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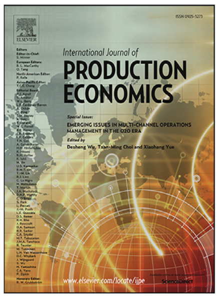

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PII: S0925-5273(22)00169-4
DOI: https://doi.org/10.1016/j.ijpe.2022.108580
Reference: PROECO 108580

To appear in: *International Journal of Production Economics*

Received date : 17 September 2021
Revised date : 10 July 2022
Accepted date : 13 July 2022

A New Cost-Sharing Preventive Maintenance Program Under Two-Dimensional Warranty

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Abstract

In this paper, we present a new cost-sharing preventive maintenance (PM) program for a product protected by a two-dimensional (2-D) warranty. The product is preventively maintained by its manufacturer in the warranty period and by a customer in the post-warranty period. To better coordinate their decisions, the cost of PM during the warranty period is borne jointly by both parties based on a fixed ratio. We propose a new way of designing such a cost-sharing PM program by using the customer's expected post-warranty cost. We find a failure intensity threshold that determines whether the customer should carry out PM at the beginning of the post-warranty period. It is shown that cost sharing can reduce the failure intensity at this point, which in turn can reduce the customer's expected total cost. Given a specific usage rate of the product, we then derive the optimal cost-sharing ratio from the customer's perspective. In the numerical study, we examine the effect of usage rate on the optimal cost-sharing ratio as well as the benefits gained from cost sharing.

Keywords: Cost sharing, Preventive maintenance, Two-dimensional warranty

1. Introduction

Many capital-intensive products, such as commercial vehicles and industrial machinery, come with a two-dimensional (2-D) warranty (Wang & Xie, 2018). This type of warranty specifies its protection limits in terms of both age and usage (e.g., three years and $36,000$ miles for some commercial vehicles); it lasts until either of these two limits is reached. Because of the imposed usage limit, a 2-D warranty can protect manufacturers against excessive failures from heavy users. If a product fails under warranty, then its manufacturer has to pay the cost of repair or replacement (Jack & Van der Duyn Schouten, 2000). Warranty redemption costs in many industries still remain high. For example, in 2019, automakers worldwide spent roughly 2.5% of their total revenue—equivalent to $49.4 billion—on warranty repairs (Warranty Week, 2020).

High warranty costs force manufacturers to improve quality and reliability of their products (Priest, 1981). This is known as the incentive mechanism of warranties on the firm side (Cooper & Ross, 1985). Hyundai Motor provides a great example. In 1999, it extended the powertrain warranty on its cars in the U.S. from 5 years/$60,000$ miles to 10 years/$100,000$ miles. To avoid paying too much for warranty claims, Hyundai invested heavily in a quality improvement program that was implemented in a top-down manner (Forbes, 2005). Another way to reduce warranty expenses is to adopt a preventive maintenance (PM) program for warranted products (Alqahtani et al., 2019). Manufacturers have a strong incentive to do so because PM can reduce the chance of a product failing in the warranty period. In practice, PM under new-car warranty, such as oil changes, is generally recommended to be performed at certain time or mileage intervals.

Proper maintenance is critical for keeping products in good working condition; however, it usually comes at a significant cost. Manufacturers can transfer a portion of the cost to customers by offering *cost-sharing* PM programs. For instance, while some auto brands, including Hyundai and BMW, offer several years of free maintenance, others, such as Audi and Lexus, only cover the first one or two maintenance visits in the warranty period[1]. Customers are required to pay for the remaining visits and adhere to the maintenance schedule specified in the owner's manual because lack of maintenance may void their warranties. To get a repair approved, customers have to go to an authorized service center or independent workshop for regular maintenance.

Motivated by the aforementioned practices, this paper aims to design a new cost-sharing PM program for products sold under 2-D warranty. In this program, the cost of PM during the warranty period is borne jointly by a manufacturer and a customer. We formulate a sequential game model in which the customer first announces a cost-sharing ratio and then the manufacturer decides on the PM schedule in the warranty period. This schedule dictates the product failure intensity at the end of the 2-D warranty, which further affects the customer's PM decision in the post-warranty period. Our model is intended to highlight the role of a cost-sharing ratio in coordinating PM decisions over the product life cycle. We show that both parties benefit from cost-sharing PM in the sense that their expected total costs are reduced relative to the scenario without cost sharing. For the manufacturer, different product usage rates lead to different percentage cost reductions.

The remainder of this paper is organized as follows. Section 2 reviews the relevant literature. Section 3 formulates the manufacturer's scheduling problem in the warranty period. Section 4 analyzes the customer's PM planning problem in the post-warranty period, given the failure intensity at the beginning of this period. Section 5 optimizes the cost-sharing ratio for the customer. Section 6 presents a numerical study to illustrate the benefits of sharing PM costs. Section 7 concludes the paper with a summary and suggestions for future research. All proofs

[1] Free car maintenance programs. http://static.ed.edmunds-media.com/unversioned/img/pdf/free.vehicle.maintenance.programs/free.vehicle.maint.5.pdf.

are relegated to Appendix A.

2. Literature review

There is extensive literature on maintenance planning problems in the context of warranties (see Shafiee & Chukova, 2013 for a detailed overview). Our paper is related to the recent literature investigating PM under 2-D warranties. A review of this stream of research can be found in Wang & Xie (2018). Most existing studies focus on designing optimal PM policies under 2-D base warranties (see, e.g., Wang & Su, 2016; Peng et al., 2021; Wang et al., 2020), while Shahanaghi et al. (2013) and Huang et al. (2017) optimize PM policies implemented under 2-D extended warranties. Several others consider two-stage maintenance problems and study PM scheduling in both base and extended warranty periods (see, e.g., Wang et al., 2015; Su & Wang, 2016; Wang et al., 2017; Li et al., 2019). The start and end points of an extended warranty period are dependent on product usage rate. In addition, following the expiration of a 2-D warranty, Park et al. (2018) determine a PM policy that minimizes the expected cost rate over the product life cycle. In this paper, our focus is on PM scheduling over base warranty and post-warranty periods.

Most research on PM planning under 2-D warranties concentrates on periodic PM policies because they are easy for manufacturers to manage and for customers to follow. Nonperiodic PM policies have been studied by Huang et al. (2013), Huang et al. (2017), and Peng et al. (2021), among others. Huang et al. (2017) establish a procedure for customizing PM services in the 2-D extended warranty period based on each customer's usage rate in the base warranty period. Peng et al. (2021) propose a usage-dependent threshold policy for PM when customer usage rates are random and dynamic. Huang et al. (2013) optimize the two parameters of a 2-D warranty when PM is performed whenever product reliability falls below a certain level. In this work, the customer's optimal PM schedule can be nonperiodic, depending on the failure intensity at the end of the 2-D warranty.

In practice, PM costs can be paid by the manufacturer (e.g., Shahanaghi et al., 2013; Peng et al., 2021), the customer (Huang & Yen, 2009), or both of them. The literature on maintenance models with cost sharing is quite limited. Cost-sharing PM typically includes a fixed sharing ratio (Huang et al., 2017; He et al., 2020) or a sequence of pro-rata shares (Wang & Su, 2016; Wang et al., 2017). Maintenance cost sharing has several advantages over the first two payment mechanisms. It can reduce the manufacturer's repair costs (Huang et al., 2017; Dai et al., 2020) and the customer's downtime losses (Zheng et al., 2020); in addition, it also helps to achieve channel coordination in the context of maintenance outsourcing (Tarakci et al., 2006).

Our work contributes to the literature in three ways. First, we develop a new cost-sharing PM model for managing the product life cycle by incorporating the customer's expected total cost over the post-warranty period. In contrast, previous research only considers the customer's profits from PM activities (Wang et al., 2017) or losses from downtime (Zheng et al., 2020) during the base warranty period. Second, little attention has been paid to how the failure intensity at the end of the base warranty affects the customer's PM schedule in the post-warranty period. We show that there exists a threshold on this failure intensity above which a PM action should be performed. Third, we find that offering customized cost-sharing PM programs to customers with distinct usage rates is a win-win strategy. Such programs can be used by manufacturers as a way of gaining competitive advantages.

3. The manufacturer's scheduling problem

We consider a manufacturer providing customized PM for a product covered by a 2-D warranty. Of the total PM cost in the warranty period, the fraction that the manufacturer pays is denoted by p so that $1-p$ is the fraction borne by the customer. When the manufacturer covers some of the maintenance visits, as in the cases discussed above, the cost-sharing ratio p is restricted to a finite set of values—that is, $p \in \{0, 1/n, 2/n, \ldots, 1\}$, where n is the total number of PM actions in the warranty period. However, to facilitate analysis, we consider p to be a real number between zero and one, inclusive. In this setting, the customer partially pays for each of the maintenance visits. The objective of the manufacturer's scheduling problem is to balance PM and repair costs by performing PM in the warranty period as cost-effectively as possible. The determination of an optimal cost-sharing ratio is deferred to Section 5. Table 1 summarizes the notation used in this paper.

3.1. Assumptions

Suppose that the rate at which product usage accumulates is a constant, denoted by r, and is known to the manufacturer. This constant usage rate assumption is reasonable for automobiles and many other products. We model product failures using a one-dimensional approach (see, e.g., Murthy et al., 1995; Wang & Xie, 2018). In this approach, the time required for a repair is assumed to be negligible relative to the mean time between failures. Therefore, under usage rate r, we can express the cumulative amount of usage at time t as rt. When a product is repairable, time refers to its age. We assume that product failures occur according to a nonhomogeneous Poisson process (NHPP) with intensity function $\lambda(t)$, which is increasing in age and usage. In

Table 1
Model notation.

Symbol	*Definition*
Parameters:	
r	Product usage rate
$\lambda(t)$	Product failure intensity at time t
θ_0, θ_1, θ_2, θ_3	Coefficients in the failure intensity function
k	Setup cost of PM
b	Variable cost of PM
c	Cost per repair
α	One plus the markup over the repair cost
β	One plus the markup over the PM cost
W, U	Age and usage limits of the 2-D warranty, respectively
T, L	Age and usage limits of the product's useful life, respectively
$\widetilde{n}^*$, $\widetilde{m}^*$	Solutions of the continuous relaxations
$\underline{m}$	Optimal number of PM actions by the customer when $\lambda(W) = \theta_0 + \theta_1 r$
λ^*	Failure intensity at the end of the 2-D warranty under the optimal PM schedule
$\overline{\lambda}$	λ^* without cost sharing
Decision variables:	
p	Cost-sharing ratio of the manufacturer
n, m	Number of PM actions performed by the manufacturer and the customer, respectively
τ_i, $\widetilde{\tau}_i$	Time of the ith PM action
δ_i, ξ_i	Time interval between the $(i-1)$th and ith PM actions
x_i, y_i	Reduction in failure intensity as a result of the ith PM action
Cost functions:	
$C(n, \boldsymbol{\delta}, \boldsymbol{x}; r, p)$	Manufacturer's cost function
$H(m, \boldsymbol{\xi}, \boldsymbol{y}; \lambda(W))$	Customer's cost function

the absence of PM, the failure intensity at time t is given by

$$\lambda(t) = \theta_0 + \theta_1 r + \theta_2 t + \theta_3 r t,$$

where θ_i's are nonnegative constant coefficients. This additive intensity function is widely adopted in the 2-D warranty literature (see, e.g., Iskandar & Murthy, 2003; Su & Wang, 2016; Huang et al., 2017) and can be viewed as an additive hazards model (Eliashberg et al., 1997; Singpurwalla & Wilson, 1998). The first three terms represent a baseline failure intensity. Were a product to undergo no use, it would fail at a time characterized by this failure intensity. Specifically, the first term is an intrinsic intensity level that is independent of age and usage. The second term captures the heterogeneity in the baseline failure intensity. The higher the usage rate, the harsher the operating condition becomes. The third term corresponds to the effect of age. The last term implies that each unit of cumulative usage increases the failure intensity by the same amount. It is worth mentioning that the effect of usage on time to failure can also be described by an accelerated failure time model (see, e.g., Shahanaghi et al., 2013; Li et al., 2019; Wang et al., 2020, 2021).

We denote τ_i as the instant at which the ith PM action is performed and δ_i as the interval between the $(i-1)$th and ith actions for $i = 1, \ldots, n$. When maintenance duration is negligible, we have $\delta_i = \tau_i - \tau_{i-1}$, where $\tau_0 = 0$. The efficiency of the ith PM action is described by the amount of failure intensity reduction x_i. We assume that the failure intensity can be restored to its lowest attainable level $\lambda(0) = \theta_0 + \theta_1 r$ every time PM is performed. For any $\tau_{i-1} \leq t < \tau_i$, the failure intensity in the presence of PM becomes

$$\lambda(t) = \theta_0 + \theta_1 r + \theta_2 t + \theta_3 rt - \sum_{j=1}^{i-1} x_j.$$

According to this intensity function, a PM action with efficiency x_i makes a product $x_i/(\theta_2+\theta_3 r)$ units of time younger.

It is assumed that the cost of a PM action is linear in the failure intensity reduction x_i and is given by $k+bx_i$, where k is the setup cost and b is the variable cost of PM. In addition to PM costs, the manufacturer bears possible repair costs during the warranty period. Under NHPP, product failures are rectified by minimal repairs; that is, the failure intensity stays unchanged after repair. The average cost of each repair is a constant c.

3.2. Optimal PM schedule

We represent the 2-D warranty by an age limit W and a usage limit U. Depending on the value of usage rate r, we have the following two cases for warranty expiration: If $r \leq U/W$, the 2-D warranty will expire at time W; if $r > U/W$, it will cease at time U/r.

We first fix the cost-sharing ratio p. The manufacturer's decision variables are the number of PM actions, the intervals between successive PM actions, and the amount of failure intensity reduction at each PM instant. We denote the manufacturer's PM schedule by $(n, \boldsymbol{\delta}, \boldsymbol{x})$, where $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. Let $C(n, \boldsymbol{\delta}, \boldsymbol{x}; r, p)$ be the manufacturer's expected total cost of honoring the 2-D warranty for a given usage rate r and cost-sharing ratio p. In the case of $r \leq U/W$, this cost function can be written as

$$C(n, \boldsymbol{\delta}, \boldsymbol{x}; r, p) = c \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \lambda(t)\,\mathrm{d}t + c \int_{\tau_n}^{W} \lambda(t)\,\mathrm{d}t + \sum_{i=1}^{n} p(k + bx_i).$$

The first term represents the expected repair cost from time zero to time τ_n, which is calculated by multiplying the average cost per repair by the expected number of failures. The second term corresponds to the expected repair cost from time τ_n to time W. In this term, $\lambda(t) =$

$\theta_0 + \theta_1 r + \theta_2 t + \theta_3 rt - \sum_{i=1}^{n} x_i$. The last term is the total PM cost borne by the manufacturer.

Following the optimization steps in Jack & Murthy (2002), we first determine the optimal amounts of failure intensity reduction for any $n \geq 1$ and $\delta_i \geq 0$ and then find the optimal PM intervals for a given $n \geq 1$. Finally, we obtain the optimal number of PM actions. In the first step, we solve the following linear program:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & C(n, \boldsymbol{\delta}, \boldsymbol{x}; r, p) \\ \text{subject to} \quad & x_i \leq \lim_{t \to \tau_i^-} \lambda(t) - \lambda(0), \quad i = 1, \ldots, n, \\ & x_i \geq 0, \qquad\qquad\qquad\qquad i = 1, \ldots, n, \end{aligned}$$

where $\lim_{t \to \tau_i^-} \lambda(t) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r) \sum_{j=1}^{i} \delta_j - \sum_{j=1}^{i-1} x_j$ and $\lambda(0) = \theta_0 + \theta_1 r$. The first constraint ensures that the failure intensity after PM is no less than its lowest attainable level. This constraint is equivalent to $\sum_{j=1}^{i} x_j \leq (\theta_2 + \theta_3 r) \sum_{j=1}^{i} \delta_j$ for any i.

Proposition 1. *Given $n \geq 1$ and $\delta_i \geq 0$ such that $\sum_{i=1}^{n} \delta_i \leq W - pb/c$, we have $x_i^* = (\theta_2 + \theta_3 r)\delta_i$ for $i = 1, \ldots, n$.*

This result indicates that the manufacturer should bring the failure intensity down to its lowest attainable level at each PM instant, which can be achieved through replacement or a major overhaul. This result is consistent with the findings in Yeh & Lo (2001), Jack & Murthy (2002), and Peng et al. (2021). If the amount of failure intensity reduction is required to be no greater than the increment in the failure intensity since the last PM action, then we can add the constraints $x_i \leq (\theta_2 + \theta_3 r)\delta_i$ for $i = 1, 2, \ldots, n$. The optimal solution remains unchanged because it still lies in the new feasible region, which is smaller than the original one.

The condition under which Proposition 1 holds has an intuitive interpretation: It specifies an interval that must include the instant of the last PM action. Otherwise, if this action is performed in $(W - pb/c, W]$, that is, if $W - pb/c < \sum_{i=1}^{n} \delta_i \leq W$, then $p(k + bx_n) > c(W - \sum_{i=1}^{n} \delta_i)x_n$ for any $x_n \geq 0$ and $p > 0$, implying that the increase in the PM cost will be greater than the decrease in the expected repair cost. Therefore, when deciding on δ_i, we restrict our attention to the region defined by $\sum_{i=1}^{n} \delta_i \leq W - pb/c$ rather than by $\sum_{i=1}^{n} \delta_i \leq W$. Note that when $W < pb/c$, any PM action is not worthwhile. In this case, the solution $n = 0$ is optimal.

Substituting x_i^* into $C(n, \boldsymbol{\delta}, \boldsymbol{x}; r, p)$ yields

$$C(n, \boldsymbol{\delta}, \boldsymbol{x}^*; r, p) = c(\theta_2 + \theta_3 r)\left(\frac{\sum_{i=1}^n \delta_i^2}{2} + \frac{(W - \sum_{i=1}^n \delta_i)^2}{2} + \frac{pb\sum_{i=1}^n \delta_i}{c}\right) + npk + c(\theta_0 + \theta_1 r)W.$$

Next, we solve the following minimization problem given $n \geq 1$:

$$\begin{aligned} \underset{\boldsymbol{\delta}}{\text{minimize}} \quad & C(n, \boldsymbol{\delta}, \boldsymbol{x}^*; r, p) \\ \text{subject to} \quad & \sum_{i=1}^n \delta_i \leq W - \frac{pb}{c}, \\ & \delta_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}$$

Proposition 2. *If $W \geq pb/c$, then we have $\delta_1^* = \delta_2^* = \ldots = \delta_n^* = \frac{W}{(n+1)} - \frac{pb}{(n+1)c}$ for any given integer $n \geq 1$.*

This result shows that the optimal PM policy exhibits *periodicity*. The optimal PM interval δ_i^* is independent of setup cost k and is less than $W/(n+1)$. Hence, our policy differs from the one that divides the warranty period into $n+1$ equal intervals, as in Dai et al. (2021), Hamidi et al. (2016), Iskandar & Husniah (2017), and Wang et al. (2020). The constraint $\sum_{i=1}^n \delta_i \leq W - pb/c$ is not binding at the optimum, and thus there is a no-maintenance interval of length $\frac{W}{n+1} + \frac{npb}{(n+1)c}$ at the end of the 2-D warranty, where only minimal repairs are carried out.

Further substituting δ_i^* into $C(n, \boldsymbol{\delta}, \boldsymbol{x}^*; r, p)$ gives

$$C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) = pkn + \frac{c(\theta_2 + \theta_3 r)(W - pb/c)^2}{2(n+1)} + c(\theta_0 + \theta_1 r)W + (\theta_2 + \theta_3 r)pbW - \frac{(\theta_2 + \theta_3 r)p^2 b^2}{2c}.$$

Note that $C(0, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) = \frac{1}{2}c(\theta_2 + \theta_3 r)W^2 + c(\theta_0 + \theta_1 r)W$. This is precisely the expected total cost for the policy of no PM actions. Although $\boldsymbol{\delta}$ and $\boldsymbol{x}$ are undefined when $W \geq pb/c$ and $n = 0$, we have included this case in the optimization model.

Proposition 3. *Let*

$$\widetilde{n}^* = \left(W - \frac{pb}{c}\right)\sqrt{\frac{c(\theta_2 + \theta_3 r)}{2pk}} - 1.$$

When $\widetilde{n}^* > 0$, the optimal number of PM actions n^* is determined by $\lfloor \widetilde{n}^* \rfloor$ or $\lceil \widetilde{n}^* \rceil$, where $\lfloor \cdot \rfloor$ is the floor function and $\lceil \cdot \rceil$ is the ceiling function. When $-1 \leq \widetilde{n}^* \leq 0$, that is, when

$pb/c \leq W \leq pb/c + \sqrt{\frac{2pk}{c(\theta_2+\theta_3 r)}}$, *we have* $n^* = 0$.

Since n is a discrete variable, its optimal value corresponds to the integer below or above $\widetilde{n}^*$, the optimal solution of the continuous relaxation. The rounding depends on the values of the objective function evaluated at these two integers. Another way to obtain n^* is to take the first difference $C(n+1, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) - C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$, which is the benefit of having one more PM action. The condition for optimality is that n be the smallest nonnegative integer such that the difference $C(n+1, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) - C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) \geq 0$. This corresponds to finding the smallest n for which

$$(n+1)(n+2) \geq \frac{c(\theta_2+\theta_3 r)(W - pb/c)^2}{2pk}. \tag{1}$$

For the case of $r > U/W$, we substitute U/r for W and simply repeat the above optimization steps. Under the optimal policy, PM is scheduled to be performed every $\frac{U}{(n+1)r} - \frac{pb}{(n+1)c}$ units of time, and at each instant of PM, the failure intensity is reduced by $(\theta_2 + \theta_3 r)(\frac{U}{(n+1)r} - \frac{pb}{(n+1)c})$. The objective function in the last step is given as follows:

$$\begin{aligned} C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) = pkn &+ \frac{c(\theta_2+\theta_3 r)(U/r - pb/c)^2}{2(n+1)} + \frac{c(\theta_0+\theta_1 r)U}{r} \\ &+ \frac{(\theta_2+\theta_3 r)pbU}{r} - \frac{(\theta_2+\theta_3 r)p^2b^2}{2c}. \end{aligned}$$

To determine n^*, we need to find the smallest n for which

$$(n+1)(n+2) \geq \frac{c(\theta_2+\theta_3 r)(U/r - pb/c)^2}{2pk}.$$

It is worth noting that the condition $U/r \geq pb/c$ is less likely to hold for higher values of r.

With p held fixed, we next perform comparative statics of the optimal PM schedule in the warranty period with respect to r.

Proposition 4. *If* $r \leq U/W$, *then* n^* *is increasing and* δ_i^* *is decreasing in* r*; if* $r > U/W$, *then* n^* *is decreasing in* r.

Despite the discrete nature of our problem, we can still use differential techniques to examine the properties of the optimal PM policy. The key to our analysis is the interaction between n and r in $C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$. Depending on the sign of the cross-partial derivative, the marginal cost of increasing the number of PM actions either increases or decreases in usage rate, meaning that the desirability of more PM actions is either decreasing or increasing in usage rate. The first part of this result can also be obtained from Inequality (1) by noting that the left-hand side of this inequality is increasing in n and the right-hand side is increasing in r. The second part follows similarly.

Table 2
Expiry and end times for different usage rates.

Case	*Expiry time of the warranty*	*End time of the useful life*
$r < U/W \leq L/T$	W	T
$U/W \leq r \leq L/T$	U/r	T
$U/W \leq L/T < r$	U/r	L/r
$r < L/T < U/W$	W	T
$L/T \leq r < U/W$	W	L/r
$L/T < U/W \leq r$	U/r	L/r

When $r \leq U/W$, the higher the usage rate, the higher the failure intensity, and thus the higher the number of PM actions. However, when $r > U/W$, the higher the usage rate, the lower the number of PM actions because the 2-D warranty ceases sooner. Although a comparative statics result does not hold for δ_i^* when $r > U/W$ and for x_i^* in both cases, we can show that for any given n, δ_i^* is constant and x_i^* is increasing in r if $r \leq U/W$; otherwise, δ_i^* and x_i^* are decreasing in r. We can also show monotonic properties with respect to r for some other quantities, such as the instant of the last PM action $n^*\delta_i^*$ and the total amount of failure intensity reduction $n^*x_i^*$.

4. The customer's scheduling problem

In this section, we specify the customer's optimal PM schedule in the post-warranty period to determine an optimal cost-sharing ratio. We assume that the product has a useful life of T units of time or L units of cumulative usage, whichever comes first. The customer will keep this product until the end of its life. The end time depends on the usage rate r and is listed in Table 2 for different values of r, along with the expiry time of the 2-D warranty. The analysis is quite similar for each case, and thus we only focus on the first case for illustration.

During the post-warranty period, the customer will carry out PM so as to reduce future repair costs (Wu et al., 2011). We denote m as the number of PM actions by the customer, ξ_i as the interval between the $(i-1)$th and ith actions, and y_i as the reduction in failure intensity due to the ith action for $i = 1, \dots, m$. Then, the customer's PM schedule can be represented as $(m, \boldsymbol{\xi}, \boldsymbol{y})$, where $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_m)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_m)$. The schedule is directly influenced by the failure intensity at the end of the 2-D warranty (i.e., $\lambda(W)$), which is also the initial failure intensity in the post-warranty period. We use $\widetilde{\tau}_i$ to represent the instant of the ith PM action during the post-warranty period, with $\widetilde{\tau}_0 = W$. It is easy to see that $\widetilde{\tau}_i = W + \sum_{j=1}^{i} \xi_j$. For any $\widetilde{\tau}_{i-1} \leq t < \widetilde{\tau}_i$, the failure intensity at time t is

$$\lambda(t) = \lambda(W) + (\theta_2 + \theta_3 r)(t - W) - \sum_{j=1}^{i-1} y_j,$$

where $\lambda(W) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)W - \sum_{i=1}^{n^*} x_i^*$.

The customer's scheduling problem is to find an optimal PM plan that minimizes the expected total cost in the post-warranty period, which is given by

$$H(m, \boldsymbol{\xi}, \boldsymbol{y}; \lambda(W)) = \alpha c \sum_{i=1}^{m} \int_{\tilde{\tau}_{i-1}}^{\tilde{\tau}_i} \lambda(t)\,\mathrm{d}t + \alpha c \int_{\tilde{\tau}_n}^{T} \lambda(t)\,\mathrm{d}t + \sum_{i=1}^{m} \beta(k + b y_i), \tag{2}$$

where $\alpha \geq 1$ is one plus the markup over the manufacturer's repair cost and $\beta \geq 1$ is one plus the markup over the PM cost. In the second term, $\lambda(t) = \lambda(W) + (\theta_2 + \theta_3 r)(t - W) - \sum_{i=1}^{n} y_i$. We can see from Equation (2) that the customer has the same cost structure as the manufacturer but performs PM over a different time horizon and faces a different initial failure intensity.

As in the manufacturer's problem, the optimal schedule can be obtained by first optimizing the amount of failure intensity reduction at each PM instant, then the intervals between successive PM actions, and finally the number of PM actions. In the first step, the customer's problem can be written as:

$$\begin{aligned} \underset{\boldsymbol{y}}{\text{minimize}} \quad & H(m, \boldsymbol{\xi}, \boldsymbol{y}; \lambda(W)) \\ \text{subject to} \quad & y_i \leq \lim_{t \to \tilde{\tau}_i^-} \lambda(t) - \lambda(0), \quad i = 1, \ldots, m, \\ & y_i \geq 0, \qquad\qquad\qquad\quad\;\; i = 1, \ldots, m, \end{aligned}$$

where $\lim_{t \to \tilde{\tau}_i^-} \lambda(t) = \lambda(W) + (\theta_2 + \theta_3 r) \sum_{j=1}^{i} \xi_j - \sum_{j=1}^{i-1} y_j$. The first constraint implies that the customer can reduce the failure intensity by an additional amount $\lambda(W) - \theta_0 - \theta_1 r$ compared to the manufacturer, which is the increment in the failure intensity from time τ_n to time W.

Proposition 5. *For any $m \geq 1$ and $\xi_i \geq 0$ such that $\sum_{i=1}^{m} \xi_i \leq T - W - \frac{\beta b}{\alpha c}$, we have*

$$y_i^* = \begin{cases} \lambda(W) - \theta_0 - \theta_1 r + (\theta_2 + \theta_3 r)\xi_i, & i = 1, \\ (\theta_2 + \theta_3 r)\xi_i, & i = 2, \ldots, m. \end{cases}$$

A common feature of the optimal policies adopted by the customer and the manufacturer is that the failure intensity should be reduced to its lowest attainable level each time PM is performed. Therefore, in terms of PM efficiency, the failure intensity at the end of the 2-D warranty only affects the customer's first PM action. Because it is not worthwhile for the customer to take PM actions in the interval $(T - \frac{\beta b}{\alpha c}, T]$, we impose the constraint $\sum_{i=1}^{m} \xi_i \leq T - W - \frac{\beta b}{\alpha c}$ on ξ_i instead of the constraint $\sum_{i=1}^{m} \xi_i \leq T - W$. Note that when $T - W < \frac{\beta b}{\alpha c}$, the solution $m = 0$ is optimal.

Substituting y_i^* into $H(m, \boldsymbol{\xi}, \boldsymbol{y}; \lambda(W))$ yields

$$H(m, \boldsymbol{\xi}, \boldsymbol{y}^*; \lambda(W)) = \alpha c(\theta_2 + \theta_3 r)\left(\frac{\sum_{i=1}^m \xi_i^2}{2} + \frac{(T - W - \sum_{i=1}^m \xi_i)^2}{2} + \frac{\beta b \sum_{i=1}^m \xi_i}{\alpha c}\right)$$
$$+ \alpha c\big(\lambda(W) - \theta_0 - \theta_1 r\big)\xi_1 + m\beta k + \alpha c(\theta_0 + \theta_1 r)(T - W)$$
$$+ \beta b\big(\lambda(W) - \theta_0 - \theta_1 r\big).$$

We next solve the following minimization problem given $m \geq 1$:

$$\begin{aligned} \underset{\boldsymbol{\xi}}{\text{minimize}} \quad & H(m, \boldsymbol{\xi}, \boldsymbol{y}^*; \lambda(W)) \\ \text{subject to} \quad & \sum_{i=1}^m \xi_i \leq T - W - \frac{\beta b}{\alpha c}, \\ & \xi_i \geq 0, \quad i = 1, \ldots, m. \end{aligned}$$

Proposition 6. *Suppose that $T - W \geq \frac{\beta b}{\alpha c}$. The following statements are true for any integer $m \geq 1$:*

(a) If $\lambda(W) \leq \theta_0 + \theta_1 r + \frac{\theta_2 + \theta_3 r}{m}\left(T - W - \frac{\beta b}{\alpha c}\right)$, then

$$\xi_i^* = \begin{cases} \frac{T-W}{m+1} - \frac{\beta b}{(m+1)\alpha c} - \frac{m(\lambda(W) - \theta_0 - \theta_1 r)}{(m+1)(\theta_2 + \theta_3 r)}, & i = 1, \\ \frac{T-W}{m+1} - \frac{\beta b}{(m+1)\alpha c} + \frac{\lambda(W) - \theta_0 - \theta_1 r}{(m+1)(\theta_2 + \theta_3 r)}, & i = 2, \ldots, m. \end{cases}$$

(b) If $\lambda(W) > \theta_0 + \theta_1 r + \frac{\theta_2 + \theta_3 r}{m}\left(T - W - \frac{\beta b}{\alpha c}\right)$, then

$$\xi_i^* = \begin{cases} 0, & i = 1, \\ \frac{T-W}{m} - \frac{\beta b}{m\alpha c}, & i = 2, \ldots, m. \end{cases}$$

The failure intensity at the end of the 2-D warranty affects not only the efficiency of the first PM action, but also the optimal PM intervals in the post-warranty period. The first case of this proposition is illustrated in Figure 1. Since in this case $\xi_i^* = \xi_1^* + \frac{\lambda(W) - \theta_0 - \theta_1 r}{\theta_2 + \theta_3 r}$ for $i = 2, \ldots, m$, we can treat time $W - \frac{\lambda(W) - \theta_0 - \theta_1 r}{\theta_2 + \theta_3 r}$ as the starting point of the customer's planning horizon. When $\lambda(W) = \theta_0 + \theta_1 r$, the optimal schedule requires equal maintenance intervals. As $\lambda(W)$ increases from $\theta_0 + \theta_1 r$ to $\theta_0 + \theta_1 r + \frac{\theta_2 + \theta_3 r}{m}(T - W - \frac{\beta b}{\alpha c})$, it becomes nonperiodic: The first PM interval gets smaller while the others get larger. At the end of the product's useful life, a no-maintenance interval of length $\frac{T-W}{m+1} + \frac{m\beta b}{(m+1)\alpha c} + \frac{\lambda(W) - \theta_0 - \theta_1 r}{(m+1)(\theta_2 + \theta_3 r)}$ is also larger. In the second case, the length of the first PM interval shrinks to zero, and it is no longer profitable for the

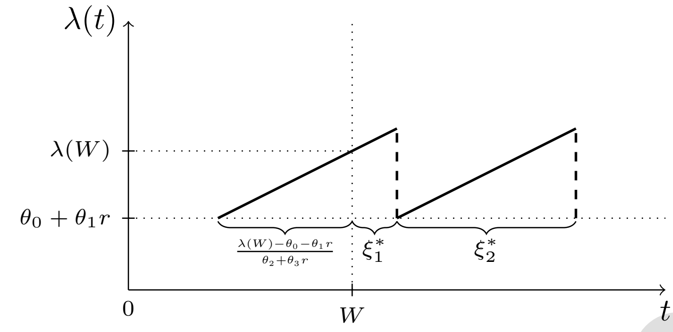

Fig. 1. Optimal PM intervals for small $\lambda(W)$.

customer to defer PM. The first PM action is thus similar to an upgrade implemented at the beginning of the post-warranty period. The remaining PM actions are performed at regular intervals. The no-maintenance interval is of length $\frac{T-W}{m} + \frac{(m-1)\beta b}{m\alpha c}$.

When $\lambda(W) \le \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{m}(T - W - \frac{\beta b}{\alpha c})$, that is, when $m \le \frac{\theta_2+\theta_3 r}{\lambda(W)-\theta_0-\theta_1 r}(T - W - \frac{\beta b}{\alpha c})$, substituting ξ_i^* into $H(m, \boldsymbol{\xi}, \boldsymbol{y}^*; \lambda(W))$ and rearranging terms gives

$$\begin{aligned} H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W)) = \beta k m &+ \frac{\alpha c(\theta_2+\theta_3 r)}{2(m+1)} \left(T - W - \frac{\beta b}{\alpha c} + \frac{\lambda(W)-\theta_0-\theta_1 r}{\theta_2+\theta_3 r} \right)^2 \\ &+ \beta b(\theta_2+\theta_3 r)(T-W) - \frac{\alpha c(\lambda(W)-\theta_0-\theta_1 r)^2}{2(\theta_2+\theta_3 r)} - \frac{\beta^2 b^2(\theta_2+\theta_3 r)}{2\alpha c} \\ &+ \alpha c(\theta_0+\theta_1 r)(T-W) + \beta b\big(\lambda(W)-\theta_0-\theta_1 r\big). \end{aligned} \tag{3}$$

From this expression, we see that $H(0, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W)) = \alpha c(T-W)\lambda(W) + \frac{1}{2}\alpha c(\theta_2+\theta_3 r)(T-W)^2$ corresponds to the expected total cost for the case of $m = 0$, which we did not include in the above analysis.

When $\lambda(W) > \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{m}(T - W - \frac{\beta b}{\alpha c})$, that is, when $m > \frac{\theta_2+\theta_3 r}{\lambda(W)-\theta_0-\theta_1 r}(T - W - \frac{\beta b}{\alpha c})$, the cost function $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ becomes

$$\begin{aligned} H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W)) = \beta k m &+ \frac{\alpha c(\theta_2+\theta_3 r)}{2m} \left(T - W - \frac{\beta b}{\alpha c} \right)^2 \\ &+ \beta b(\theta_2+\theta_3 r)(T-W) - \frac{\beta^2 b^2(\theta_2+\theta_3 r)}{2\alpha c} \\ &+ \alpha c(\theta_0+\theta_1 r)(T-W) + \beta b\big(\lambda(W)-\theta_0-\theta_1 r\big). \end{aligned} \tag{4}$$

When m is regarded as a continuous variable, we can show that $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ is convex in each of the two intervals. It is also continuous and differentiable at $m = \frac{\theta_2+\theta_3 r}{\lambda(W)-\theta_0-\theta_1 r}(T - W - \frac{\beta b}{\alpha c})$. The derivative at this point is $\beta k - \frac{\alpha c(\lambda(W)-\theta_0-\theta_1 r)^2}{2(\theta_2+\theta_3 r)}$. The following proposition characterizes the optimal m that minimizes the cost function $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$.

Proposition 7. *Suppose that* $T-W > \frac{\beta b}{\alpha c}$ *and let* $\underline{m}$ *be the optimal number of PM actions when* $\lambda(W) = \theta_0 + \theta_1 r$. *Then, as* $\lambda(W)$ *increases from* $\theta_0 + \theta_1 r$ *to* $\theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{\underline{m}}(T - W - \frac{\beta b}{\alpha c})$, m^* *increases from* $\underline{m}$ *to* $\underline{m}+1$. *When* $\lambda(W) > \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{\underline{m}}(T - W - \frac{\beta b}{\alpha c})$, *we have* $m^* = \underline{m}+1$. *When* $T - W = \frac{\beta b}{\alpha c}$, *we have* $m^* = 0$.

As $\lambda(W)$ increases, m^* increases by at most one, indicating a limited effect of $\lambda(W)$ on the customer's optimal PM schedule. The reason is that the failure process will start anew once a PM action is performed at the beginning of the post-warranty period. Thus, in this case, the optimal number of PM actions is one plus $\underline{m}$, where $\underline{m}$ is the smallest nonnegative integer such that

$$(m+1)(m+2) \geq \frac{\alpha c(\theta_2 + \theta_3 r)\big(T - W - \beta b/(\alpha c)\big)^2}{2\beta k}.$$

If $\theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{\underline{m}+1}(T - W - \frac{\beta b}{\alpha c}) < \lambda(W) \leq \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{\underline{m}}(T - W - \frac{\beta b}{\alpha c})$, the functional form of $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ we use to derive m^* is different for $m = \underline{m}$ and $m = \underline{m}+1$. The point at which the jump in m^* occurs can lie on either side of $\theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{\underline{m}+1}(T - W - \frac{\beta b}{\alpha c})$. By Propositions 5 and 6, we can show that y_1^* is increasing and ξ_1^* is decreasing in $\lambda(W)$.

5. Optimal cost-sharing ratio

In the automobile industry, although lack of maintenance usually leads to coverage denials, the balance of power can still shift from the manufacturer to the customer. When auto sales decline, the industry becomes a buyer's market, which means that the manufacturer is more willing to negotiate and needs better sales and service approaches for winning customers. Therefore, we can view the customer as having more bargaining power than the manufacturer.

We formulate the problem of designing a cost-sharing PM program as a sequential game. The customer moves first and chooses p at time zero to maximize his or her surplus. The manufacturer responds by deciding whether to accept this ratio. If yes, then the manufacturer specifies a PM schedule in the warranty period, and the customer pays all uncovered PM costs; otherwise, the product is subject to the optimal schedule without cost-sharing. In the post-warranty period, the customer makes the PM decision upon observing the failure intensity at the end of the 2-D warranty. To obtain the optimal cost-sharing ratio, we next investigate how the manufacturer and the customer benefit from sharing PM costs.

Proposition 8. *The following results hold:*

(a) $C(n^*, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ *is increasing in* p.

(b) n^* *is decreasing in* p *and goes to infinity as* $p \to 0^+$.

(c) *Let* $\lambda^* = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)W - \sum_{i=1}^{n^*} x_i^*$ *denote the failure intensity at the end of the 2-D warranty under the optimal PM schedule. Then,* λ^* *is increasing in* p *and approaches to* $\theta_0 + \theta_1 r$ *as* $p \to 0^+$.

(d) $H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ *is increasing in* $\lambda(W)$.

Part (a) shows that cost sharing reduces the manufacturer's expected total cost over the warranty period. Consequently, the manufacturer will accept any cost-sharing ratio from the customer. In part (b), as would be expected, the greater the proportion paid by the manufacturer, the smaller the number of PM actions performed. By parts (c) and (d), the optimal PM schedule without cost-sharing results in the highest failure intensity at the end of the 2-D warranty, which in turn leads to the highest expected total cost in the post-warranty period. Therefore, the customer may be willing to incur some costs to reduce this failure intensity.

We define customer surplus as the difference between the reduction in the post-warranty cost and the shared PM cost. Customers will be better off if they are left with positive surplus, and the customer surplus maximization problem can be described as follows:

$$\begin{aligned} \underset{p}{\text{maximize}} \quad & H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \overline{\lambda}) - H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda^*) - (1-p)\sum_{i=1}^{n^*}(k + bx_i^*) \\ \text{subject to} \quad & C(n^*, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) \le C(n^*, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, 1), \\ & 0 \le p \le 1, \end{aligned}$$

where $\overline{\lambda}$ is the failure intensity at the end of the 2-D warranty under the optimal PM schedule without cost sharing. The first constraint is the participation constraint of the manufacturer. This constraint always holds by part (a) of Proposition 8. Note that maximizing the customer surplus is equivalent to minimizing the expected total cost $H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda^*) + (1-p)\sum_{i=1}^{n^*}(k + bx_i^*)$. We find the optimal solution using a grid search algorithm (see Appendix B for details). When $n^* = 0$, the maximizing value of p is not unique. In this case, we take $p^* = 1$. For the other cases in Table 2, we simply replace W by the corresponding expiry time of the 2-D warranty and T by the corresponding end time of the useful life.

6. Numerical study

In this section, we show how to find the optimal cost-sharing ratio and demonstrate the benefits of cost-sharing PM. We consider a product covered by a 2-D warranty with $W = 3$ years and $U = 6 \times 10^4$ miles. The product can be used for $T = 6$ years or $L = 15 \times 10^4$ miles, whichever comes first. As $U/W \le L/T$, the results of this numerical study correspond to the

Table 3
Optimal PM schedules for different usage rates and cost-sharing ratios.

r	p	n^*	δ_i^*	x_i^*	$C(n^*, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$	m^*	ξ_1^*	ξ_i^*	y_1^*	y_i^*	$H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$
1	0.1	14	0.19	0.27	623.64	3	0.28	0.57	0.80	0.80	2911.87
	0.3	7	0.34	0.47	988.46	3	0.02	0.66	0.92	0.92	3161.42
	0.5	5	0.42	0.58	1266.25	3	0.00	0.67	1.28	0.93	3337.50
	0.7	3	0.57	0.80	1494.82	3	0.00	0.67	1.78	0.93	3563.25
	0.9	3	0.52	0.73	1681.42	3	0.00	0.67	2.00	0.93	3657.75
2	0.1	17	0.16	0.34	919.02	4	0.19	0.45	0.95	0.95	4215.18
	0.3	9	0.27	0.57	1434.28	4	0.00	0.50	1.20	1.05	4528.65
	0.5	6	0.36	0.75	1837.50	4	0.00	0.50	1.80	1.05	4800.00
	0.7	4	0.46	0.97	2175.92	4	0.00	0.50	2.44	1.05	5086.20
	0.9	3	0.52	1.10	2459.14	4	0.00	0.50	2.99	1.05	5336.62
3	0.1	13	0.14	0.38	796.10	4	0.21	0.45	1.25	1.25	5466.80
	0.3	6	0.24	0.68	1203.60	5	0.00	0.40	1.52	1.12	5883.00
	0.5	4	0.30	0.84	1504.00	5	0.00	0.40	2.24	1.12	6207.00
	0.7	3	0.32	0.91	1735.65	5	0.00	0.40	2.87	1.12	6490.50
	0.9	2	0.37	1.03	1905.20	5	0.00	0.40	3.55	1.12	6795.00

Notes. The subscript i for ξ_i^* and y_i^* is greater than one.

first three cases in Table 2. The coefficients in the failure intensity function are set to $\theta_0 = 0.1$, $\theta_1 = 0.2$, $\theta_2 = 0.7$, and $\theta_3 = 0.7$. The product, upon failure, is minimally repaired at cost $c = 300$. The setup cost for each PM action is 80 and the variable cost is 300. The markups over PM and repair costs are both 0.5; thus, $\alpha = \beta = 1.5$.

Table 3 reports the optimal PM schedules in the warranty and post-warranty periods for various values of r and p. One can see that the optimal number of PM actions n^* is decreasing in p, while its post-warranty counterpart m^* is increasing in p because the failure intensity at the end of the 2-D warranty is increasing in p. However, m^* increases by at most one (see Proposition 7). One can also observe that ξ_1^* is decreasing and y_1^* is increasing in p. The customer postpones the first PM action when p is fairly small because of a low failure intensity at the end of the 2-D warranty. As long as $\xi_1^* = 0$, ξ_i^* and y_i^* do not change with p. It is therefore the efficiency of the first PM action that drives the increase in the post-warranty cost.

Although $H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ increases with p in Table 3, the customer would not like p to be as small as possible. The choice of p depends on the reduction in the post-warranty cost and the shared PM cost, both of which are plotted as functions of the cost-sharing ratio p in Figure 2(a). As can be seen, they are decreasing in p and become zero at $p = 1$. Due to the discrete nature of our problem, the discontinuities occur when n^* changes. As p approaches zero, the shared PM cost goes to infinity because n^* goes to infinity (see Proposition 8). The failure intensity during the warranty period will eventually be maintained at $\theta_0 + \theta_1 r$ so that the cost reduction will converge to a finite limit. To illustrate how to find p^* for a given r, Figure 2(b)

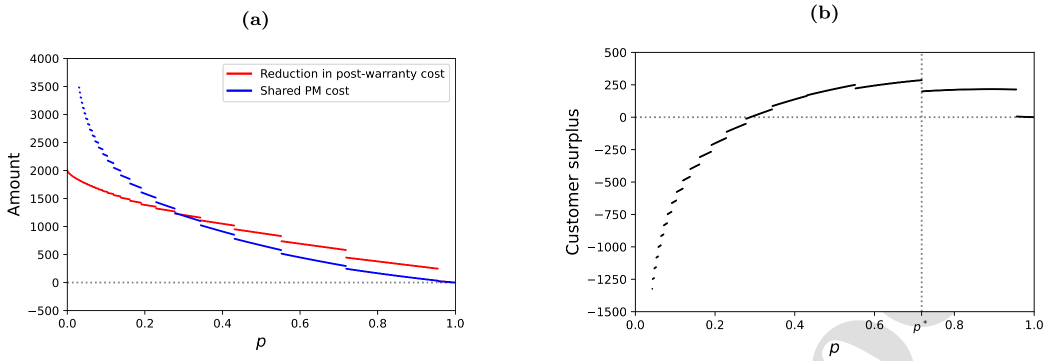

Fig. 2. Identification of the optimal cost-sharing ratio p^* when $r = 3$.

plots the difference between the two quantities. One can observe a nonnegative difference for any p in $[0.289, 1]$, which indicates that the customer is better off. Since $C(n^*, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ increases with p in Table 3, providing cost-sharing PM is a win-win strategy. From the customer's perspective, we choose $p^* = 0.718$, corresponding to the point at which the largest difference occurs.

Figure 3(a) shows the optimal number of PM actions, n^*, for a range of values of r. One can see that n^* is increasing in r when $r \leq U/W$ and decreasing otherwise. The curve of the optimal cost-sharing ratio p^* in Figure 3(b) is discontinuous because n is restricted to be an integer. When n^* changes, p^* will move in the opposite direction. With n^* held fixed, p^* increases with r when $r \leq U/W$ and decreases otherwise. This can be attributed to the impact of usage rate on the manufacturer's total PM cost. Note that no cost sharing is sometimes optimal. Figures 3(c) and 3(d) further show the cost benefits of cost sharing over no cost sharing for the manufacturer and the customer, respectively. Both parties are more likely to experience a greater percentage cost reduction for large r. Moreover, the manufacturer usually benefits more than the customer. This result is intuitive because the former enjoys cost advantages (specifically, α, $\beta \geq 1$).

In Figure 3, we also conduct sensitivity analysis with respect to β. As β increases, the benefit of reducing the failure intensity at the end of the 2-D warranty increases. Consequently, the customer will share more of the PM cost for a larger reduction in this failure intensity, which increases n^* and the percentage reduction in the manufacturer's cost. The customer's percentage cost reduction is also larger, leading to an increase in customer surplus. Figure 4 plots p^* against r for different values of c and b. The analysis is performed by keeping n^* fixed. An increase in c has a positive impact on p^* due to an increase in the manufacturer's total PM cost. However, there is an opposite effect of increasing the variable cost b because the customer has more incentive to share the cost of PM. Figure 5 shows the customer's expected total cost for three values of k. Increasing the setup cost k leads to an increase in the customer's expected

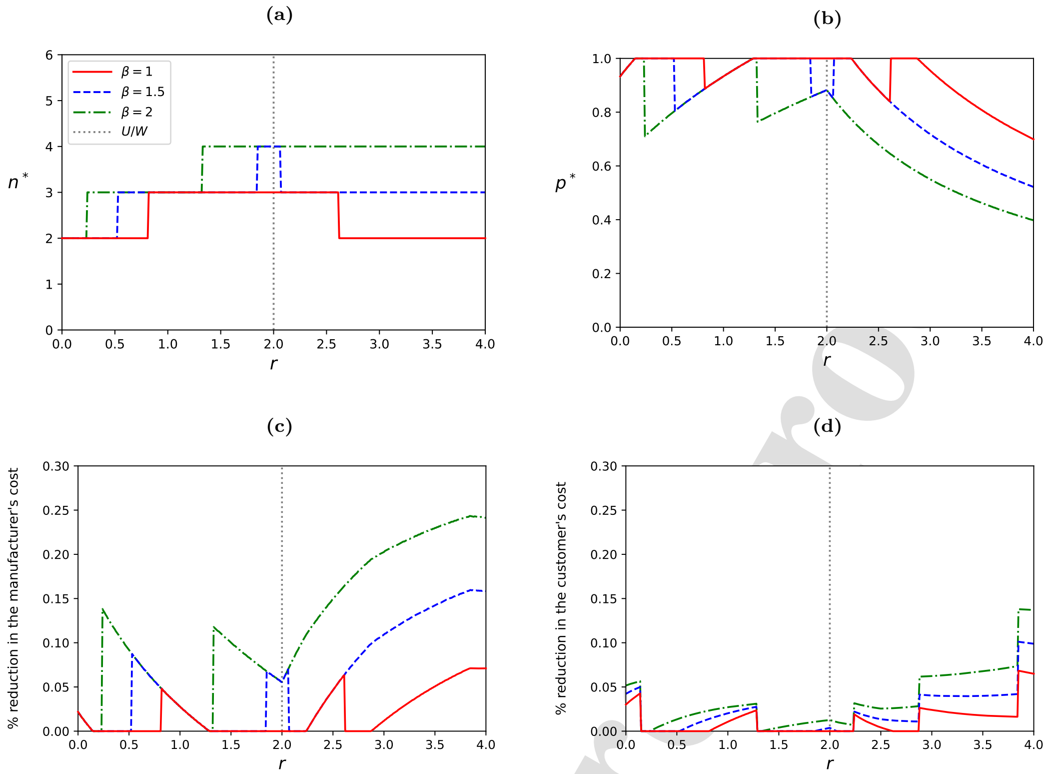

Fig. 3. Optimal cost-sharing ratio and percentage cost reductions.

total cost. Note that all the curves reach a peak at $r = L/T$.

The experimental results yield the following managerial insights for manufacturers. First, offering cost-sharing PM programs is a win-win strategy, although in some cases customers are not willing to share. A uniform cost-sharing PM program may prevent manufacturers from establishing a competitive edge because some customers receive negative surplus. Second, different levels of usage rates have distinct cost implications for manufacturers. In terms of percentage cost reduction, more benefits are more likely to be gained from customers with high usage rates. Finally, to reduce the cost-sharing ratio, manufacturers can increase the markup on PM costs. Given the number of PM actions during the warranty period, changes in the cost parameters also lead to changes in the cost-sharing ratio.

7. Concluding remarks

This paper studies a new cost-sharing PM program for products sold with a 2-D warranty, and the customer is the one who decides on the cost-sharing ratio. We construct two cost models, one for the manufacturer's PM scheduling problem in the warranty period and the other for the customer's PM scheduling problem in the post-warranty period. The optimal PM

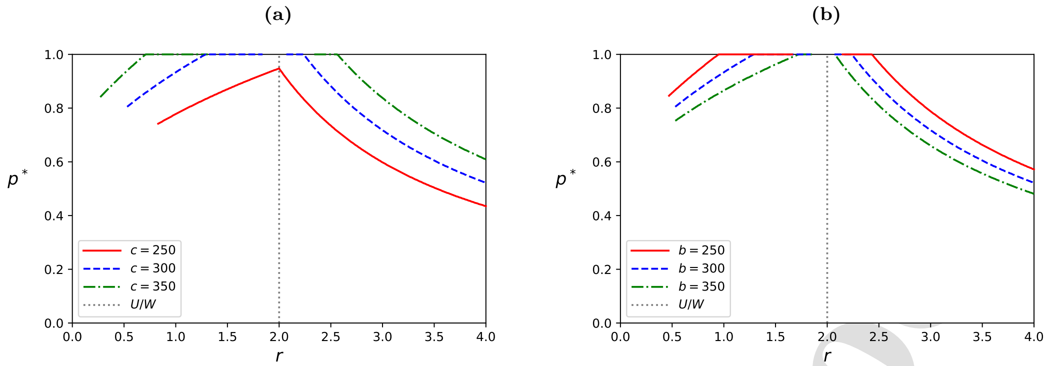

Fig. 4. Optimal cost-sharing ratio when $n^* = 3$.

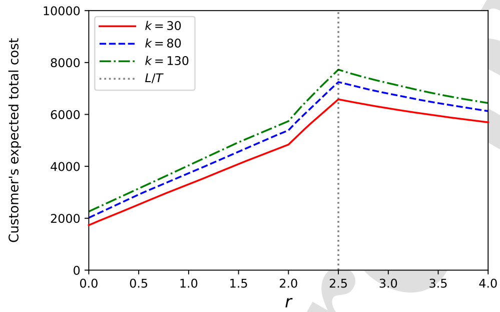

Fig. 5. Customer's expected total cost for different values of k.

policies in these two models indicate that the failure intensity should be reduced to its lowest attainable level at each instant of PM. One difference is that the optimal PM schedule in the post-warranty period may be nonperiodic: The customer will defer the first PM action if the failure intensity at the end of the 2-D warranty is less than or equal to a certain threshold and otherwise perform PM immediately. We find that the customer's expected total cost is related to the cost-sharing ratio through this failure intensity. In the numerical study, we examine how the optimal cost-sharing ratio and the cost benefits vary with product usage rate, highlighting that offering customized cost-sharing PM programs is a win-win strategy.

This study can be extended in several directions. First, we focus only on designing cost-sharing PM for a given usage rate. In many applications, however, a uniform cost-sharing ratio might be of interest to manufacturers for ease of implementation. An important extension would be to investigate the optimal PM schedule and cost-sharing ratio at the population level. In this scenario, a customer who is left with negative surplus has no choice but to pay the shared cost. Another interesting extension would be to design a menu of PM schedules and cost-sharing ratios to serve a population of customers when their usage rates are unknown to manufacturers.

Second, we assume that PM can restore a product to an as-good-as-new condition. This assumption may not hold for a complex product whose replacement cost is highly prohibitive. New models need to be developed for cases where only imperfect PM is allowed. For example, under the current cost structure, we could assume that the lower bound on the failure intensity increases with time or that the PM efficiency decreases with the number of PM actions. Optimal policies in those situations might be nonperiodic. In addition, because of safety concerns, some regulations require manufacturers to conduct at least one annual safety inspection of their products. A direction for future research is to consider an upper bound on the intervals between successive PM actions to ensure safety.

Third, automakers often provide free maintenance programs as an incentive for customers to purchase their cars. It would thus be interesting to study a competition model that includes cost-sharing PM. In recent years, however, some automakers have begun to offer a variety of prepaid maintenance programs. Further research is needed to investigate customer purchase behavior under these two types of programs. In addition, other cost-sharing rules and the manufacturer's decisions on the cost markups would also be worth exploring.

Acknowledgements

We are grateful to the two anonymous referees for their constructive comments. Part of this research was conducted while the first author was visiting the Department of Industrial Systems Engineering and Management at National University of Singapore. This research was supported by the National Science Foundation of China [grant numbers 72122013, 71801151] and the University Start-up Fund for RAPs under the Strategic Hiring Scheme [grant number P0036168].

Appendix A. Proofs

Proof of Proposition 1. By substituting for $\lambda(t)$ and rearranging terms, we have

$$
\begin{aligned}
& C(n, \boldsymbol{\delta}, \boldsymbol{x}; r, p) \\
&= \frac{c}{2} \sum_{i=1}^{n} \left(\theta_0 + \theta_1 r + (\theta_2 + \theta_3 r) \sum_{j=1}^{i-1} \delta_j - \sum_{j=1}^{i-1} x_j + \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r) \sum_{j=1}^{i} \delta_j - \sum_{j=1}^{i-1} x_j \right) \delta_i \\
&\quad + \frac{c}{2} \left(\theta_0 + \theta_1 r + (\theta_2 + \theta_3 r) \sum_{i=1}^{n} \delta_i - \sum_{i=1}^{n} x_i + \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r) W - \sum_{i=1}^{n} x_i \right) \\
&\quad \times \left(W - \sum_{i=1}^{n} \delta_i \right) + \sum_{i=1}^{n} p(k + b x_i) \\
&= c(\theta_2 + \theta_3 r) \left(\sum_{i=1}^{n} \delta_i \sum_{j=1}^{i-1} \delta_j + \frac{1}{2} \sum_{i=1}^{n} \delta_i^2 + \frac{W^2}{2} - \frac{1}{2} \left(\sum_{i=1}^{n} \delta_i \right)^2 \right) + npk \\
&\quad - c \left(\sum_{i=1}^{n} \delta_i \sum_{j=1}^{i-1} x_j + \left(W - \sum_{i=1}^{n} \delta_i - \frac{pb}{c} \right) \sum_{i=1}^{n} x_i \right) + c(\theta_0 + \theta_1 r) W.
\end{aligned}
$$

When $\sum_{i=1}^{n} \delta_i \leq W - pb/c$, the coefficients of $\sum_{j=1}^{i} x_j$ are all nonpositive. To minimize $C(n, \boldsymbol{\delta}, \boldsymbol{x}; r, p)$, we choose $\sum_{j=1}^{i} x_j$ as large as possible, that is, $\sum_{j=1}^{i} x_j = (\theta_2 + \theta_3 r) \sum_{j=1}^{i} \delta_i$. Solving for x_i gives $x_i^* = (\theta_2 + \theta_3 r)\delta_i$. □

Proof of Proposition 2. We first prove that $C(n, \boldsymbol{\delta}, \boldsymbol{x}^*; r, p)$ is a convex function of the vector $\boldsymbol{\delta}$. Its Hessian matrix is given by

$$
\mathrm{Hess}(C) = \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{bmatrix}_{n \times n}
$$

with eigenvalues 1 of multiplicity $n-1$ and $n+1$ of multiplicity 1. Since all the eigenvalues are greater than zero, the Hessian matrix is positive definite.

We then solve the optimization problem using Lagrange multipliers. The Lagrangian function is defined as

$$
\mathcal{L}(\boldsymbol{\delta}, \boldsymbol{\mu}) = C(n, \boldsymbol{\delta}, \boldsymbol{x}^*; r, p) - \mu_0 \left(W - \frac{pb}{c} - \sum_{i=1}^{n} \delta_i \right) - \sum_{i=1}^{n} \mu_i \delta_i,
$$

where $\mu_i \geq 0$, $i = 0, \dots, n$, are a set of Lagrange multipliers. The optimal solution must satisfy the following first-order conditions:

$$
\begin{gathered}
c(\theta_2+\theta_3 r)\left(\delta_i - W + \sum_{i=1}^{n}\delta_i + \frac{pb}{c}\right) + \mu_0 - \mu_i = 0 \text{ for } i=1,\ldots,n,\\
\mu_0\left(W - \frac{pb}{c} - \sum_{i=1}^{n}\delta_i\right) = 0,\ \mu_i\delta_i = 0 \text{ for } i=1,\ldots,n,\\
\mu_i \geq 0 \text{ for } i=0,\ldots,n,\\
\sum_{i=1}^{n}\delta_i \leq W - \frac{pb}{c},\ \delta_i \geq 0 \text{ for } i=1,\ldots,n.
\end{gathered}
$$

There are four cases to consider.

(i) If $\mu_0 = 0$ and some $\delta_i > 0$, then $\mu_i = 0$. For any $j \neq i$, we have

$$
\delta_j = W - \sum_{i=1}^{n}\delta_i - \frac{pb}{c} + \frac{\mu_j}{c(\theta_2+\theta_3 r)} = \delta_i + \frac{\mu_j}{c(\theta_2+\theta_3 r)} \geq \delta_i > 0.
$$

Therefore, $\mu_j = 0$, and thus $\delta_j = \delta_i = \frac{W}{n+1} - \frac{pb}{(n+1)c}$. Since $\delta_i > 0$, we have $W > pb/c$.

(ii) If $\mu_0 = 0$ and all $\delta_i = 0$, then all $\mu_i = c(\theta_2+\theta_3 r)(pb/c - W)$. Since $\mu_i \geq 0$, it follows that $W \leq pb/c$. Moreover, because $\sum_{i=1}^{n}\delta_i = 0 \leq W - pb/c$, we must have $W = pb/c$ and all $\mu_i = 0$.

(iii) If $\mu_0 > 0$ and some $\delta_i > 0$, then $\mu_0 = -c(\theta_2+\theta_3 r)\delta_i < 0$, which contradicts the inequality $\mu_0 > 0$.

(iv) If $\mu_0 > 0$ and all $\delta_i = 0$, then $W - pb/c - \sum_{i=1}^{n}\delta_i = W - pb/c = 0$, and thus $\mu_i = \mu_0 > 0$. So this solution is feasible. □

Proof of Proposition 3. We first treat n as continuous. Differentiating $C(n,\boldsymbol{\delta}^*,\boldsymbol{x}^*;r,p)$ with respect to n yields

$$
\frac{\partial C(n,\boldsymbol{\delta}^*,\boldsymbol{x}^*;r,p)}{\partial n} = pk - \frac{c(\theta_2+\theta_3 r)(W-pb/c)^2}{2(n+1)^2}.
$$

We then calculate the second derivative as

$$
\frac{\partial^2 C(n,\boldsymbol{\delta}^*,\boldsymbol{x}^*;r,p)}{\partial n^2} = \frac{c(\theta_2+\theta_3 r)(W-pb/c)^2}{(n+1)^3},
$$

which is nonnegative. Therefore, the cost function $C(n,\boldsymbol{\delta}^*,\boldsymbol{x}^*;r,p)$ is convex in n. When $W > pb/c$, $\widetilde{n}^*$ is the solution to the equation $\partial C(n,\boldsymbol{\delta}^*,\boldsymbol{x}^*;r,p)/\partial n = 0$. When $\widetilde{n}^* > 0$, the optimal integer value of n is either $\lfloor\widetilde{n}^*\rfloor$ or $\lceil\widetilde{n}^*\rceil$. When $-1 < \widetilde{n}^* \leq 0$, we have $n^* = 0$ because $C(n,\boldsymbol{\delta}^*,\boldsymbol{x}^*;r,p)$ is increasing over $[0,+\infty)$. When $W = pb/c$, that is, when $\widetilde{n}^* = -1$, we also have $n^* = 0$ because $C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ increases linearly with n. $\square$

Proof of Proposition 4. We use lattice-theoretic methods to show monotone comparative statics. A sufficient condition is the submodularity or supermodularity of the cost function. A twice differentiable function is submodular if it has a negative cross derivative. (Supermodularity is defined by the opposite inequality.) We take n as a continuous variable and regard δ_i^*, n^*, and $\widetilde{n}^*$ as functions of the usage rate r. When $r \leq U/W$, the cross-partial derivative of the cost function with respect to n and r is

$$\frac{\partial^2 C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)}{\partial n \partial r} = -\frac{c\theta_3(W - pb/c)^2}{2(n+1)^2} \leq 0.$$

Therefore, $C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ is submodular in (n, r), and thus $\widetilde{n}^*(r)$ is increasing in r (Topkis, 1998). Next, we show that the monotonicity still holds for $n^*(r)$. Suppose that $r' < r$ and that $\widetilde{n}^*(r')$ lies in the interval $(\ell, \ell+1]$, where ℓ is a nonnegative integer. When $\widetilde{n}^*(r) \geq \ell + 1$, it is easy to see that $n^*(r) \geq n^*(r')$. When $\widetilde{n}^*(r') \leq \widetilde{n}^*(r) < \ell + 1$ and $n^*(r') = \ell$, the inequality is satisfied because $n^*(r)$ is either ℓ or $\ell + 1$. When $\widetilde{n}^*(r') \leq \widetilde{n}^*(r) < \ell + 1$ and $n^*(r') = \ell + 1$, we have $C(\ell, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r', p) > C(\ell + 1, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r', p)$. Moreover, by submodularity, we have

$$C(\ell, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r', p) + C(\ell + 1, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) \leq C(\ell, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) + C(\ell + 1, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r', p).$$

Therefore, $C(\ell, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) > C(\ell+1, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$, which implies that $n^*(r) = \ell+1$. In summary, $n^*(r)$ is increasing in r. Since $\delta_i^*(r) = \frac{W}{(n+1)} - \frac{pb}{(n+1)c}$, it is decreasing in r.

When $r > U/T$, taking the cross-partial derivative, we have the following:

$$\frac{\partial^2 C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)}{\partial n \partial r} = \frac{c}{2(n+1)^2} \left(\frac{U}{r} - \frac{pb}{c} \right) \left(\frac{\theta_3 pb}{c} + \frac{2\theta_2 U}{r^2} + \frac{\theta_3 U}{r} \right) \geq 0,$$

where the inequality holds because $U/r \geq pb/c$. Therefore, $C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ is supermodular in (n, r). Using a similar argument as above, we can show that $n^*(r)$ is decreasing in r. $\square$

Proof of Proposition 5. Substituting for $\lambda(t)$ and rearranging terms gives

$$\begin{aligned}
&H(m,\boldsymbol{\xi},\boldsymbol{y};\lambda(W))\\
&=\frac{\alpha c}{2}\sum_{i=1}^{m}\left(\lambda(W)+(\theta_2+\theta_3 r)\sum_{j=1}^{i-1}\xi_j-\sum_{j=1}^{i-1}y_j+\lambda(W)+(\theta_2+\theta_3 r)\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i-1}y_j\right)\xi_i\\
&\quad+\frac{\alpha c}{2}\left(\lambda(W)+(\theta_2+\theta_3 r)\sum_{i=1}^{m}\xi_i-\sum_{i=1}^{m}y_i+\lambda(W)+(\theta_2+\theta_3 r)(T-W)-\sum_{i=1}^{m}y_i\right)\\
&\quad\times\left(T-W-\sum_{i=1}^{m}\xi_i\right)+\sum_{i=1}^{m}\beta(k+by_i)\\
&=\alpha c(\theta_2+\theta_3 r)\left(\sum_{i=1}^{m}\xi_i\sum_{j=1}^{i-1}\xi_j+\frac{1}{2}\sum_{i=1}^{m}\xi_i^2+\frac{(T-W)^2}{2}-\frac{1}{2}\left(\sum_{i=1}^{m}\xi_i\right)^2\right)+m\beta k\\
&\quad-\alpha c\left(\sum_{i=1}^{m}\xi_i\sum_{j=1}^{i-1}y_j+\left(T-W-\sum_{i=1}^{m}\xi_i-\frac{\beta b}{\alpha c}\right)\sum_{i=1}^{m}y_i\right)+\alpha c(T-W)\lambda(W).
\end{aligned}$$

When $\sum_{i=1}^{m}\xi_i\le T-W-\frac{\beta b}{\alpha c}$, the coefficients of $\sum_{j=1}^{i}y_j$ are all nonpositive. Hence, we choose $\sum_{j=1}^{i}y_j$ as large as possible, that is, $\sum_{j=1}^{i}y_j=\lambda(W)-\theta_0-\theta_1 r+(\theta_2+\theta_3 r)\sum_{j=1}^{i}\xi_i$. It then follows that $y_1^*=\lambda(W)-\theta_0-\theta_1 r+(\theta_2+\theta_3 r)\xi_1$ and all other $y_i^*=(\theta_2+\theta_3 r)\xi_i$. □

Proof of Proposition 6. It is straightforward to show that $H(m,\boldsymbol{\xi},\boldsymbol{y}^*;\lambda(W))$ is a convex function of the vector $\boldsymbol{\xi}$. To find the optimal value of $\boldsymbol{\xi}$, we write the Lagrangian of the minimization problem as:

$$\mathcal{L}(\boldsymbol{\xi},\boldsymbol{\mu})=H(m,\boldsymbol{\xi},\boldsymbol{y}^*;\lambda(W))-\mu_0\left(T-W-\frac{\beta b}{\alpha c}-\sum_{i=1}^{m}\xi_i\right)-\sum_{i=1}^{m}\mu_i\xi_i,$$

where $\mu_i\ge 0$, $i=0,\dots,m$, are a set of Lagrange multipliers. The first-order conditions for optimality are

$$\alpha c(\theta_2+\theta_3 r)\left(\xi_1-T+W+\sum_{i=1}^{m}\xi_i+\frac{\beta b}{\alpha c}\right)+\alpha c\big(\lambda(W)-\theta_0-\theta_1 r\big)+\mu_0-\mu_1=0, \tag{A.1}$$

$$\alpha c(\theta_2+\theta_3 r)\left(\xi_i-T+W+\sum_{i=1}^{m}\xi_i+\frac{\beta b}{\alpha c}\right)+\mu_0-\mu_i=0 \text{ for } i=2,\dots,m, \tag{A.2}$$

$$\mu_0\left(T-W-\frac{\beta b}{\alpha c}-\sum_{i=1}^{m}\xi_i\right)=0,\ \mu_i\xi_i=0 \text{ for } i=1,\dots,m,$$

$$\mu_i\ge 0 \text{ for } i=0,\dots,m,$$

$$\sum_{i=1}^{m}\xi_i\le T-W-\frac{\beta b}{\alpha c},\ \xi_i\ge 0 \text{ for } i=1,\dots,m.$$

There are two cases to consider.

(i) $\mu_1 = 0$. Suppose that $\mu_i > 0$ for some $i = 2, \ldots, m$. Then we have $\xi_i = 0$. From Equations (A.1) and (A.2), it follows that $\mu_i = -\alpha c(\theta_2 + \theta_3 r)\xi_1 - \alpha c\big(\lambda(W) - \theta_0 - \theta_1 r\big) \leq 0$, which contradicts $\mu_i > 0$. Hence, $\mu_i = 0$ for all $i = 2, \ldots, m$. If $\mu_0 > 0$, then $\sum_{i=1}^m \xi_i = T - W - \frac{\beta b}{\alpha c}$, and from Equation (A.2) we have $\mu_0 = -\alpha c(\theta_2 + \theta_3 r)\xi_i \leq 0$, which is also a contradiction. Therefore, $\mu_0 = 0$. By solving the first-order conditions, we get

$$\xi_i^* = \begin{cases} \frac{T-W}{m+1} - \frac{\beta b}{(m+1)\alpha c} - \frac{m(\lambda(W)-\theta_0-\theta_1 r)}{(m+1)(\theta_2+\theta_3 r)}, & i = 1, \\ \frac{T-W}{m+1} - \frac{\beta b}{(m+1)\alpha c} + \frac{\lambda(W)-\theta_0-\theta_1 r}{(m+1)(\theta_2+\theta_3 r)}, & i = 2, \ldots, m. \end{cases}$$

This solution is feasible if and only if $T - W - \frac{\beta b}{\alpha c} \geq \frac{m(\lambda(W)-\theta_0-\theta_1 r)}{\theta_2+\theta_3 r}$, which is equivalent to $\lambda(W) \leq \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{m}\left(T - W - \frac{\beta b}{\alpha c}\right)$.

(ii) $\mu_1 > 0$. In this case, $\xi_1 = 0$. Suppose that $\mu_0 > 0$, which implies that $\sum_{i=1}^m \xi_i = T - W - \frac{\beta b}{\alpha c}$. By Equation (A.2), we have $\mu_i = \mu_0 + \alpha c(\theta_2 + \theta_3 r)\xi_i > 0$ for $i = 2, \ldots, m$. So all $\xi_i = 0$ and $T - W = \frac{\beta b}{\alpha c}$. If $T - W > \frac{\beta b}{\alpha c}$, we obtain a contradiction, and thus $\mu_0 = 0$. Suppose that $\mu_i > 0$ for some $i = 2, \ldots, m$. Then $\xi_i = 0$. It follows from Equation (A.2) that $\mu_i = \alpha c(\theta_2 + \theta_3 r)\left(-T + W + \sum_{i=1}^m \xi_i + \frac{\beta b}{\alpha c}\right) \leq 0$, which contradicts the assumption. Hence, we have $\mu_i = 0$ for $i = 2, \ldots, m$. Solving the first-order conditions yields

$$\xi_i^* = \begin{cases} 0, & i = 1, \\ \frac{T-W}{m} - \frac{\beta b}{m\alpha c}, & i = 2, \ldots, m. \end{cases}$$

Notice that with $T - W = \frac{\beta b}{\alpha c}$, this solution coincides with the solution $\xi_i^* = 0$. Substituting ξ_i^* into Equation (A.1) gives $\mu_1 = \alpha c\big(\lambda(W) - \theta_0 - \theta_1 r\big) + \alpha c(\theta_2 + \theta_3 r)\left(\frac{\beta b}{m\alpha c} - \frac{T-W}{m}\right)$. To ensure $\mu_1 > 0$, we need $\lambda(W) > \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{m}\left(T - W - \frac{\beta b}{\alpha c}\right)$. □

Proof of Proposition 7. Differentiating $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ with respect to m yields

$$\frac{\partial H}{\partial m} = \begin{cases} \beta k - \frac{\alpha c(\theta_2+\theta_3 r)}{2(m+1)^2}\left(T - W - \frac{\beta b}{\alpha c} + \frac{\lambda(W)-\theta_0-\theta_1 r}{\theta_2+\theta_3 r}\right)^2, & \text{if } m \leq \frac{\theta_2+\theta_3 r}{\lambda(W)-\theta_0-\theta_1 r}\left(T - W - \frac{\beta b}{\alpha c}\right), \\ \beta k - \frac{\alpha c(\theta_2+\theta_3 r)}{2m^2}\left(T - W - \frac{\beta b}{\alpha c}\right)^2, & \text{otherwise.} \end{cases}$$

Denote $\widetilde{m}^*$ as the solution to the continuous relaxation. When $T - W > \frac{\beta b}{\alpha c}$ and $\frac{\partial H}{\partial m}\big|_{m=0} \leq 0$, $\widetilde{m}^*$ solves the equation $\partial H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))/\partial m = 0$. When $T - W > \frac{\beta b}{\alpha c}$ and $\frac{\partial H}{\partial m}\big|_{m=0} > 0$, $\widetilde{m}^* = 0$. From the second derivative, it follows that m^* is determined by $\lfloor \widetilde{m}^* \rfloor$ or $\lceil \widetilde{m}^* \rceil$. Because $\partial H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))/\partial m$ is nonincreasing in $\lambda(W)$, $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ is submodular, and thus $\widetilde{m}^*$ is increasing in $\lambda(W)$. Using a similar argument as in the proof of Proposition 4, we can show that m^* is also increasing in $\lambda(W)$. If $\lambda(W) = \theta_0 + \theta_1 r$, then by definition, $m^* = \underline{m}$. For any $m \geq \underline{m}$, if $\lambda(W) > \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{\underline{m}}\left(T - W - \frac{\beta b}{\alpha c}\right)$, then by Proposition 6, a PM action occurs at time W, reducing the failure intensity to $\theta_0 + \theta_1 r$. Since the optimal number of subsequent PM actions is $\underline{m}$, we have $m^* = \underline{m} + 1$. If $\theta_0 + \theta_1 r < \lambda(W) \leq \theta_0 + \theta_1 r + \frac{\theta_2+\theta_3 r}{\underline{m}}\left(T - W - \frac{\beta b}{\alpha c}\right)$, then m^* increases from $\underline{m}$ to $\underline{m} + 1$. When $T - W = \frac{\beta b}{\alpha c}$, it is easy to see that $m^* = 0$. □

Proof of Proposition 8. To prove part (a), we first note that when $p \leq cW/b$,

$$\frac{\partial C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)}{\partial p} = kn + \frac{bn(\theta_2 + \theta_3 r)}{n+1}\left(W - \frac{pb}{c}\right) \geq 0.$$

Hence, $C(n, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ is increasing in p. We emphasize the dependence of $n^*(r)$ on the cost-sharing ratio p using the notation $n^*(r, p)$. For any $p' < p$, we have

$$C(n^*(r,p), \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p) \geq C(n^*(r,p), \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p') \geq C(n^*(r,p'), \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p'),$$

which implies that $C(n^*, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ is also increasing in p. When $p > cW/b$, since $n^*(r,p) = 0$, $C(n^*, \boldsymbol{\delta}^*, \boldsymbol{x}^*; r, p)$ does not change with p. The second part holds because the right-hand side of Inequality (1) is decreasing in p when $p \leq cW/b$ and its left-hand side is increasing in n. To show $\lim_{p\to 0^+} n^*(r,p) = +\infty$, note that $\lim_{p\to 0^+} \frac{c(\theta_2+\theta_3 r)(W-pb/c)^2}{2pk} = +\infty$. For part (c), it is clear that

$$\lambda^* = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)W - (\theta_2 + \theta_3 r)\left(W - \frac{pb}{c}\right)\left(1 - \frac{1}{n^*(r,p)+1}\right)$$

is increasing in p, and by part (b), $\lim_{p\to 0^+} \lambda(W) = \theta_0 + \theta_1 r$. Finally, consider part (d). For any $\lambda' < \lambda$, we have $\frac{\theta_2+\theta_3 r}{\lambda-\theta_0-\theta_1 r}\left(T - W - \frac{\beta b}{\alpha c}\right) \leq \frac{\theta_2+\theta_3 r}{\lambda'-\theta_0-\theta_1 r}\left(T - W - \frac{\beta b}{\alpha c}\right)$. When $0 \leq m \leq \frac{\theta_2+\theta_3 r}{\lambda-\theta_0-\theta_1 r}\left(T - W - \frac{\beta b}{\alpha c}\right)$, differentiating $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))$ with respect to $\lambda(W)$ yields

$$\frac{\partial H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda(W))}{\partial \lambda(W)} = \beta b + \frac{\alpha c}{m+1}\left(T - W - \frac{\beta b}{\alpha c} - \frac{m(\lambda(W) - \theta_0 - \theta_1 r)}{\theta_2 + \theta_3 r}\right) \geq 0$$

for $\lambda(W) \leq \lambda$. By the mean value theorem, $H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda) - H(m, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda') \geq 0$. When

$\frac{\theta_2+\theta_3 r}{\lambda-\theta_0-\theta_1 r}\left(T-W-\frac{\beta b}{\alpha c}\right) < m \leq \frac{\theta_2+\theta_3 r}{\lambda'-\theta_0-\theta_1 r}\left(T-W-\frac{\beta b}{\alpha c}\right)$, we have

$$
\begin{aligned}
& H(m,\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda) - H(m,\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda') \\
&= \frac{\alpha c(\theta_2+\theta_3 r)}{2m}\left(T-W-\frac{\beta b}{\alpha c}\right)^2 - \frac{\alpha c(\theta_2+\theta_3 r)}{2(m+1)}\left(T-W-\frac{\beta b}{\alpha c}+\frac{\lambda'-\theta_0-\theta_1 r}{\theta_2+\theta_3 r}\right)^2 \\
&\quad + \frac{\alpha c(\lambda'-\theta_0-\theta_1 r)^2}{2(\theta_2+\theta_3 r)} + \beta b(\lambda-\lambda') \\
&= \frac{\alpha c(\theta_2+\theta_3 r)}{2m(m+1)}\left(T-W-\frac{\beta b}{\alpha c}-\frac{m(\lambda'-\theta_0-\theta_1 r)}{\theta_2+\theta_3 r}\right)^2 + \beta b(\lambda-\lambda') \\
&\geq 0.
\end{aligned}
$$

When $\frac{\theta_2+\theta_3 r}{\lambda'-\theta_0-\theta_1 r}\left(T-W-\frac{\beta b}{\alpha c}\right) < m$, we also have $H(m,\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda) - H(m,\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda') \geq 0$. It then follows that for any $\lambda' < \lambda$,

$$H(m^*(\lambda),\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda) \geq H(m^*(\lambda),\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda') \geq H(m^*(\lambda'),\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda'),$$

where we use the notation $m^*(\lambda(W))$ to make explicit the dependence on $\lambda(W)$. □

Appendix B. Algorithm to determine the optimal cost-sharing ratio

Input: r, W, U, T, L, θ_0, θ_1, θ_2, θ_3, c, k, b, α, β ($r < U/W \leq L/T$)

Output: p^*

$p \leftarrow 1$
$\Delta p \leftarrow 0.01$
$min \leftarrow +\infty$
$\underline{m} \leftarrow 0$
while $(\underline{m}+1)(\underline{m}+2) < \frac{\alpha c(\theta_2+\theta_3 r)(T-W-\beta b/(\alpha c))^2}{2\beta k}$ **do**
  $\underline{m} \leftarrow \underline{m}+1$
end while
while $p > 0$ **do**
  $n^* \leftarrow 0$
  while $(n^*+1)(n^*+2) < \frac{c(\theta_2+\theta_3 r)(W-pb/c)^2}{2pk}$ **do**
    $n^* \leftarrow n^*+1$
  end while
  $x_i^* \leftarrow (\theta_2+\theta_3 r)(\frac{W}{n^*+1} - \frac{pb}{(n^*+1)c})$
  $\lambda^* \leftarrow \theta_0+\theta_1 r+(\theta_2+\theta_3 r)W - \sum_{i=1}^{n^*} x_i^*$
  calculate $H(\underline{m},\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda^*)$ and $H(\underline{m}+1,\boldsymbol{\xi}^*,\boldsymbol{y}^*;\lambda^*)$ using Equations (3) and (4)

$H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda^*) \leftarrow \min\{H(\underline{m}, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda^*), H(\underline{m}+1, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda^*)\}$

if $H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda^*) + (1-p)\sum_{i=1}^{n^*}(k + bx_i^*) < min$ **then**

  $min \leftarrow H(m^*, \boldsymbol{\xi}^*, \boldsymbol{y}^*; \lambda^*) + (1-p)\sum_{i=1}^{n^*}(k + bx_i^*)$

  $p^* \leftarrow p$

end if

$p \leftarrow p - \Delta p$

end while

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