STAT0008 Lecture 1

SLIDE 18, EXAMPLE 1

Let X_i = Lifetime of i^{th} kettle

$$X_i \sim \text{Exp}(\lambda)$$

We seek

$$\mathbb{P}(\text{All kettles have a lifetime} > 2)$$

$$= \mathbb{P}(X_1 > 2 \cap X_2 > 2 \cap \dots \cap X_n > 2)$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{n} X_i > 2\right)$$

$$= \prod_{i=1}^{n} \mathbb{P}(X_i > 2) \quad \text{because } X_1, \dots, X_n \text{ are independent.}$$

$$= \prod_{i=1}^{n} \left[1 - \mathbb{P}(X_i \leqslant 2)\right]$$

Now, $\mathbb{P}(X_i \leqslant 2) = F_{X_i}(2) \leftarrow$ CDF of X_i

$$= 1 - \exp(-2\lambda)$$

$$\therefore \mathbb{P}(\text{All kettles have a lifetime} > 2) = \prod_{i=1}^{n} \left[1 - (1 - \exp(-2\lambda))\right]$$

$$= \exp(-2n\lambda).$$

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SLIDE 23, EXAMPLE 2

Let $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ sampled voter votes Conservative} \\ 0 & \text{otherwise} \end{cases}$

and $X_i \sim Bin(1, p) \equiv Bernoulli(p)$.

Here p = Probability of voting Conservative (within this London borough)

$\therefore$ ESTIMAND = p (the parameter we wish to estimate).

To estimate p, we construct our ESTIMATOR

$\hat{p} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \leftarrow$ Random variable.

Given that we observe 400 responses where the voter will vote Conservative, our ESTIMATE of p is

$$\hat{p} = \frac{400}{1000} = 0 \cdot 4.$$

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SLIDE 27, EXAMPLE 3

Let X = Number of road accidents at the junction in one year.

$$X \sim \mathrm{Poi}(\lambda)$$

Given that x accidents are observed, the likelihood function for λ is

$$\mathcal{L}(\lambda \mid X = x) = \mathbb{P}(X = x; \lambda)$$
$$= \frac{\lambda^x e^{-\lambda}}{x!}.$$

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SLIDE 29, EXAMPLE 4

The likelihood function for $(\mu, \sigma^2)^T$ is given by

$$L(\mu, \sigma^2 \mid \underline{x}) = f(\underline{x} \mid \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} f(x_i \mid \mu, \sigma^2) \quad \text{(independence)}$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right\}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}.$$

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SLIDE 33, EXAMPLE 5

Let X_i = Result of i^{th} trial (1 = 'Success', 0 = 'Failure')

$X_i \sim \text{Bern}(p)$

But $Y = \sum_{i=1}^{n} X_i$ total number of successes in n trials

is such that $Y \sim \text{Bin}(n, p)$

Then, the likelihood function for p, given that r successes are observed is:

$$L(p \mid Y=r) = \mathbb{P}(Y=r \mid p)$$
$$= \binom{n}{r} p^r (1-p)^{n-r}$$

The log-likelihood function is

$$\ell(p \mid Y=r) = \log L(p \mid Y=r)$$
$$= \log \binom{n}{r} + r\log p + (n-r)\log(1-p)$$

To find the mle, $\hat{p}$, we solve

$$\left.\frac{\partial \ell}{\partial p}\right|_{p=\hat{p}} = 0$$

$$\Rightarrow \quad \frac{r}{p} - \frac{(n-r)}{1-p} = 0 \quad \Rightarrow \frac{r(1-p)-(n-r)p}{p(1-p)} = 0$$

$$\Rightarrow \quad r - np = 0$$

Which is solved where $p = \frac{r}{n}$.

$\therefore$ The maximum likelihood estimate is $\hat{p} = \frac{r}{n}$.