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Filtering and Spectral Analysis

10. Task for Model Analysis I, 2023/24

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1 Introduction

Today we're taking a look at the filtering and spectral analysis of signals. Filtering is a process of removing unwanted parts of a signal, while spectral analysis is a process of decomposing a signal into its frequency components. Both of these processes are crucial in signal processing and would not be possible without the Fourier transform. The Fourier transform is a mathematical operation that transforms a function of time into a function of frequency. It is used to represent the signal as a sum of sinusoidal functions from which we can extract important frequency information. The equation for the Fourier transform and its inverse are given by:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \mathrm{d}t \,, \tag{1}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \mathrm{d}\omega \,. \tag{2}$$

The Fourier transform has various properties that we've discussed in other tasks. What will turn out to be significant in this task is the fact that the Fourier transform *imagines/expects* that the input signal is periodic. This is important because the Fourier transform of a signal that is not periodic will be subject to various effects of aliasing and leakage. These effects can be mitigated by windowing the signal before applying the Fourier transform. Windowing is a process of multiplying the signal by a window function that is zero outside of a certain interval. This effectively makes the signal periodic and reduces the effects of aliasing and leakage. Figure 1 shows some common window functions that are used in signal processing.

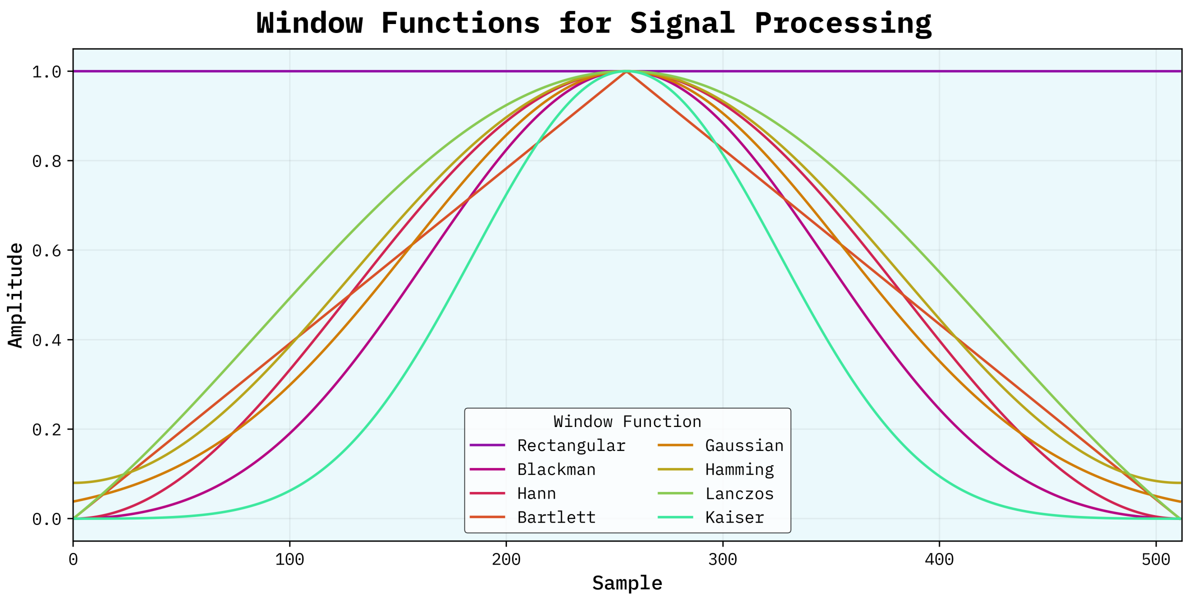

Figure 1: Common window functions used in signal processing.

An important operation in signal processing is the convolution. Convolution is a mathematical operation that combines two signals to produce a third signal. It is used to model the effect of one signal on another signal. The convolution of two signals $f(t)$ and $g(t)$ is given by:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau \, . \tag{3}$$

The convolution operation is commonly used in filtering. Filtering is a process of removing unwanted parts of a signal. Filters can be divided into roughly two categories: low-pass filters and high-pass filters. Low-pass filters allow (or pass) low-frequency signals and block high-frequency signals, while high-pass filters do the opposite. Filters can be implemented in the time domain or in the frequency domain. In the time domain, filters are implemented as convolution operations, while in the frequency domain, filters are implemented as multiplication operations. For todays task we'll take a look at **Wiener's (Optimal) Filter**. Wiener's filter is an optimal filter that minimizes the mean square error between the estimated random process (noise) and the desired process (signal). Imagine we have a signal $u(t)$ which we measure using a sensor with the transfer function $r(t)$. The signal with the addition of noise $n(t)$ is then given by:

$$c(t) = u(t) * r(t) + n(t) = s(t) + n(t) \, , \tag{4}$$

where $*$ denotes the convolution operation. From the measured quantity $c(t)$ we want to reconstruct the signal $u(t)$, given the fact that we have some information on the noise $n(t)$ and the sensors response $r(t)$. Following analogously to the Least Squares method, Wiener proposed a filter in which we have to multiply the Fourier transform of the measured signal $\hat{c}(\omega)$ with:

$$\Phi(\omega) = \frac{|\hat{s}(\omega)|^2}{|\hat{s}(\omega)|^2 + |\hat{n}(\omega)|^2} \, . \tag{5}$$

We can also perform the so-called Wiener deconvolution using the Wiener filter and a convolution kernel (which is the transfer function of the sensor). So in the case of image restoration we can use the Wiener filter to remove the noise from the image if we know the transfer function of the sensor, which could for example be the point spread function of the camera with leads to blurring of the image. There are many other methods for image restoration, but Wiener's filter is a good starting point and we'll limit ourselves to this method in this task.

2 Task

2.1 Spectra of Signals

In the first subtask, the instructions want us to calculate the spectra of signals, that were provided in `val2.dat` and `val3.dat`. We should try out different windowing functions to see how they affect the spectra. We can also try and see what happens if we only select a part of the signal and calculate the spectrum of that part.

2.2 Wiener Filtering

We have signals `signal{0,1,2,3}.dat` provided for the second task, each 512 samples long. Using Wiener's Filter we should try and remove the noise from the signals. `signal0.dat` represents the noiseless signal while the other signals have increasing levels of noise added to them. The transfer function of the sensor is given by:

$$r(t) = \frac{1}{2\tau} e^{-|t|/\tau} \, , \qquad \text{where} \quad \tau = 16 \, . \tag{6}$$

2.3 Wieners Deconvolution

For the last subtask we've received (cropped) images of Playboy model Lena Forsen (previously Soderberg). Her portrait called `Lenna` has become the standard test for various image processing algorithms and techniques. We've been given images of Lena that have been damaged by the addition of one of three convolution kernels and increasing levels of noise. The instructions want us to use Wiener's deconvolution to restore the images as best we can making sure to take care of artifacting due to a non-periodic signal by using either windowing or zero-padding. For the final challenge we we're also given images that have an additional periodic perturbation to them. We should try and remove the periodic perturbation from the images using some form of frequency domain filtering.

3 Solution Overview

4 Results

5 Conclusion and Comments