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Advanced Particle Detectors and Data Analysis

Notes for Exercises

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1 Interactions of Particles with Photons

1.1 Bethe-Bloch Equation

The Bethe-Bloch equation describes the mean energy loss per distance traveled while traversing through matter. We generally use the Bethe-Bloch equation when we are dealing with **thick absorbers**, such as the ones in calorimeters. Do note that the Bethe-Bloch equation does not accurately describe the energy loss of **electrons** and **positrons** due to their small mass and the fact that they suffer from much larger energy losses due to bremsstrahlung and pair production. For a particle with charge z and velocity $\beta = v/c$, the Bethe-Bloch equation is given as:

$$-\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle = 2\pi N_a r_e^2 m_e c^2 \rho \frac{Z}{A}\frac{z^2}{\beta^2}\left[\ln\left(\frac{2m_e c^2\beta^2\gamma^2 W_{\max}}{I^2}\right) - 2\beta^2 - \delta - 2\frac{C}{Z}\right], \tag{1.1}$$

where δ is the **density effect correction** and C is the **shell correction**. The rest is as follows:

$$N_a = 6.022\times10^{23}\,\mathrm{mol}^{-1}\,,\quad r_e = 2.818\times10^{-15}\,\mathrm{m}\,,\quad m_e = 9.11\times10^{-31}\,\mathrm{kg}\,,\quad c = 3\times10^{8}\,\mathrm{m/s}\,,$$
$$\rho = \text{density of the material}\,,\quad A = \text{atomic mass of the material}\,,\quad Z = \text{atomic number of the material}\,,$$
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}\,,\quad W_{\max} = \text{maximum energy transfer in a single collision}\,,\quad I = \text{mean excitation energy}\,.$$

The constant factor in the equation can be written as:

$$\Xi = \boldsymbol{2\pi N_a r_e^2 m_e c^2 = 0.1535\,\mathrm{MeV\ cm^2 mol^{-1}}}\,, \tag{1.2}$$

where I've chosen to mark this constant factor as Ξ for easier reference in further calculations. We can find the mean excitation energy I from the following experimentally determined formula:

$$I = \begin{cases} Z(12 + \frac{7}{Z})\,\mathrm{eV} & \text{for } Z < 13\,, \\ Z(9.76 + 58.8Z^{-1.19})\,\mathrm{eV} & \text{for } Z \geq 13\,. \end{cases} \tag{1.3}$$

The maximum energy transfer in a single collision $W_{\max}$ can be calculated as:

$$W_{\max} = \frac{2m_e c^2\beta^2\gamma^2}{1 + 2\gamma\left({}^{m_e}/_{M}\right) + \left({}^{m_e}/_{M}\right)^2} \approx 2m_e c^2\beta^2\gamma^2\,. \tag{1.4}$$

For our purposes we will ignore the density effect correction δ and the shell correction C.

1.1.1 Energy Loss of Charged Kaons

Let us calculate the energy losses for charged kaons K^+ and K^- with a rest mass of 0.493 GeV and momentum of 2.5 GeV in copper which has the following properties:

$$\rho = 8.92\,\mathrm{g/cm^3}\,,$$
$$Z = 29\,,$$
$$A = 63.5\,\mathrm{g/mol}\,.$$

First let us calculate the velocity β and the Lorentz factor γ. We know that

$$\beta = \frac{pc}{E} = \frac{pc}{\sqrt{(pc)^2 + (Mc^2)^2}}\,, \tag{1.5}$$

where M is the mass of the particle. Thus:

$$\beta = \frac{2.5\,\frac{\text{GeV}}{c}\,c}{\sqrt{\left(2.5\,\frac{\text{GeV}}{c}\,c\right)^2 + \left(0.493\,\frac{\text{GeV}}{c^2}\,c^2\right)^2}} \approx 0.981031\,. \tag{1.6}$$

Remember to take at least 4 significant digits for the velocity β! This is due to the logarithm in the Bethe-Bloch equation. The Lorentz factor γ is then:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 5.159\,. \tag{1.7}$$

Next let us calculate the maximum energy transfer in a single collision W_{max}:

$$W_{\text{max}} = 2m_e c^2\beta^2\gamma^2 = 2\cdot 0.511\,\frac{\text{MeV}}{c^2}\,c^2\cdot(0.981031)^2(5.159)^2 = 26.7\,\text{MeV}\,. \tag{1.8}$$

Last prerequisite is the mean excitation energy I which we can calculate using the formula (1.3) for $Z \geq 13$:

$$I = 29(9.76 + 58.8\cdot 29^{-1.19})\,\text{eV} = 313.9\,\text{eV}\,. \tag{1.9}$$

Now all that is left is to plug in the values into the Bethe-Bloch equation (1.1):

$$\begin{aligned}
-\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle &= \Xi\,\rho\frac{Z}{A}\frac{z^2}{\beta^2}\left[\ln\left(\frac{W_{\text{max}}^2}{I^2}\right) - 2\beta^2\right] \\
&= 0.1535\,\frac{\text{MeV cm}^2}{\text{mol}}\cdot 8.92\,\frac{\text{g}}{\text{cm}^3}\cdot\frac{29}{63.5}\,\frac{\text{mol}}{\text{g}}\cdot\frac{1}{0.981031^2}\cdot\left[\ln\left(\frac{(26.7\cdot 10^6\,\text{eV})^2}{(313.9\,\text{eV})^2}\right) - 2\cdot(0.981031)^2\right] \\
&= 13.47\,\frac{\text{MeV}}{\text{cm}}\,.
\end{aligned} \tag{1.10}$$

Thus the energy loss of charged kaons K^+ and K^- with a momentum of 2.5 GeV in copper is 13.47 MeV/cm.

1.1.2 What is the Energy Resolution of the Detector from the Previous Example?

Let's calculate the energy resolution of the detector from the previous example, assuming that the length of the particle track through the detector is $d = 5\,\text{cm}$ and that energy is measured based on all deposited energy without any additional losses. Using the result from the previous example (1.10), we can calculate the average energy deposited in the detector as:

$$\Delta E = \bar{\Delta} = -\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle\cdot d = 13.47\,\frac{\text{MeV}}{\text{cm}}\cdot 5\,\text{cm} = 67.35\,\text{MeV}\,. \tag{1.11}$$

This is an approximation since we are assuming that β is constant throughout the detector, which is not true. In reality we'd have to integrate the energy loss over the path of the particle, however at $p \sim$ GeV additional losses of $\sim$ MeV are negligible. Measurements of energy are dependant on the energy resolution R which is defined as:

$$R = \frac{\sigma_E}{\bar{\Delta}}\,, \tag{1.12}$$

where σ_E is the standard deviation of the energy measurement which we assume to have a Gaussian distribution like such:

$$p(\Delta) = \frac{1}{\sqrt{2\pi}\sigma_E}\exp\left(-\frac{\Delta - \bar{\Delta}^2}{2\sigma_E^2}\right)\,. \tag{1.13}$$

σ_E is determined empirically. For **non-relativistic** particles it can be calculated as the variance of the Bethe-Bloch equation as:

$$\sigma_0^2 = 4\pi N_a r_e^2(m_e c^2)^2\rho\frac{Z}{A}\,\Delta x\,. \tag{1.14}$$

For **relativistic** particles we can correct the variance from (1.14) as such:

$$\sigma_E^2 = \sigma_0^2\,\frac{1-\frac{1}{2}\beta^2}{1-\beta^2}\,. \tag{1.15}$$

In our case this gives us:

$$\begin{aligned}\sigma_E^2 &= 2 \cdot 0.511\, \frac{\text{MeV}}{c^2}\, c^2 \cdot 0.1535\, \frac{\text{MeVcm}^2}{\text{mol}} \cdot 8.92\, \frac{\text{g}}{\text{cm}^3} \cdot \frac{29}{63.5}\, \frac{\text{mol}}{\text{g}} \cdot 5\,\text{cm} \cdot \frac{1 - \frac{1}{2}(0.981031)^2}{1 - (0.981031)^2} \\ &= 44.12\,\text{MeV}^2\,. \end{aligned} \tag{1.16}$$

Thus the energy resolution of the detector is:

$$R = \frac{\sqrt{44.12\,\text{MeV}^2}}{67.35\,\text{MeV}} = 9.9\%\,. \tag{1.17}$$

1.1.3 What if the Detector is Made of a Molecule?

Let's assume now that our detector is made of lead(II) fluoride PbF_2 in a cubic crystal form which has the following properties:

$$\begin{aligned} \rho &= 7.77\,\text{g/cm}^3\,, & Z_{\text{Pb}} &= 82\,, \\ Z &= 100\,, & Z_{\text{F}} &= 9\,, \\ A &= 245.2\,\text{g/mol}\,. & \rho_{\text{Pb}} &= 11.34\,\text{g/cm}^3\,, \\ A_{\text{Pb}} &= 207.2\,\text{u}\,, & \rho_{\text{F}} &= 0.001696\,\text{g/cm}^3\,. \\ A_{\text{F}} &= 19\,\text{u}\,, \end{aligned}$$

We are interested in the energy loss of protons with a momentum of 3 GeV in such a detector. The difference between calculating the energy loss in a compound material is that we have to calculate the energy loss for each element in the compound. This sum is weighted by the fraction of the element in the compound. As such:

$$\frac{1}{\rho}\left\langle\frac{\text{d}E}{\text{d}x}\right\rangle_{\text{compound}} = \frac{w_1}{\rho_1}\left\langle\frac{\text{d}E}{\text{d}x}\right\rangle_1 + \frac{w_2}{\rho_2}\left\langle\frac{\text{d}E}{\text{d}x}\right\rangle_2 + \dots\,, \tag{1.18}$$

where we calculate w_i as:

$$w_i = \frac{a_i \cdot A_i}{\sum a_i \cdot A_i}\,, \tag{1.19}$$

here a_i is the number of atoms of the element in the compound and A_i is the atomic mass of the element. Our professor stated that such problems will not be present on the exam and that we should not worry about them. However it is still good to know how to calculate the energy loss in a compound. In our case we can expect to get effective values if the detector is made of a compound. If we calculate the weights for lead and fluorine in lead(II) fluoride we get:

$$\begin{aligned} w_{\text{Pb}} &= \frac{1 \cdot 207.2\,\text{u}}{1 \cdot 207.2\,\text{u} + 2 \cdot 19\,\text{u}} = 0.845\,, \\ w_{\text{F}} &= \frac{2 \cdot 19\,\text{u}}{1 \cdot 207.2\,\text{u} + 2 \cdot 19\,\text{u}} = 0.154\,, \end{aligned}$$

where we used the atomic masses of lead and fluoride in atomic mass units. Next we need to calculate the velocity β and the Lorentz factor γ for protons. So using (1.5) we get:

$$\beta = \frac{3\,\frac{\text{GeV}}{c}\,c}{\sqrt{\left(3\,\frac{\text{GeV}}{c}\,c\right)^2 + \left(0.938\,\frac{\text{GeV}}{c^2}\,c^2\right)^2}} \approx 0.95443\,, \tag{1.20}$$

which gives us a Lorentz factor of:

$$\gamma = \frac{1}{\sqrt{1 - (0.95443)^2}} \approx 3.351\,. \tag{1.21}$$

Next we need to calculate the maximum energy transfer in a single collision $W_{\max}$ using (1.4) as:

$$W_{\max} = 2 \cdot 0.511\,\frac{\text{MeV}}{c^2}\,c^2 \cdot (0.95443)^2 (3.351)^2 = 10.5\,\text{MeV}\,, \tag{1.22}$$

and the mean excitation energy I using (1.3) for each component:

$$I_{\mathrm{Pb}} = 82\left(9.76 + 58.8 \cdot 82^{-1.19}\right)\mathrm{eV} = 825.8\,\mathrm{eV}\,,$$
$$I_{\mathrm{Cu}} = 9\left(12 + \frac{7}{9}\right)\mathrm{eV} = 115\,\mathrm{eV}\,.$$

Now we can calculate the energy loss for each component using the Bethe-Bloch equation (1.1) and sum them up:

$$\begin{aligned}
-\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{Pb}} &= \Xi\,\rho_{\mathrm{Pb}}\frac{Z_{\mathrm{Pb}}}{A_{\mathrm{Pb}}}\frac{1}{0.95443^2}\left[\ln\left(\frac{(10.5\cdot 10^6\,\mathrm{eV})^2}{(825.8\,\mathrm{eV})^2}\right) - 2\cdot(0.95443)^2\right] \\
&= \Xi\,\rho_{\mathrm{Pb}}\frac{Z_{\mathrm{Pb}}}{A_{\mathrm{Pb}}}\cdot 18.7488 \\
&= 0.1535\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{mol}}\cdot 11.34\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot\frac{82}{207.2}\,\frac{\mathrm{mol}}{\mathrm{g}}\cdot 18.7488 \\
&= 12.9\,\frac{\mathrm{MeV}}{\mathrm{cm}}\,, && (1.23)\\
-\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{F}} &= \Xi\,\rho_{\mathrm{F}}\frac{Z_{\mathrm{F}}}{A_{\mathrm{F}}}\frac{1}{0.95443^2}\left[\ln\left(\frac{(10.5\cdot 10^6\,\mathrm{eV})^2}{(115\,\mathrm{eV})^2}\right) - 2\cdot(0.95443)^2\right] \\
&= \Xi\,\rho_{\mathrm{F}}\frac{Z_{\mathrm{F}}}{A_{\mathrm{F}}}\cdot 23.0773 \\
&= 0.1535\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{mol}}\cdot 0.001696\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot\frac{9}{19}\,\frac{\mathrm{mol}}{\mathrm{g}}\cdot 23.0773 \\
&= 0.002846\,\frac{\mathrm{MeV}}{\mathrm{cm}}\,. && (1.24)
\end{aligned}$$

Now all that is left is to compute the weighted sum as stated in (1.18):

$$\begin{aligned}
-\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{compound}} &= -\frac{\rho\cdot w_{\mathrm{Pb}}}{\rho_{\mathrm{Pb}}}\cdot\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{Pb}} - \frac{\rho\cdot w_{\mathrm{F}}}{\rho_{\mathrm{F}}}\cdot\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{F}} \\
&= \frac{7.77\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot 0.845}{11.34\,\frac{\mathrm{g}}{\mathrm{cm^3}}}\cdot 12.9\,\frac{\mathrm{MeV}}{\mathrm{cm}} + \frac{7.77\,\frac{\mathrm{g}}{\mathrm{cm^3}}\cdot 0.154}{0.001696\,\frac{\mathrm{g}}{\mathrm{cm^3}}}\cdot 0.002846\,\frac{\mathrm{MeV}}{\mathrm{cm}} \\
&= 9.47\,\frac{\mathrm{MeV}}{\mathrm{cm}}\,. && (1.25)
\end{aligned}$$

1.2 Landau Distribution

For detectors of moderate thickness, which we can consider as **thin absorbers**, we can use a highly-skewed Landau-Vavilov distribution to describe the energy loss of particles. The most probable energy loss Δ_p is given as:

$$\bar{\Delta} = \Delta_p = \xi\left[\ln\frac{2mc^2\beta^2\gamma^2}{I} + \ln\frac{\xi}{I} + j - \beta^2 - \delta(\beta\gamma)\right]\,, \tag{1.26}$$

where $\delta(\beta\gamma)$ represents corrections due to the density effect, $j = 0.200$ and ξ is given as:

$$\xi = \frac{K}{2}\left\langle\frac{Z}{A}\right\rangle\frac{x}{\beta^2}\,\mathrm{MeV}\,, \tag{1.27}$$

for x in $\mathrm{g/cm^2}$ and $K = 0.3\,\mathrm{MeV\,cm^2/g}$. **Warning:** x here is normalized with density $\mathrm{g/cm^2}$. This means that $x = \rho\cdot d$ where d is the thickness of the detector. To know which distribution to use, we can use the following rule of thumb:

$$\kappa = \frac{\bar{\Delta}}{W_{\max}}\begin{cases} > 10 & \text{use Bethe-Bloch}\,,\\ < 0.01 & \text{use Landau}\,.\end{cases} \tag{1.28}$$

To determine the energy resolution of such a detector we can use the following formula:

$$R_{\mathrm{FWHM}} = \frac{4\xi}{\Delta_p}\,. \tag{1.29}$$

1.2.1 What is the Most Probable Energy Loss of a Charged Pion in Silicon?

Let's calculate the most probable energy loss of a charged pion with a rest mass of 139.57 MeV and momentum of 0.5 GeV in a silicon based detector which has a 320 μm thick silicon layer. Silicon has the following properties:

$$\begin{aligned}\rho &= 2.32\,\mathrm{g/cm}^3\,,\\ Z &= 14\,,\\ A &= 28\,\mathrm{g/mol}\,.\end{aligned}$$

As before we first calculate the velocity β and the Lorentz factor γ for pions. Using (1.5) we get:

$$\beta = \frac{0.5\,\frac{\mathrm{GeV}}{c}\,c}{\sqrt{\left(0.5\,\frac{\mathrm{GeV}}{c}\,c\right)^2 + \left(0.13957\,\frac{\mathrm{GeV}}{c^2}\,c^2\right)^2}} \approx 0.96318\,, \tag{1.30}$$

which gives us a Lorentz factor of:

$$\gamma = \frac{1}{\sqrt{1-(0.96318)^2}} \approx 3.72\,. \tag{1.31}$$

Likewise as before we want to calculate the mean excitation energy I using (1.3) for $Z \geq 13$:

$$I = 14\left(9.76 + 58.8 \cdot 14^{-1.19}\right)\,\mathrm{eV} = 172.3\,\mathrm{eV}\,. \tag{1.32}$$

We can also calculate our approximation for the maximum energy transfer in a single collision W_{max} using (1.4), since we can spot it in the Landau distribution (1.26):

$$W_{\mathrm{max}} = 2 \cdot 0.511\,\frac{\mathrm{MeV}}{c^2}\,c^2 \cdot (0.96318)^2(3.72)^2 = 13.12\,\mathrm{MeV}\,. \tag{1.33}$$

Next we calculate ξ using (1.27):

$$\xi = \frac{0.3\,\frac{\mathrm{MeV\,cm}^2}{\mathrm{g}}}{2}\,\frac{14}{28}\,\frac{320 \cdot 10^{-4}\,\mathrm{cm} \cdot 2.32\,\frac{\mathrm{g}}{\mathrm{cm}^3}}{(0.96318)^2} = 0.0060\,\mathrm{MeV}\,. \tag{1.34}$$

Now we can calculate the most probable energy loss using the Landau distribution (1.26):

$$\begin{aligned}\Delta_p &= 0.006\,\mathrm{MeV}\left[\ln\frac{13.12 \cdot 10^6\,\mathrm{eV}}{172.3\,\mathrm{eV}} + \ln\frac{0.006 \cdot 10^6\,\mathrm{eV}}{172.3\,\mathrm{eV}} + 0.2 - (0.96318)^2\right]\\ &= 0.0844\,\mathrm{MeV}\,.\end{aligned} \tag{1.35}$$

From this we can now also calculate the energy resolution of the detector using the formula (1.29):

$$R_{\mathrm{FWHM}} = \frac{4 \cdot 0.006\,\mathrm{MeV}}{0.0844\,\mathrm{MeV}} = 28.5\%\,. \tag{1.36}$$

If we're paranoid if we've used the right distribution, we can calculate κ as:

$$\kappa = \frac{0.0844\,\mathrm{MeV}}{13.12\,\mathrm{MeV}} = 0.0064\,, \tag{1.37}$$

which is less than 0.01 so we've used the right distribution. Alternatively if we magically procure the result from the Bethe-Bloch equation, we'd get $\bar{\Delta} = 126\,\mathrm{keV}$ and $\sigma = 402\,\mathrm{keV}$ which would give us $\kappa = 0.0105$ which still hints that we should use the Landau distribution.