11.7 The QR Algorithm for Real Hessenberg Matrices

To complete the strategy for real, nonsymmetric matrices that was laid out in §11.6, we need to compute the eigenvalues and eigenvectors of a real Hessenberg matrix. Recall the following relations for the QR algorithm with shifts:

$$\mathbf{Q}_s \cdot (\mathbf{A}_s - k_s \mathbf{1}) = \mathbf{R}_s \tag{11.7.1}$$

where $\mathbf{Q}$ is orthogonal and $\mathbf{R}$ is upper triangular, and

$$\begin{aligned} \mathbf{A}_{s+1} &= \mathbf{R}_s \cdot \mathbf{Q}_s^T + k_s \mathbf{1} \\ &= \mathbf{Q}_s \cdot \mathbf{A}_s \cdot \mathbf{Q}_s^T \end{aligned} \tag{11.7.2}$$

The QR transformation preserves the upper Hessenberg form of the original matrix $\mathbf{A} \equiv \mathbf{A}_1$, and the workload on such a matrix is $O(n^2)$ per iteration as opposed to $O(n^3)$ on a general matrix. As $s \to \infty$, $\mathbf{A}_s$ converges to a form where the eigenvalues are either isolated on the diagonal or are eigenvalues of a 2×2 submatrix on the diagonal.

As we pointed out in §11.4, shifting is essential for rapid convergence. A key difference here is that a nonsymmetric real matrix can have complex eigenvalues. This means that good choices for the shifts k_s may be complex, apparently necessitating complex arithmetic.

Complex arithmetic can be avoided, however, by a clever trick. This trick, plus a detailed description of how the QR algorithm is used, is described in a Webnote [1].

The operation count for the QR algorithm for Hessenberg matrices is $\sim 10k^2$ per iteration, where k is the current size of the matrix. The typical average number of iterations per eigenvalue is about two, so the total operation count for all the eigenvalues is $\sim 10n^3$. The total operation count for both eigenvalues and eigenvectors is $\sim 25n^3$.

The routines `hqr` for the eigenvalues only, and `hqr2`, which computes both eigenvalues and eigenvectors, are given in full in a Webnote [2], along with a few `Unsymmeig` utility routines not already listed. The implementations are based algorithmically on the above description, in turn following the implementations in [3,4].

CITED REFERENCES AND FURTHER READING:

Numerical Recipes Software 2007, "Description of the QR Algorithm for Hessenberg Matrices," *Numerical Recipes Webnote No. 16*, at http://numerical.recipes/webnotes?16 [1]

Numerical Recipes Software 2007, "Implementations in Unsymmeig," *Numerical Recipes Webnote No. 17*, at http://numerical.recipes/webnotes?17 [2]

Wilkinson, J.H., and Reinsch, C. 1971, *Linear Algebra*, vol. II of *Handbook for Automatic Computation* (New York: Springer).[3]

Golub, G.H., and Van Loan, C.F. 1996, *Matrix Computations*, 3rd ed. (Baltimore: Johns Hopkins University Press), §7.5.

Smith, B.T., et al. 1976, *Matrix Eigensystem Routines — EISPACK Guide*, 2nd ed., vol. 6 of Lecture Notes in Computer Science (New York: Springer).[4]