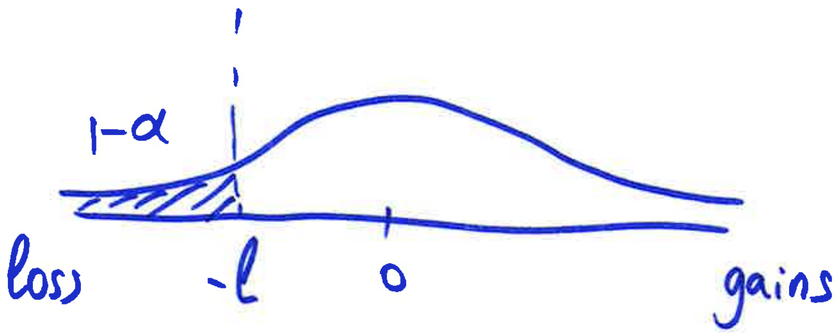
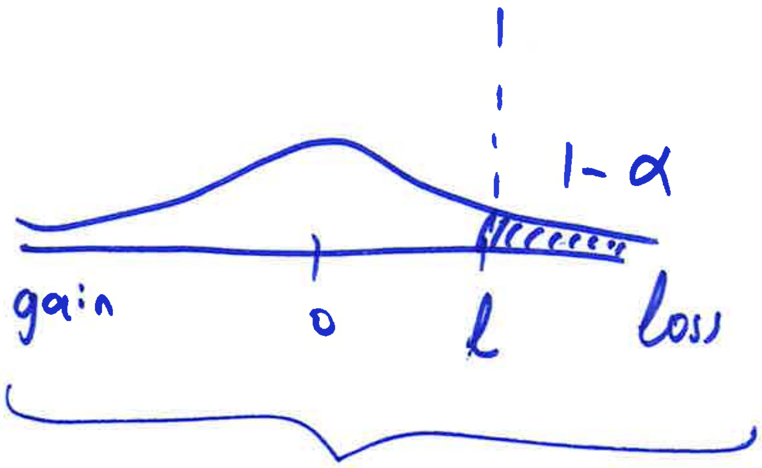

1-α
loss
-l
0
gains


1-α
gain
0
l
loss

$$= \exp\left(-\frac{\mu^2}{2}\underbrace{\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)}_{=a}+\mu\underbrace{\left(\frac{\mu_0}{\sigma_0^2}+\frac{\sum_n^{i=1} y_i}{\sigma^2}\right)}_{=b}-\underbrace{\left(\frac{\mu_0^2}{2\sigma_0^2}+\frac{\sum_{i=1}^n y_i^2}{2\sigma^2}\right)}_{=C}\right)$$

$$= \exp\left\{-\frac{\mu^2}{2}a+\mu b-c\right\} = \exp\left\{-\frac{\mu^2}{2}a+\mu b\right\}\cdot \underbrace{\exp\{-c\}}$$

this is a constant which we can drop and use the proportionality symbol "$\propto$"

$$\propto \exp\left\{-\frac{\mu^2}{2}a+\mu b\right\} = \exp\left\{-\frac{a}{2}\left(\mu^2-\frac{2b}{a}\mu\right)\right\}$$

$$\propto \exp\left\{-\frac{a}{2}\left(\mu^2-\frac{2b}{a}\mu\right)\right\}\cdot \underbrace{\exp\left\{-\frac{a}{2}\left(\frac{b}{a}\right)^2\right\}}$$

this is a constant that we can introduce in order to "complete the square", but we have to use the proportionality symbol again

$$= \exp\left\{-\frac{a}{2}\left(\mu^2-\frac{2b}{a}\mu\right)-\frac{a}{2}\left(\frac{b}{a}\right)^2\right\} = \exp\left\{-\frac{a}{2}\left(\mu^2-\frac{2b}{a}\mu+\left(\frac{b}{a}\right)^2\right)\right\} = \left\{-\frac{a}{2}\left(\mu-\frac{b}{a}\right)^2\right\}$$

$$= \exp\left\{-\frac{1}{2\frac{1}{a}}\left(\mu-\frac{b}{a}\right)^2\right\}$$

this looks like the part of the normal distribution with mean $=\frac{b}{a}$ and variance $=\frac{1}{a}$

Hence $\mu \mid y, \sigma^2 \sim N\left(\frac{b}{a}, \frac{1}{a}\right) = N\left(\frac{\frac{\mu_0}{\sigma_0^2}+\frac{\sum y_i}{\sigma^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}, \frac{1}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}\right)$

$$p(\tilde{y}|y) = N(\mu_n, \sigma_n^2 + \sigma^2)$$

1. $\tilde{y} \sim N(\mu, \sigma^2)$ $\qquad$ $Y_i \sim N(\mu, \sigma^2)$

$$\frac{\tilde{y} - \mu}{\sigma} = \underline{z \sim N(0,1)}$$

$$\tilde{y} = \mu + \sigma z$$

$$E(\tilde{y}|y, \sigma^2) = E(\mu|y, \sigma^2) + \cancel{E(\sigma z|y, \sigma^2)}$$

$$= \mu_n \qquad\qquad = \sigma E(z|y, \sigma^2)$$

$$= \sigma E(z)$$

$$= \sigma \cdot 0$$

$$\tilde{y}|y \sim N(\mu_n, \sigma_n^2 + \sigma^2)$$

$$\mathrm{var}(\tilde{y}|y, \sigma^2) = \mathrm{var}(\mu|\sigma^2, y) + \mathrm{Var}(\sigma z|y, \sigma^2)$$

$$= \sigma^2 \mathrm{var}(z)$$

$$= \sigma_n^2 + \sigma^2 \qquad = \sigma^2$$

$$E(\theta) = E_y\left[E(\theta \mid y)\right]$$

$$\mathrm{var}(\theta) = E_y\left[\mathrm{var}(\theta \mid y)\right] + \mathrm{var}\left(E(\theta \mid y)\right)$$

$$E(\tilde{y} \mid y) = E_\mu\left[\underbrace{E(\tilde{y} \mid y, \mu)}_{E(\tilde{y} \mid \mu)} \mid y\right] = E_\mu\left[\mu \mid y\right] = \mu_n$$

$$\mathrm{var}(\tilde{y} \mid y) = E_\mu\left[\underbrace{\mathrm{var}(\tilde{y} \mid y, \mu)}_{\mathrm{var}(\tilde{y} \mid \mu)} \mid y\right] + \mathrm{var}\left(\underbrace{E(\tilde{y} \mid y, \mu)}_{E(\tilde{y} \mid \mu) = \mu} \mid y\right)$$

$$= E_\mu\left[\sigma^2 \mid y\right] + \mathrm{var}(\mu \mid y)$$

$$= \sigma^2 + \sigma_n^2$$