)\Big)$$

[4]

ii) How does the variance of the above formula contrast with the one corresponding to *independent* observations? [2]

(a) $\sigma^2/(1-\phi^2)$.

(b) i)

ii) The difference is term $\frac{\sigma_X^2}{T} \cdot \Big(2\sum_{k=1}^{T-1}(1 - \frac{k}{T})\rho_X(k)\Big)$

**B1** In the UK, many sea birds rely on small fish called sandeels for their food. Let $M_t$ and $N_t$ denote respectively the population sizes of sea birds and sandeels (in tens of thousands), at the beginning of year $t$, and let $Y_t$ denote a corresponding survey-based estimate of the sea bird population. Suppose that these quantities are related via the equations:

$$\begin{aligned} Y_t &= M_t + \epsilon_t \\ M_t &= -7 + 0.7M_{t-1} + 0.1N_{t-1} + h_t \\ N_t &= 70 - 2M_{t-1} + 0.5N_{t-1} + z_t \ , \end{aligned}$$

where $(\epsilon_t)$, $(h_t)$ and $(z_t)$ are independent white noise sequences with variances $\sigma_\epsilon^2$, $\sigma_h^2$ and $\sigma_z^2$ respectively.

(a) You are given that both $M_t$ and $N_t$ are stationary processes. Calculate the long term average population sizes of the sea birds and the sandeels. [4]

(b) Define $\mathbf{S}_t = (1,\ M_t,\ N_t)^\top$. Show that the system can be described by the two equations

$$\begin{aligned} Y_t &= \mathbf{B}^\top \mathbf{S}_t + \epsilon_t \\ \mathbf{S}_t &= \mathbf{C}\,\mathbf{S}_{t-1} + \mathbf{H}_t \ , \end{aligned}$$

where $\mathbf{B}$ is a vector, $\mathbf{C}$ is a matrix and $(\mathbf{H}_t)$ is an uncorrelated sequence of random vectors with mean $\mathbf{0}$ and variance-covariance matrix $\mathbf{V}$. Give the values of $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{V}$. [5]

(c) The Kalman Filter specifies that, if $\hat{\mathbf{S}}_{t|t-\ell}$ is the best estimator of $\mathbf{S}_t$ based on the observed $Y$'s up to time $t-\ell$, and $\mathbf{P}_{t|t-\ell}$ is the covariance matrix of the associated forecast error, then we have that $\hat{\mathbf{S}}_{t|t-1} = \mathbf{C}\,\hat{\mathbf{S}}_{t-1|t-1}$ and $\mathbf{P}_{t|t-1} = \mathbf{C}\,\mathbf{P}_{t-1|t-1}\mathbf{C}^\top + \mathbf{V}$.

Suppose that at the start of year zero we have $\hat{\mathbf{S}}_{0|0} = (1,\ 8,\ 60)^\top$, with $\mathbf{P}_{0|0} = 2\mathbf{V}$. Suppose also that $\sigma_\epsilon^2 = 1$, $\sigma_h^2 = 1$ and $\sigma_z^2 = 25$. Use the Kalman Filter to forecast the population sizes of both sea birds and sandeels one year later, and give the covariance matrix of the associated forecast errors. [5]

(d) The Kalman Filter also gives that

$$\begin{aligned} \hat{\mathbf{S}}_{t|t} &= \hat{\mathbf{S}}_{t|t-1} + \mathbf{P}_{t|t-1}\,\mathbf{B}\,(Y_t - \mathbf{B}^\top\,\hat{\mathbf{S}}_{t|t-1})/f_t \\ \text{and } \mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}\,\mathbf{B}\mathbf{B}^\top\mathbf{P}_{t|t-1}/f_t \ , \\ \text{where } f_t &= \mathbf{B}^\top\,\mathbf{P}_{t|t-1}\,\mathbf{B} + \sigma_\epsilon^2 \ . \end{aligned}$$

CONTINUED

Subsequently, the survey-based estimate of sea bird population becomes available: $Y_1 = 5$. You are also given that $f_1 = 3.48$ and $\mathbf{P}_{1|0}\,\mathbf{B} = (0,\ 2.48,\ -0.3)^\top$.

i) Use this information to calculate a new estimate of $\mathbf{S}_1$, along with its error covariance matrix. [5]

ii) Give also an estimate of $M_1+0.1N_1$ together with its variance. [3]

(e) Notice that $N_t$ can be written as $(1-0.5B)^{-1}\,[70-2M_{t-1}+z_t]$, where $B$ is the backshift operator. *Without attempting any series expansions*, use this to show that $M_t$ can be written as

$$M_t = 3.5 + 1.2M_{t-1} - 0.55M_{t-2} + \lambda_t + \xi_{t-1},$$

where $(\lambda_t)$ and $(\xi_t)$ are white noise sequences that can be constructed from $(h_t)$ and $(z_t)$.

i) Give the variances of $(\lambda_t)$ and $(\xi_t)$ in terms of $\sigma_h^2$ and $\sigma_z^2$. [5]

ii) Which ARMA model does the $M_t$ process correspond to? Explain your answer. [3]

(a) $\mu_M = 10$, $\mu_N = 100$.

(b)

$$\mathbf{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ -7 & 0.7 & 0.1 \\ 70 & -2 & 0.5 \end{pmatrix}$$

and

$$\mathbf{V} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_h^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{pmatrix}$$

(c)

$$\hat{\mathbf{S}}_{1|0} = \begin{pmatrix} 1 \\ 4.6 \\ 84 \end{pmatrix}$$

$$\mathbf{P}_{1|0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2.48 & -0.3 \\ 0 & -0.3 & 45.5 \end{pmatrix}$$

(d) i)

$$\hat{\mathbf{S}}_{1|1} = \begin{pmatrix} 1 \\ 4.89 \\ 83.97 \end{pmatrix}$$

$$\mathbf{P}_{1|1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.71 & -0.09 \\ 0 & -0.09 & 45.47 \end{pmatrix}$$

ii) Estimate of $M_1 + 0.1N_1$ is 13.29; variance is 1.15.

(e) i)

ii) ARMA$(2, 1)$

**B2** The monthly sales of a retail shop in thousands of pounds are modeled according to a SARMA$(0, 2)\times(1, 0)_4$ process of the form

$$(1 - \tfrac{1}{4}B^4)X_t = \tfrac{150}{2} + (1 - \tfrac{1}{2}B - \tfrac{1}{4}B^2)\epsilon_t$$

where $\epsilon_t \sim WN(0, \sigma^2)$ with $\sigma^2 = 2$.

(a) Calculate the long term average sales $\mathbb{E}[\,X_t\,]$ and the long term variance $\text{Var}[\,X_t\,]$. [4]

(b) The table below shows the observations for the last 6 months of available data, and all corresponding estimated residuals except for the one for the last day:

| Month | Sales | Residuals ($e_t$) |
|---|---|---|
| 1 | 99 | -1 |
| 2 | 103 | 4 |
| 3 | 97 | -3 |
| 4 | 100 | 4 |
| 5 | 96 | -2 |
| 6 | 104 | |

i) Calculate the missing residual at time $T = 6$ and show that it is equal to $\frac{13}{4}$. [3]

ii) Produce forecasts for times $T = 7$ and $T = 8$. [4]

iii) Give a general expression for the forecasts at times $t = 9+4k$, for all $k \geq 0$. [3]

(c) Derive in detail the variance for the error of the estimate for time $T = 8$ and calculate a 95% prediction interval. [5]

(d) The practitioner carrying out this study used the BIC criterion to choose an appropriate model. For the SARMA$(0,2)\times(1,0)_4$ model BIC was found to be 205, whereas for SARMA$(0,3)\times(1,0)_4$ it was found to be 207. The size of the data was 240. You are reminded of the following expressions for BIC and AIC from the textbook:

$$\begin{aligned} \text{BIC}: &\quad T\log(s_e^2) + k\log T \\ \text{AIC}: &\quad T\log(s_\epsilon^2) + 2k \end{aligned}$$

i) Define all involved terms in the above given expressions for AIC and BIC. Did the practitioner make the correct decision to choose SARMA$(0,2)\times(1,0)_4$ over SARMA$(0,3)\times(1,0)_4$ according to the BIC criterion? [3]

ii) Would the decision be the same if the AIC criterion was used instead? Explain your answer. [4]

(e) Assume that in some other scenario, one claims that the $\epsilon_t$ process appearing in the definition of the model is not white noise, but it is in fact such that $\nabla\epsilon_t$ is a stationary ARMA$(1,1)$ process. Determine which model would the $(X_t)$ process correspond to in such a case. [4]

(a) $\mu = 100;\quad \text{var} = 42/15.$

(b) i)

ii)

$$\hat{X}_{T+2} = 99.1875$$

iii)

$$\hat{X}_{9+4k} = 100 - \tfrac{1}{4^k}$$

(c)

$$\text{var} = \tfrac{5}{2}$$

(d) i) Practitioner made a correct choice.

ii) More complex model would be preferred.

(e) $X_t$ is now SARIMA$(1,1,3)\times(1,0,0)_4$.

END OF PAPER