University *of Ljubljana*
Faculty *of Mathematics and Physics*

Department of Physics

Advanced Particle Detectors and Data Analysis

Notes for Exercises

Author: Marko Urbanč
Professor: prof. dr. Peter Križan
Assistant: doc. dr. Rok Dolenec

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1 Interactions of Particles with Photons

1.1 Bethe-Bloch Equation

The Bethe-Bloch equation describes the mean energy loss per distance traveled while traversing through matter. We generally use the Bethe-Bloch equation when we are dealing with **thick absorbers**, such as the ones in calorimeters. Do note that the Bethe-Bloch equation does not accurately describe the energy loss of **electrons** and **positrons** due to their small mass and the fact that they suffer from much larger energy losses due to bremsstrahlung and pair production. For a particle with charge z and velocity $\beta = v/c$, the Bethe-Bloch equation is given as:

$$-\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle = 2\pi N_a r_e^2 m_e c^2 \rho \frac{Z}{A} \frac{z^2}{\beta^2} \left[\ln\left(\frac{2m_e c^2 \beta^2 \gamma^2 W_{\max}}{I^2} \right) - 2\beta^2 - \delta - 2\frac{C}{Z} \right] , \tag{1.1}$$

where δ is the **density effect correction** and C is the **shell correction**. The rest is as follows:

$$N_a = 6.022 \times 10^{23}\,\mathrm{mol}^{-1} , \quad r_e = 2.818 \times 10^{-15}\,\mathrm{m} , \quad m_e = 9.11 \times 10^{-31}\,\mathrm{kg} , \quad c = 3 \times 10^8\,\mathrm{m/s} ,$$
$$\rho = \text{density of the material} , \quad A = \text{atomic mass of the material} , \quad Z = \text{atomic number of the material} ,$$
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} , \quad W_{\max} = \text{maximum energy transfer in a single collision} , \quad I = \text{mean excitation energy} .$$

The constant factor in the equation can be written as:

$$\boldsymbol{\Xi = 2\pi N_a r_e^2 m_e c^2 = 0.1535\,\mathrm{MeV\ cm^2 mol^{-1}}} , \tag{1.2}$$

where I've chosen to mark this constant factor as Ξ for easier reference in further calculations. We can find the mean excitation energy I from the following experimentally determined formula:

$$I = \begin{cases} Z(12 + \frac{7}{Z})\,\mathrm{eV} & \text{for } Z < 13 , \\ Z(9.76 + 58.8 Z^{-1.19})\,\mathrm{eV} & \text{for } Z \geq 13 . \end{cases} \tag{1.3}$$

The maximum energy transfer in a single collision W_{max} can be calculated as:

$$W_{\text{max}} = \frac{2m_e c^2 \beta^2 \gamma^2}{1 + 2\gamma \left({}^{m_e}\!/\!_{M}\right) + \left({}^{m_e}\!/\!_{M}\right)^2} \approx 2m_e c^2 \beta^2 \gamma^2 \,. \tag{1.4}$$

For our purposes we will ignore the density effect correction δ and the shell correction C.

1.1.1 Energy Loss of Charged Kaons

Let us calculate the energy losses for charged kaons K^+ and K^- with a rest mass of 0.493 GeV and momentum of 2.5 GeV in copper which has the following properties:

$$\begin{aligned} \rho &= 8.92\,\text{g/cm}^3 \,, \\ Z &= 29 \,, \\ A &= 63.5\,\text{g/mol} \,. \end{aligned}$$

First let us calculate the velocity β and the Lorentz factor γ. We know that

$$\beta = \frac{pc}{E} = \frac{pc}{\sqrt{(pc)^2 + (Mc^2)^2}} \,, \tag{1.5}$$

where M is the mass of the particle. Thus:

$$\beta = \frac{2.5\,\dfrac{\text{GeV}}{c}\,c}{\sqrt{\left(2.5\,\dfrac{\text{GeV}}{c}\,c\right)^2 + \left(0.493\,\dfrac{\text{GeV}}{c^2}\,c^2\right)^2}} \approx 0.981031 \,. \tag{1.6}$$

Remember to take at least 4 significant digits for the velocity β! This is due to the logarithm in the Bethe-Bloch equation. The Lorentz factor γ is then:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 5.159 \,. \tag{1.7}$$

Next let us calculate the maximum energy transfer in a single collision W_{max}:

$$W_{\text{max}} = 2m_e c^2 \beta^2 \gamma^2 = 2 \cdot 0.511\,\frac{\text{MeV}}{c^2}\,c^2 \cdot (0.981031)^2 (5.159)^2 = 26.7\,\text{MeV} \,. \tag{1.8}$$

Last prerequisite is the mean excitation energy I which we can calculate using the formula (1.3) for $Z \geq 13$:

$$I = 29(9.76 + 58.8 \cdot 29^{-1.19})\,\text{eV} = 313.9\,\text{eV} \,. \tag{1.9}$$

Now all that is left is to plug in the values into the Bethe-Bloch equation (1.1):

$$\begin{aligned} -\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle &= \Xi\,\rho \frac{Z}{A} \frac{z^2}{\beta^2} \left[\ln\left(\frac{W_{\text{max}}^2}{I^2}\right) - 2\beta^2 \right] \\ &= 0.1535\,\frac{\text{MeV}\,\text{cm}^2}{\text{mol}} \cdot 8.92\,\frac{\text{g}}{\text{cm}^3} \cdot \frac{29}{63.5}\,\frac{\text{mol}}{\text{g}} \cdot \frac{1}{0.981031^2} \cdot \left[\ln\left(\frac{(26.7 \cdot 10^6\,\text{eV})^2}{(313.9\,\text{eV})^2}\right) - 2 \cdot (0.981031)^2 \right] \\ &= 13.47\,\frac{\text{MeV}}{\text{cm}} \,. \end{aligned} \tag{1.10}$$

Thus the energy loss of charged kaons K^+ and K^- with a momentum of 2.5 GeV in copper is 13.47 MeV/cm.

1.1.2 What is the Energy Resolution of the Detector from the Previous Example?

Let's calculate the energy resolution of the detector from the previous example, assuming that the length of the particle track through the detector is $d = 5\,\text{cm}$ and that energy is measured based on all deposited energy without any additional losses. Using the result from the previous example (1.10), we can calculate the average energy deposited in the detector as:

$$\Delta E = \bar{\Delta} = -\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle \cdot d = 13.47\,\frac{\text{MeV}}{\text{cm}} \cdot 5\,\text{cm} = 67.35\,\text{MeV} \,. \tag{1.11}$$

This is an approximation since we are assuming that β is constant throughout the detector, which is not true. In reality we'd have to integrate the energy loss over the path of the particle, however at $p \sim$ GeV additional losses of $\sim$ MeV are negligible. Measurements of energy are dependant on the energy resolution R which is defined as:

$$R = \frac{\sigma_E}{\bar{\Delta}} \,, \tag{1.12}$$

where σ_E is the standard deviation of the energy measurement which we assume to have a Gaussian distribution like such:

$$p(\Delta) = \frac{1}{\sqrt{2\pi}\sigma_E} \exp\left(-\frac{\Delta - \bar{\Delta}^2}{2\sigma_E^2}\right) . \tag{1.13}$$

σ_E is determined empirically. For **non-relativistic** particles it can be calculated as the variance of the Bethe-Bloch equation as:

$$\sigma_0^2 = 4\pi N_a r_e^2 (m_e c^2)^2 \rho \frac{Z}{A} \Delta x \,. \tag{1.14}$$

For **relativistic** particles we can correct the variance from (1.14) as such:

$$\sigma_E^2 = \sigma_0^2 \frac{1 - \frac{1}{2}\beta^2}{1 - \beta^2} \,. \tag{1.15}$$

In our case this gives us:

$$\begin{aligned} \sigma_E^2 &= 2 \cdot 0.511 \, \frac{\text{MeV}}{c^2} \, c^2 \cdot 0.1535 \, \frac{\text{MeVcm}^2}{\text{mol}} \cdot 8.92 \, \frac{\text{g}}{\text{cm}^3} \cdot \frac{29}{63.5} \, \frac{\text{mol}}{\text{g}} \cdot 5 \, \text{cm} \cdot \frac{1 - \frac{1}{2}(0.981031)^2}{1 - (0.981031)^2} \\ &= 44.12 \, \text{MeV}^2 \,. \end{aligned} \tag{1.16}$$

Thus the energy resolution of the detector is:

$$R = \frac{\sqrt{44.12 \, \text{MeV}^2}}{67.35 \, \text{MeV}} = 9.9\% \,. \tag{1.17}$$

1.1.3 What if the Detector is Made of a Molecule?

Let's assume now that our detector is made of lead(II) fluoride PbF_2 in a cubic crystal form which has the following properties:

$$\begin{aligned} \rho &= 7.77 \, \text{g/cm}^3 \,, & Z_{\text{Pb}} &= 82 \,, \\ Z &= 100 \,, & Z_{\text{F}} &= 9 \,, \\ A &= 245.2 \, \text{g/mol} \,. & \rho_{\text{Pb}} &= 11.34 \, \text{g/cm}^3 \,, \\ A_{\text{Pb}} &= 207.2 \, \text{u} \,, & \rho_{\text{F}} &= 0.001696 \, \text{g/cm}^3 \,. \\ A_{\text{F}} &= 19 \, \text{u} \,, \end{aligned}$$

We are interested in the energy loss of protons with a momentum of 3 GeV in such a detector. The difference between calculating the energy loss in a compound material is that we have to calculate the energy loss for each element in the compound. This sum is weighted by the fraction of the element in the compound. As such:

$$\frac{1}{\rho} \left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_{\text{compound}} = \frac{w_1}{\rho_1} \left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_1 + \frac{w_2}{\rho_2} \left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_2 + \dots \,, \tag{1.18}$$

where we calculate w_i as:

$$w_i = \frac{a_i \cdot A_i}{\sum a_i \cdot A_i} \,, \tag{1.19}$$

here a_i is the number of atoms of the element in the compound and A_i is the atomic mass of the element. Our professor stated that such problems will not be present on the exam and that we should not worry about them. However it is still good to know how to calculate the energy loss in a compound. In our case we can expect to get effective values if the detector is made of a compound. If we calculate the weights for lead and fluorine in lead(II) fluoride we get:

$$\begin{aligned} w_{\text{Pb}} &= \frac{1 \cdot 207.2 \, \text{u}}{1 \cdot 207.2 \, \text{u} + 2 \cdot 19 \, \text{u}} = 0.845 \,, \\ w_{\text{F}} &= \frac{2 \cdot 19 \, \text{u}}{1 \cdot 207.2 \, \text{u} + 2 \cdot 19 \, \text{u}} = 0.154 \,, \end{aligned}$$

where we used the atomic masses of lead and fluoride in atomic mass units. Next we need to calculate the velocity β and the Lorentz factor γ for protons. So using (1.5) we get:

$$\beta = \frac{3\,\frac{\mathrm{GeV}}{c}\,c}{\sqrt{\left(3\,\frac{\mathrm{GeV}}{c}\,c\right)^2 + \left(0.938\,\frac{\mathrm{GeV}}{c^2}\,c^2\right)^2}} \approx 0.95443\,, \tag{1.20}$$

which gives us a Lorentz factor of:

$$\gamma = \frac{1}{\sqrt{1-(0.95443)^2}} \approx 3.351\,. \tag{1.21}$$

Next we need to calculate the maximum energy transfer in a single collision W_{max} using (1.4) as:

$$W_{\mathrm{max}} = 2\cdot 0.511\,\frac{\mathrm{MeV}}{c^2}\,c^2\cdot(0.95443)^2(3.351)^2 = 10.5\,\mathrm{MeV}\,, \tag{1.22}$$

and the mean excitation energy I using (1.3) for each component:

$$\begin{aligned} I_{\mathrm{Pb}} &= 82\left(9.76 + 58.8\cdot 82^{-1.19}\right)\mathrm{eV} = 825.8\,\mathrm{eV}\,,\\ I_{\mathrm{Cu}} &= 9\left(12+\frac{7}{9}\right)\mathrm{eV} = 115\,\mathrm{eV}\,. \end{aligned}$$

Now we can calculate the energy loss for each component using the Bethe-Bloch equation (1.1) and sum them up:

$$\begin{aligned} -\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{Pb}} &= \Xi\,\rho_{\mathrm{Pb}}\frac{Z_{\mathrm{Pb}}}{A_{\mathrm{Pb}}}\frac{1}{0.95443^2}\left[\ln\left(\frac{(10.5\cdot10^6\,\mathrm{eV})^2}{(825.8\,\mathrm{eV})^2}\right) - 2\cdot(0.95443)^2\right]\\ &= \Xi\,\rho_{\mathrm{Pb}}\frac{Z_{\mathrm{Pb}}}{A_{\mathrm{Pb}}}\cdot 18.7488\\ &= 0.1535\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{mol}}\cdot 11.34\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot\frac{82}{207.2}\,\frac{\mathrm{mol}}{\mathrm{g}}\cdot 18.7488\\ &= 12.9\,\frac{\mathrm{MeV}}{\mathrm{cm}}\,, \end{aligned} \tag{1.23}$$

$$\begin{aligned} -\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{F}} &= \Xi\,\rho_{\mathrm{F}}\frac{Z_{\mathrm{F}}}{A_{\mathrm{F}}}\frac{1}{0.95443^2}\left[\ln\left(\frac{(10.5\cdot10^6\,\mathrm{eV})^2}{(115\,\mathrm{eV})^2}\right) - 2\cdot(0.95443)^2\right]\\ &= \Xi\,\rho_{\mathrm{F}}\frac{Z_{\mathrm{F}}}{A_{\mathrm{F}}}\cdot 23.0773\\ &= 0.1535\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{mol}}\cdot 0.001696\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot\frac{9}{19}\,\frac{\mathrm{mol}}{\mathrm{g}}\cdot 23.0773\\ &= 0.002846\,\frac{\mathrm{MeV}}{\mathrm{cm}}\,. \end{aligned} \tag{1.24}$$

Now all that is left is to compute the weighted sum as stated in (1.18):

$$\begin{aligned} -\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{compound}} &= -\frac{\rho\cdot w_{\mathrm{Pb}}}{\rho_{\mathrm{Pb}}}\cdot\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{Pb}} - \frac{\rho\cdot w_{\mathrm{F}}}{\rho_{\mathrm{F}}}\cdot\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{F}}\\ &= \frac{7.77\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot 0.845}{11.34\,\frac{\mathrm{g}}{\mathrm{cm^3}}}\cdot 12.9\,\frac{\mathrm{MeV}}{\mathrm{cm}} + \frac{7.77\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot 0.154}{0.001696\,\frac{\mathrm{g}}{\mathrm{cm^3}}}\cdot 0.002846\,\frac{\mathrm{MeV}}{\mathrm{cm}}\\ &= 9.47\,\frac{\mathrm{MeV}}{\mathrm{cm}}\,. \end{aligned} \tag{1.25}$$

1.2 Landau Distribution

For detectors of moderate thickness, which we can consider as **thin absorbers**, we can use a highly-skewed Landau-Vavilov distribution to describe the energy loss of particles. The most probable energy loss Δ_p is given as:

$$\bar{\Delta} = \Delta_p = \xi\left[\ln\frac{2m_ec^2\beta^2\gamma^2}{I} + \ln\frac{\xi}{I} + j - \beta^2 - \delta(\beta\gamma)\right]\,, \tag{1.26}$$

where $\delta(\beta\gamma)$ represents corrections due to the density effect, $j = 0.200$ and ξ is given as:

$$\xi = \frac{K}{2} \left\langle \frac{Z}{A} \right\rangle \frac{x}{\beta^2} \,\mathrm{MeV}\,, \tag{1.27}$$

for x in $\mathrm{g/cm^2}$ and $K = 0.3\,\mathrm{MeV\,cm^2/g}$. **Warning:** x here is normalized with density $\mathrm{g/cm^2}$. This means that $x = \rho \cdot d$ where d is the thickness of the detector. To know which distribution to use, we can use the following rule of thumb:

$$\kappa = \frac{\bar{\Delta}}{W_{\mathrm{max}}} \begin{cases} > 10 & \text{use Bethe-Bloch}\,, \\ < 0.01 & \text{use Landau}\,. \end{cases} \tag{1.28}$$

To determine the energy resolution of such a detector we can use the following formula:

$$R_{\mathrm{FWHM}} = \frac{4\xi}{\Delta_p}\,. \tag{1.29}$$

1.2.1 What is the Most Probable Energy Loss of a Charged Pion in Silicon?

Let's calculate the most probable energy loss of a charged pion with a rest mass of 139.57 MeV and momentum of 0.5 GeV in a silicon based detector which has a 320 μm thick silicon layer. Silicon has the following properties:

$$\begin{aligned} \rho &= 2.32\,\mathrm{g/cm^3}\,, \\ Z &= 14\,, \\ A &= 28\,\mathrm{g/mol}\,. \end{aligned}$$

As before we first calculate the velocity β and the Lorentz factor γ for pions. Using (1.5) we get:

$$\beta = \frac{0.5\,\frac{\mathrm{GeV}}{c}\,c}{\sqrt{\left(0.5\,\frac{\mathrm{GeV}}{c}\,c\right)^2 + \left(0.13957\,\frac{\mathrm{GeV}}{c^2}\,c^2\right)^2}} \approx 0.96318\,, \tag{1.30}$$

which gives us a Lorentz factor of:

$$\gamma = \frac{1}{\sqrt{1-(0.96318)^2}} \approx 3.72\,. \tag{1.31}$$

Likewise as before we want to calculate the mean excitation energy I using (1.3) for $Z \geq 13$:

$$I = 14\left(9.76 + 58.8 \cdot 14^{-1.19}\right)\,\mathrm{eV} = 172.3\,\mathrm{eV}\,. \tag{1.32}$$

We can also calculate our approximation for the maximum energy transfer in a single collision W_{max} using (1.4), since we can spot it in the Landau distribution (1.26):

$$W_{\mathrm{max}} = 2 \cdot 0.511\,\frac{\mathrm{MeV}}{c^2}\,c^2 \cdot (0.96318)^2 (3.72)^2 = 13.12\,\mathrm{MeV}\,. \tag{1.33}$$

Next we calculate ξ using (1.27):

$$\xi = \frac{0.3\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}}}{2}\,\frac{14}{28}\,\frac{320 \cdot 10^{-4}\,\mathrm{cm} \cdot 2.32\,\frac{\mathrm{g}}{\mathrm{cm^3}}}{(0.96318)^2} = 0.0060\,\mathrm{MeV}\,. \tag{1.34}$$

Now we can calculate the most probable energy loss using the Landau distribution (1.26):

$$\begin{aligned} \Delta_p &= 0.006\,\mathrm{MeV}\left[\ln\frac{13.12 \cdot 10^6\,\mathrm{eV}}{172.3\,\mathrm{eV}} + \ln\frac{0.006 \cdot 10^6\,\mathrm{eV}}{172.3\,\mathrm{eV}} + 0.2 - (0.96318)^2\right] \\ &= 0.0844\,\mathrm{MeV}\,. \end{aligned} \tag{1.35}$$

From this we can now also calculate the energy resolution of the detector using the formula (1.29):

$$R_{\mathrm{FWHM}} = \frac{4 \cdot 0.006\,\mathrm{MeV}}{0.0844\,\mathrm{MeV}} = 28.5\,\%\,. \tag{1.36}$$

If we're paranoid if we've used the right distribution, we can calculate κ as:

$$\kappa = \frac{0.0844\,\text{MeV}}{13.12\,\text{MeV}} = 0.0064\,, \tag{1.37}$$

which is less than 0.01 so we've used the right distribution. Alternatively if we magically procure the result from the Bethe-Bloch equation, we'd get $\bar{\Delta} = 126\,\text{keV}$ and $\sigma = 402\,\text{keV}$ which would give us $\kappa = 0.0105$ which still hints that we should use the Landau distribution.

1.3 Cherenkov Radiation

Charged particles moving through a medium with a velocity greater than the speed of light in that medium emit Cherenkov radiation. The angle of the emitted radiation is given by the Cherenkov angle which is defined as:

$$\cos\theta = \frac{1}{\beta n}\,, \tag{1.38}$$

where n is the refractive index of the medium. The threshold velocity for Cherenkov radiation is given as:

$$\beta_{\text{thr}} = \frac{1}{n}\,. \tag{1.39}$$

We can calculate the number of produced Cherenkov photons per unit length x with the following formula:

$$\frac{\mathrm{d}^2 N}{\mathrm{d}E\,\mathrm{d}x} = \frac{\alpha z^2}{\hbar c}\sin^2\theta\,, \tag{1.40}$$

which we can approximate for $z = 1$ as:

$$\frac{\mathrm{d}^2 N}{\mathrm{d}E\,\mathrm{d}x} = \frac{370}{\text{eV cm}}\sin^2\theta\,. \tag{1.41}$$

if we assume that $\beta \approx$ const. and that the refractive index is not a function of the wavelength $n \neq n(\lambda)$. Thus if we'd like to calculate the total number of Cherenkov photons produced in our detector we can use the following formula:

$$N_{\text{Cherenkov}} = \frac{370}{\text{eV cm}}\,\Delta x\,\Delta E\left(1 - \frac{1}{\beta^2 n^2}\right)\,, \tag{1.42}$$

where Δx is the thickness of the detector and ΔE is the energy of an emitted Cherenkov photon which is simply calculated as:

$$\Delta E = h\nu = \frac{hc}{\lambda} = \frac{1240\,\text{eV nm}}{\lambda}\,, \tag{1.43}$$

where λ is the wavelength of the emitted Cherenkov photon.

1.3.1 How many Cherenkov Photons are Produced in Water by a Proton?

Let's calculate the number of Cherenkov photons produced in 1 cm of water by a proton with a rest mass of 0.938 GeV and a momentum of 2 GeV. The refractive index of water is $n = 1.33$. How many photons are detected with a photodetector which is sensitive to light between 250 nm and 800 nm with an average efficiency of 10%?

Using the formula (1.5) we can calculate the velocity β for the proton:

$$\beta = \frac{2\,\frac{\text{GeV}}{c}\,c}{\sqrt{\left(2\,\frac{\text{GeV}}{c}\,c\right)^2 + \left(0.938\,\frac{\text{GeV}}{c^2}\,c^2\right)^2}} \approx 0.90537\,. \tag{1.44}$$

The Cherenkov angle θ can be calculated using the formula (1.38):

$$\theta = \arccos\frac{1}{0.90537\cdot 1.33} = 33.85^\circ\,. \tag{1.45}$$

From this we can calculate the number of Cherenkov photons produced per unit length using the formula (1.41):

$$\frac{\mathrm{d}^2 N}{\mathrm{d}E\,\mathrm{d}x} = \frac{370}{\text{eV cm}}\sin^2(33.85^\circ) = 114.8\,\text{eV}^{-1}\text{cm}^{-1}\,. \tag{1.46}$$

Now we can calculate the total number of Cherenkov photons produced in 1 cm of water using the formula (1.42):

$$N_{\text{min}} = \frac{114.8}{\text{eV cm}} \cdot 1\ \text{cm} \cdot \frac{1240\,\text{eV nm}}{250\ \text{nm}} = 569.4\,, \tag{1.47}$$

$$N_{\text{max}} = \frac{114.8}{\text{eV cm}} \cdot 1\ \text{cm} \cdot \frac{1240\,\text{eV nm}}{800\ \text{nm}} = 177.9\,. \tag{1.48}$$

Using these two values we can very roughly estimate the number of detected Cherenkov photons in the range 250 nm to 800 nm. Essentially we use a linear approximation of the integral of the number of Cherenkov photons produced per unit length over the range of wavelengths and multiply it by the efficiency of the photodetector. Thus the number of detected Cherenkov photons is:

$$N_{\text{det}} = 0.1 \cdot (N_{\text{max}} - N_{\text{min}}) \approx 39\,. \tag{1.49}$$

1.4 Neutrinos and Dark Matter

Neutrinos are elementary particles that interact only very weakly with matter. They are produced in nuclear reactions and in the decay of particles. Neutrinos are classified into three types: electron neutrinos ν_e, muon neutrinos ν_μ and tau neutrinos ν_τ. There are three possible interactions we can detect:

$$\begin{aligned} &\nu_x + e^- \to \nu_x + e^- && \text{elastic scattering}\,, \\ &\nu_e + n \to p + e^- && \text{charged current interaction}\,, \\ &\overline{\nu}_e + p \to n + e^+ && \text{inverse beta decay}\,. \end{aligned}$$

All of these interactions produce electrons/positrons which emit Cherenkov radiation. For an incoming flux of neutrinos F we can calculate the interaction cross-section as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = F\,\frac{\mathrm{d}N}{\mathrm{d}\Omega}\,, \tag{1.50}$$

where $\frac{\mathrm{d}N}{\mathrm{d}\Omega}$ is the number of Cherenkov photons produced per unit angle. We often measure cross-sections in barns where $1\,\text{b} = 10^{-28}\,\text{m}^2$. For neutrinos that only interact via the weak force the interaction cross-section is about:

$$\boldsymbol{\sigma_\nu = 10^{-20}\,\textbf{b} = 10^{-44}\,\textbf{cm}^2 = 10^{-48}\,\textbf{m}^2}\,. \tag{1.51}$$

The number of interactions can be calculated as:

$$N = t\phi\sigma_\nu N_{\text{sc}}\,, \tag{1.52}$$

where t is the time of exposure, ϕ is the flux of neutrinos, σ_ν is the interaction cross-section and N_{sc} is the number of target particles (scattering centers). Both σ_ν and N_{sc} are in general dependant on the type of interaction. We can calculate the number of number of scattering centers as:

$$N_{\text{sc}} = \frac{mN_a}{A}\,, \tag{1.53}$$

if we assume that the number of scattering centers is the same as the number of nucleons in the target material.

1.4.1 What mass of pure atomic hydrogen is needed to detect 1000 solar neutrinos per year?

Lets assume a neutrino flux of $\phi = 6 \cdot 10^{14}\,\text{m}^{-2}\text{s}^{-1}$, an interaction cross-section of $\sigma_\nu = 10^{-48}\,\text{m}^2$ and that the number of scattering centers is the same as the number of nucleons in the hydrogen molecule. Thus the number of detected neutrinos per year is given as:

$$N = t\phi\sigma\,\frac{mN_a}{A}\,. \tag{1.54}$$

If we want to detect 1000 neutrinos per year we can calculate the mass of hydrogen as:

$$\begin{aligned} m &= \frac{A}{N_a t\phi\sigma} \cdot 1000 \\ &= \frac{1\,\frac{\text{g}}{\text{mol}}}{\left[6.02 \cdot 10^{23}\,\frac{1}{\text{mol}}\right](3.15 \cdot 10^{7}\ \text{s})\left[6 \cdot 10^{14}\,\frac{1}{\text{m}^2\text{s}}\right](10^{-48}\ \text{m}^2)} \cdot 1000 \\ &= 87.9\ \text{kg}\,. \end{aligned} \tag{1.55}$$

1.4.2 Detection of WIMP's in a germanium detector

WIMP's (Weakly Interacting Massive Particles) are hypothetical particles that are thought to make up dark matter. Let's say that we are trying to detect them through elastic scattering with germanium nuclei. We take the mass of a WIMP to be 100 GeV and presume that they are stationary in intergalactic space. Our solar system is moving through intergalactic space at a velocity of $2.2 \cdot 10^5$ m/s. We first need to estimate the maximum energy transfer in one collision with a germanium nucleus from which we can then calculate the needed mass of germanium in the detector to get one event per year. The estimated cross-section for WIMP-nucleus scattering is $\sigma_{\text{WIMP}} = 10^{-45}\,\text{cm}^2$. The estimated flux of WIMP's is $\phi = 10^5\,\text{cm}^{-2}\text{s}^{-1}$.

We can directly use equation (1.52) to calculate the mass of germanium needed in the detector:

$$\begin{aligned} m &= \frac{NA}{N_a t \phi \sigma} \\ &= \frac{1 \cdot 72.6\,\frac{\text{g}}{\text{mol}}}{\left[6.02 \cdot 10^{23}\,\frac{1}{\text{mol}}\right](3.15 \cdot 10^7\,\text{s})\left[10^5\,\frac{1}{\text{cm}^2\text{s}}\right](10^{-45}\,\text{cm}^2)} \\ &= 3.826 \cdot 10^7\,\text{t} = 38\,\text{kt}\,. \end{aligned} \tag{1.56}$$

where we used the atomic mass of germanium $A = 72.6\,\text{g/mol}$ and $N = 1$. To calculate the maximum energy transfer in one collision we'd need to calculate the kinematics of the problem along with taking into account conservation laws. This would be best done in the center of mass frame and then transformed back to the lab frame. This is a bit too much for this exercise so we can try to estimate the maximum energy transfer semi-classically. The kinetic energy the germanium nucleon receives is:

$$W_2 = \frac{4mM}{(m+M)^2}\,W_1\,, \tag{1.57}$$

where M is the mass of the WIMP, m is the mass of the germanium nucleus and W_1 is the kinetic energy of the WIMP before the collision. The germanium nucleus is made of 72 nucleons and the mass of a nucleon is $\sim 1\,\text{GeV}$. Thus the mass of the germanium nucleus is $m = 72\,\text{GeV}$. We can calculate β for the WIMP as:

$$\beta = \frac{v}{c} = \frac{2.2 \cdot 10^5\,\frac{\text{m}}{\text{s}}}{3 \cdot 10^8\,\frac{\text{m}}{\text{s}}} = 0.733 \cdot 10^{-3}\,. \tag{1.58}$$

From this we can calculate the initial kinetic energy of the WIMP as:

$$W_1 = \frac{1}{2}Mv^2 = \frac{1}{2}Mc^2\beta^2 = 0.5 \cdot 100 \cdot 10^9\,\frac{\text{eV}}{c^2}\,c^2 \cdot (0.733 \cdot 10^{-3})^2 = 27.0\,\text{keV}\,. \tag{1.59}$$

Thus the maximum energy transfer in one collision is:

$$\begin{aligned} W_2 &= \frac{4 \cdot 72 \cdot 100}{(72+100)^2}\,27.0\,\text{keV} \\ &= 0.9735 \cdot 27.0\,\text{keV} \\ &= 26.3\,\text{keV}\,. \end{aligned} \tag{1.60}$$

2 Detectors

2.1 Semiconductor Detectors

Semiconductor detectors are most commonly placed close to the interaction point of a collider or experiment since their purpose is to measure the positions of various output particles without disturbing them. The most common semiconductor detectors are silicon and germanium detectors. They are made of a p-n junction which is reverse biased. When a charged particle passes through the detector it creates electron-hole pairs which are then separated by the electric field of the reverse biased p-n junction. The electrons and holes are then collected at the electrodes of the detector. The average number of detected electron-hole pairs is given as:

$$\langle N \rangle = \frac{\Delta E}{w}\,, \tag{2.1}$$

where ΔE is the energy deposited in the detector and w is the energy needed to create an electron-hole pair in the semiconductor. Some common values for w are:

$$
\begin{aligned}
w_{\mathrm{Si}}(300\,\mathrm{K}) &= 3.6\,\mathrm{eV}\,, \\
w_{\mathrm{Si}}(77\,\mathrm{K}) &= 3.7\,\mathrm{eV}\,, \\
w_{\mathrm{Ge}}(77\,\mathrm{K}) &= 2.9\,\mathrm{eV}\,,
\end{aligned}
$$

The energy resolution of a semiconductor detector is given as:

$$
R = \frac{\sigma_N}{\langle N \rangle} = \frac{\sqrt{F \cdot \langle N \rangle}}{\langle N \rangle}\,, \tag{2.2}
$$

where F is the Fano factor which is a measure of the fluctuations in the number of detected electron-hole pairs. The Fano factor is usually around:

$$
F = \begin{cases} 0.086 \;-\; 0.16 & \text{for silicon}\,, \\ 0.06 \;-\; 0.13 & \text{for germanium}\,. \end{cases} \tag{2.3}
$$

Using the equation (2.1) we can calculate the energy resolution directly as:

$$
R = \sqrt{\frac{F \cdot w}{\Delta E}} = \frac{R_{\mathrm{FWHM}}}{2.35}\,, \tag{2.4}
$$

where R_{FWHM} is the full width at half maximum of the energy resolution.

2.1.1 What is the energy resolution of a silicon detector at 300 K?

Consider a detector with a 1 mm thick silicon layer at 300 K. In the case of a perpendicularly crossing kaon with a rest mass of 0.493 GeV and momentum of 4 GeV what is the energy resolution? Silicon has the following properties:

$$
\begin{aligned}
&w_{\mathrm{Si}}(300\,\mathrm{K}) = 3.6\,\mathrm{eV}\,, \\
&\rho = 2.32\,\mathrm{g/cm}^3\,, \\
&Z = 14\,, \\
&A = 28\,\mathrm{g/mol}\,.
\end{aligned}
$$

We can calculate the energy loss of the kaon in the silicon detector using the Landau distribution (1.26). Before that we need to calculate the velocity β and the Lorentz factor γ for the kaon. Using (1.5) we get:

$$
\beta = \frac{4\,\frac{\mathrm{GeV}}{c}\,c}{\sqrt{\left(4\,\frac{\mathrm{GeV}}{c}\,c\right)^2 + \left(0.493\,\frac{\mathrm{GeV}}{c^2}\,c^2\right)^2}} \approx 0.99249\,, \tag{2.5}
$$

which gives us a Lorentz factor of:

$$
\gamma = \frac{1}{\sqrt{1 - (0.99249)^2}} \approx 8.175\,. \tag{2.6}
$$

Next we need to calculate the mean excitation energy I using (1.3) for $Z \geq 13$:

$$
I = 14\left(9.76 + 58.8 \cdot 14^{-1.19}\right)\,\mathrm{eV} = 172.1\,\mathrm{eV}\,, \tag{2.7}
$$

and the maximum energy transfer in a single collision $W_{\max}$ using (1.4):

$$
W_{\max} = 2 \cdot 0.511\,\frac{\mathrm{MeV}}{c^2}\,c^2 \cdot (0.99249)^2(8.175)^2 = 67.3\,\mathrm{MeV}\,. \tag{2.8}
$$

We also need to calculate ξ using (1.27):

$$
\begin{aligned}
\xi &= \frac{0.3\,\frac{\mathrm{MeV\,cm}^2}{\mathrm{g}}}{2}\,\frac{14}{28}\,\frac{0.1\,\mathrm{cm} \cdot 2.32\,\frac{\mathrm{g}}{\mathrm{cm}^3}}{(0.99249)^2} \\
&= 0.0178\,\mathrm{MeV}\,.
\end{aligned} \tag{2.9}
$$

Now all that is left to get the most probable energy loss is to use the Landau distribution (1.26):

$$\begin{aligned}\Delta_p &= 0.0178\,\text{MeV}\left[\ln\frac{67.3\cdot 10^6\,\text{eV}}{172.1\,\text{eV}} + \ln\frac{0.0178\cdot 10^6\,\text{eV}}{172.1\,\text{eV}} + 0.2 - (0.99249)^2\right]\\ &= 0.298\,\text{MeV}\,. \end{aligned} \tag{2.10}$$

From this we can calculate the energy resolution of the detector using the formula (2.4):

$$\begin{aligned} R &= \sqrt{\frac{0.086\cdot 3.6\,\text{eV}}{0.298\cdot 10^6\,\text{eV}}}\\ &= 0.11\%\,, \end{aligned} \tag{2.11}$$

$$R_{\text{FWHM}} = 2.35\cdot 0.11\% = 0.26\%\,. \tag{2.12}$$

For the purpose of education if we were to repeat the entire calculation while taking values for germanium we'd get:

$$\begin{aligned} I &= 342\,\text{eV}\,,\\ \xi &= 35.77\,\text{keV}\,,\\ \Delta_p &= 573\,\text{keV}\,, \end{aligned}$$

which would give us an energy resolution of $R_{\text{Ge}} = 0.072\%$ and $R_{\text{FWHM}_{\text{Ge}}} = 0.17\%$.

2.1.2 What voltage is needed to get a 1 mm thick depletion region in the detector from the previous exercise?

Let's say that the impurity concentration is $N_D = 6\cdot 10^{14}\,\text{cm}^{-3}$ and the dielectric constant of silicon is $\varepsilon = 11.7$. The equation for the depletion region width is:

$$d = \sqrt{\frac{2\varepsilon\varepsilon_0 U}{e_0 N_D}}\,. \tag{2.13}$$

We can use this equation to find the voltage U needed to get a 1 mm thick depletion region:

$$\begin{aligned} U &= \frac{d e_0 N_D}{2\varepsilon\varepsilon_0}\\ &= \frac{(0.1\,\text{cm})^2\left[1.6\cdot 10^{-19}\,\text{As}\right]\left(6\cdot 10^{14}\frac{1}{\text{cm}^3}\right)}{2\cdot 11.7\left(8.85\cdot 10^{-12}\cdot 10^2\,\frac{\text{As}}{\text{V cm}}\right)}\\ &= 45.4\,\text{V}\,. \end{aligned} \tag{2.14}$$

2.2 Ionization Detectors

Ionization Chambers Gas ionization chambers are detectors that are filled with a gas and have an electric field applied to them. When a charged particle passes through the gas it ionizes the gas atoms and the electrons and ions are collected at the electrodes of the detector. Without multiplication of the signal the energy resolution is given as:

$$R = \sqrt{\frac{F\cdot w}{\Delta E}}\,, \tag{2.15}$$

where F is the Fano factor which is around $F = 0.2$ for gas ionization chambers and where we already took into account the number of created electron-ion pairs and that they are Poisson distributed:

$$N = \frac{\Delta E}{w}\,, \tag{2.16}$$

$$\sigma_N = \sqrt{F\cdot N}\,. \tag{2.17}$$

$$\tag{2.18}$$

The Fano factor is needed to correct the fact that subsequent electron-ion pairs are not entirely statistically independent. w is the energy needed to create an electron-ion pair in the gas.

Proportional Counters Proportional counters are gas ionization chambers with a multiplication factor. Since the created electron-ion pairs are hard to detect we use the Townsend avalanche effect to amplify the signal, which gives us a cascade of electron-ion pairs. For argon Ar it is $w_{\mathrm{Ar}} = 26\,\mathrm{eV}$. The amount of charge we collect is:

$$Q = N \cdot e \qquad \text{without multiplication}\,, \tag{2.19}$$
$$Q_{\mathrm{mult}} = N \cdot e \cdot M \qquad \text{with multiplication}\,, \tag{2.20}$$

where M is the multiplication factor. The energy resolution of a gas ionization chamber with multiplication is **lower** than without multiplication. The energy resolution with multiplication is given as:

$$R_{\mathrm{mult}} = \sqrt{\frac{w\,(F+b)}{\Delta E}}\,, \tag{2.21}$$

where b is a constant that depends on the detector, usually between 0.4 and 0.7.

2.2.1 What is the energy resolution of a gas ionization chamber?

What is the energy resolution of an ionization chamber with a thickness of $d = 10\,\mathrm{cm}$, for a MIP particle if the gas used is the so-called *magic gas* which has the properties:

$$\begin{aligned}
&Q_{\mathrm{Ar}} = 75\%\,,\\
&Q_{\mathrm{C_4H_{10}}} = 25\%\,,\\
&\rho_{\mathrm{Ar}} = 1.66\,\mathrm{g/L}\,,\\
&\rho_{\mathrm{C_4H_{10}}} = 2.5\,\mathrm{g/L}\,,\\
&w_{\mathrm{Ar}} = 26\,\mathrm{eV}\,,\\
&w_{\mathrm{C_4H_{10}}} = 23\,\mathrm{eV}\,,\\
&F_{\mathrm{Ar}} = 0.2\,,\\
&F_{\mathrm{C_4H_{10}}} \approx 0.2\,,
\end{aligned}$$

where C_4H_{10} is isobutane and Q represents percentage by volume in the gas mixture. Since we're dealing with a MIP (Minimum Ionizing Particle) we can assume that the energy deposited is:

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = 2\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}} \tag{2.22}$$

We need to calculate the deposited energy by the individual gas components. So the energy deposited in the argon is:

$$\begin{aligned}
\Delta E_{\mathrm{Ar}} &= Q_{\mathrm{Ar}} \cdot \rho_{\mathrm{Ar}} \cdot d \left(-\frac{\mathrm{d}E}{\mathrm{d}x}\right)\\
&= 0.75 \cdot 1.66 \cdot 10^{-3}\,\frac{\mathrm{g}}{\mathrm{cm^3}} \cdot 10\,\mathrm{cm} \cdot 2\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}}\\
&= 0.0249\,\mathrm{MeV}\,.
\end{aligned} \tag{2.23}$$

Likewise for isobutane we get:

$$\begin{aligned}
\Delta E_{\mathrm{C_4H_{10}}} &= Q_{\mathrm{C_4H_{10}}} \cdot \rho_{\mathrm{C_4H_{10}}} \cdot d \left(-\frac{\mathrm{d}E}{\mathrm{d}x}\right)\\
&= 0.25 \cdot 2.5 \cdot 10^{-3}\,\frac{\mathrm{g}}{\mathrm{cm^3}} \cdot 10\,\mathrm{cm} \cdot 2\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}}\\
&= 0.014\,\mathrm{MeV}\,.
\end{aligned} \tag{2.24}$$

Now we can calculate the number of created electron-ion pairs for argon and isobutane using the formula (2.17):

$$N_{\mathrm{Ar}} = \frac{0.0249 \cdot 10^6\,\mathrm{eV}}{26\,\mathrm{eV}} = 958\,, \tag{2.25}$$
$$N_{\mathrm{C_4H_{10}}} = \frac{0.014 \cdot 10^6\,\mathrm{eV}}{23\,\mathrm{eV}} = 609\,, \tag{2.26}$$

from which we can calculate the standard deviation using the formula (2.18):

$$\sigma_{N_{\mathrm{Ar}}} = \sqrt{0.2 \cdot 958} = 13.84\,, \tag{2.27}$$

$$\sigma_{N_{\mathrm{C_4H_{10}}}} = \sqrt{0.2 \cdot 609} = 11.03\,. \tag{2.28}$$

The trick here is how to combine the two deviations and number of created electron-ion pairs. We know from statistics that we can sum the squares of the deviations and then take the square root of the sum to get the total deviation. The number of pairs is simply the sum of the number of pairs created in argon and isobutane. From this we can calculate the energy resolution as:

$$\begin{aligned} R &= \frac{\sqrt{{\sigma_{N_{\mathrm{Ar}}}}^2 + {\sigma_{N_{\mathrm{C_4H_{10}}}}}^2}}{N_{\mathrm{Ar}} + N_{\mathrm{C_4H_{10}}}} \\ &= \frac{\sqrt{(13.84)^2 + (11.03)^2}}{958 + 609} \\ &= 1.1\%\,. \end{aligned} \tag{2.29}$$

2.2.2 What is the energy resolution of a proportional counter?

For educational purposes let's calculate how the energy resolution worsens with multiplication in a proportional counter. Let's keep the rest of the data as in the previous exercise and assume that the multiplication factor is $M = 900$ and that the constant $b = 0.5$. To get the new energy resolution we can recycle our previous result and add an additional term to the resolution that is due to multiplication. The combined resolution is:

$$R = \sqrt{R_N^2 + R_M^2}\,, \tag{2.30}$$

where R_N is the resolution without multiplication and R_M is the resolution decrease due to multiplication. The resolution decrease due to multiplication is given as:

$$\begin{aligned} R_M &= \sqrt{\frac{b}{N}} \\ &= \sqrt{\frac{0.5}{958 + 609}} \\ &= 1.8\%\,, \end{aligned} \tag{2.31}$$

where we took the **primary number of created pairs**, not the number of pairs after multiplication. Thus the total resolution is:

$$R = \sqrt{(0.011)^2 + (0.018)^2} = 2.9\%\,. \tag{2.32}$$

2.2.3 Why the cylindrical geometry? What voltage would be needed to achieve the same electric field in a parallel plate capacitor?

Let's imagine that the structure of our detector is analogous to a Geiger-Muller tube, so a cylinder with a wire in the middle. This is only an approximation since actual ionization chambers are more complex and are often made of thousands of parallel wires inside a cylindrical gas chamber. The reason for the cylindrical geometry is exactly the properties of the electric field. Taking the center wire thickness to be $a = 0.008\,\mathrm{cm}$ and the radius of the cylinder to be $b = 1\,\mathrm{cm}$, the electric field inside the cylinder is given as:

$$E(r) = \frac{U}{r \ln \frac{b}{a}}\,, \tag{2.33}$$

which at $U = 2000\,\mathrm{V}$ yields an electric field at the center wire of the cylinder:

$$\begin{aligned} E(r = b) &= \frac{2000\ \mathrm{V}}{1\ \mathrm{cm}\ \ln \frac{1\ \mathrm{cm}}{0.008\ \mathrm{cm}}} \\ &= 5.2 \cdot 10^6\ \mathrm{V/m}\,. \end{aligned} \tag{2.34}$$

In comparison, if we wanted to achieve the same electric field inside a planar capacitor with a distance of $d = 1\,\mathrm{cm}$ we'd need a voltage of:

$$U = E \cdot d = 5.2 \cdot 10^6\ \mathrm{V/m} \cdot 1\ \mathrm{cm} = 52\,\mathrm{kV}\,. \tag{2.35}$$

A power supply that can provide $52\,\mathrm{kV}$ is much more expensive and harder to maintain than a power supply that can provide $2\,\mathrm{kV}$, hence the practical choice of a cylindrical geometry.