Design of Usage-Based Preventive Maintenance Under Two-Dimensional Warranties

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Abstract

Motivated by the Internet of Things (IoT) applications in the after-sales market, this paper considers the problem of designing a usage-based preventive maintenance (PM) service in the context of a two-dimensional (2-D) warranty. Most of the 2-D warranty models in the literature have concentrated on the constant rate of product usage. In contrast, we consider random and dynamic usage rates obtained from IoT applications. To minimize the manufacturer's warranty servicing cost, we use dynamic programming to show that a policy characterized by a time threshold and a sequence of failure rate thresholds is optimal. Specifically, the manufacturer should provide PM services up to the time threshold and perform PM to bring the failure rate back to its lowest level once it exceeds the corresponding failure rate threshold. Interestingly, the time threshold is a random variable, which depends on the usage rate, and the failure rate thresholds are not constant, but are functions of the cumulative usage. Hence, such PM is called usage-based. Our numerical experiments show a no-maintenance region due to the time threshold policy and evaluate the randomness in the optimal PM actions.

Keywords: preventive maintenance; two-dimensional warranty; state-dependent threshold policies; dynamic programming

1. Introduction

Through sensors embedded in many modern products, real-time data about product performance and customer behavior are now available to the manufacturer after a product is sold. Such useful information forces changes in maintenance services in the after-sales market, thereby allowing the manufacturer to have excellent response capabilities and to reap substantial cost savings. Also, customers are able to get the best experience out of their products.

Maintenance services can be split into three categories based on their levels of market segmentation. The first is the traditional maintenance service, where the manufacturer provides the same support to all customers. There may be a potential revenue loss for lack of segmentation. The second category is the flexible maintenance service, where the manufacturer offers a menu of service plans to customers who then self-select the best fit. Although more flexibility is available in this case, each plan in the menu is designed based on the typical characteristics of the corresponding market segment, which implies that a homogeneous service is offered to this group. Moreover, there may well exist situations where customers are unclear about their own need so that picking the most suitable option is pretty tough. To overcome these shortcomings, we introduce the last category, called personalized maintenance services, which are enabled by sensor technology. During product usage, data from customers are collected and analyzed to gain a deep understanding of customer behavior patterns. By doing so, the manufacturer can benefit in several ways including maintaining continuous contact with customers and satisfying their various needs. Finally, a closed-loop system to enhance the manufacturer-customer relationship is hoped to be built.

In the real world, many companies explore ways to use environmental and operational sensor data, such as temperature, humidity, acceleration, and tilt, to provide personalized maintenance services. For example, KONE, a global leader in the elevator and escalator industry, has started putting sensors in its products recently. In 2017, KONE launched the 24/7 Connected Services project in which products are remotely monitored, and their health conditions are evaluated in real time. The ultimate goal of this project is to take maintenance actions proactively before a shutdown takes place (see KONE 2019). Another example is Pivotal's cutting-edge business model in the automobile industry. It looks at the data streamed from vehicles, such as diagnostic trouble codes, and then predicts when automobile components are going to fail and when maintenance jobs should be initiated. This model enables the company to move quickly in response to component failures (see Ramanujam 2016). Both companies benefit significantly from sensor technology, with which data collection is facilitated, and new maintenance strategies are implemented.

In this paper, we tackle the design challenges for a usage-based PM service, aimed at minimizing a manufacturer's expected total cost associated with providing a 2-D warranty. We expect to use the case of usage-based PM to provide a new perspective on how to design personalized maintenance services in today's IoT era. According to Djamaludin *et al.* (2001), the actions for this type of PM are based on the information of product usage, and they are appropriate for products subject to wear, such as automobile tires and aircraft engines. The PM in this research is perceived to be usage-based since it crucially depends on the cumulative amount of usage and the rate of usage. Commonly such information is available at the time of product repair or return, but now we can continuously track it over time with IoT sensors. Combined with the auxiliary usage information, our model would be more accurate than the ones using age information alone.

Given the uncertainty in the environment, a product's usage rate is usually random and dynamic. For example, a customer who faces sporadic demand has to make plans accordingly, resulting in frequently changing needs for the product. Another example is the products operating in the natural environment, such as wind turbines in a wind farm. Making long-term predictions of their usage is extremely hard. The dynamic usage rate includes the shifts in customer behavior over multiple periods, to which the manufacturer has to respond with dynamic maintenance planning. To model these dynamics, we divide the planning horizon into equal periods, and the usage-based PM is carried out based on each period's usage information, much like the periodic PM. Thus, it differs significantly from the usage-based PM in Wang and Su (2016) where the maintenance is initiated deterministically every specific amount of usage.

The reason why our usage-based PM belongs to the personalized maintenance service is twofold. First, each customer in our model has an associated usage rate distribution, and the service is tailored to that distribution, while it is often assumed in the literature that the usage rate for a specific customer is sampled from the same distribution. Second, even if two customers have very similar distributions, PM actions can be performed differently depending on the usage rate realizations. This fact indicates that the plan for a personalized maintenance service can be dynamically adjusted and fine-tuned during the service, as opposed to being set up at the beginning for the entire time horizon.

After-sales support spans the longest part of a product's life and hence is a rich source of revenue to the manufacturer. Nonetheless, managing it, especially in a random and dynamic system, is quite challenging. As the provider of usage-based PM, the manufacturer needs to schedule the PM as cost-effectively as possible to balance the maintenance cost and the repair cost. At least three questions of interest naturally arise from this consideration, and they are described as follows:

- Is it optimal for the manufacturer to maintain the product in each time period?
- What is the optimal PM effort when a maintenance event occurs?
- How does usage affect the decisions about PM times and efforts?

To answer these research questions, we first untangle the relationship between usage rate and failure rate. Then we present a dynamic programming model for the usage-based PM problem. In the face of the random usage rate, we provide two state-dependent maintenance policies, that is, a time threshold policy and a failure rate threshold policy. We show the existence and the optimality of each policy.

Moreover, we derive the exact probability distribution for the time threshold and identify three regions of the cumulative usage within which the failure rate threshold exhibits different structural properties.

Our paper makes the following contributions to the emerging literature on the PM under 2-D warranties. First, we relax the common assumption of constant usage rate and apply the ideas of dynamic programming to model the random and dynamic usage rate, which leads to the randomness of the optimal PM actions. Second, no prior research addresses the question of whether or not a PM job should be initiated in consideration of the setup cost. Understanding the link between PM decisions and such cost is essential since many companies perform routine maintenance more often than necessary. Our failure rate threshold policy, resulting in nonconstant maintenance efforts, offers a direct answer to this question. Third, several attempts have been made to understand the impact of warranty coverage on PM scheduling (see, Jack and Murthy 2002, Iskandar and Husniah 2017). We use the time threshold policy to identify a no-maintenance region due to the 2-D warranty explicitly. More attention in the PM literature should be placed on this finding.

The outline of this paper is as follows. We devote Section 2 to a review of the literature on PM and 2-D warranty management. We discuss the dynamics of the system and formulate the stochastic model of usage-based PM in Section 3, and in Section 4, we demonstrate that two threshold policies, driven by product usage information, are optimal. In Section 5, we complement the above findings with a computational study. Finally, we conclude the paper in Section 6.

2. Literature Review

The interplay between PM and 2-D warranty has attracted considerable interest in the past few years. For detailed overviews, readers are referred to Shafiee and Chukova (2013) and Wang and Xie (2018). In this literature, papers that establish the optimal PM strategies in the 2-D warranty context are related to our work. Wang *et al.* (2015) seek to optimize the numbers of PM activities in the base and extended warranty periods based on a failure rate reduction model. Wang *et al.* (2017a) analyze the periodic PM implemented in these two periods, with each PM service in the extended warranty having a chance of being rejected by the customer. Wang *et al.* (2017b) consider reliability improvement programs involving either upgrade or PM for the second-hand products sold with 2-D warranties. A new PM strategy, called 2-D PM, is proposed by Wang and Su (2016) and Su and Wang (2016). Following this strategy, the next PM will occur after an increment of K units in age or L units in cumulative usage, whichever comes first. Unlike these authors, we incorporate nonconstant maintenance efforts into the optimal PM policy, which is to say that the effort or the level of each PM action is not captured by the same variable.

A few papers examine such PM policies in the 2-D warranty setting. Shahanaghi *et al.* (2013) obtain the optimal number of periodic PM and the optimal effort of each maintenance activity after constructing finite usage rate scenarios. In contrast, the nonconstant maintenance efforts in our paper are driven by the state-dependent threshold policies. Additionally, some of these efforts might be zero, which leads to irregular schedules. Huang *et al.* (2017) establish a procedure for the customization of the 2-D extended warranty with PM. All customers are first classified according to their usage rates in the base warranty period. Then, the corresponding periodic or nonperiodic PM is arranged for them in the extended warranty period. Therefore, this service closely resembles a maintenance menu. Our work differs from theirs in the sense that we treat customers individually based on the personal usage rate distribution, and the exact functional forms of PM efforts are not assumed. Iskandar and Husniah (2017) study a 2-D lease contract under periodic PM. They jointly optimize the number of PM and the PM interval for every possible usage rate. The intervals between the last PM and the end of the contract form an irregular no-maintenance region in the plane. Our inclusion of the random and dynamic usage rate leads to different no-maintenance regions. Moreover, we perform nonperiodic PM and highlight the role of usage information in PM decisions.

Among studies on the design of the 2-D warranty under PM, Huang and Yen (2009) are marked the first known effort. To maximize the manufacturer's profit, they optimize the two parameters of a 2-D warranty in the presence of a prespecified PM policy. Huang *et al.* (2013) create a similar setting with consideration of reliability-based PM, which is performed whenever product reliability falls below a given threshold. These two models are extended by Huang *et al.* (2015) to incorporate a bivariate Weibull process and the PM causing both age and usage reductions.

Practically all the work in mathematical modeling of PM and 2-D warranty assumes that the usage rate is constant over time; that is, it remains the same for a specific customer throughout the product lifetime (see, for example, Murthy *et al.* 1995, Chun and Tang 1999). In contrast, our focus is on the nonconstant usage rate. As noted by Lawless *et al.* (2009), there are short-term variations in the usage rate, which make usage paths not precisely linear. There have been several methods developed for modelling the nonconstant usage rate. Eliashberg *et al.* (1997) choose a specific form of the logistic function to model the accumulation of usage when calculating the reserve for the 2-D warranty. One appealing advantage of this functional form is that it naturally represents the decline in the usage rate as a product ages. The accelerated failure time model can also be employed to represent the failure rate under piecewise constant usage rates over different periods, e.g., burn-in followed by a warranty period for repairable products (Ye *et al.* 2013) and multiple lease periods for leased equipment (Wang *et al.* 2018). Tong *et al.* (2017) use a weighted prediction model to obtain the future average usage rate and optimize the levels of repair.

Very few modelling efforts on the random and dynamic usage rate have appeared. An exception is the work of Singpurwalla and Wilson (1993), who propose a framework to describe an ideal product usage process by three sets of nonnegative random variables, that is, the lengths of periods of nonuse and use, together with the usage rate during each period of use. This method, however, seems analytically intractable, and their ideas do not appear to have been developed subsequently. To include the random and dynamic usage rate in our PM problem, we reasonably approximate the stochastic process specified above by dividing the planning horizon into periods of equal length and considering the usage rate of each period as the only source of uncertainty.

Some research that uses the methodology of dynamic programming to solve PM problems is also relevant to ours. Anderson (1981) formulates the maintenance of a machine operating over an infinite horizon as a continuous-time Markov decision process. A control-limit rule for replacement and monotone PM policies are obtained. In a finite-horizon setting, Chen *et al.* (2003) propose a state- and time-dependent PM policy for Markovian deteriorating systems. Both papers show that the optimal PM decisions are influenced by the machine state, whereas we characterize the effect of a time and usage indexed warranty on such decisions.

We note that machine replacement is often seen as a means of PM (see, e.g., Rust 1987, Barbera *et al.* 1996, Hartman 2001). Typically, such models entail deciding when a machine should be replaced by an identical one in light of uncertain machine breakdowns. Similar to our work in model formulation, Barbera *et al.* (1996) consider a machine with random deterioration during fixed inspection intervals and show through simulation a control-limit policy. One major difference is that they have a fixed replacement cost, while we assume a setup cost, such as the cost of sending out a technician, plus a variable cost associated with the maintenance effort. The latter cost structure allows for a more detailed analysis of the maintenance effects on a product.

The closest theoretical work to ours is Jack and Murthy (2002), who optimize the maintenance times and efforts in a one-dimensional, continuous-time, deterministic version of our problem. They also establish a time threshold after which no PM action should be taken. Although the decisions are quite similar, we consider a multi-period model under the 2-D warranty, which is developed over a random planning horizon since the 2-D warranty may terminate in any period. Moreover, we specify thresholds in the usage dimension. Table 1 summarizes this and the above papers.

3. Problem Formulation

This section introduces the notation, the assumptions, and the model of usage-based PM. We consider a maintenance planning problem for a manufacturer who carries out PM for products sold with a 2-D warranty. This type of warranty usually comes with two attributes, namely usage limit U and age limit T (see Table 2 for a summary of notation). It expires when either the cumulative usage or the product age reaches their corresponding limits. After that, the manufacturer is not obliged to repair the faulty product and PM is also no longer needed. Therefore, under the 2-D warranty, the manufacturer

Table 1. Comparison of our paper with the key literature.

Papers	*Problem*			*Maintenance type*				*Usage rate*		*Setup cost?*
	PM	*PMuW*	*WuPM*	*PC*	*PN*	*NC*	*NN*	*C*	*D*	
Wang *et al.* (2015)		✓		✓				✓		✓
Wang *et al.* (2017a)		✓		✓				✓		✓
Wang *et al.* (2017b)		✓		✓					✓	
Wang and Su (2016)		✓		✓				✓		
Su and Wang (2016)		✓		✓				✓		
Shahanaghi *et al.* (2013)		✓			✓			✓		✓
Huang *et al.* (2017)		✓					✓	✓		
Iskandar and Husniah (2017)		✓			✓			✓		✓
Huang and Yen (2009)			✓			✓		✓		
Huang *et al.* (2013)			✓			✓		✓		
Huang *et al.* (2015)			✓			✓		✓		
Barbera *et al.* (1996)	✓						✓			✓
Jack and Murthy (2002)	✓						✓			✓
This paper		✓					✓		✓	✓

Notes. PMuW = PM under 2-D warranty, WuPM = 2-D warranty under PM, PC = periodic PM with constant maintenance effort, PN = periodic PM with nonconstant maintenance effort, NC = nonperiodic PM with constant maintenance effort, NN = nonperiodic PM with nonconstant maintenance effort, C = constant, D = dynamic.

Table 2. Summary of notation.

Symbol	*Definition*
U	Usage limit of 2-D warranty
T	Age limit of 2-D warranty
R_t	Random usage rate of period t
$f(r_t)$	Density function of R_t
r	Expected value of R_t
λ	Base-Line failure rate
u_t	Cumulative usage at the beginning of period t
λ_t	Failure rate at the beginning of period t
θ_t	Failure rate after the PM of period t (decision variable)
η	Coefficient of the cumulative usage in the additive proportional hazed model
k	Setup cost for PM
b	Marginal maintenance cost
c	Average cost of each repair
$L(\theta_t, u_t)$	Expected one-period repair cost
$J_t(\lambda_t, u_t)$	Manufacturer's minimum expected total cost over $\{t, t+1, \ldots, T+1\}$, given that the PM decision of period t has not been made yet
$W_t(\theta_t, u_t)$	Manufacturer's minimum expected total cost over $\{t, t+1, \ldots, T+1\}$, given that the PM decision of period t has just been made
$G_t(\theta_t, u_t)$	$W_t(\theta_t, u_t) - b\theta_t$
$H_t(\lambda_t, u_t)$	$\min\{G_t(\lambda_t, u_t), \min_{0\le\theta_t\le\lambda_t} k + G_t(\theta_t, u_t)\}$
u_t^*	Cumulative usage threshold in period t
t^*	Time threshold
$s_t(u_t)$	Failure rate threshold in period t
$\gamma(u_t)$	Slope of $L(\theta_t, u_t)$ given u_t
$\rho(u_t)$	Intercept of $L(\theta_t, u_t)$ given u_t
$\gamma(u_t) + \alpha_t(u_t) - b$	Slope of $G_t(\theta_t, u_t)$ given any $u_{t+1}^* \le u_t \le U$
$\beta_t(u_t)$	Intercept of $G_t(\theta_t, u_t)$ given any $u_{t+1}^* \le u_t \le U$

is protected against an unusually high amount of usage, which would lead to quite frequent failures.

3.1. Model Assumptions

Below, we present the assumptions underlying the model.

A1. Denote the random usage rate of period t as R_t with density function $f(\cdot)$ and expected value r. Its uncertainty is assumed to be resolved right after the action of PM. Moreover, we assume that the usage rate remains unchanged in each period.

A2. In the additive proportional hazard model (see, for example, Eliashberg *et al.* 1997), the base-line failure rate defines the time to failure were the product to remain unused. We assume that it is always a constant λ, which means the failure rate of an unused product stays the same as time goes on. To put it another way, the deterioration of the product is solely attributed to its use. We can justify this assumption by converting the intensity function in our model to the widely used one in Iskandar and Murthy (2003) after a transformation of the random variable R_t.

A3. Let λ_t be the failure rate at the beginning of period t. We assume instantaneous PM after observing λ_t.

A4. Let θ_t be the failure rate *right after* the PM of period t. Let us assume that the failure rate for each period can be reduced to zero.

A5. We make the same assumption as Jack and Murthy (2002) and Yeh and Chang (2007) that the cost of reducing the failure rate by an amount of $\lambda_t - \theta_t$ is given by $k + b(\lambda_t - \theta_t)$ for positive constants k and b, where k is the setup cost and b is the marginal maintenance cost. As we will see later, working with this simplified assumption allows for a convenient expression of the fundamental tradeoff between maintenance cost and repair cost.

A6. We assume that failures follow a nonhomogeneous Poisson process, and they are corrected by minimal repairs. Both assumptions are quite common in the literature. By minimal, we mean that the failure rate of the product stays unchanged after repair. Thus, PM is the only cause of failure rate reduction.

3.2. System States

There are two states of this dynamic system over the T-period horizon: failure rate and cumulative usage, both of which are intimately related to the random usage rate. Let u_t be the cumulative usage at the beginning of period t. Then after one time unit, we have that $u_{t+1} = u_t + R_t$. See Figure 1(a) for two typical usage trajectories whose usage rates have different expected values. It can be observed that, during each period, both lines grow at fixed but stochastic rates, with the slope of each piece being the usage rate realization r_t. This piecewise linearity is a clear manifestation of the random and dynamic usage rate. The figure also shows that, for the trajectory with the smaller expected value, the

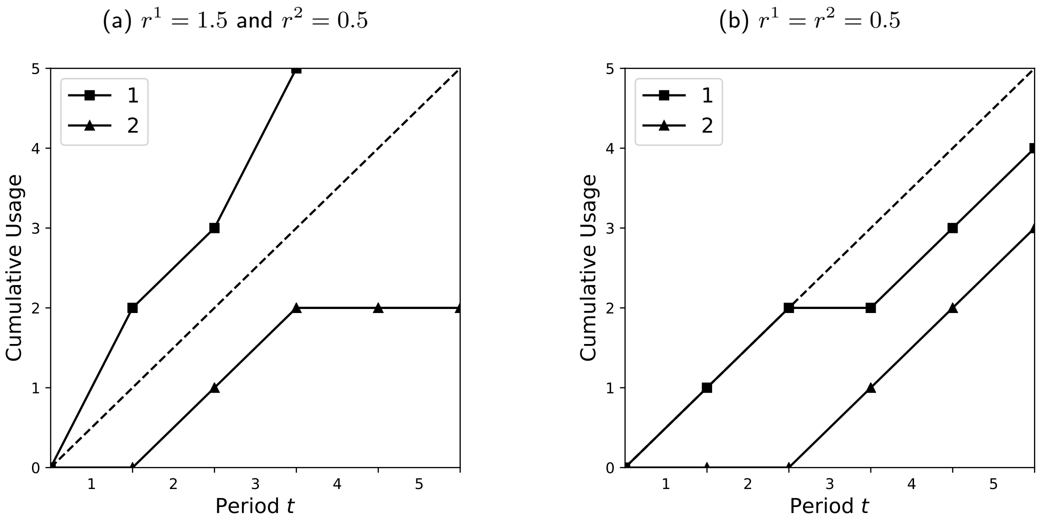

Figure 1. Usage trajectories under 2-D warranty with $U = 5$ and $T = 5$.

2-D warranty probably lasts longer.

Cumulative usage not only directly determines the expiration of the 2-D warranty, but also has a decisive effect on the failure rate. By the additive proportional hazard model in Eliashberg *et al.* (1997), the failure rate at time t is $\lambda(t) = r(t) + \eta M(t)$, where $r(t)$ is the base-line failure rate, and $M(t)$ is the cumulative usage. The positive coefficient η implies that every one-unit increase in the cumulative usage is associated with an increase in the failure rate by the same amount. Unfortunately, this model is built up in a continuous-time setting and has to be adapted to a discrete-time one for our use. By Assumption A2, the transition equation of the failure rate can be written as $\lambda_{t+1} = \lambda + \eta u_{t+1} = \lambda + \eta(u_t + R_t) = \lambda_t + \eta R_t$. After a period, the failure rate grows by a random amount ηR_t.

When we incorporate PM into our analysis, minor alterations are made to the dynamics of the failure rate. Its transition equation under PM is given by $\lambda_{t+1} = \lambda_t - (\lambda_t - \theta_t) + \eta R_t = \theta_t + \eta R_t$, where $\lambda_t - \theta_t$ is the effect of PM and referred to as the maintenance effort. In this equation, the deterioration of the product is entirely determined by the dynamic usage rate. The estimated aging speed should be adjusted downwards if $r_t/r_{t-1} < 1$ or upwards otherwise, where r_{t-1} and r_t are the usage rate realizations of period $t-1$ and period t, respectively. Although there are other ways to model the effect of PM, e.g., age or usage reduction models, we choose this failure rate reduction model for the ease of model specification (see, for example, Wang *et al.* 2017a).

Finally, we explain the difference between the constant and dynamic usage rate in terms of warranty servicing cost. One particular sequence of usage rate realizations $\{r_t\}$ associates with itself a failure rate curve, which in turn dictates the total cost. Figure 1(b) shows two usage trajectories simulated using the same usage rate distribution. The 2-D warranty region under Line 1 is much larger than that under Line 2. By the above transition equation, we know that, for Line 1, the area under the failure rate curve is also larger than that for Line 2, and so is the cost of honoring the warranty. Nonetheless, under

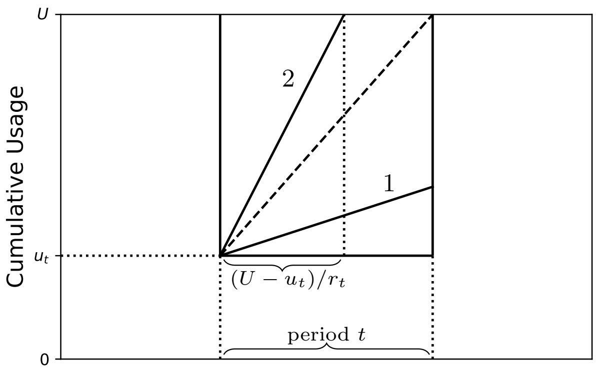

Figure 2. Typical scenarios when calculating the expected repair cost.

the assumption of the constant usage rate, the warranty costs would be calculated as if these two lines increased at the same rate r, which gives rise to the underestimation or overestimation of the true costs.

3.3. Dynamic Program

The third part of this section is devoted to the model formulation of the dynamic programming. To minimize the manufacturer's expected total cost over the remainder of the horizon, we choose the values of the reduce-down-to level θ_t at the beginning of each period. Note that this is equivalent to optimizing over the maintenance effort. To facilitate our discussion, in the rest of the paper, we regard θ_t rather than the maintenance effort as the decision variable.

We denote the expected repair cost of period t by $L(\theta_t, u_t)$ as a function of the failure rate θ_t and the cumulative usage u_t. Both states are observed at the beginning of period t. The first argument is set to θ_t instead of λ_t since the failure rate has already been reduced to θ_t before all the potential failures in this period. The cost of each repair is a constant c that is large enough to make the PM worthwhile. From the reliability theory, we have that

$$L(\theta_t, u_t) = \int_0^{U-u_t} cf(r_t) \int_0^1 (\theta_t + \eta r_t s)\,\mathrm{d}s\,\mathrm{d}r_t + \int_{U-u_t}^{+\infty} cf(r_t) \int_0^{\frac{U-u_t}{r_t}} (\theta_t + \eta r_t s)\,\mathrm{d}s\,\mathrm{d}r_t. \tag{1}$$

For each double integral appearing in Equation (1), we first integrate with respect to s, which represents the time passed since the start of period t, and then r_t, which is the possible usage rate. The two inner integrals, corresponding to Lines 1 and 2 in Figure 2, respectively, are the expected number of failures in period t. To see this, we note that if the usage rate realization r_t is greater than $U - u_t$, then after $(U - u_t)/r_t$ time units, the 2-D warranty is invalid, and only the failures occurring in $[0, (U - u_t)/r_t]$ are counted. Otherwise, it is carried over to the next period, and the counting process continues.

Denote $J_t(\lambda_t, u_t)$ as the manufacturer's minimum expected total cost over $\{t, t+1, \ldots, T+1\}$, given that the 2-D warranty is still valid at the beginning of period t and that the PM decision has *not* been made yet. At this point, we have the information about λ_t and u_t. Denote $W_t(\theta_t, u_t)$ as the manufacturer's minimum expected total cost over $\{t, t+1, \ldots, T+1\}$, given that the 2-D warranty is still valid at the beginning of period t and that the PM decision has *just* been made. Note that we know θ_t and u_t now. Then, we have

$$J_t(\lambda_t, u_t) = \min\left\{W_t(\lambda_t, u_t), \min_{0\le\theta_t\le\lambda_t} k + b(\lambda_t - \theta_t) + W_t(\theta_t, u_t)\right\} \tag{2}$$

and

$$W_t(\theta_t, u_t) = L(\theta_t, u_t) + \int_0^{U-u_t} J_{t+1}(\theta_t + \eta r_t, u_t + r_t) f(r_t)\,\mathrm{d}r_t. \tag{3}$$

Equation (2) shows the manufacturer's decision of whether to maintain the product in period t. If no maintenance action is taken, then the first variable of $W_t(\cdot, u_t)$ gets the value λ_t. On the other hand, if the failure rate is reduced to θ_t, a maintenance cost is incurred immediately plus an additional expected total cost from the moment right after the PM to the end of period T. By Assumption A4, we know that $0 \le \theta_t \le \lambda_t$. Then, the best of the above two options is selected. For emphasis, given the context we are analyzing, we do not model the decision of product replacement in any detail. That is because the product being maintained is assumed as a large repairable system whose replacement cost is prohibitive. Moreover, within the warranty duration, the failure rate of such a system never escalates to the replacement point.

In Equation (3), the value function $W_t(\theta_t, u_t)$ is the sum of the one-period expected repair cost and the expected total cost from the next period onward. Only when R_t is less than $U - u_t$ does $J_{t+1}(\lambda_{t+1}, u_{t+1})$ exist. Otherwise, all the costs in later periods are zero for the manufacturer. That is precisely why the upper limit of the integral on the right-hand side is $U - u_t$ rather than positive infinity. In essence, this formulation describes a random horizon problem because the usage rate adds uncertainty to the expiration of the 2-D warranty.

For convenience, let $G_t(\theta_t, u_t)$ and $H_t(\lambda_t, u_t)$ be defined as

$$G_t(\theta_t, u_t) = W_t(\theta_t, u_t) - b\theta_t \tag{4}$$

and

$$H_t(\lambda_t, u_t) = \min\left\{G_t(\lambda_t, u_t), \min_{0\le\theta_t\le\lambda_t} k + G_t(\theta_t, u_t)\right\}. \tag{5}$$

Then, Equation (2) becomes

$$J_t(\lambda_t, u_t) = b\lambda_t + H_t(\lambda_t, u_t) \tag{6}$$

with boundary conditions $J_t(\cdot, U) = 0$ and $J_{T+1}(\cdot, \cdot) = 0$.

For the derivation of Equation (6), Assumption A5 (the linear maintenance cost) is used. The above transformations are useful since an optimal policy can be found, as shown in Equation (5), by comparing

$G_t(\theta_t, u_t)$ at $\theta_t = \lambda_t$ with its minimum within $[0, \lambda_t]$ plus a constant k. This comparison can be made using the concept of k-convexity in the literature on inventory management (see, Porteus 2002). However, it turns out that $G_t(\theta_t, u_t)$ has relatively simpler structures than general k-convex functions.

4. The Optimal Maintenance Policy

In this section, we first present two lemmas that help obtain the most results in the paper. Then, we specify some structural properties of the value function $G_t(\theta_t, u_t)$ and show that these properties are preserved through the dynamic programming recursion. For the optimal PM actions, we characterize two threshold policies, that is, a time threshold policy and a failure rate threshold policy. To obtain additional insights, important properties of these thresholds are identified.

Lemma 1. *Given a fixed $u_t \in [0, U]$, the one-period expected repair cost function $L(\theta_t, u_t)$ increases linearly in θ_t and can be expressed as follows*

$$L(\theta_t, u_t) = \gamma(u_t)\theta_t + \rho(u_t),$$

where

$$\gamma(u_t) = c\int_0^{U-u_t} f(r_t)\,\mathrm{d}r_t + c(U-u_t)\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\,\mathrm{d}r_t$$

and

$$\rho(u_t) = \frac{c\eta}{2}\int_0^{U-u_t} r_t f(r_t)\,\mathrm{d}r_t + \frac{c\eta(U-u_t)^2}{2}\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\,\mathrm{d}r_t.$$

Moreover, both the slope $\gamma(u_t)$ and the intercept $\rho(u_t)$ are decreasing in u_t.

All proofs are given in the online appendix. For any given cumulative usage, the higher the failure rate, the larger the expected repair cost because of more potential failures. As the cumulative usage gets larger, it becomes more likely the 2-D warranty becomes expired in this period. Consequently, an increment of the failure rate appears to be less severe for the manufacturer, as reflected by the decreases in the slope and the intercept.

Lemma 2. *Consider the recursive equation for any $0 \le x \le U$,*

$$\alpha_t(x) = \int_0^{U-x} \big(\gamma(x+r_t) + \alpha_{t+1}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t \tag{7}$$

with the terminal condition $\alpha_{T+1}(x) = -\gamma(x)$. Let

$$u_t^* = \min\{0 \le x \le U \mid \gamma(x) + \alpha_t(x) \le b\}.$$

Then, we have

(a) $\gamma(x) + \alpha_t(x)$ *is decreasing in* x*.*

(b) $\alpha_t(x) \geq \alpha_{t+1}(x)$*.*

(c) $u_t^* \geq u_{t+1}^*$*.*

(d) If $0 \leq x < u_{t+1}^*$*, then* $\gamma(x) + \int_{u_{t+1}^*-x}^{U-x} \big(\gamma(x+r_t) + \alpha_{t+1}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t - b\int_{u_{t+1}^*-x}^{+\infty} f(r_t)\,\mathrm{d}r_t \geq 0$*.*

By part (a), the existence of the decreasing sequence $\{u_t^*\}$ is guaranteed. We refer to u_t^* as the cumulative usage threshold of period t for the remainder of the paper. It plays a vital role in our analysis since it allows us to partition the interval $[0, U]$ into the non-overlapping subintervals, i.e., $[0, u_{t+1}^*)$, $[u_{t+1}^*, u_t^*)$, and $[u_t^*, U]$.

4.1. Time Threshold Policy

Let us now consider the case $u_{t+1}^* \leq u_t \leq U$ and defer all the analysis for the other case $0 \leq u_t < u_{t+1}^*$ to the next subsection. For the first case, the following proposition proves that $G_t(\theta_t, u_t)$ is a linear function that incorporates the critical elements defined in Lemmas 1 and 2.

Proposition 1. *For* $t = 1, 2, \ldots, T$ *and any fixed* $u_t \in [u_{t+1}^*, U]$*,* $G_t(\theta_t, u_t)$ *is the linear function of the form* $\big(\gamma(u_t) + \alpha_t(u_t) - b\big)\theta_t + \beta_t(u_t)$*, where*

$$\beta_t(u_t) = \rho(u_t) + \int_0^{U-u_t} \Big(\big(\gamma(u_t+r_t) + \alpha_{t+1}(u_t+r_t)\big)\eta r_t + \beta_{t+1}(u_t+r_t)\Big) f(r_t)\,\mathrm{d}r_t \tag{8}$$

with the terminal condition $\beta_{T+1}(u_{T+1}) = 0$*. Moreover,*

(a) if $u_t^* \leq u_t \leq U$*, then* $G_t(\theta_t, u_t)$ *decreases linearly with* θ_t*;*

(b) if $u_{t+1}^* \leq u_t < u_t^*$*, then* $G_t(\theta_t, u_t)$ *increases linearly with* θ_t*.*

By the linear form in this proposition, Equation (3) can be further expressed as $W_t(\theta_t, u_t) = \big(\gamma(u_t) + \alpha_t(u_t)\big)\theta_t + \rho(u_t)$. Accordingly, $\gamma(u_t)$ represents the marginal increase in the one-period expected repair cost, and for any $u_t \geq u_{t+1}^*$, $\alpha_t(u_t)$ captures the marginal increase in the expected total cost in future periods. Moreover, according to parts (a) and (b) of Lemma 2, these marginal changes are smaller when we move closer to the boundaries of the 2-D warranty.

Recall that the cumulative usage threshold u_t^* is defined on the basis of $\gamma(u_t)$ and $\alpha_t(u_t)$. Then, it can be appropriately interpreted as the minimum amount of usage below which the benefit of a one-unit reduction in the failure rate will be greater than the marginal maintenance cost. Since u_t^* is decreasing with respect to b, the larger the marginal maintenance cost is, the more reluctant the manufacturer will be to perform PM.

We are now in a position to present the time threshold policy for the usage-based PM, which provides the solution to the question about the optimal maintenance times.

Theorem 1. *There exists a time threshold* $t^* = \max\{t = 1, 2, \ldots, T \mid u_t < u_t^*\}$ *such that it is suboptimal for the manufacturer to perform PM when* $t > t^*$*.*

As the building block for the time threshold policy, the cumulative usage threshold u_t^* is independent of the system state and thus can be pre-computed. Once the observed the cumulative usage u_t is greater than or equal to u_t^*, no matter how high the failure rate goes, PM is not carried out in the current and subsequent periods. In addition, this decision rule becomes more stringent over time since u_t^* decreases in t. From a managerial perspective, it tells us that the information beyond the time threshold has no influence on the optimal maintenance effort as a consequence of a sufficiently high probability of warranty expiration.

By definition, the time threshold t^* depends on when the increasing path of cumulative usage meets the decreasing path of u_t^* for the first time. Hence, it involves the randomness arising from u_t. We characterize its probability distribution in the following proposition.

Proposition 2. *At the start of period t, if u_t falls into the interval $[u_{i+1}^*, u_i^*)$, where $i \in \{t+1, t+2, \dots, T\}$, then the distribution of the time threshold t^* is given by*

$$\Pr(t^* = j) = \begin{cases} 1 - \Pr\left(R_t < u_{t+1}^* - u_t\right), & \text{if } j = t, \\ \Pr\left(\sum_{z=t}^{j-1} R_z < u_j^* - u_t\right) - \Pr\left(\sum_{z=t}^{j} R_z < u_{j+1}^* - u_t\right), & \text{if } j = t+1, \dots, i-1, \\ \Pr\left(\sum_{z=t}^{i-1} R_z < u_i^* - u_t\right), & \text{if } j = i. \end{cases}$$

This result indicates that the distribution of t^* hinges on the planning period, the cumulative usage, and the usage rate distribution. Hence, it is required that the cumulative usage be updated at the beginning of each period to recalculate this distribution. However, when we replace the stochastic process of usage rates by their expected value r, the time threshold t^*, seen from the first period, is degenerate and equal to $\max\{t = 1, 2, \dots, T \mid (t-1)r < u_t^*\}$ with probability one. Under this constant usage rate system, the time threshold policy still exits, but is of a deterministic nature.

4.2. Failure Rate Threshold Policy

The previous threshold policy gives us the peace of mind after the time threshold t^*, but it does not provide any insight into how to maintain the product when $t \leq t^*$, i.e., $u_t < u_t^*$. For this reason, we develop a second threshold policy, which is characterized by a sequence of failure rate thresholds. From Proposition 1, we know that $G_t(\theta_t, u_t)$ is a linear function of θ_t for any $u_{t+1}^* \leq u_t < u_t^*$. All that remains is to specify its structures with respect to θ_t for any $0 \leq u_t < u_{t+1}^*$. We will see how these structured value functions lead us to the second PM policy.

Proposition 3. *For $t = 1, 2, \dots, T-1$ and any fixed vector (θ, u), $G_t(\theta, u) \geq G_{t+1}(\theta, u)$.*

This horizon result states that the value function G_t increases as the number of planning periods increases since J_t is a decreasing function in t. Intuitively, providing the service of usage-based PM for one extra period requires an additional cost.

Recall that our decision variable is the reduce-down-to level θ_t. To calculate its optimal value, let us define

$$s_t(u_t) = \max\{\theta_t \geq 0 \mid G_t(\theta_t, u_t) \leq k + G_t(0, u_t)\}.$$

Then, we can ground our failure rate threshold policy in the sequence $\{s_t(u_t)\}$.

Theorem 2. *The following statements are true for any fixed $u_t \in [0, u_t^*)$:*

(a) $G_t(\theta_t, u_t)$ is an increasing function of θ_t, and $\lim_{\theta_t \to +\infty} G_t(\theta_t, u_t) = +\infty$.

(b) $s_t(u_t)$ is finite.

(c) There exists an optimal failure rate threshold policy such that

$$\theta_t = \begin{cases} 0, & \text{if } \lambda_t > s_t(u_t), \\ \lambda_t, & \text{otherwise,} \end{cases}$$

where the policy parameter depends on the cumulative usage.

Despite the nonlinear stricture of $G_t(\theta_t, u_t)$ when u_t is in $[0, u_{t+1}^*)$, the form of the optimal policy can still be characterized. As $G_t(\theta_t, u_t)$ grows to infinity, the failure rate threshold $s_t(u_t)$ is guaranteed to exist by definition. Part (a) also suggests that the cost-to-go function $W_t(\theta_t, u_t)$ increases faster than $b\theta_t$ under the condition of this theorem. Let us now consider a special case of our usage-based PM problem with $k = 0$. Then, in the absence of the setup cost, it pays to reduce the failure rate to zero since the marginal decrease in future costs is always higher than the marginal maintenance cost.

However, this trivial policy no longer applies to the problems with nonzero setup costs. For the optimal policy in such cases, we reduce the failure rate to zero, once it exceeds the state-dependent threshold. The underlying idea is to defer PM if a small amount of maintenance effort is required. Instead, it is only when considerable effort is needed that the manufacturer chooses to bear a setup cost.

The above result is similar to Yeh and Chang (2007) in that the product should be maintained and restored to its original condition when the failure rate reaches a threshold value. A major difference is that they try to optimize the constant threshold and the constant maintenance effort as well as the number of PM actions. In contrast, PM decisions in our model are made at the given equidistant inspection times based on a sequence of state-dependent thresholds. Those policy parameters, as functions of the cumulative usage, represent the core of the usage-based PM. To look further into them, we need to demonstrate another property of $G_t(\theta_t, u_t)$.

Proposition 4. *For any fixed $0 \leq u_t < u_t^*$, the straight line $\big(\gamma(u_t) + \alpha_t(u_t) - b\big)\theta_t + \beta_t(u_t)$ increases faster than $G_t(\theta_t, u_t)$.*

If the product were henceforth to be left unmaintained, the value function $G_t(\theta_t, u_t)$ would have the same structure as this straight line (an argument similar to the one used in Proposition 1). The slope, in this case, is larger than the derivative of $G_t(\theta_t, u_t)$ obtained when the failure rate threshold policy is adopted.

Proposition 5. *The policy parameter $s_t(u_t)$ has the following properties:*

(a) for $u_t \in [u_t^*, U]$, $s_t(u_t) = +\infty$;

(b) for $u_t \in [u_{t+1}^*, u_t^*)$, $s_t(u_t) = \frac{k}{\gamma(u_t)+\alpha_t(u_t)-b}$ and is increasing with u_t;

(c) for $u_t \in [0, u_{t+1}^*)$, $s_t(u_t) \geq \frac{k}{\gamma(u_t)+\alpha_t(u_t)-b}$.

Part (a) shows that management can ignore PM if the cumulative usage is sufficiently high. Part (b) allows us to find a closed-form expression for $s_t(u_t)$. Also, it shows that the failure rate threshold increases as the system has higher cumulative usage owing to the consideration of warranty expiration. Nevertheless, the monotonicity does not hold in part (c), where a lower bound is provided instead. We observe that, as u_t increases to u_{t+1}^*, $G_t(\theta_t, u_t)$ approaches the straight line $\big(\gamma(u_t)+\alpha_t(u_t)-b\big)\theta_t+\beta_t(u_t)$ from below, whose slope is, however, decreasing in u_t. Therefore, we are unclear about the monotonic property of $s_t(u_t)$ when u_t is less than u_{t+1}^*.

Finally, we summarize our optimal policy for the usage-based PM as the following steps. At the beginning of period t, u_t is compared to u_t^*. If $u_t \geq u_t^*$, then according to the time threshold policy, no maintenance action is taken in the current and future periods. Otherwise, the failure rate threshold policy is applied immediately. As stated earlier, if $\lambda_t \leq s_t(u_t)$, then it is uneconomical to maintain the product in this period. On the other hand, if $\lambda_t > s_t(u_t)$, then the failure rate is reduced all the way to zero.

5. Numerical Experiments

In this section, we first report on the results on a base case of the usage-based PM problem. Then, we study how sensitive the optimal PM actions are to the model parameters. Since both the state and decision variables are continuous, our model is approximately solved. The basic idea is to discretize dynamics. We replace the continuous, random usage rate with a finite set of permissible values and split the state space into an equally spaced rectangular grid. By this discretization, we can use a backward induction algorithm to evaluate the value functions at the grid points. A bilinear interpolation method is used for the points that are off the grid.

5.1. *Base Case*

We begin by describing a numerical example of the problem to illustrate the analytical results discussed earlier. Throughout the computation, we choose the following parameters: $T = 12$, $U = 12$, $c = 300$, $k = 100$, $b = 1200$, $\eta = 0.1$, and a normally distributed random usage rate R_t truncated on the interval [0.6,1.8] with mean $r = 1.2$ and standard deviation $\sigma = 0.4$.

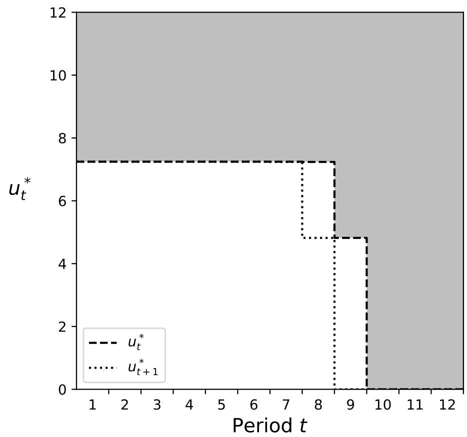

Figure 3. Trajectories of u_t^* and u_{t+1}^*.

We show how the cumulative usage threshold u_t^* evolves in Figure 3 since it can be pre-determined. Surprisingly, it remains constant up to period 8 and then drops dramatically to zero. Its trajectory divides the 2-D warranty into two parts: a no-maintenance region, which is represented by the shaded area (56.4% of the 2-D warranty coverage), and a maintenance region. The former is associated with the time threshold policy, while the latter is covered by the failure rate threshold policy. Notice that the area between the two stepwise curves of u_t^* and u_{t+1}^* is where the closed-form failure rate threshold can be derived. As seen from the figure, the effect of a 2-D warranty on PM is to confine PM within a region smaller than the rectangular warranty. PM, no matter what type it is, should not be scheduled over the entire warranty plane. This 2-D effect deserves more attention from both industry and academia.

We next turn to the value iteration. Figure 4 depicts how the value function $G_t(\theta_t, u_t)$ changes in response to the reduced failure rate for three fixed levels of the cumulative usage. Consistent with Theorem 2, it is increasing with respect to θ_t when u_t is less than u_{t+1}^*, though not linearly. Hence, the failure rate threshold exists in the sense that solving Equation (5) amounts to finding the point where $G_t(\theta_t, u_t)$ increases by k from the origin. Also, we observe that the gap between $G_t(\theta_t, u_t)$ and the straight line widens as θ_t gets larger, as predicted by Proposition 4.

Figure 5 depicts the threshold level $s_t(u_t)$ as a function of the cumulative usage. Although having some ups and downs, it is always higher than $k/(\gamma(u_t)+\alpha_t(u_t)-b)$, which is shown in part (c) of Proposition 5. Moreover, the difference between these two curves is almost imperceptible after u_t increases beyond a certain level. This observation can be explained by the fact that $G_t(\theta_t, u_t)$, in Figure 4, overlaps the pertinent straight line in different ranges of θ_t depending on u_t.

To illustrate the workability of our model, we examine the two threshold policies in light of a sequence of usage rates shown in Table 3. The cumulative usage and the failure rate are initially set to zero. As depicted in Figure 6, their paths are both piecewise linear. Given our setup, in the first three periods, it is more beneficial to do nothing than to initiate the PM. Subsequently, when the failure

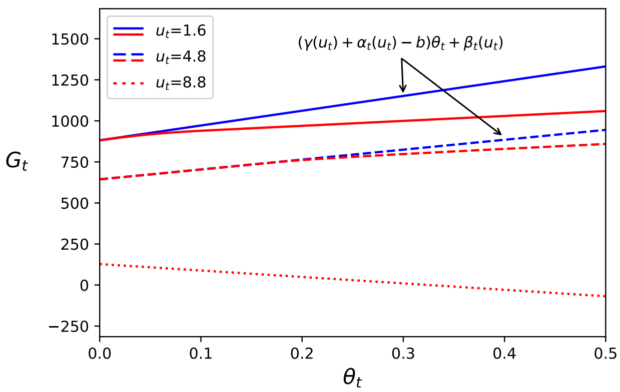

Figure 4. (Color online) Value function $G_t(\theta_t, u_t)$ (red) with $t = 6$ and $u_t = 1.6, 4.8, 8.8$.

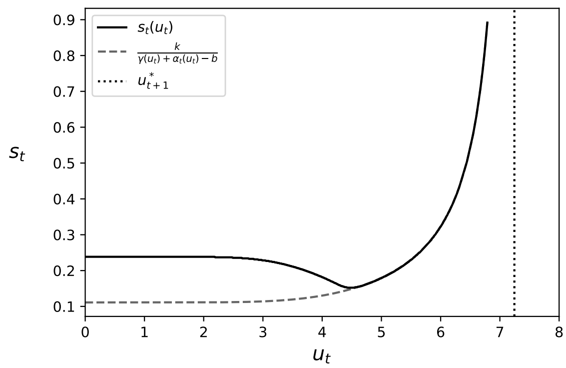

Figure 5. Threshold level $s_t(u_t)$ with $t = 6$ and $u^*_{t+1} = 7.24$.

rate is high enough to justify the setup cost, a PM job must be performed, as represented by the dotted line in Figure 6(b). Finally, after period 7 (the shaded region), the time threshold policy is in place. This numerical experiment indicates that both threshold policies are easy to implement and illustrate. Management should concentrate on the PM decisions under low cumulative usage, knowing that sometimes the setup cost may dominate the positive benefit of failure rate reduction.

5.2. Sensitivity Analysis

Some insight is gained by considering the sensitivity of the optimal PM execution to the variations in the parameters. We use the following input values: $b = 300, 1200, 2100, 3000$; $\sigma = 0.05, 0.4, 0.9$; and $k = 35, 100, 200$. Then, we vary these parameters one at a time with all the others still set to their base case levels. For the last two parameters, we analyze the behavior of our dynamic system over $1,000$

Table 3. Usage rate data drawn from a truncated normal distribution on [0.6,1.8] with mean 1.2 and standard deviation 0.4.

r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
1.45	0.65	1.31	1.42	1.19	0.94	0.92	1.43	0.88	0.77	1.36	1.70

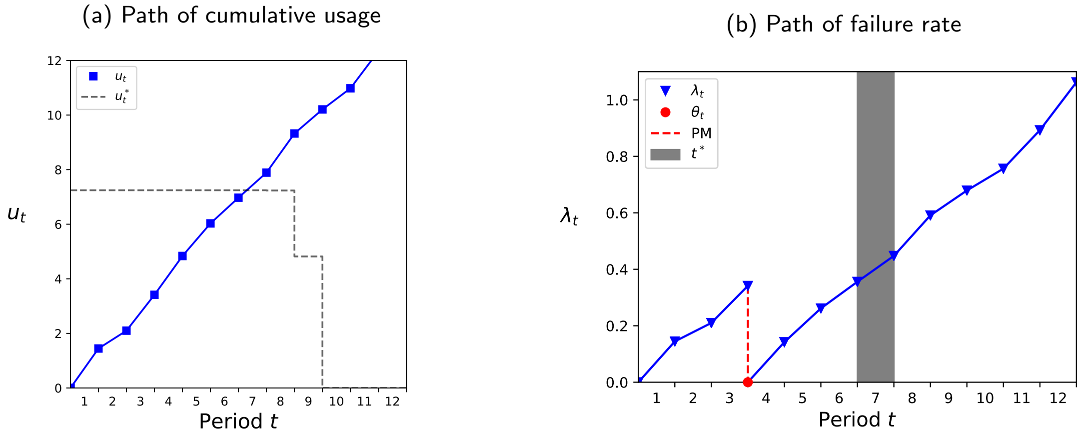

Figure 6. (Color online) Schematic representation of the two threshold maintenance policies.

Table 4. Values of u_t^*.

b/c	u_1^*	u_2^*	u_3^*	u_4^*	u_5^*	u_6^*	u_7^*	u_8^*	u_9^*	u_{10}^*	u_{11}^*	u_{12}^*
1	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.84	10.2
4	7.24	7.24	7.24	7.24	7.24	7.24	7.24	7.24	4.82	0	0	0
7	3.64	3.64	3.64	3.64	3.61	0	0	0	0	0	0	0
10	0.04	0	0	0	0	0	0	0	0	0	0	0

problem instances, each of which has 12 simulated values of R_t. The optimal PM actions directly depend on the usage rate realizations.

We first examine the impact of the cost ratio b/c on u_t^*. As illustrated in Table 4, an increase in the cost ratio decreases the cumulative usage threshold and makes it equivalent to zero for longer, resulting in a larger no-maintenance region. A low value of the cost ratio means that the marginal maintenance cost is close to the repair cost. Therefore, the manufacturer has an incentive to carry out PM. However, for a high cost ratio, e.g., when b is ten times larger than c, repair costs outweigh the benefits from PM.

Figure 7 shows a histogram of the time threshold t^* for different values of the standard deviation σ. We can observe that the effect of increasing σ is to increase the variance in t^*. Surprisingly, the time threshold behaves like a Bernoulli random variable even if the variability of R_t is extremely low. This is because the variances add for the independent random usage rates. We know from Proposition 2 that the availability of usage information can help reduce the degree of policy uncertainty. Therefore, the manufacturer should adopt IoT technology to get more information about product use in the field.

Another interesting quantity is the setup cost k. It is varied from 35 to 200 to show how the number of PM actions is affected. As expected, an increase in k leads to less frequent PM. Eventually, the manufacturer would rarely perform PM since the failure rate threshold is relatively high.

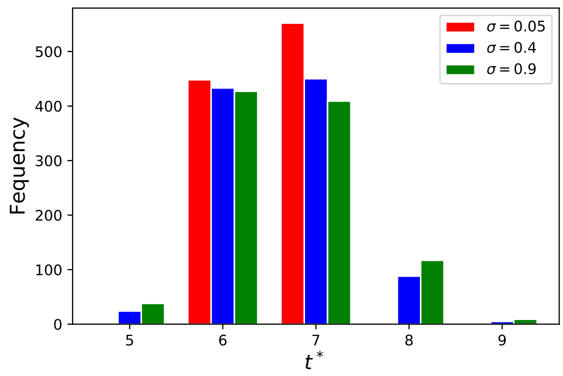

Figure 7. (Color online) Histogram of the time threshold t^* for 1,000 simulated usage paths with $\sigma = 0.1, 0.4, 0.7$.

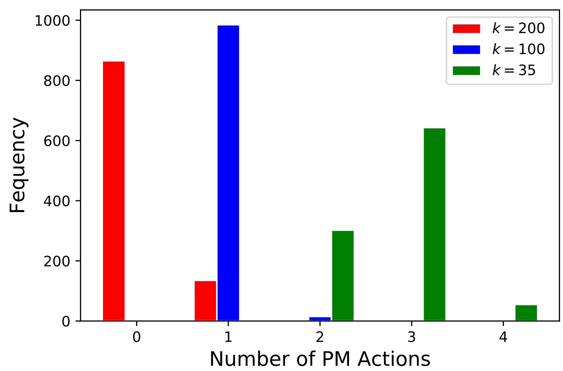

Figure 8. (Color online) Histogram of the number of PM actions for $1,000$ simulated usage paths with $k = 35, 100, 200$.

6. Conclusions and Future Research

This paper contributes to the burgeoning literature on PM and 2-D warranty using dynamic programming. In view of ample research opportunities in the after-sales market, we highlight the importance of the dynamic usage rate as a means of personalization. In our setting, the failure rate and the cumulative usage are tracked in the system state. Then, they are used to determine the optimal policy for the usage-based PM model. The term usage-based here refers to the role of the parameter u_t. Depending upon whether or not u_t is less than u_t^*, the optimal policy takes one of the following two forms: the failure rate threshold policy or the time threshold policy. Our numerical study shows some important results graphically and illustrates the workability of the model. Through the sensitivity analysis, we show that the optimal policy involves a lot of uncertainty due to the variance of the cumulative usage.

There are several limitations to our stylized PM model. First, we assume that the maintenance cost is linear for tractability. An important extension would be to consider more complex cost functions to incorporate diminishing PM effect and regular replacement schedules. It may require that certain structural properties of the value function be preserved under dynamic programming recursions. We conjecture that the time threshold still exists, but the optimal reduce-down-to level is not necessarily zero.

Second, we assume that the manufacturer possesses the complete information of the usage rate distribution. However, in many real situations, customer uncertainty may unfold over time, and it may not be learned all at once. Therefore, investigating the learning process is an appealing research direction. This may require a Bayesian approach to make inferences about distribution parameters.

Third, we neglect the customer's problem. Our model could be extended to a setup similar to Huang *et al.* (2017), where the PM cost is partially paid by the customer to compensate the manufacturer for performing PM. This contract is termed cost-sharing maintenance by which strategic customer behavior could be induced (see, for example, Gallego *et al.* 2014, 2015). Another interesting avenue for further research is to take into account customers' risk preferences, especially loss aversion since Jindal (2015) finds that loss aversion significantly drives customer demand for extended warranties.

By exploring the above issues, we wish to enhance the dynamic paradigm developed in this research further and to shed more light on the use of dynamic programming on PM and warranty problems.

Acknowledgements

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Appendix

Proof of Lemma 1. After some algebra, we have

$$\begin{aligned} L(\theta_t, u_t) &= \int_0^{U-u_t} \Big(\theta_t + \frac{\eta r_t}{2}\Big) c f(r_t)\,\mathrm{d}r_t + \int_{U-u_t}^{+\infty} \frac{2\theta_t(U-u_t) + \eta(U-u_t)^2}{2r_t} \cdot c f(r_t)\,\mathrm{d}r_t \\ &= \left(c\int_0^{U-u_t} f(r_t)\,\mathrm{d}r_t + c(U-u_t)\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\,\mathrm{d}r_t\right)\theta_t + \frac{c\eta}{2}\int_0^{U-u_t} r_t f(r_t)\,\mathrm{d}r_t \\ &\quad + \frac{c\eta(U-u_t)^2}{2}\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\,\mathrm{d}r_t. \end{aligned}$$

From the last equation, we know that $L(\theta_t, u_t)$ is a linear function of θ_t for a fixed u_t. Since $\gamma(u_t) \geq 0$ for any $0 \leq u_t \leq U$, $L(\theta_t, u_t)$ is increasing in θ_t. By differentiating $\gamma(u_t)$ and $\rho(u_t)$ with respect to u_t, we obtain $\frac{\mathrm{d}\gamma(u_t)}{\mathrm{d}u_t} = -c\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\,\mathrm{d}r_t \leq 0$ and $\frac{\mathrm{d}\rho(u_t)}{\mathrm{d}u_t} = -c\eta(U-u_t)\int_{U-u_t}^{+\infty} \frac{1}{r_t} f(r_t)\,\mathrm{d}r_t \leq 0$. $\square$

Proof of Lemma 2. We will prove parts (a) and (b) of this lemma by induction and part (c) by contradiction. As for part (d), it follows immediately from these results.

(a) For period $T+1$, part (a) is satisfied since $\gamma(x)+\alpha_{T+1}(x)=0$. Suppose $\gamma(x)+\alpha_{t+1}(x)$ is decreasing in x. Then for any $0 \le x' < x \le U$, we have

$$\begin{aligned}
&\gamma(x)+\alpha_t(x)-\gamma(x')-\alpha_t(x') \\
&= \int_0^{U-x} \big(\gamma(x+r_t)+\alpha_{t+1}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t - \int_0^{U-x'} \big(\gamma(x'+r_t)+\alpha_{t+1}(x'+r_t)\big) f(r_t)\,\mathrm{d}r_t \\
&\quad + \gamma(x)-\gamma(x') \\
&= \int_0^{U-x} \Big(\big(\gamma(x+r_t)+\alpha_{t+1}(x+r_t)\big) - \big(\gamma(x'+r_t)+\alpha_{t+1}(x'+r_t)\big)\Big) f(r_t)\,\mathrm{d}r_t \\
&\quad - \int_{U-x}^{U-x'} \big(\gamma(x'+r_t)+\alpha_{t+1}(x'+r_t)\big) f(r_t)\,\mathrm{d}r_t + \gamma(x)-\gamma(x') \\
&\le \int_0^{U-x} \Big(\big(\gamma(x+r_t)+\alpha_{t+1}(x+r_t)\big) - \big(\gamma(x'+r_t)+\alpha_{t+1}(x'+r_t)\big)\Big) f(r_t)\,\mathrm{d}r_t \\
&\le 0.
\end{aligned}$$

The first inequality holds because $\gamma(x)-\gamma(x') \le 0$ from Lemma 1 and $\gamma(x'+r_t)+\alpha_{t+1}(x'+r_t) \ge \gamma(U)+\alpha_{t+1}(U)=0$ for any $r_t \in [U-x, U-x']$. The second inequality holds because $x+r_t > x'+r_t$ and thus $\gamma(x+r_t)+\alpha_{t+1}(x+r_t) \le \gamma(x'+r_t)+\alpha_{t+1}(x'+r_t)$ by the induction hypothesis. Therefore, $\gamma(x)+\alpha_t(x)$ is decreasing in x.

(b) For $t=T$, we have that $\alpha_T(x)=0 \ge \alpha_{T+1}(x) = -\gamma(x)$. Suppose that $\alpha_{t+1}(x) \ge \alpha_{t+2}(x)$ for any $0 \le x \le U$. Then,

$$\begin{aligned}
&\alpha_t(x)-\alpha_{t+1}(x) \\
&= \int_0^{U-x} \big(\gamma(x+r_t)+\alpha_{t+1}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t \\
&\quad - \int_0^{U-x} \big(\gamma(x+y)+\alpha_{t+2}(x+r_{t+1})\big) f(r_{t+1})\,\mathrm{d}r_{t+1} \\
&= \int_0^{U-x} \big(\alpha_{t+1}(x+r_t)-\alpha_{t+2}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t \\
&\ge 0.
\end{aligned}$$

The inequality holds because $\alpha_{t+1}(x+r_t) \ge \alpha_{t+2}(x+r_t)$ for any $r_t \in [0, U-x]$. Therefore, $\alpha_t(x)$ is decreasing in t.

(c) Assume for a contradiction that there exists some t such that $0 \le u_t^* < u_{t+1}^*$. Notice that

$$\gamma(u_t^*)+\alpha_t(u_t^*) \ge \gamma(u_t^*)+\alpha_{t+1}(u_t^*) > b.$$

The last inequality holds because $u_t^* < u_{t+1}^*$ by assumption and $\gamma(u_{t+1})+\alpha_{t+1}(u_{t+1}) > b$ for any $u_{t+1} < u_{t+1}^*$. It, however, contradicts the fact that $\gamma(u_t^*)+\alpha_t(u_t^*) \le b$ according to the definition of u_t^*.

(d) Notice that

$$
\begin{aligned}
&\gamma(x) + \int_{u_{t+1}^*-x}^{U-x} \big(\gamma(x+r_t) + \alpha_{t+1}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t - b\int_{u_{t+1}^*-x}^{+\infty} f(r_t)\,\mathrm{d}r_t \\
&= \gamma(x) + \alpha_t(x) - \int_0^{u_{t+1}^*-x} \big(\gamma(x+r_t) + \alpha_{t+1}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t - b\int_{u_{t+1}^*-x}^{+\infty} f(r_t)\,\mathrm{d}r_t \\
&= \left(\int_0^{u_{t+1}^*-x} f(r_t)\,\mathrm{d}r_t + \int_{u_{t+1}^*-x}^{+\infty} f(r_t)\,\mathrm{d}r_t\right)\big(\gamma(x)+\alpha_t(x)\big) \\
&\quad - \int_0^{u_{t+1}^*-x} \big(\gamma(x+r_t) + \alpha_{t+1}(x+r_t)\big) f(r_t)\,\mathrm{d}r_t - b\int_{u_{t+1}^*-x}^{+\infty} f(r_t)\,\mathrm{d}r_t \\
&= \int_0^{u_{t+1}^*-x} \Big(\big(\gamma(x)+\alpha_t(x)\big) - \big(\gamma(x+r_t) + \alpha_{t+1}(x+r_t)\big)\Big) f(r_t)\,\mathrm{d}r_t \\
&\quad + \big(\gamma(x)+\alpha_t(x) - b\big)\int_{u_{t+1}^*-x}^{+\infty} f(r_t)\,\mathrm{d}r_t \\
&\geq 0.
\end{aligned}
$$

The first equality follows from Equation (7). The inequality holds because $\gamma(x) + \alpha_t(x) \geq \gamma(x + r_t) + \alpha_t(x+r_t) \geq \gamma(x+r_t) + \alpha_{t+1}(x+r_t)$ and $\gamma(x) + \alpha_t(x) - b > 0$ for any $0 \leq x < u_{t+1}^* \leq u_t^*$. □

Proof of Proposition 1. For $t = T$, $G_T(\theta_T, u_T) = \big(\gamma(u_T) - b\big)\theta_T + \rho(u_T)$ with $\alpha_T(u_T) = 0$ and $\beta_T(u_T) = \rho(u_T)$ when $0 = u_{T+1}^* \leq u_T \leq U$. Assume by induction that $G_{t+1}(\theta_{t+1}, u_{t+1}) = \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)\theta_{t+1} + \beta_{t+1}(u_{t+1})$ for any $u_{t+1} \in [u_{t+2}^*, U]$. Then for period t and $u_t \in [u_{t+1}^*, U]$,

$$
\begin{aligned}
&G_t(\theta_t, u_t) \\
&= -b\theta_t + L(\theta_t, u_t) + \int_0^{U-u_t} \big(b(\theta_t + \eta r_t) + H_{t+1}(\theta_t + \eta r_t, u_t + r_t)\big) f(r_t)\,\mathrm{d}r_t \\
&= -b\theta_t + L(\theta_t, u_t) + \int_0^{U-u_t} \big(b(\theta_t + \eta r_t) + G_{t+1}(\theta_t + \eta r_t, u_t + r_t)\big) f(r_t)\,\mathrm{d}r_t \\
&= -b\theta_t + L(\theta_t, u_t) + \int_0^{U-u_t} \Big(\big(\gamma(u_t + r_t) + \alpha_{t+1}(u_t+r_t)\big)(\theta_t + \eta r_t) + \beta_{t+1}(u_t + r_t)\Big) f(r_t)\,\mathrm{d}r_t \\
&= \left(\gamma(u_t) + \int_0^{U-u_t} \big(\gamma(u_t + r_t) + \alpha_{t+1}(u_t+r_t)\big) f(r_t)\,\mathrm{d}r_t - b\right)\theta_t + \rho(u_t) \\
&\quad + \int_0^{U-u_t} \Big(\big(\gamma(u_t + r_t) + \alpha_{t+1}(u_t+r_t)\big)\eta r_t + \beta_{t+1}(u_t + r_t)\Big) f(r_t)\,\mathrm{d}r_t.
\end{aligned}
$$

The second equality holds because $u_{t+2}^* \leq u_{t+1}^* \leq u_t \leq u_t + r_t \leq U$ for any $r_t \in [0, U - u_t]$ and hence $G_{t+1}(\theta_{t+1}, u_t + r_t)$ is a linear function of θ_{t+1} by assumption. Moreover, it is decreasing in θ_{t+1} from the definition of u_{t+1}^*. This implies that

$$
\begin{aligned}
&H_{t+1}(\theta_t + \eta r_t, u_t + r_t) \\
&= \min\left\{G_{t+1}(\theta_t + \eta r_t, u_t + r_t), \min_{0 \leq \theta_{t+1} \leq \theta_t + \eta r_t} k + G_{t+1}(\theta_{t+1}, u_t + r_t)\right\} \\
&= G_{t+1}(\theta_t + \eta r_t, u_t + r_t).
\end{aligned}
$$

After substituting the linear expression of $G_{t+1}(\theta_{t+1}, u_t + r_t)$ at $\theta_{t+1} = \theta_t + \eta r_t$, we arrive at the third equation. From Equations (7) and (8), we finally show that $G_t(\theta_t, u_t) = \big(\gamma(u_t) + \alpha_t(u_t) - b\big)\theta_t + \beta_t(u_t)$ for any $u^*_{t+1} \le u_t \le U$.

Next we prove the monotonicity of $G_t(\theta_t, u_t)$ which depends on the sign of its slope. Since $u^*_{t+1} \le u^*_t$, the interval $[u^*_{t+1}, U]$ can be further divided into two subintervals, that is, $[u^*_{t+1}, u^*_t)$ and $[u^*_t, U]$. When $u^*_t \le u_t \le U$, the definition of u^*_t implies that $\gamma(u_t) + \alpha_t(u_t) \le b$. Consequently, $G_t(\theta_t, u_t)$ is a decreasing linear function of θ_t. The proof for the other subinterval is entirely similar. □

Proof of Theorem 1. Take $t^* = 0$ if the set in the definition of t^* is empty, i.e., when $u^*_1 = 0$. For any $t > t^*$, we have that $u^*_t \le u^*_{t^*+1} \le u_{t^*+1} \le u_t$. The first inequality follows from part (c) of Lemma 2 and the second inequality is due to the definition of t^*. The third inequality holds because u_t is increasing in t. Then by part (a) of Proposition 1, we know that $G_t(\theta_t, u_t)$ is a decreasing linear function for $t = t^* + 1, t^* + 2, \dots, T$. Consequently, $H_t(\lambda_t, u_t) = \min\{G_t(\lambda_t, u_t), \min_{0 \le \theta_t \le \lambda_t} k + G_t(\theta_t, u_t)\} = G_t(\lambda_t, u_t)$, which implies that the manufacturer chooses not to perform PM in these periods. □

Proof of Proposition 2. If u_t falls into the interval $[u^*_{i+1}, u^*_i)$, then the time threshold t^* ranges over $\{t, t+1, \dots, i\}$. In order to calculate the probability of t^* at each value, we first prove by induction that $\Pr(t^* \ge j) = \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_j\right)$ for $j = t+1, t+2, \dots, i$. When $j = i$, we have that $\Pr(t^* \ge i) = \Pr(t^* = i) = \Pr\left(u_t + \sum_{z=t}^{i-1} R_z < u^*_i\right)$. Suppose that $\Pr(t^* \ge j+1) = \Pr\left(u_t + \sum_{z=t}^{j} R_z < u^*_{j+1}\right)$. This equation can be expanded into

$$
\begin{aligned}
&\Pr(t^* \ge j+1) \\
&= \Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z\right) \\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right) \Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z \,\middle|\, u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right) \\
&\quad + \Pr\left(u_t + \sum_{z=t}^{j-1} R_z \ge u^*_{j+1}\right) \Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z \,\middle|\, u_t + \sum_{z=t}^{j-1} R_z \ge u^*_{j+1}\right) \\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right) \Pr\left(R_j < u^*_{j+1} - u_t - \sum_{z=t}^{j-1} R_z \,\middle|\, u_t + \sum_{z=t}^{j-1} R_z < u^*_{j+1}\right).
\end{aligned}
$$

The second equality is due to the law of total probability. Notice that the last conditional probability in this equality is zero, since R_j is positive. Then,

$$
\begin{aligned}
\Pr(t^* \geq j) &= \Pr(t^* \geq j+1) + \Pr(t^* = j) \\
&= \Pr(t^* \geq j+1) + \Pr\left(u_{j+1}^* \leq u_t + \sum_{z=t}^{j-1} R_z < u_j^*\right) \\
&\quad + \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u_{j+1}^*\right) \Pr\left(R_j \geq u_{j+1}^* - u_t - \sum_{z=t}^{j-1} R_z \;\middle|\; u_t + \sum_{z=t}^{j-1} R_z < u_{j+1}^*\right) \\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u_{j+1}^*\right) + \Pr\left(u_{j+1}^* \leq u_t + \sum_{z=t}^{j-1} R_z < u_j^*\right) \\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u_j^*\right).
\end{aligned}
$$

The last two summands in the second equality correspond to the probability of observing event $u_{j+1}^* \leq u_j < u_j^*$ and observing events $u_j < u_{j+1}^*$ and $u_{j+1} \geq u_{j+1}^*$, respectively. They add up to $\Pr(t^* = j)$. To get rid of the conditional probability, we proceed as follows:

$$
\begin{aligned}
\Pr(t^* = j) &= \Pr(t^* \geq j) - \Pr(t^* \geq j+1) \\
&= \Pr\left(u_t + \sum_{z=t}^{j-1} R_z < u_j^*\right) - \Pr\left(u_t + \sum_{z=t}^{j} R_z < u_{j+1}^*\right) \\
&= \Pr\left(\sum_{z=t}^{j-1} R_z < u_j^* - u_t\right) - \Pr\left(\sum_{z=t}^{j} R_z < u_{j+1}^* - u_t\right).
\end{aligned}
$$

We conclude the proof by checking that

$$
\Pr(t^* = t) = \Pr\left(R_t \geq u_{i+1}^* - u_t\right) = 1 - \Pr\left(R_t < u_{i+1}^* - u_t\right). \qquad \square
$$

Proof of Proposition 3. We first argue inductively that $J_t(\lambda, u) \geq J_{t+1}(\lambda, u)$ for any fixed vector (λ, u). Then, as we will see later, it follows that $G_t(\theta, u) \geq G_{t+1}(\theta, u)$. For $t = T$,

$$
\begin{aligned}
J_T(\lambda, u) - J_{T+1}(\lambda, u) &= b\lambda + H_T(\lambda, u) \\
&= b\lambda + \min\left\{G_T(\lambda, u), \min_{0 \leq \theta \leq \lambda} k + G_T(\theta, u)\right\} \\
&= \min\left\{L(\lambda, u), \min_{0 \leq \theta \leq \lambda} k + b(\lambda - \theta) + L(\theta, u)\right\} \\
&\geq 0.
\end{aligned}
$$

The third equality holds because $G_T(\theta, u) = L(\theta, u) - b\theta$. Now, assume by induction that this inequality is true for period $t+1$. Note that $G_t(\theta, u) = L(\theta, u) + \int_0^{U-u} J_{t+1}(\theta + \eta r_t, u + r_t) f(r_t)\, \mathrm{d}r_t - b\theta$. Then $G_t(\theta, u) - G_{t+1}(\theta, u) = \int_0^{U-u} \left(J_{t+1}(\theta + \eta r_t, u + r_t) - J_{t+2}(\theta + \eta r_t, u + r_t)\right) f(r_t)\, \mathrm{d}r_t \geq 0$. We now establish that $J_t(\lambda,u) \geq J_{t+1}(\lambda,u)$ holds for t. For any fixed vector (λ,u),

$$
\begin{aligned}
& J_t(\lambda,u) - J_{t+1}(\lambda,u) \\
&= H_t(\lambda,u) - H_{t+1}(\lambda,u) \\
&= \min\left\{G_t(\lambda,u), \min_{0\leq\theta\leq\lambda} k + G_t(\theta,u)\right\} - \min\left\{G_{t+1}(\lambda,u), \min_{0\leq\theta\leq\lambda} k + G_{t+1}(\theta,u)\right\} \\
&\geq \min\left\{G_{t+1}(\lambda,u), \min_{0\leq\theta\leq\lambda} k + G_{t+1}(\theta,u)\right\} - \min\left\{G_{t+1}(\lambda,u), \min_{0\leq\theta\leq\lambda} k + G_{t+1}(\theta,u)\right\} \\
&= 0.
\end{aligned}
$$

Denote $\theta_t^* = \arg\min_{0\leq\theta\leq\lambda} k + G_t(\theta,u)$. The above inequality follows from $G_t(\lambda,u) \geq G_{t+1}(\lambda,u)$ and $\min_{0\leq\theta\leq\lambda} k + G_t(\theta,u) = k + G_t(\theta_t^*,u) \geq k + G_{t+1}(\theta_t^*,u) \geq \min_{0\leq\theta\leq\lambda} k + G_{t+1}(\theta,u)$. This concludes the induction argument and thus the inequality of this proposition holds. $\square$

Proof of Theorem 2. The proof of this theorem is based on an induction argument built around part (a). For $t = T$, $G_T(\theta_T,u_T) = \big(\gamma(u_T) - b\big)\theta_T + \rho(u_T)$ is strictly increasing in θ_T for any $0 \leq u_T < u_T^*$ and hence $\lim_{\theta_T\to+\infty} G_T(\theta_T,u_T) = +\infty$. Suppose that the first part of the theorem is true for period $t+1$. This clearly implies the second part. To obtain the optimality of such a state-dependent failure rate threshold policy, we examine $H_{t+1}(\lambda_{t+1},u_{t+1})$ in the following two cases: (i) for $\lambda_{t+1} > s_{t+1}(u_{t+1})$, we have

$$
\begin{aligned}
H_{t+1}(\lambda_{t+1},u_{t+1}) &= \min\{G_{t+1}(\lambda_{t+1},u_{t+1}), \min_{0\leq\theta_{t+1}\leq\lambda_{t+1}} k + G_{t+1}(\theta_{t+1},u_{t+1})\} \\
&= \min\{G_{t+1}(\lambda_{t+1},u_{t+1}), k + G_{t+1}(0,u_{t+1})\} \\
&= k + G_{t+1}(0,u_{t+1}).
\end{aligned}
$$

The second equality follows from the inductive hypothesis that $G_{t+1}(\theta_{t+1},u_{t+1})$ is increasing in θ_{t+1}. The last equality is due to the definition of $s_{t+1}(u_{t+1})$. (ii) For $\lambda_{t+1} \leq s_{t+1}(u_{t+1})$, similarly we have $H_{t+1}(\lambda_{t+1},u_{t+1}) = G_{t+1}(\lambda_{t+1},u_{t+1})$.

Next, we establish that $H_{t+1}(\lambda_{t+1},u_{t+1})$ is increasing in λ_{t+1} for $0 \leq u_{t+1} < u_{t+1}^*$. To do this, suppose that $\lambda'_{t+1} < \lambda_{t+1}$. Then, there are three cases to consider.

Case 1. If $s_{t+1}(u_{t+1}) < \lambda'_{t+1} < \lambda_{t+1}$, then $H_{t+1}(\lambda_{t+1},u_{t+1}) - H_{t+1}(\lambda'_{t+1},u_{t+1}) = k + G_{t+1}(0,u_{t+1}) - k - G_{t+1}(0,u_{t+1}) = 0$.

Case 2. If $\lambda'_{t+1} \leq s_{t+1}(u_{t+1}) < \lambda_{t+1}$, then $H_{t+1}(\lambda_{t+1},u_{t+1}) - H_{t+1}(\lambda'_{t+1},u_{t+1}) = k + G_{t+1}(0,u_{t+1}) - G_{t+1}(\lambda'_{t+1},u_{t+1}) \geq 0$ by the definition of $s_{t+1}(u_{t+1})$.

Case 3. If $\lambda'_{t+1} < \lambda_{t+1} \leq s_{t+1}(u_{t+1})$, then $H_{t+1}(\lambda_{t+1},u_{t+1}) - H_{t+1}(\lambda'_{t+1},u_{t+1}) = G_{t+1}(\lambda_{t+1},u_{t+1}) - G_{t+1}(\lambda'_{t+1},u_{t+1}) \geq 0$ from the induction hypothesis.

We now argue that $G_t(\theta_t,u_t)$ is increasing in θ_t by analyzing separately the case when $0 \leq u_t < u_{t+1}^*$

and when $u_{t+1}^* \le u_t < u_t^*$. For the first case, suppose that $\theta_t' < \theta_t$. Then

$$
\begin{aligned}
& G_t(\theta_t, u_t) - G_t(\theta_t', u_t) \\
&= \big(\gamma(u_t) - b\big)(\theta_t - \theta_t') + \int_0^{U-u_t} \big(J_{t+1}(\theta_t + \eta r_t, u_t + r_t) - J_{t+1}(\theta_t' + \eta r_t, u_t + r_t)\big) f(r_t)\, \mathrm{d}r_t \\
&= \big(\gamma(u_t) - b\big)(\theta_t - \theta_t') + \int_{u_{t+1}^*-u_t}^{U-u_t} \big(J_{t+1}(\theta_t + \eta r_t, u_t + r_t) - J_{t+1}(\theta_t' + \eta r_t, u_t + r_t)\big) f(r_t)\, \mathrm{d}r_t \\
&\quad + \int_0^{u_{t+1}^*-u_t} \big(J_{t+1}(\theta_t + \eta r_t, u_t + r_t) - J_{t+1}(\theta_t' + \eta r_t, u_t + r_t)\big) f(r_t)\, \mathrm{d}r_t \\
&= \big(\gamma(u_t) - b\big)(\theta_t - \theta_t') + \int_{u_{t+1}^*-u_t}^{U-u_t} \big(\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t)\big)(\theta_t - \theta_t') f(r_t)\, \mathrm{d}r_t \\
&\quad + \int_0^{u_{t+1}^*-u_t} \big(b(\theta_t - \theta_t') + H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta_t' + \eta r_t, u_t + r_t)\big) f(r_t)\, \mathrm{d}r_t \\
&= \left(\gamma(u_t) + \int_{u_{t+1}^*-u_t}^{U-u_t} \big(\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t)\big) f(r_t)\, \mathrm{d}r_t - b \int_{u_{t+1}^*-u_t}^{+\infty} f(r_t)\, \mathrm{d}r_t\right)(\theta_t - \theta_t') \\
&\quad + \int_0^{u_{t+1}^*-u_t} \big(H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta_t' + \eta r_t, u_t + r_t)\big) f(r_t)\, \mathrm{d}r_t \qquad \text{(A1)} \\
&\ge 0.
\end{aligned}
$$

The third equality holds because $J_{t+1}(\lambda_{t+1}, u_{t+1}) = b\lambda_{t+1} + H_{t+1}(\lambda_{t+1}, u_{t+1}) = b\lambda_{t+1} + G_{t+1}(\lambda_{t+1}, u_{t+1})$ for any $u_{t+1}^* \le u_{t+1} \le U$. The inequality is due to part (d) of Lemma 2 and the fact that $H_{t+1}(\lambda_{t+1}, u_{t+1})$ is increasing with respect to λ_{t+1} for $0 \le u_{t+1} < u_{t+1}^*$. For part (b) and any $0 \le u_t < u_{t+1}^*$, we have that $\lim_{\theta_t \to +\infty} G_t(\theta_t, u_t) \ge \lim_{\theta_t \to +\infty} G_{t+1}(\theta_t, u_t) = +\infty$ by Proposition 3. For the case when $u_{t+1}^* \le u_t < u_t^*$, part (a) is satisfied by the linearity of $G_t(\theta_t, u_t)$. Thus, we conclude that the first part of this theorem is also true for period t. Since the proofs for parts (b) and (c) are similar to the arguments given for period $t+1$, they are omitted. □

Proof of Proposition 4. Equivalently, we show that for any $0 \le u_t < u_t^*$ and $\theta_t' < \theta_t$,

$$
\begin{aligned}
& G_t(\theta_t, u_t) - G_t(\theta_t', u_t) - \Big(\big(\gamma(u_t) + \alpha_t(u_t) - b\big)\theta_t + \beta_t(u_t) - \big(\gamma(u_t) + \alpha_t(u_t) - b\big)\theta_t' - \beta_t(u_t)\Big) \\
&= G_t(\theta_t, u_t) - G_t(\theta_t', u_t) - \big(\gamma(u_t) + \alpha_t(u_t) - b\big)(\theta_t - \theta_t') \\
&\le 0.
\end{aligned}
$$

In period T, $G_T(\theta_T, u_T) - G_T(\theta_T', u_T) - \big(\gamma(u_T) + \alpha_T(u_T) - b\big)(\theta_T - \theta_T') = 0$ when $0 \le u_T < u_T^*$ and $\theta_T' < \theta_T$. Suppose the inequality is true for period $t+1$. We now demonstrate that $H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda_{t+1}', u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} - \lambda_{t+1}') \le 0$ for any $0 \le u_{t+1} < u_{t+1}^*$ and $\lambda_{t+1}' < \lambda_{t+1}$ by investigating the following three cases.

Case 1. If $s_{t+1}(u_{t+1}) < \lambda_{t+1}' < \lambda_{t+1}$, then $H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda_{t+1}', u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} - \lambda_{t+1}') = k + G_{t+1}(0, u_{t+1}) - k - G_{t+1}(0, u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} -$

$\lambda'_{t+1}) = -\big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} - \lambda'_{t+1}) \leq 0$ by the definition of u^*_{t+1}.

Case 2. If $\lambda'_{t+1} \leq s_{t+1}(u_{t+1}) < \lambda_{t+1}$, then

$$
\begin{aligned}
& H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} - \lambda'_{t+1}) \\
&= k + G_{t+1}(0, u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} - \lambda'_{t+1}) \\
&\leq G_{t+1}(s_{t+1}(u_{t+1}), u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(s_{t+1}(u_{t+1}) - \lambda'_{t+1}) \\
&\leq 0.
\end{aligned}
$$

The first inequality holds because $k + G_{t+1}(0, u_{t+1}) = G_{t+1}(s_{t+1}(u_{t+1}), u_{t+1})$ and $\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b > 0$ when $0 \leq u_{t+1} < u^*_{t+1}$. The second inequality is from the induction hypothesis.

Case 3. If $\lambda'_{t+1} < \lambda_{t+1} \leq s_{t+1}(u_{t+1})$, then $H_{t+1}(\lambda_{t+1}, u_{t+1}) - H_{t+1}(\lambda'_{t+1}, u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} - \lambda'_{t+1}) = G_{t+1}(\lambda_{t+1}, u_{t+1}) - G_{t+1}(\lambda'_{t+1}, u_{t+1}) - \big(\gamma(u_{t+1}) + \alpha_{t+1}(u_{t+1}) - b\big)(\lambda_{t+1} - \lambda'_{t+1}) \leq 0$ from the induction hypothesis.

In period t, we examine separately the case where $0 \leq u_t < u^*_{t+1}$ and the case where $u^*_{t+1} \leq u_t < u^*_t$. For the first case and any $\theta'_t < \theta_t$,

$$
\begin{aligned}
& G_t(\theta_t, u_t) - G_t(\theta'_t, u_t) - \big(\gamma(u_t) + \alpha_t(u_t) - b\big)(\theta_t - \theta'_t) \\
&= \left(\gamma(u_t) + \int_{u^*_{t+1}-u_t}^{U-u_t} \big(\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t)\big) f(r_t)\,\mathrm{d}r_t - b\int_{u^*_{t+1}-u_t}^{+\infty} f(r_t)\,\mathrm{d}r_t\right)(\theta_t - \theta'_t) \\
&\quad + \int_0^{u^*_{t+1}-u_t} \big(H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta'_t + \eta r_t, u_t + r_t)\big) f(r_t)\,\mathrm{d}r_t \\
&\quad - \big(\gamma(u_t) + \alpha_t(u_t) - b\big)(\theta_t - \theta'_t) \\
&= \int_0^{u^*_{t+1}-u_t} \Big(H_{t+1}(\theta_t + \eta r_t, u_t + r_t) - H_{t+1}(\theta'_t + \eta r_t, u_t + r_t) \\
&\quad - \big(\gamma(u_t + r_t) + \alpha_{t+1}(u_t + r_t) - b\big)(\theta_t - \theta'_t)\Big) f(r_t)\,\mathrm{d}r_t \\
&\leq 0.
\end{aligned}
$$

The first equality is from Equation (A1) and the second equality is from Equation (7). The inequality is substantiated by the previous statement about H_{t+1}. The proof for the case where $u^*_{t+1} \leq u_t < u^*_t$ is fairly trivial and thus omitted. □

Proof of Theorem 5. Part (a) follows from Proposition 1. As for part (b), we know that $G_t(\theta_t, u_t)$ is a strictly increasing linear function of θ_t when $u_t \in [u^*_{t+1}, u^*_t)$. Thus, an increment of k in $G_t(\theta_t, u_t)$ corresponds to an increment of $k/(\gamma(u_t) + \alpha_t(u_t) - b)$ in θ_t. If starting from $\theta_t = 0$, then we arrive at the desired $s_t(u_t)$. Moreover, by taking the derivative of $s_t(u_t)$ with respect to u_t, we have $\frac{\mathrm{d}s_t(u_t)}{\mathrm{d}u_t} = \frac{-k}{(\gamma(u_t)+\alpha_t(u_t)-b)^2} \cdot \frac{\mathrm{d}(\gamma(u_t)+\alpha_t(u_t))}{\mathrm{d}u_t} \geq 0$, since part (a) of Lemma 2 implies that $\frac{\mathrm{d}(\gamma(u_t)+\alpha_t(u_t))}{\mathrm{d}u_t} \leq 0$. Next, we establish the third part of this proposition. By Proportion 4, we have $G_t(\theta_t, u_t) - G_t(0, u_t) - \big(\gamma(u_t) +$

$\alpha_t(u_t) - b\big)\theta_t \leq 0$ for any $0 \leq u_t < u^*_{t+1}$. Taking $\theta_t = s_t(u_t)$, we have $G_t(s_t(u_t), u_t) - G_t(0, u_t) = k \leq \big(\gamma(u_t) + \alpha_t(u_t) - b\big)s_t(u_t)$. This leads us to conclude that $s_t(u_t) \geq k/(\gamma(u_t) + \alpha_t(u_t) - b)$. $\square$