5.6.:

merjenje: • koliko enot obsega
• Oceniti "obseg velikosti"

Fizikalna količina:

-

$$\vec{p} = m \cdot \vec{v}$$

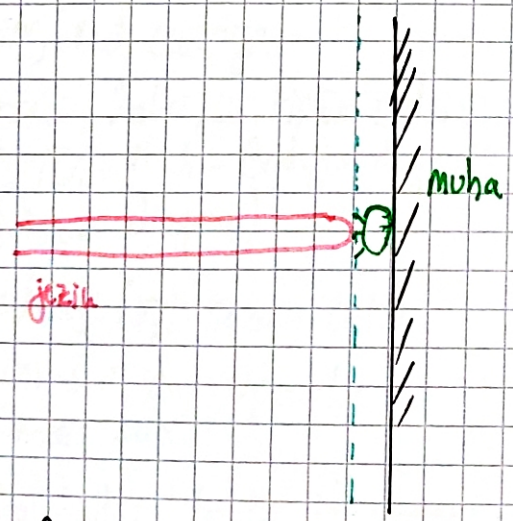
- s predpisom povezano

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{\Delta\vec{x}}{\Delta t} \begin{matrix} \to \text{Umeritev z metrom} \\ \to 1s \end{matrix}$$

Ocenjevanje Razdalje:

- Astro
- Nano
- Mikro
- Bio (plenilec → plen)

Dober kameleon. Njegov plen je muha.

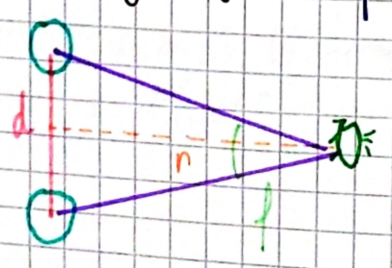


$\Delta x = 100\,\mu m$

Kaj mu to omogoča?

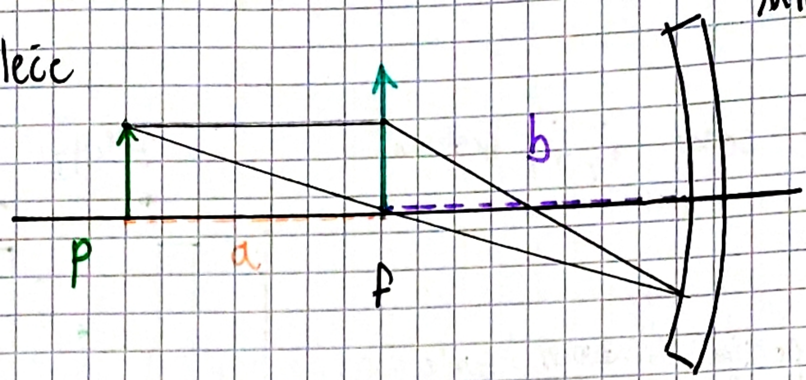
1.) Stereoskopsko gledanje

$$r = \left(\frac{d}{\varphi}\right)$$



Zaprli so mu z zaslonko eno oko in še vedno je zadel. Torej to ni mehanizem.

2.) Akomodacija leče

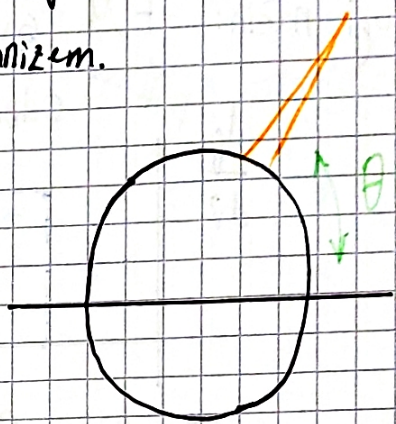


$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$$

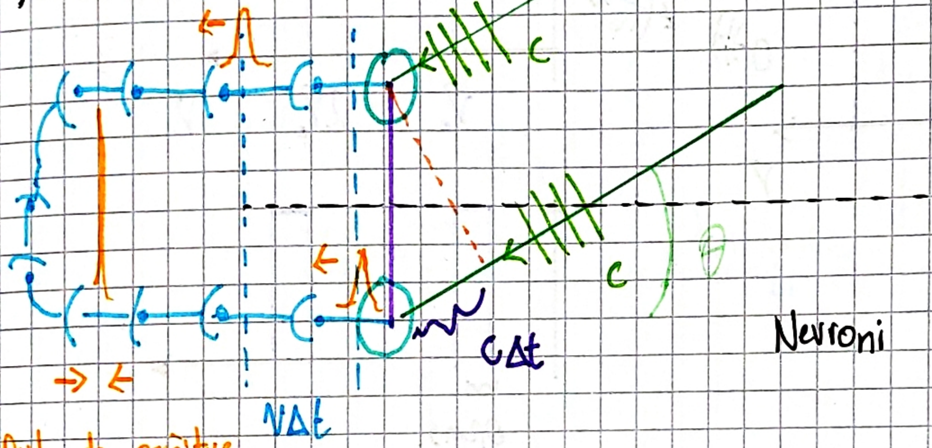
$$\Rightarrow \frac{1}{a} = \frac{1}{f} - \frac{1}{b}$$

↑ čuti ↑ fiksirano v očeh

Dali so mu leže preko oči in je zgrešil. To je mehanizem.

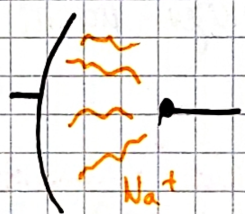


3.) Stereoskopsko poslušanje lege



Pride do gnečitve ne simetrično

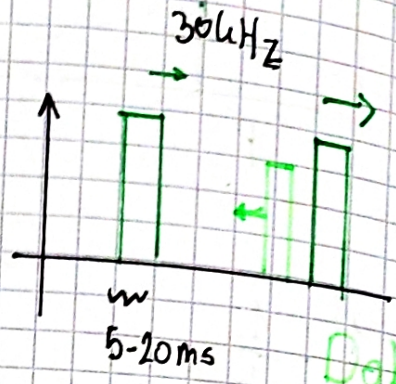
Nevroni



Nevroni imajo zelo nelinearen odziv. Dve hkratni vzbuditvi dajo 10^3 močnejši kem. signal.

¿ Kaj pa netopir?

Ima aktivni sonar $\vec{r}, \vec{v}$, sestava

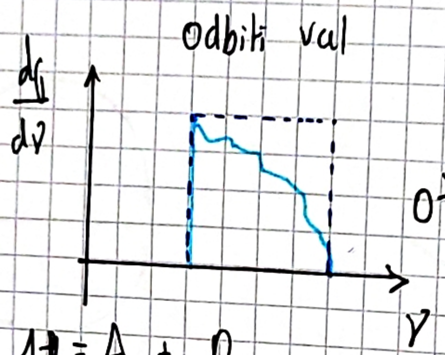


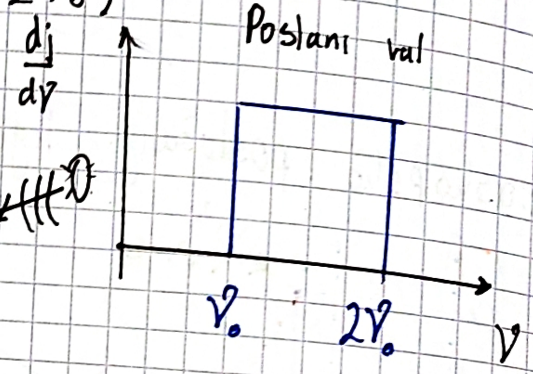
Prvi način:

V prostor pošilja ultrazvočne sunke
Okoli 200 na sekundo. To je
t.i. tipanje na daleč

Odbi lego

Drugi način: FM sunki ($\nu_0 \leftrightarrow 2\nu_0$)

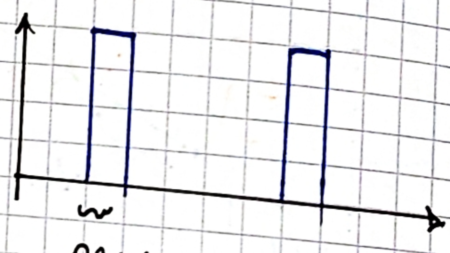




$1 = A + R$

adsorpcija moč na
funkcija ν

Odbi sestavo

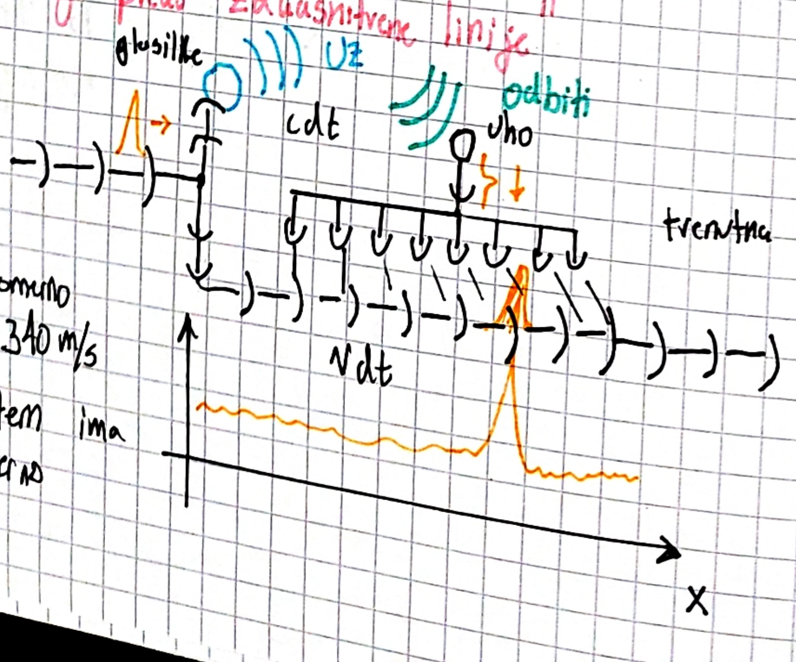


Tretji način: Prvi in drugi hkrati

Četrti način: Dopplerjev premik

$\nu = \nu_s(1 + 2v/c)$ Odbi Hitrost

Meritev razdalje preko "Zakasnitvene linije"



Imamo enakomerno gibanje $c = 340$ m/s
Modelski sistem ima tudi enakomerno gibanje.

Realni sistem $\qquad\qquad$ Modelski sistem

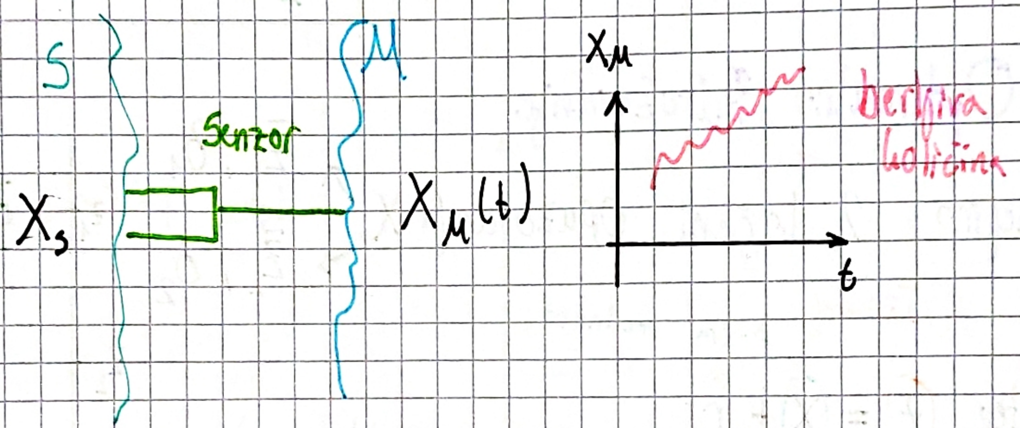
$$S = ct \quad \xleftrightarrow[\text{povezuje}]{\text{Senzor}} \quad S_M = v \cdot t$$

UHO

S_m je berljiva količina (ob vsakem primeru lahko pogledamo koliko je)
Tu senzor nima znatnega vpliva na realni sistem.

Optimalno filtriranje

Iščemo optimalen predpis za optimizacijo ~~modela~~ realnega sistema S na modelski sistem M.



Zahteve:

i) Šibka sklopitev S in M (čim manj vpliva)

ii) X_M mora biti berljiva količina

iii) Ocena stopnje usklajenosti

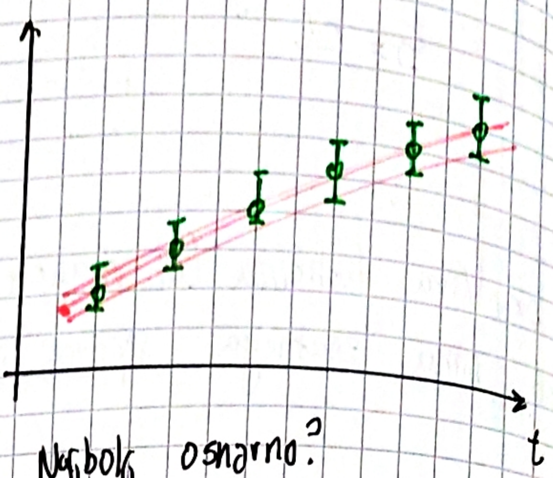
$$\lim_{t \to \infty} \langle (X_M - X_S)^2 \rangle = ? \qquad \langle \dots \rangle \text{ ensumbelsko povprečje}$$

iv) Dinamika za X_s in X_M naj bo kar se da podobna

↳ Linearne diferencialne (diferenčne) enačbe

Zgled: [Premoenakomerno gibanje telesa]

10, N = konst.



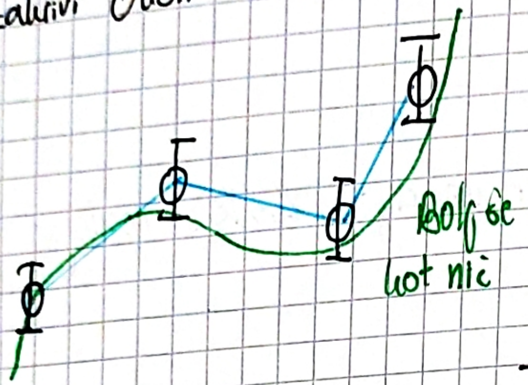
neki neskončen

↓ pospešek

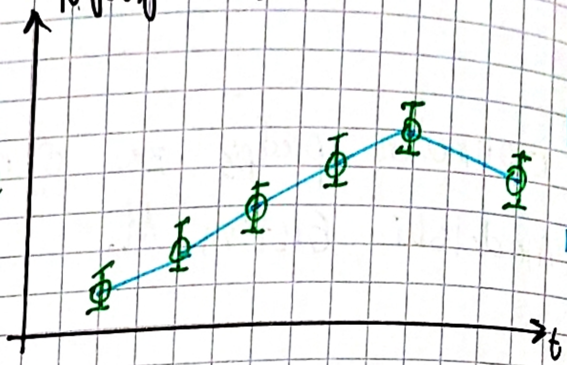
Že iz tega da $a < a_0$ vemo

da takih nezvezno skokov ne dela

ampak zakrivi okoli točk.



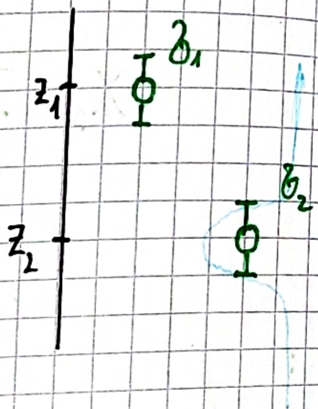
Najbolj osnarno?



Najbolj slabo nič ne vemo dinamike

① Optimalno združevanje

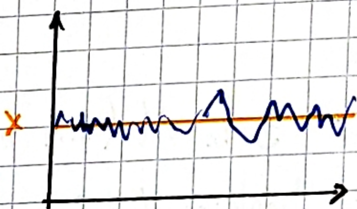
Imejmo 2 ločeni opazovanji X $\nearrow \bar{Z}_1, \sigma_1$ $\searrow \bar{Z}_2, \sigma_2$



Izmerek, prava vrednost

Meritev $Z = X + r$

naključna spremenljivka merilni šum



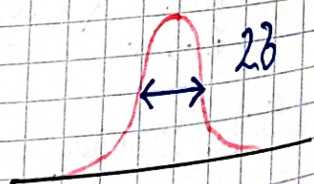
Zanima nas po kakšni porazdelitvi

$$\frac{dP}{dr} = N(0, \sigma)$$

Po Gaussovi porazdelitvi:

$$\Rightarrow \frac{dp}{dr} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-r^2/2\sigma^2}$$

disperzija



soda, liha

$$\langle r \rangle = \int_{-\infty}^{\infty} \frac{dp}{dr}\, r \, dr = 0$$

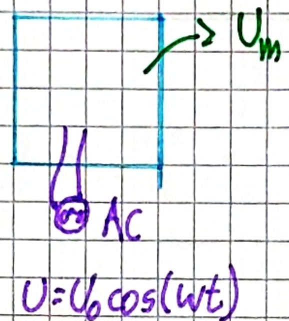
$$\langle r^2 \rangle \neq 0 \Rightarrow \langle r^2 \rangle = \sigma^2$$

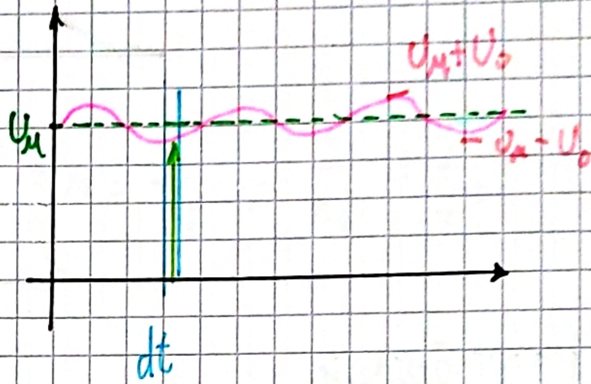
Vsota neodvisnih naključnih spremenljivk teži k normalni porazdelitvi.

Pomembno je da k šumu prispeva veliko malih prispevkov (skupaj Gauss) namesto enega dominantnega ne Gaussovskega

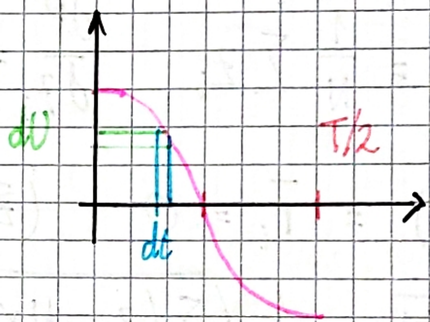
Brumov šum

Motnje zaradi napajanja z AC napetostjo





$$\frac{dP}{dt} = \frac{1}{T/2} \quad \text{na } [0, T/2] \text{ konst.}$$



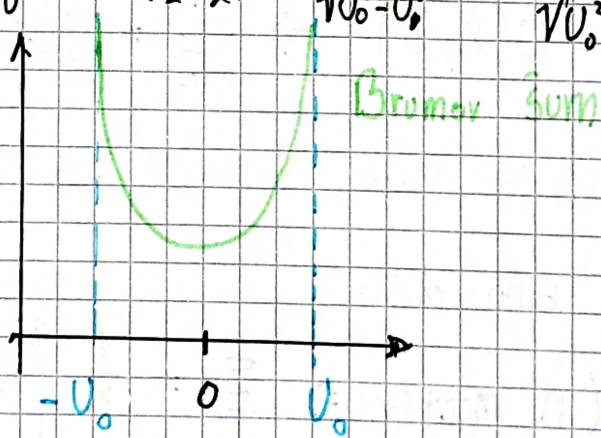
$$\frac{dP}{dU} \neq \text{konst}$$

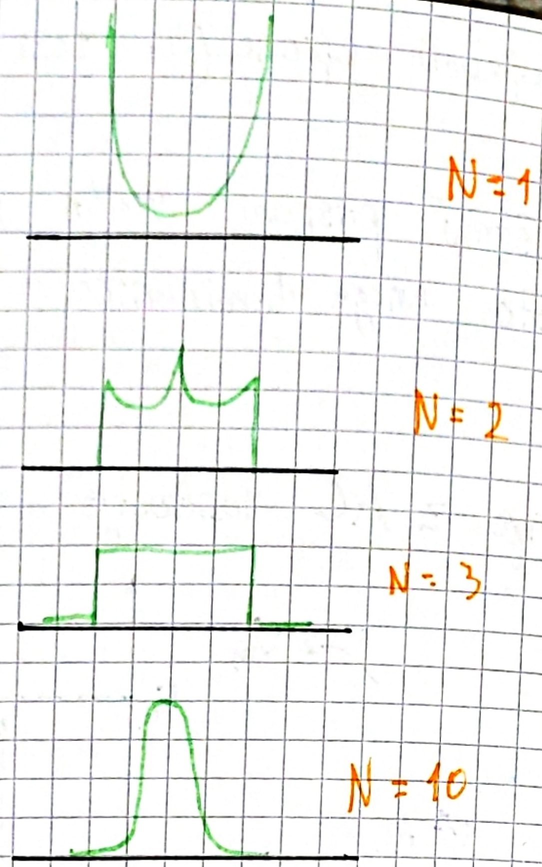
$$dU = -U_0\omega\sin(\omega t)dt$$

$$\frac{dP}{dt}\left(\left|\frac{dt}{dU}\right|\right) = \frac{dP}{dt}\left|\left(\frac{1}{-U_0\sqrt{\sin^2\omega t}\,\omega}\right)\right| = \frac{dP}{dt}\frac{1}{U_0\sqrt{1-\cos^2\omega t}\,\omega} =$$

$$= \frac{dP}{dt}\frac{T}{2\pi\sqrt{U_0^2-U^2}} = \frac{1}{T/2}\frac{1}{2\pi}\frac{T}{\sqrt{U_0^2-U_0^2}} = \frac{1}{\sqrt{U_0^2-U^2}}$$

Daleč od Gaussa, CLT nas reši. Imamo več prispevkov po brumu in ~~[illegible]~~. Če jih je veliko gre proti Gauss.





a) Povprečevanje

$$\bar{z} = \frac{1}{N}\sum z_i \qquad \frac{dP}{dz_i} = N(x, \sigma)$$

$$\overline{(z_i - x)} = \bar{r}_i = 0 \qquad \overline{(\bar{z} - x)} = 0$$

$$\overline{(\bar{z}_i - x^2)} = \frac{1}{N^2}\left(\overline{\sum (z_i - Nx)^1}\right)^2 = \frac{1}{N^2}\overline{\left(\sum (z_i - x)\right)^2} =$$

$$= \frac{1}{N^2}\left[\overline{\sum (z_i - x)^2} + \cancel{\overline{\sum_{i,j} (z_i - x)(z_j - x)}}\right] = \frac{1}{N^2} N\sigma^2 = \frac{\sigma^2}{N}$$

0 če so meritve neodvisne

$$\Rightarrow \quad \frac{dP}{dz_i} = N(x, \sigma) \quad \Rightarrow \quad \frac{dP}{d\bar{z}} = N\left(x, \frac{\sigma}{\sqrt{N}}\right)$$

Porazdeljeno okoli istega x z ožjo Gaussovko

Recimo da smo delali

1. meritev (n meritev $\Rightarrow \bar{z}_1 = \frac{1}{n}\sum_1^n z_i \; ; \; \sigma_1 = \frac{\sigma}{\sqrt{n}}$)
2. meritev (m meritev $\Rightarrow \bar{z}_2 = \frac{1}{m}\sum_1^m z_i \; ; \; \sigma_2 = \frac{\sigma}{\sqrt{m}}$)

Kaj pa če je nekdo naredil n+m meritev

$$n+m \Rightarrow \bar{Z}_3 = \frac{1}{n+m}\sum_{1}^{n+m} z_i \qquad \sigma_3^2 = \frac{\sigma^2}{n+m}$$

$$= \frac{1}{n+m}\left[\sum_{1}^{n} z_i + \sum_{n+1}^{n+m} z_i\right] =$$

$$\bar{Z}_3 = \underbrace{\left(\frac{n}{n+m}\right)} \bar{Z}_1 + \underbrace{\left(\frac{m}{n+m}\right)} \bar{Z}_2$$

Dve različni uteži

Izrazimo n, m z sigmami:

$$n = \frac{\sigma^2}{\sigma_1^2} \qquad m = \frac{\sigma^2}{\sigma_2^2} \qquad n+m = \frac{\sigma^2}{\sigma_3^2} = \underbrace{\frac{\sigma^2}{\sigma_1^2} + \frac{\sigma^2}{\sigma_2^2}}$$

ostranje

$$\frac{1}{\sigma_3^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

Torej:

$$\bar{Z}_3 = \frac{\sigma_3^2}{\sigma_1^2}\bar{Z}_1 + \frac{\sigma_3^2}{\sigma_2^2}\bar{Z}_2 =$$

$$\boxed{\bar{Z}_3 = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)\bar{Z}_1 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\bar{Z}_2}$$

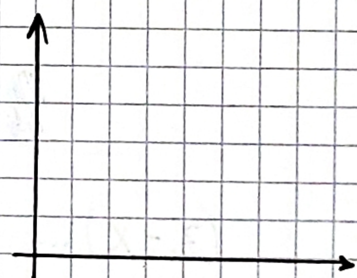
Zastopa

Izmerek, ki je bolj natančen je bolj (upoštevan) v optimalni združitveni oceni.

Merilni šum

- mnogo neporavnih prispevkov sestavlja šum (CLI)

$$\frac{dp}{dr} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}e^{-r^2/2\sigma^2} \qquad z_i = x + r_i$$



$$\langle r^2 \rangle = \int \frac{dp}{dr} r^2 dr = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\int e^{-r^2/2\sigma^2} r^2 dr = \qquad u = r^2/2\sigma^2$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\int e^{-u^2} 2\sigma^2 u^2 \sqrt{2}\sigma = \frac{1}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-u^2}u^2 du}_{\frac{\sqrt{\pi}}{2}}(2\sigma^2) = \sigma^2$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \quad \Big/ \cdot \frac{d}{da}$$

$$\int_{-\infty}^{\infty} (-x^2) e^{-ax^2}\,dx = \sqrt{\pi}\left(-\frac{1}{2}\right) a^{-3/2}$$

$$a \to 1$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Pričakovano odstopanje od povprečja

$$\sigma = \sqrt{\langle r^2 \rangle}$$

Konec ponavljanja

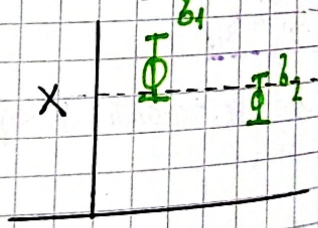
Z_3 lahko zapišemo rekurzivno:

$$Z_3 = \underbrace{Z_1}_{\text{stara meritev}} + \underbrace{\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}}_{\text{Utež}} \overbrace{(Z_2 - Z_1)}^{\text{inovacija}}$$

b) Kvadratna forma

Do tega lahko pridemo tudi preko kvadratne forme $2J(x)$

$$2J(x) = \frac{(z_1 - x)^2}{\sigma_1^2} + \frac{(z_2 - x)^2}{\sigma_2^2}$$



$(z_1 - x)$ ... por. po $\mathcal{N}(0, \sigma_1)$

$(z_2 - x)$ ... por. po $\mathcal{N}(0, \sigma_2)$

$\frac{(z_1 - x)}{\sigma_1}$ ... por. po $\mathcal{N}(0, 1)$

$\frac{(z_2 - x)}{\sigma_2}$ ... por. po $\mathcal{N}(0, 1)$

Normirani Gaussov šum

Hočemo minimalni seštevek, torej zahtevamo $\frac{d}{dx}\chi^2(x) = 0$

$$\frac{-2(z_1-x)}{\sigma_1^2} - \frac{2(z_2-x)}{\sigma_2^2} = 0$$

$$x\left[\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right] = \frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}$$

optimalen:

$$x = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1}\left[z_1\frac{1}{\sigma_1^2} + z_2\frac{1}{\sigma_2^2}\right] = z_3$$

c) Disperzija optimalno združene ocene

Ali je optimalno združevanje res optimalno?

$$(z_1, \sigma_1), \quad z_1 = x + r_1 \qquad \langle r_1\rangle = 0, \quad \langle r_1^2\rangle = \sigma_1^2$$

$$(z_2, \sigma_2), \quad z_2 = x + r_2 \qquad \langle r_2\rangle = 0, \quad \langle r_2^2\rangle = \sigma_2^2$$

Sestavimo z_3 kot linearno kombinacijo:

$$z_3 = \alpha z_1 + \beta z_2 = x + r$$

$z_3 = \hat{z}$

$$\Rightarrow \quad z_3 = \alpha(x+r_1) + \beta(x+r_2) = x + r$$

$$= \underbrace{(\alpha+\beta)}_{1}x + \underbrace{\alpha r_1 + \beta r_2}_{r} = 1x + r$$

$$r = \alpha r_1 + (1-\alpha)r_2$$

$$\langle r\rangle = 0$$

$$\langle r^2\rangle = \alpha^2\underbrace{\langle r_1^2\rangle}_{\sigma_1^2} + (1-\alpha)^2\underbrace{\langle r_2^2\rangle}_{\sigma_2^2} + 2\alpha(1-\alpha)\cancel{\langle r_1 r_2\rangle}$$

0 če nista korelirani

$$= \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2$$

Minimiziramo to po α $\quad \frac{d}{d\alpha}\langle r^2\rangle = 0$

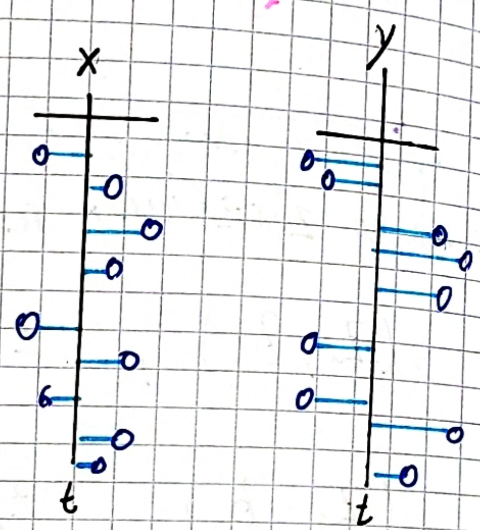
$$2\alpha\sigma_1^2 + 2(1-\alpha)(-1)\sigma_2^2 = 0$$

$$\alpha(\sigma_1^2+\sigma_2^2) = \sigma_2^2$$

$$\alpha = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \qquad \beta = 1-\alpha = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$$

2) Korelacija med izmerki (Ocenami)

Imamo 2 seta meritev x, y



N meritev $\{x_i\}_N$ $\qquad$ N meritev $\{y_i\}_N$

indeks je prav isti

$$\bar{r}_x = \bar{r}_y = 0$$

$$\sigma_x^2 = \overline{r_x^2} \neq 0 \qquad \overline{r_x r_y} \neq 0$$

$$\sigma_y^2 = \overline{r_y^2} \neq 0$$

Obstaja korelacija

(povezava med šumoma x in y meritev)

Definiramo kovarianco

$$\sigma_{xy} = \overline{(x-\bar{x})(y-\bar{y})} =$$

$$= \rho\,\sigma_x\sigma_y$$

korelacijski koeficient

$$|\rho| \le 1$$

Negativen ρ pomeni antikorelacijo

To lahko izvedemo tudi drugače (vstavimo povprečenje)

$$\sigma_{xy} = \frac{1}{N}\sum_i (x-\bar{x})(y-\bar{y}) =$$

$$= \frac{1}{N}\sum_i (xy - \bar{x}y - x\bar{y} + \bar{y}\bar{x}) =$$

$$= \frac{1}{N}\sum xy - \frac{1}{N}\bar{x}\sum y - \frac{1}{N}\bar{y}\sum x + \frac{1}{N}\sum 1\,\bar{x}\bar{y} =$$

$$= \overline{xy} - \bar{x}\bar{y} - \bar{y}\cdot\bar{x} + \bar{x}\bar{y}$$

$$\sigma_{xy} = \overline{xy} - \bar{x}\cdot\bar{y}$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$ Korelacijski koeficient

③ Združevanje koreliranih meritev/ocen

W... je merilni šum (v vlogi tistega kar je bilo prej r)

$$\left.\begin{matrix}(z_1, \sigma_1) \\ (z_2\ \sigma_2)\end{matrix}\right\}\rho$$

$$z_1 = x + w_1 \qquad \langle w_1^2 \rangle = \sigma_1^2$$

$$z_2 = x + w_2 \qquad \langle w_2^2 \rangle = \sigma_2^2$$

$$\langle w_1 w_2 \rangle \neq 0$$

$$\langle w_1 w_2 \rangle = \langle (z_1 - x)(z_2 - x) \rangle = \rho\,\sigma_1 \sigma_2$$ kovarianca

Šum ene meritve zapišemo kot lin. kombinacijo šuma druge meritve in neodvisnega dela (šum ubistvu razstavimo)

$$w_1 = \alpha w_2 + w$$ neodvisni šum $\leftrightarrow$ $(z_1 - x) = \alpha(z_2 - x) + w$

$$\langle w^2 \rangle = \sigma_w^2$$

$$\langle w \rangle = 0$$

$$\langle w w_2 \rangle = 0$$ (nekorelirano)

Izrazimo s tem kovarianco:

$$\rho\,\sigma_1\sigma_2 = \langle w_1 w_2 \rangle = \langle (\alpha w_2 + w) w_2 \rangle =$$

$$= \alpha\langle w_2^2 \rangle + \langle w w_2 \rangle$$

$$\rho \sigma_1 \sigma_2 = \alpha \sigma_2^2$$

$$\alpha = \rho \frac{\sigma_1}{\sigma_2}$$

Še iz disperzije prvega šuma:

$$\sigma_1^2 = \langle w_1^2 \rangle = \langle (\alpha w_2 + w)^2 \rangle = \alpha^2 \langle w_2^2 \rangle + \langle w^2 \rangle + \alpha \langle w_2 w \rangle$$

$$\sigma_1^2 = \alpha^2 \sigma_2^2 + \sigma_w^2$$

$$= \rho^2 \frac{\sigma_1^2}{\sigma_2^2} \sigma_2^2 + \sigma_w^2$$

$$\sigma_1^2 (1 - \rho^2) = \sigma_w^2$$

Sestavimo sedaj kvadratno formo $\underbrace{2\mathcal{J}(x)}_{\text{ime}}$:

(ne moreš pokrajšat 2)

$$2\mathcal{J}(x) = \left(\frac{w_2}{\sigma_2}\right)^2 + \left(\frac{w}{\sigma_w}\right)^2 =$$

$$= \frac{w_2^2}{\sigma_2^2} + \frac{(w_1 - \alpha w_2)^2}{\sigma_w^2} = \frac{w_2^2}{\sigma_2^2} + \frac{w_1^2 - 2\alpha w_1 w_2 + \alpha^2 w_2^2}{\sigma_w^2} =$$

Vstavimo:

$$= \frac{w_2^2}{\sigma_2^2} + \frac{w_2^2 \rho^2 \sigma_1^2/\sigma_2^2}{\sigma_1^2(1-\rho^2)} + \frac{w_1^2}{\sigma_1^2(1-\rho^2)} - \frac{2\rho \frac{\sigma_1}{\sigma_2} w_1 w_2}{\sigma_1^2(1-\rho^2)} =$$

$$= \frac{w^2}{\sigma_2^2}\underbrace{\left(1 + \frac{\rho^2}{1-\rho^2}\right)}_{1/(1-\rho^2)} + \frac{w_1^2}{\sigma_1^2(1-\rho^2)} - \frac{2\rho w_1 w_2}{\sigma_1 \sigma_2 (1-\rho^2)} =$$

$(z_2 - x)^2$ $\quad$ $(z_1 - x)^2$ $\quad$ $(z_1 - x)(z_2 - x)$

$$2\mathcal{J}(x) = \left(\frac{w_2^2}{\sigma_2^2} + \frac{w_1^2}{\sigma_1^2} - \frac{2\rho w_1 w_2}{\sigma_1 \sigma_2}\right)\frac{1}{1-\rho^2}$$

Da je združevanje optimalno $\frac{d}{dx}(2J(x)) = 0$

$$+\frac{\cancel{2}(z_2-x)}{\sigma_2^2} + \frac{\cancel{2}(z_1-x)}{\sigma_1^2} + \frac{\cancel{2}\rho}{\sigma_1\sigma_2}\left(2x-(z_1+z_2)\right) = 0$$

$$\left[\frac{z_2}{\sigma_2^2}+\frac{z_1}{\sigma_1^2}-\frac{\rho(z_1+z_2)}{\sigma_1\sigma_2}\right] = x\left[\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}-\frac{2\rho}{\sigma_1\sigma_2}\right]$$

Dobimo, da je optimalen $\hat{z} = x$

$$\hat{z} = \left[\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}-\frac{2\rho}{\sigma_1\sigma_2}\right]^{-1}\left(\frac{z_1}{\sigma_1^2}+\frac{z_2}{\sigma_2^2}-\frac{\rho(z_1+z_2)}{\sigma_1\sigma_2}\right)$$

$$\hat{\sigma}^2 = (1-\rho^2)\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}-\frac{2\rho}{\sigma_1\sigma_2}\right)^{-1}$$

ta del smo pri odvodu pokrajšali

Mejni primeri:

1.) $\rho = 0$ $\Rightarrow$ dobimo prejšnjo formulo

2.) $\rho = 1$ (popolna korelacija $z_1 = z_2$)

$$x = \left(\frac{\sigma_2^2+\sigma_1^2-2\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\right)^{-1}\left[\frac{z_1\sigma_2^2+z_2\sigma_1^2-(z_1+z_2)\sigma_1\sigma_2}{\cancel{\sigma_1^2\sigma_2^2}}\right]$$

$$= (\sigma_2-\sigma_1)^{-2}\left[z_1\sigma_2(\sigma_2-\sigma_1) - z_2\sigma_1(\sigma_2-\sigma_1)\right]$$

$$= (\sigma_2-\sigma_1)^{-2}(\sigma_2-\sigma_1)^2 z_2 = z_2$$

$$\hat{\sigma}^2 = \sigma_2^2 = \sigma_1^2$$

3) Enaka disperzija $\sigma_1 = \sigma_2$

ρ kurkoli

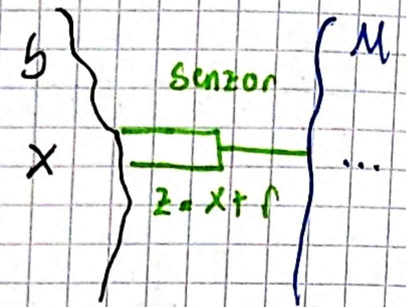
$\sigma_1 = \sigma_2 = \sigma$

$$x = (2-2\rho)^{-1}\left[(z_1+z_2)-\rho(z_1+z_2)\right]$$

$$= \frac{1}{2(1-\rho)}(z_1+z_2)(1-\rho) = \frac{z_1+z_2}{2}$$

Običajno povprečevanje

Sledenje (merjenje) konstantni skalarni količini



Kako dotok novih informacij preko meritev

$$Z_i = X + r_i$$

Sinhronizira modelski sistem z realnim.

$\hat{X}$ ... Ocena k X

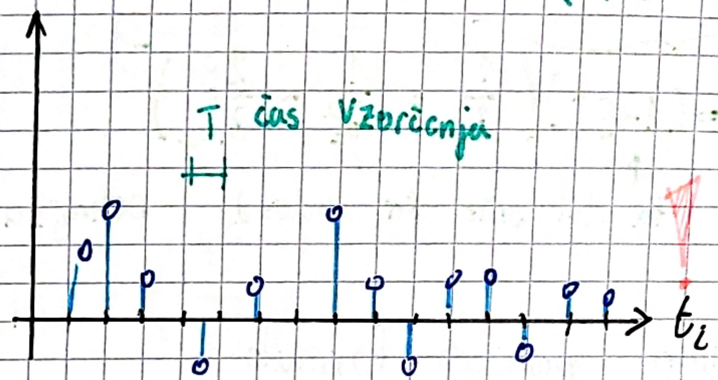
Lastnosti merilnega šuma: $Z_i = X + r_i$

r_i ... merilni šum

$\langle r_i \rangle = 0$

$\langle r_i^2 \rangle = \sigma^2$

Naključna spremenljivka por. po $N(0,\sigma)$



Merilni šum je nekoreliran.

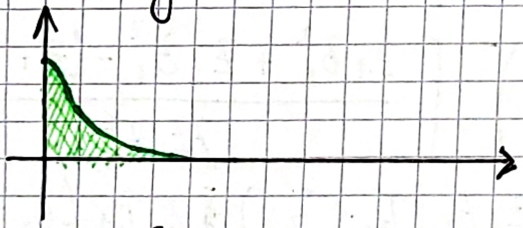
Avtokorelacijska funkcija

$$\langle r_i r_j \rangle = \delta_{ij}\, \sigma^2$$

To je idealizacija

← (če bi čas vmes → 0 ima neka fluktuacija neko vrednost)

Fizikalno gledano ni δ ampak

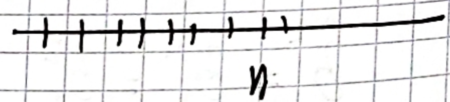


Šum v vsakem trenutku je popolnoma neporezan s šumom v prejšnjih trenutkih.

$$\langle (\hat{x} - x) \rangle = \sigma^2$$

↳ Ocena sinhronizacijo

Shema za sledenje



$$(n \cdot T): \quad \hat{X}_n = \frac{1}{n} \sum_{i=1}^{N} Z_i \qquad \text{(že imamo } n \text{ meritev)}$$

Korak

$$\hat{\sigma}^2_{\sigma n} = \frac{\sigma^2}{n}$$

$$((n+1)\cdot T): \quad \hat{X}_{n+1} = \frac{1}{n+1} \sum_{i}^{n+1} Z_i = \frac{1}{n+1}\left(\sum_{i=1}^{N} Z_i + Z_{n+1} \right)$$

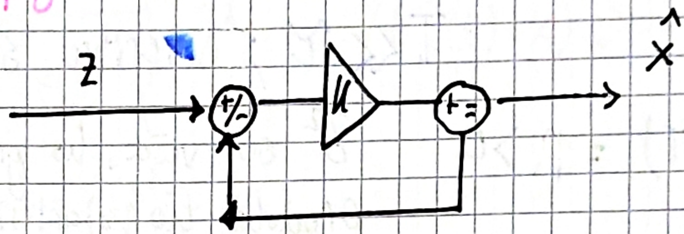
$$= \frac{1}{n+1}\hat{X}_n + \frac{1}{n+1} Z_{n+1} = \hat{X}_n + \frac{1}{n+1}\left(Z_{n+1} - \hat{X}_n\right)$$

Povratna zanka.

$$\hat{X}_{n+1} = \hat{X}_n + \frac{1}{n+1}\left(Z_{n+1} - \hat{X}_n\right)$$

$$\hat{X}_{n+1} = \hat{X}_n + \underbrace{\frac{\hat{\sigma}^2_{n+1}}{\sigma^2}}_{\text{utež}} \underbrace{\left(Z_{n+1} - \hat{X}_n\right)}_{\text{inovacija}}$$

$$\sigma^{-2}_{n+1} = \sigma_n^{-2} + \sigma^{-2}$$



Shema sledenja konstanti

$$K(t_i) = \frac{\hat{\sigma}^2_{n+1}}{\sigma^2}$$

Ocena konvergence $\hat{x} \to x$?

Vzemimo čas vzorčenja $T \to 0$. V limiti $\lim T \to 0$ ratajo vse diskretne spremenljivke zvezne.

$$\hat{X}_n \to \hat{X}(t) \qquad Z_n \to Z(t)$$

$$\hat{\sigma}_n^2 \to \hat{\sigma}_x^2(t)$$

Poglejmo:

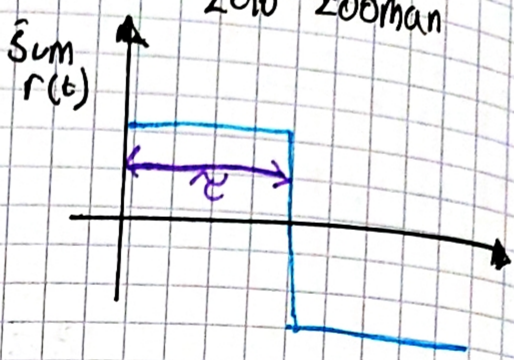
$$\lim_{T \to 0} \frac{\hat{X}_{n+1} - \hat{X}_n}{T} = \dot{\hat{X}}(t) = \frac{\hat{\sigma}^2_{n+1}}{\sigma^2 T}\left(Z_n - \hat{X}_n\right)$$

Vstavimo "limite"

$$\Rightarrow \quad \dot{\hat{X}}(t) = \frac{\hat{\sigma}_x^2}{(\sigma^2 T)}\left(Z(t) - \hat{X}(t)\right)$$

$$\lim_{T\to 0}(\sigma^2 T) = R(t) > 0$$ Zahtevamo, da je to tako!

$$\Rightarrow \quad \dot{\hat{x}}(t) = \frac{\hat{\sigma}_x^2}{R(t)}(z - x)$$



τ ... čas minimalnih fluktuacij

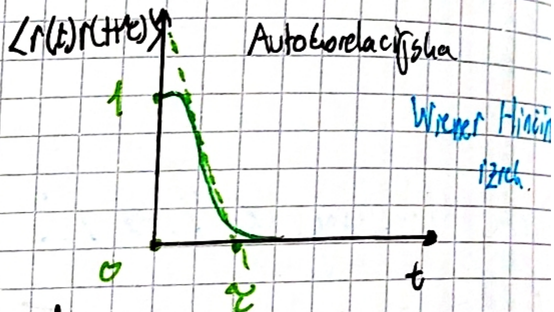
Kako je z nekoreliranostjo?

$T \gg \tau$; meritve so nekorelirane

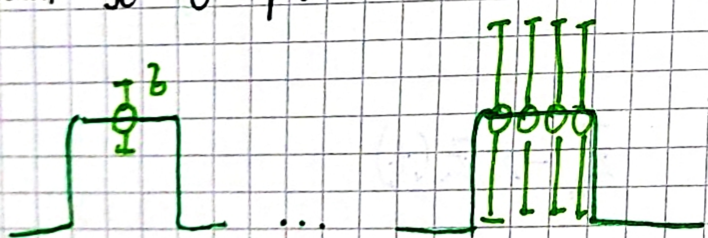
$T \ll \tau$; meritve so korelirane

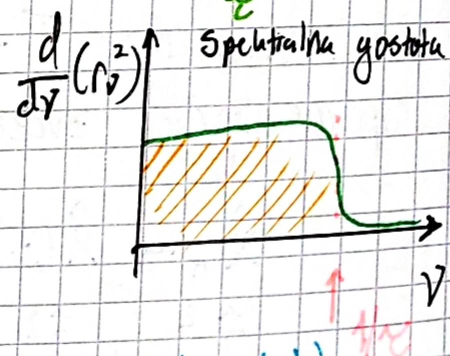
$$\lim_{T\to 0}(\sigma^2 T) = R > 0$$ σ^2 se veča, ko gre $T \to 0$, tako, da je produkt konstanten.

Približanje pri zelo majhnih časih vzorčenja



Torej če znotraj ene fluktuacije vzorčimo večkrat se σ poveča.





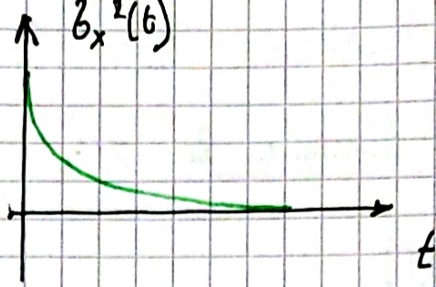
Konvergenca disperzije združene meritve (v kontinuumskih slikah)

Poglejmo si:
$$\frac{1}{T}\left(\hat{\sigma}^2_{n+1} - \hat{\sigma}^2_n\right) = \left(\frac{\hat{\sigma}^2_n \sigma^2}{\hat{\sigma}^2_n - \sigma^2} - \hat{\sigma}^2_n\right)\frac{1}{T} =$$

$$= \frac{-(\hat{\sigma}^2_n)^2}{(\hat{\sigma}^2_n T + \sigma^2 T)} = \dot{\hat{\sigma}}^2_x$$

in tu sedaj limitiramo $T \to 0$.

$$\dot{\hat{\sigma}}_x^2 = -\frac{(\hat{\sigma}_x^2)^2}{R}$$



$$\hat{\sigma}_x^2 \to 0 = \langle(\hat{x}-x)^2\rangle$$

Ocena konvergira k pravi vrednosti, ko dotekamo informacije. Na koncu dobimo točno sinhronizacijo med realnim in modelskim sistemom.

S

$X = \text{konst.}$

$z(t) = x + r(t)$

$\langle r(t) r(t') \rangle = \delta(t-t')R$

M

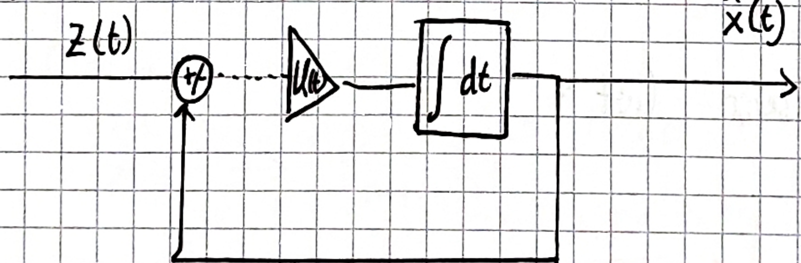
$\hat{X}(t)$ ← berljiva količina

$\hat{\sigma}^2(t) = \langle (\hat{x}(t) - x)^2 \rangle$

$$\dot{\hat{X}} = \frac{\hat{\sigma}_x^2}{R}(z - \hat{X})$$

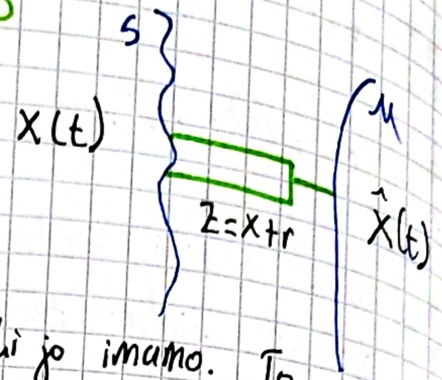
$$\dot{\hat{\sigma}}_x^2 = -\frac{(\hat{\sigma}_x^2)^2}{R}$$

Shema sledenja konstanti:



Merjenje skalarne spremenljivke

1.) Če ne poznamo dinamike za $x(t)$ v S

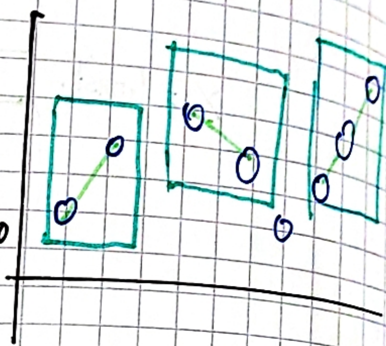


$$\hat{x} = Z(t)$$

$$\hat{\sigma}^2 = R(t)$$

V vsakem trenutku je meritev edina ocena, ki jo imamo. To je kot, da bi na starejše/predhodne meritve.

2.) Nekaj gotovo vemo o dinamiki $x(t)$



Na dovolj majhnih opazovalnih oknih lahko rečemo, da je približno linearno. Tu manj pozabljamo na prejšnje meritve.

3.) Kako opišemo dinamiko v S?

Opišemo jo z linearno diferencialno enačbo 1. reda

$$\dot{x}(t) = A x(t) + c(t)$$

Diskretno to zapišemo kot:

$$\dot{x} = \frac{x_{n+1} - x_n}{T}$$

$$\Rightarrow x_{n+1} = (1 + A(t_n) T) x_n + C(t_n) T$$

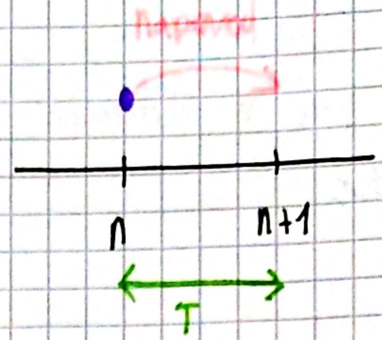
$$x_{n+1} = \Phi_n x_n + c_n$$ → Linearna diferenčna enačba (1. reda seveda)

$$\Phi_n = 1 + A(t_n) T$$

$$c_n = C(t_n) T$$

Postopek optimalne sinhronizacije

Imejmo v trenutku (nekem) $\hat{x}_n, \hat{\sigma}_n^2$



napovedna ocena: $\underline{\underline{\bar{x}_{n+1} = \phi_n \hat{x}_n + c_n}}$

napoved disperzije za (n+1). T : $\underline{\underline{\bar{\sigma}_{n+1}^2 = \phi_n^2 \hat{\sigma}_n^2}}$

$\downarrow$ Def

$$\langle (\bar{x}_{n+1} - x_{n+1})^2 \rangle = \langle (\phi_n \hat{x}_n + c_n - \phi_n x_n - c_n)^2 \rangle =$$

$$= \phi_n^2 \langle (\hat{x}_n - x_n)^2 \rangle = \phi_n^2 \hat{\sigma}_n^2$$

Sedaj pa dobimo nov izmerek v času n+1 z_{n+1}, σ. Naredimo izostreno oceno po znanem postopku:

$$\left.\begin{aligned} \hat{x}_{n+1} &= \bar{x}_{n+1} + \frac{\hat{\sigma}_{n+1}^2}{\sigma^2}(z_{n+1} - \bar{x}_{n+1}) \\ \hat{\sigma}_{n+1}^{-2} &= \bar{\sigma}_{n+1}^{-2} + \sigma^{-2} \end{aligned}\right\} \text{Postopek optimalnega sledenja}$$

Uvedemo nove oznake:

$\hat{\sigma}_{n+1}^2 \rightarrow P_{n+1}$ Kovarianca izostrene ocene (kovariančna matrika ocene)

$\bar{\sigma}_{n+1}^2 \rightarrow M_{n+1}$ Kovarianca napovedi (kovariančna matrika napovedi)

$K_{n+1} = \frac{\hat{\sigma}_{n+1}^2}{\sigma^2} = \frac{P_{n+1}}{\sigma^2}$ Ojačevalni faktor inovacije

$$M_{n+1} = \phi_n^2 P_n \qquad P_{n+1} = \frac{M_{n+1}\sigma^2}{M_{n+1}+\sigma^2} = M_{n+1} - \frac{M_{n+1}^2}{M_{n+1}+\sigma^2}$$

$$\bar{x}_{n+1} = \phi_n \hat{x}_n + c_n$$

Dinamični šum

Φ_n in C_n sta v diferenčni sliki nepopolna. Dodati moramo še nekaj. ~~To je~~

$$X_{n+1} = \Phi_n X_n + C_n + \Gamma_n W_n \rightarrow \text{Dinamični šum}$$

multiplikativni faktor

W_n obravnavajmo to kot šum. Tisto kar ne poznamo o dinamiki skrijemo v dinamični šum (isto Gaussovsko porazdeljen nekoreliran (bel) šum).

$$\langle W_n W_{n'} \rangle = \delta_{nn'} \underline{\underline{Q_n}}$$

$Q = 0 \Rightarrow$ natančno znana dinamika

Ta ne vpliva na našo oceno $\hat{X}_{n+1}$. Vpliva pa na kovarianco:

$$M_{n+1} = \langle (\bar{X}_{n+1} - X_{n+1})^2 \rangle = \langle (\Phi_n \hat{X}_n + \cancel{C_n} - \Phi_n X_n - \cancel{C_n} - \Gamma_n W_n)^2 \rangle$$

$$= \langle (\Phi_n (\hat{X}_n - X_n) - \Gamma_n W_n)^2 \rangle =$$

$$= \Phi_n^2 \langle (\hat{X}_n - X_n)^2 \rangle + \Gamma_n^2 \langle W_n^2 \rangle - 2\Gamma_n \Phi_n \cancelto{0}{\langle (\hat{X}_n - X_n) W_n \rangle} =$$

popolnoma nepovezani

$$= \Phi_n^2 P_n + \Gamma_n^2 Q_n$$

$$\Rightarrow M_{n+1} = \Phi_n^2 P_n + \Gamma_n^2 Q_n$$

Zaradi nepoznavanje dinamike se nam kovarianca lahko le poveča.

Prehod v kontinuumsko sliko:

V sistemu M smo rekli:
$$\hat{X}_{n+1} = \bar{X}_{n+1} + K_{n+1}(Z_{n+1} - \bar{X}_{n+1})$$
$$\bar{X}_{n+1} = \Phi_n \hat{X}_n + C_n$$

Poglejmo si $\lim\limits_{T\to 0} \frac{\hat{x}_{n+1}-\hat{x}_n}{T} = \underbrace{\frac{(\phi_n - 1)\hat{x}_n}{T}}_{A(t)} + \underbrace{\frac{c_n}{T}}_{C(t)} + \frac{P_{n+1}}{T\delta^2}\left(z_{n+1} - \hat{x}_{n+1}\right) =$

$\phi_n = 1 + A(nT)T$

$c_n = C(nT)T$

$\Rightarrow \quad \dot{\hat{x}}(t) = A(t)\hat{x}(t) + C(t) + \frac{P(t)}{R}\left(z(t) - \hat{x}(t)\right)$

Poglejmo prehod še za kovarianco:

$$\underset{\text{izostrena}}{P_{n+1}} = \underset{\text{napoved}}{M_{n+1}} - \frac{M_{n+1}^2}{M_{n+1} + \delta^2}$$

$$P_{n+1} = \underset{\text{to } 1}{\phi_n^2} P_n + \Gamma_n^2 Q_n - \frac{\left(\phi_n^2 P_n + \Gamma_n^2 Q_n\right)^2}{\left(M_{n+1} + \delta^2\right)}$$

$(\phi_n + 1)(\phi_n - 1)$

$\frac{\Gamma_n^4}{T^4} \quad Q_n^2 T^2 \cdot T^2 \to 0$

Spet poglejmo limito

Podoben razlog $\to 0$

$$\lim \frac{P_{n+1} - P_n}{T} = \frac{(\phi_n^2 - 1)P_n}{T} + \frac{\Gamma^2 Q_n}{T} - \frac{\left(\phi_n^4 P_n^2 + \cancel{\Gamma_n^4 Q_n^2} + \cancel{2\Gamma_n^2 Q_n \phi_n^2 P_n}\right)}{M_{n+1}T + \delta^2 T} =$$

$$= 2AP(t) + \ldots =$$

$$\lim \frac{\Gamma^2 (Q_n \cdot T)}{T^2} \quad \begin{cases} \lim (Q_n \cdot T) \to Q(t) \\ \left(\frac{\Gamma}{T}\right)^2 \to \Gamma^2(t) \end{cases}$$

$$= 2AP(t) + \Gamma^2 Q - \frac{P^2(t)}{R}$$

Ostrenje, zaradi dotoka novih meritev

$$\Rightarrow \quad \underset{\text{sprememba}}{\dot{P}(t)} = \underbrace{2AP}_{\text{kovariance zaradi znane dinamike}} + \underbrace{\Gamma^2 Q}_{\text{povečevanje dinamičnega}} - \underset{\text{zaradi šuma}}{\frac{P^2}{R}}$$

sprememba kovariance zaradi znane dinamike

povečevanje dinamičnega šuma

zaradi šuma

Komentar k limitam:

$$\lim_{T \to 0} \frac{\Gamma_n}{T} = \Gamma(t) \rightarrow$$ Bolj kot znatost zmanjšamo manjši je

$$\lim_{T \to 0} (Q_n T) = Q(t) \rightarrow$$ Bolj ko je znato večja je negotovost večja kot šum.

$\Downarrow$

Skupaj je njun produkt $\Gamma \cdot Q$

Vektorske Spremenljivke

S $\quad \vec{x}$

M $\quad \hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$

$$\sum F = m\dot{x} \quad \Rightarrow \quad \dot{x} = v \qquad \text{sistem dif. 1. reda}$$

Dif. 2 reda

$$\dot{v} = \sum F/m$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \vec{x}$$

$\hat{\vec{x}}$ – ocena k vektorju $\vec{x}$

$$\langle (\hat{x}_i - x_i)^2 \rangle = \sigma_i^2$$

Med komponentami so lahko korelacije

$$\langle (\hat{x}_i - x_i)(\hat{x}_j - x_j) \rangle$$

Zgled: $\bar{x} = x + m_x \leftarrow$ šum

$\bar{v} = v + m_v \leftarrow$ šum

$t \rightarrow t + T \Rightarrow x(t+T) = x(t) + v\,T$

Ostrenje vektorske spremenljivke

$z = x + r$ $\quad r$ - merilni šum $\quad$ (merimo samo lego) $\quad \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$

$\langle r^2 \rangle = \sigma_r^2$

$\bar{x} = x + m_x$

$\bar{v} = v + m_v$

$\langle m_x r \rangle = \langle m_v r \rangle = 0$

$\langle m_x m_v \rangle \neq 0$

$$\hat{x} = x + \hat{p}_x = a_{xx} \underline{\underline{\bar{x}}} + a_{xv} \underline{\underline{\bar{v}}} + b_x \underline{\underline{z}}$$

$$\hat{v} = v + p_v = a_{vx} \underline{\underline{\bar{x}}} + a_{vv} \underline{\underline{\bar{v}}} + b_v \underline{\underline{z}}$$

$$\hat{x} = a_{xx}(x + m_x) + a_{xv}(v + m_v) + b_x(x + r) =$$

$$= x(a_{xx} + b_x) + v a_{xv} + \underbrace{a_{xx} m_x + a_{xv} m_v + b_x r}_{\hat{p}_x} =$$

$a_{xx} + b_x = 1$

$a_{xv} = 0$

$$\hat{p}_x = a_{xx} m_x + (1 - a_{xx}) \cdot r$$

$$\langle \hat{p}_x^2 \rangle = a_{xx}^2 \underbrace{\langle m_x^2 \rangle}_{\sigma_x^2} + (1 - a_{xx})^2 \underbrace{\langle r^2 \rangle}_{\sigma_r^2} + a_{xx}(1 - a_{xx}) \langle m_x \cdot r \rangle$$

$$\langle p_x^2 \rangle = a_{xx}^2 \sigma_x^2 + (1 - a_{xx})^2 \sigma_r^2 \quad \Big/ \frac{d}{d a_{xx}} = 0 \quad \text{minimum}$$

$$0 = 2 a_{xx} \sigma_x^2 - (1 - a_{xx}) 2 \sigma_r^2$$

$$\Rightarrow a_{xx} = \frac{\sigma_r^2}{\sigma_x^2 + \sigma_r^2} \qquad b_x = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2}$$

Isto naredimo še za hitrost:

$$\hat{V} = a_{vx}(x+m_x) + a_{vv}(v+m_v) + b_v(x+r) =$$

$$= \underbrace{a_{vx}}_{=1} v + x\underbrace{(a_{vx}+b_v)}_{=0} + a_{vx}m_x + a_{vv}m_v + b_v r$$

$a_{vv} = 1$

$a_{vx} = -b_v$

$$= v + \underbrace{a_{vx}m_x + m_v + (- a_{vx})r}_{\hat{p}_v}$$

$$\langle \hat{p}_v^2 \rangle = a_{vx}^2 \langle m_x^2\rangle + \langle m_v^2\rangle + 2a_{vx}\langle m_x m_v\rangle + (a_{vx})^2\langle r^2\rangle \quad / \frac{d}{da_{vx}}$$

$$0 = 2a_{vx}\sigma_x^2 + 2\langle m_x m_v\rangle + 2(a_{vx})\sigma_r^2$$

D $$a_{vx}(\sigma_x^2+\sigma_r^2) = -\langle m_x m_v\rangle$$

$$a_{vx} = -\frac{\langle m_x m_v\rangle}{\sigma_x^2+\sigma_r^2} \qquad b_v = -a_{vx}$$

$$\hat{x} = \frac{\sigma_r^2+\sigma_x^2-\sigma_r^2}{\sigma_x^2+\sigma_r^2}\bar{x} + \frac{\sigma_x^2}{\sigma_x^2+\sigma_r^2} z$$

$$\boxed{\hat{x} = \bar{x} + \frac{\sigma_x^2}{\sigma_x^2+\sigma_r^2}(z-\bar{x})}$$

Korelacija da odziv tudi komponent ki jih ne merimo

$$\hat{v} = \bar{v} - \frac{\langle m_x m_v\rangle}{\sigma_x^2+\sigma_r^2}\bar{x} + \frac{\langle m_x m_v\rangle}{\sigma_x^2+\sigma_r^2} z = \bar{v} + \frac{\langle m_x m_v\rangle}{\sigma_x^2+\sigma_r^2}(z-\bar{x})$$

Kovariančna matrika ocen

Def:

$$M = \langle(\bar{x}-x)(\bar{x}-x)^T\rangle$$

$\bar{x} = \vec{\bar{x}}$

$x = \hat{x}$

Kovariančna matrika izstopnih ocen

$$P = \langle(\hat{x}-x)(\hat{x}-x)^T\rangle$$

Za verjetnostno gostoto po prevzamemo večrazsežno Gaussovo porazdelitev

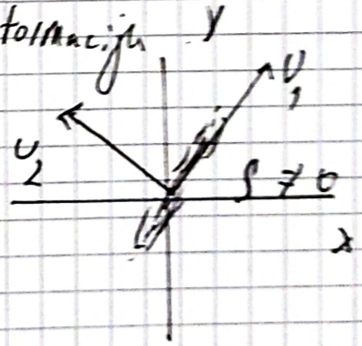
$$p(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \det M}} \exp\left[-\frac{1}{2}(\bar{x}-x)^T M^{-1}(\bar{x}-x)\right]$$

Primer:

$$p(\bar{x},\bar{y}) = \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[\frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y}\right]$$

$\bar{x}, \bar{y} \to u_1, u_2 \qquad \vec{u} = \Omega\vec{\bar{x}}$

ortogonalna transformacija



$$M_0 = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$$

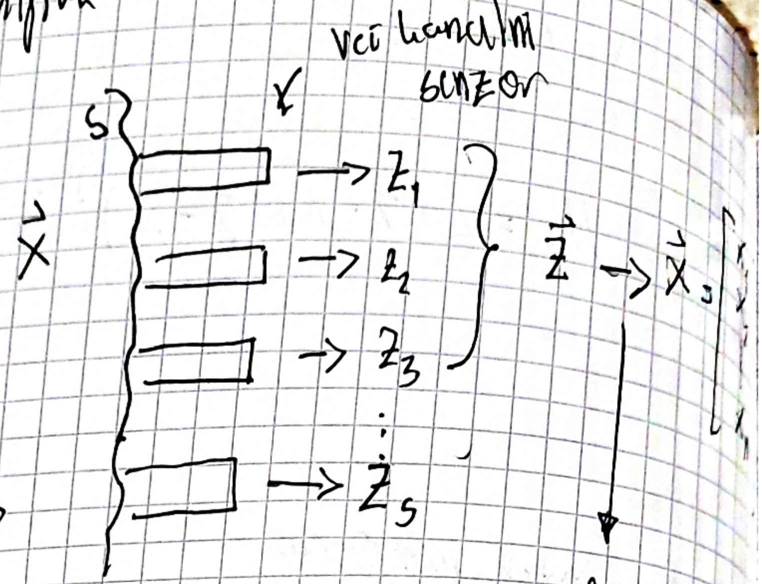
$$p(u_1,u_2) = \prod_{i=1,2} \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma} e^{-\frac{1}{2}(u_i-\bar{u}_i)^2/\sigma_i^2}$$

$\Downarrow$

Produkt dveh Gaussovk.

$$M_u = \Omega M \Omega^T$$

Merjenje več spremenljivk



Povezava je matrika senzorjev H (okvirno)

$$\vec{z} = H\vec{x} + \vec{r}$$

merilni šum

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} \quad s?$$

$$R = \langle \vec{r} \cdot \vec{r}^{\,T} \rangle$$

kovariančna matrika senzorskega šuma

Primer:

$$\begin{bmatrix} x \\ v \end{bmatrix} \quad H = [1, 0]$$

$$z = Hx + r$$

$$= [1, 0]\begin{bmatrix} x \\ v \end{bmatrix} = x$$

$$H = [0, 1] \to \text{sam hitrost}$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ 2 senzorja → lega → hitrost

$$H = \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}$$ → slabo. načeloma (meri malo obojega)

Manjkal ker punila!

$\hat{\vec{x}}$ ... izostrena ocena

$\bar{\vec{x}}$ ... napoved

$\langle (\hat{\vec{x}} - \vec{x})(\hat{\vec{x}} - \vec{x})^T \rangle = P$ kovariančna matrika izostrene ocene

$\langle (\bar{\vec{x}} - \vec{x})(\bar{\vec{x}} - \vec{x})^T \rangle = M$ kovariančna matrika napovedi

Počasi spuščamo velikosti znaka $\hat{x} \to x$

$$\hat{\vec{x}} = \bar{\vec{x}} + PH^T R^{-1} (\vec{z} - H\bar{\vec{x}})$$

$$P^{-1} = M^{-1} + H^T R^{-1} H \to P = M - MH^T (R + HMH^T)^{-1} HM$$

analog ostrenja:

$$\hat{\sigma}_x^{-2} = \bar{\sigma}_x^{2} - \sigma^{-2}$$

I $\quad P = M - MH^T(R + HMH^T)^{-1}HM$ ~~$\cdot P$ iz leve~~ $\rightarrow P = M - PH^TR^{-1}HM$

$P^{-1} = M^{-1} + H^TR^{-1}H$ / $\cdot P$ iz leve, M iz desne

$$PP^{-1} = I = PM^{-1} + PH^TR^{-1}H$$

$$M = P + PH^TR^{-1}HM$$

$$= P(I + H^TR^{-1}HM) \quad / \cdot H^T$$

$$MH^T = P(H^T + H^TR^{-1}HMH^T)$$

$$= PH^T(\underbrace{I}_{R^{-1}R} + R^{-1}HMH^T)$$

$$= PH^TR^{-1}(R + HMH^T)$$

$$\Rightarrow \quad PH^TR^{-1} = MH^T(R + HMH^T)^{-1}$$

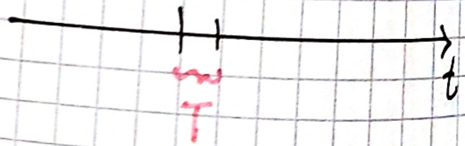
$$\Rightarrow \quad P = M - \underbrace{MH^T(R + HMH^T)^{-1}HM}$$

Zaradi meritve ($R < \infty$) se kovarianca zmanjšuje $\equiv$ Ostrenje

Dinamika + dinamični šum:

- Diskreten primer

$(nT) \rightarrow (n+1)T$; T... čas vzorčenja



Sistem:

S; $\underline{x}_{n+1} = \Phi_n \underline{x}_n + \underline{c}_n + \Gamma_n \underline{w}_n$ vsi vektorji

dinamični šum

M; $\underline{\bar{x}}_{n+1} = \Phi_n \underline{\hat{x}}_n + \underline{c}_n$

$$M_{n+1} = \langle (\bar{x}_{n+1} - x_{n+1})(\bar{x}_{n+1} - x_{n+1})^T \rangle$$

Vstavimo predpise:

isto

$$= \langle (\Phi_n \hat{x}_n c_n - \Phi_n x_n - c_n - \Gamma_n w_n)(\Phi_n(\hat{x}_n - x_n) - \Gamma_n w_n)^T \rangle =$$

$$= \langle (\Phi_n(\hat{x}_n - x_n) - \Gamma_n w_n)((\hat{x}_n - x_n)^T \Phi_n^T - w_n^T \Gamma_n^T) \rangle =$$

$$\langle w_n w_{n'} \rangle = \delta_{nn'} Q_n$$

$$= \Phi_n \underbrace{\langle (\hat{x}_n - x_n)(\hat{x}_n - x_n)^T \rangle}_{P_n} \Phi_n^T + \Gamma_n \underbrace{\langle w_n w_n^T \rangle}_{Q_n} \Gamma_n^T +$$

$$+ \Phi_n \underbrace{\langle \cancel{(\hat{x}_n - x_n) w_n} w_n^T \rangle}_{0} \Phi_n^T + \dots \underbrace{\langle \cancel{\dots} \rangle}_{0} \dots$$

$$\Rightarrow \quad M_{n+1} = \Phi_n P_n \Phi_n^T + \Gamma_n Q_n \Gamma_n^T$$

Kovariančna matrika napovedi

Od prej. šč.: $P_{n+1}^{-1} = M_{n+1}^{-1} + H^T R^{-1} H$

$$P_{n+1} = M_{n+1} - M_{n+1} H^T (R_{n+1} + H M_{n+1} H^T)^{-1} H M_{n+1}$$

To je Kalmanov optimalen filter za diskretno vektorsko spremenljivko.

Prehod v kontinuumsko (zvezno) sliko:

- dinamični šum w
- merilni šum r

} Obravnavamo kot nekorelirana

Ojačevalni faktor

$$\dot{\hat{x}} = \frac{\hat{x}_{n+1} - \hat{x}_n}{T} = \frac{\Phi_n \hat{x}_n + c_n - \hat{x}_n}{T} + K_{n+1}(z_{n+1} - H\bar{x}_{n+1}) \cdot \frac{1}{T} =$$

T diskret	lim $T \to 0$
$\hat{x}_{n+1}$	$\hat{x}(t)$
$\bar{x}_{n+1}$	$\bar{x}(t)$
TR_n	$R(t)$
z_n	$z(t)$
P_n	$P(t)$
M_n	~~[illegible]~~ $P(t)$

$$= \frac{(\phi_n - I)}{T}\hat{x}_n + \frac{C_n}{T} + \frac{P_{n+1} H^T R_{n+1}^{-1}}{T}(z_{n+1} - H\bar{x}_{n+1}) =$$

$$= A(t)\hat{x}(t) + c(t) + PH^T R^{-1}(t)\,(z(t) - Hx(t))$$

$\phi_n = I + AT$

$\dot{x} = Ax + c$

$$\Rightarrow \quad \dot{\hat{x}} = A(t)\hat{x}(t) + c(t) + PH^T R^{-1}(t)(z(t) - H\hat{x}(t))$$

Kaj pa je $\dot{P}$?

$$P_{n+1} - P_n = \underbrace{\phi_n P_n \phi_n^T + \Gamma_n Q_n \Gamma_n^T}_{M_{n+1}} - M_{n+1}H^T(HM_{n+1}H^T + R_{n+1})^{-1}HM_{n+1} - P_n$$

$$= (I + AT)P_n(I + AT)^T + \Gamma_n Q_n \Gamma^T - M_{n+1}H^T(HM_{n+1}H^T + R_{n+1})^{-1}HM_{n+1} - P_n$$

$$= P_n + ATP_n + P_n TA^{-1} + AP_n A^T T^2 + \Gamma_n Q_n \Gamma_n^T - M_{n+1}H^T(HM_{n+1}H^T + R_{n+1})^{-1}HM_{n+1}$$

To delimo s časom T in $\lim T \to 0$:

$$\dot{P} = AP_n + P_n A^T + \emptyset + \left(\frac{\Gamma_n}{t}\right)(Q_n t)\left(\frac{\Gamma_n^T}{T}\right) - PH^T R^{-1}(t) HP$$

$$\frac{\Gamma_n}{t} \downarrow \Gamma(t) \qquad Q_n t \downarrow Q(t) \qquad \frac{\Gamma_n^T}{T} \downarrow \Gamma^T(t)$$

$$\Rightarrow \quad \dot{P} = AP + PA^T + \Gamma Q \Gamma^T - PH^T R^{-1} HP$$

dinamična razvoj kovariančne matrike v zveznem primeru

Riccatijeva enačba

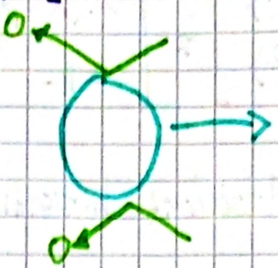
$AP + PA^T$ ↓ odvisno od primera (npr. upor ima to negativno)

$\Gamma Q \Gamma^T$ ↓ dinamični šum vedno veča

$PH^T R^{-1} HP$ ↓ meritev vedno manjša

$R \to \infty$, nič ne merimo

Primer: [Brownovo gibanje koloidnega delca v raztopini]



Dinamika (1D) / Stokesov linearni zakon upora

$$m\ddot{x} = -6\pi r\eta \cdot \dot{x} + F_x(t)$$

naključne sile zaradi diskretnih trkov

$$\left\langle \frac{F_x(t)}{m} \cdot \frac{F_x(t')}{m} \right\rangle = Q\,\delta(t-t')$$

Opišemo z dinamičnim šumom

Kahlmanov filter za sledenje delca: $\vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$

Ob $t=0$;

$$P(0) = P_0$$

$$\bar{x}(0) = 0$$ → na začetku v izhodišču

Ob $t>0$;

Dinamika:

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{1}{\tau} v + \frac{F_x(t)}{m} \end{aligned}\right\}$$ sistem 1. reda

$6\pi r\eta$

Dinamični šum na 2. komponenti za $\vec{x}$

$$\dot{\vec{x}} = A\vec{x} + \vec{c} + \Gamma w$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + 0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{F_x(t)}{m}$$

Kaj pa P?

$(AP)^T$

$$\dot{P} = \begin{bmatrix} \dot{P}_{xx} & \dot{P}_{xv} \\ \dot{P}_{vx} & \dot{P}_{vv} \end{bmatrix} = AP + PA^T + \Gamma Q \Gamma^T$$

$$\dot{P} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle vx\rangle & \langle v^2\rangle \end{bmatrix} + PA^T + \begin{bmatrix} 0 \\ 1 \end{bmatrix} Q[0,1] =$$

$$= \begin{bmatrix} \langle xv\rangle, & \langle v^2\rangle \\ -\frac{1}{\tau}\langle xv\rangle, & -\frac{1}{\tau}\langle v^2\rangle \end{bmatrix} + \begin{bmatrix} \langle xv\rangle & -\frac{1}{\tau}\langle xv\rangle \\ \langle v^2\rangle & -\frac{1}{\tau}\langle v^2\rangle \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} =$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle vx\rangle & \langle v^2\rangle \end{bmatrix} = \begin{bmatrix} 2\langle xv\rangle , & \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle \\ -\shortparallel- , & -\frac{2}{\tau}\langle v^2\rangle + Q \end{bmatrix}$$

Zanima nas a se nedoločenost hitrosti kje ustali (stacionarne rešitve):

$$\frac{d}{dt}\langle v^2\rangle = -\frac{2}{\tau}\langle v^2\rangle + Q$$

$$\frac{d}{dt}\langle v^2\rangle = 0 ; \; t \to \infty$$

$$\Rightarrow \quad \langle v^2\rangle_\infty = \frac{Q\tau}{2}$$

Ubistu iščemo termodinamsko ravnovesje:

$$\frac{1}{2}m\langle v^2\rangle = \langle W_k\rangle = \frac{1}{2}k_B T$$

$$\Rightarrow \quad \langle v^2\rangle_\infty = \frac{kT}{m}$$

Torej:

$$\frac{Q\tau}{2} = \frac{kT}{m} \quad \Rightarrow \quad Q = \frac{2kT}{\tau m}$$

Korelacijski člen?

$$\frac{d}{dt}\langle xv\rangle = \langle v^2\rangle = \frac{1}{\tau}\langle xv\rangle$$

$$\Rightarrow \langle v^2\rangle_\infty = \frac{1}{\tau}\langle xv\rangle_\infty \Rightarrow \langle xv\rangle_\infty = \frac{\tau \cdot kT}{m}$$

Še za tretjo komponento:

$$\frac{d}{dt}\langle x^2\rangle = 2\langle xv\rangle \quad \Big/ \cdot \int dt$$

Ne obstaja stacionarna rešitev za lego.

$$\Rightarrow \quad \langle x^2(t)\rangle - \langle x^2(0)\rangle = \underbrace{\frac{2\tau kT}{m}}_{D}\, t$$

$$\to \quad X^2 - X_0^2 = 2Dt$$

To je difuzijski zakon!

Sedaj vključimo še meritev lege ($R < \infty$)

$$\dot{P} = \dots - PH^TR^{-1}HP$$

$Z = Hx + r$

$Z = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}x\\v\end{bmatrix}$

$\langle rr^T\rangle = R$ = skalar σ^2

$$-\begin{bmatrix}\langle x^2\rangle & \langle xv\rangle\\ \langle xv\rangle & \langle v^2\rangle\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\frac{1}{R}[1,0]\begin{bmatrix}\langle x^2\rangle & \langle xv\rangle\\ \langle xv\rangle & \langle v^2\rangle\end{bmatrix} =$$

$$= -\frac{1}{R}\begin{bmatrix}\langle x^2\rangle\\ \langle xv\rangle\end{bmatrix}[\langle x^2\rangle\ \langle xv\rangle] = -\frac{1}{R}\begin{bmatrix}\langle x^2\rangle^2 & \langle x^2\rangle\langle xv\rangle\\ \langle xv\rangle\langle x^2\rangle & \langle xv\rangle^2\end{bmatrix}$$

To moramo še dodati prejšnjemu $\dot{P}$. Iščemo spet stacionarne rešitve.

1.) $2\langle xv\rangle - \frac{1}{R}\langle x^2\rangle^2 = 0$

2.) $\langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle - \frac{1}{R}\langle x^2\rangle\langle xv\rangle = 0$

3.) $-\frac{2}{\tau}\langle v^2\rangle - \frac{1}{R}\langle xv\rangle^2 + Q = 0$

$y = \langle x^2\rangle \frac{\tau}{R}$

Iz 3.) $\Rightarrow \langle v^2\rangle = \frac{\tau Q}{2} - \frac{\tau}{2R}\langle xv\rangle^2$

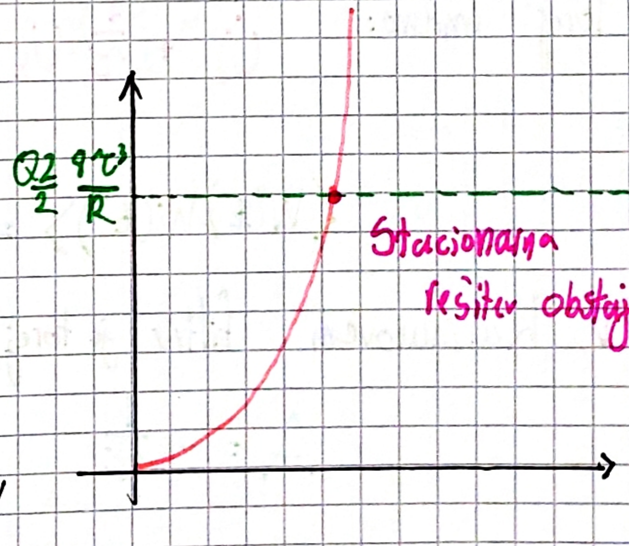
Uvedemo y

$$\frac{Q\tau}{2} - \frac{\tau}{2R}\langle xv\rangle^2 - \frac{1}{\tau}\langle xv\rangle - \frac{1}{R}\langle x^2\rangle\langle xv\rangle = 0$$

$$\frac{Q\tau}{2} - \frac{\tau}{2R}\langle x^2\rangle^4\frac{1}{4R^2} - \frac{1}{\tau}\frac{1}{2R}\langle x^2\rangle^2 - \frac{1}{R}\langle x^2\rangle\frac{1}{2R}\langle x^2\rangle^2$$

$$\frac{Q\tau}{2} - \frac{y^4 R}{\tau^3\cdot 8} - \frac{4y^2 R}{8\tau^3} - \frac{y^3 R\cdot 4}{8\tau^3}$$

$$\Rightarrow \frac{Q\tau}{2}\frac{8\tau^3}{R} = y^4 + 4y^2 + 4y^3$$



Kakršnakoli meritev vodi do omejevanja v prostoru.

Primer "grobe meritve"; R velik → y majhen

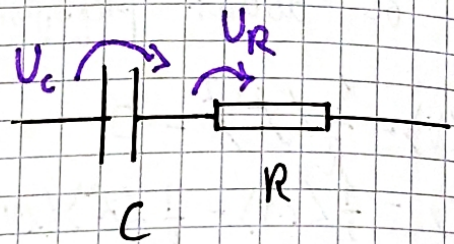
$$y \gg y^2 \gg y^3 \gg y^4$$

$$\Rightarrow \quad \frac{Q\tau}{2}\,\frac{8\tau^3}{R} = 4y^2 = \frac{4Q\tau^4}{R}$$

$$\Rightarrow \quad y = \sqrt{\frac{Q}{R}}\,\tau^2 = \langle x^2\rangle \cdot \frac{\tau}{R}$$

$$\Rightarrow \quad \langle x^2\rangle_\infty = \sqrt{\frac{Q}{R}}\,\tau R = \sqrt{QR}\cdot\tau$$

Primer: [Merjenje napetosti na RC členu]



Napišemo lahko Kirchoffov zakon:

$$\sum U_i = 0$$

$$U_R + U_C = 0$$

$$-IR - \frac{e}{C} = 0$$

$$-\frac{e}{C} = U_C = -U_R = IR$$

$$\dot U_C = -\frac{I}{C} = +\frac{U_R}{RC} = -\frac{U_C}{\tau}; \quad \tau = RC$$

Torej imamo: $\dot U_c + \frac{1}{\tau} U_c = W(t)$ — Preimenujemo $U_c \to U$; dinamični šum

$$\langle W(t) W(t')\rangle = Q\,\delta(t-t')$$

V Kalmanovem filtru je torej

$$A = -\frac{1}{\tau} \qquad \Gamma = 1$$

Meritev napetosti na C:

$$z = u + r \quad ; \quad \langle r(t) r(t') \rangle \underline{\underline{R}} \, \delta(t-t')$$

V sistemu M:

$\hat{u}$ ocena za U v sistemu S

Kovarianca ocene

$$\langle (\hat{U} - U)^2 \rangle = P$$

Torej je Kalman:

$$\dot{P}(t) = 2AP + \Gamma^2 Q - P^2/R$$

i) Stacionarna rešitev $t \to \infty$; $P(t\to\infty) = P_\infty$

$$\dot{P} = 0 \quad \Rightarrow \quad -P^2/R + 2AP + \Gamma^2 Q = 0$$

Vstavimo A in Γ:

$$-\frac{P^2}{R} - \frac{2}{\tau} P + Q = 0$$

$$\frac{1}{R} (P - P_{1_\infty})(P - P_{2_\infty}) = 0$$

$$P_{1,2_\infty} = -\frac{2R}{\tau} \frac{1}{2} \pm \sqrt{\left(\frac{2R}{\tau}\right)^2 + 4QR} \cdot \frac{1}{2} =$$

$$= -\frac{R}{\tau} \pm \alpha \; ; \qquad \alpha = \frac{R}{\tau} \sqrt{1 + \frac{QR\tau^2}{R^2}}$$

ii) Splošna rešitev

Za $\forall t$:

$$\frac{dP}{\left(P^2/R + 2P/\tau - Q\right)} = -dt$$

$$\frac{dP \cdot R}{\left(P + \frac{R}{\tau} - \alpha\right)\left(P + \frac{R}{\tau} + \alpha\right)} = -dt$$

Razbijemo na parcialne ulomke

$$\Rightarrow \quad -dt = R\left[\frac{dP\, B}{\left(p+\frac{R}{\tau}+\alpha\right)} + \frac{dP\, D}{\left(p+\frac{R}{\tau}-\alpha\right)}\right]$$

$$Bp + B\frac{R}{\tau} - \alpha B + pD + D\frac{R}{\tau} + \alpha D = 1$$

$$p(B+D)=0 \quad \Rightarrow \quad B=-D$$

$$B\frac{R}{\tau} - \alpha B + \frac{R}{\tau}D + \alpha D = 1$$

$$\alpha(-B+D) = 1$$

$$2D\alpha = 1 \quad \Rightarrow \quad D = \frac{1}{2\alpha} \qquad B = -\frac{1}{2\alpha}$$

Tako imamo enačbo:

$$-dt = \frac{R}{2\alpha}\left[\frac{dp}{\left(p+\frac{R}{\tau}-\alpha\right)} - \frac{dp}{\left(p+R/\tau+\alpha\right)}\right] \Big/ \int$$

$$\left.\frac{-2\alpha t}{R}\right|_0^t = \ln \left.\frac{\left(p+\frac{R}{\tau}-\alpha\right)}{\left(p+\frac{R}{\tau}+\alpha\right)}\right|_{P_0}^{P(t)}$$

$$\Rightarrow \quad \frac{(p+R/\tau-\alpha)}{(p+R/\tau+\alpha)} = \underbrace{\left(\frac{P_0+\frac{R}{\tau}-\alpha}{P_0+\frac{R}{\tau}+\alpha}\right)}_{P_0 \to \infty} e^{-2\alpha t/R}$$

ko $(t=0)$ sej še nič ne vemo o sistemu $\Rightarrow (\dots)=1$

Ostane:

$$(p+R/\tau-\alpha) = (p+R/\tau+\alpha)e^{-2\alpha t/R}$$

$$P\left(1-e^{-2\alpha t/R}\right) + \frac{R}{\tau}\left(1-e^{-2\alpha t/R}\right) = \alpha\left(1+e^{-2\alpha t/R}\right)$$

Tako dobimo končno rešitev:

$$P(t) = -\frac{R}{\tau} + \alpha\left(\frac{1+e^{-2\alpha t/R}}{1-e^{-2\alpha t/R}}\right)$$

Limitni primer $t\to\infty$, da vidimo, če se ujema

$$P_\infty = -\frac{R}{\tau} + \alpha$$

$$= -\frac{R}{\tau} + \sqrt{\left(\frac{R}{\tau}\right)^2 + QR}$$

$$P_\infty = -\frac{R}{\tau} + \frac{R}{\tau}\sqrt{1+\frac{Q\tau^2}{R}}$$

$$\approx -\frac{R}{\tau}\left(1-\left(1+\frac{1}{2}\frac{Q\tau^2}{R}\right)\right) \quad ; \quad Q \text{ majhen}$$

$$\Rightarrow P_\infty = +\frac{R}{\tau}\frac{Q\tau^2}{2R} \Rightarrow P_\infty = \frac{Q\tau}{2}$$

V sistemu $\mathcal{M}$:

$$\dot{\hat{U}} = -\frac{1}{\tau}\hat{U} + K(t)[z-\hat{U}]$$

$$K(t) = \frac{P_\infty}{R} = \frac{Q\tau}{2R}$$

$$\dot{\hat{U}} = -\frac{1}{\tau}\hat{U} + \frac{Q\tau}{2R}(z-\hat{U}) = \underbrace{\left(-\frac{1}{\tau} - \frac{Q\tau}{2R}\right)}_{-1/\tau_{eff}}\hat{U} + \frac{Q\tau}{2R}z$$

$$\Rightarrow \frac{1}{\tau_{eff}} = \frac{1}{\tau} + \frac{Q\tau}{2R} \quad ; \quad Q=0 \quad \tau_{eff}=\tau$$

$$Q\to\infty \quad \tau_{eff}\to 0$$

Npr. eksponentno padajoč sunek

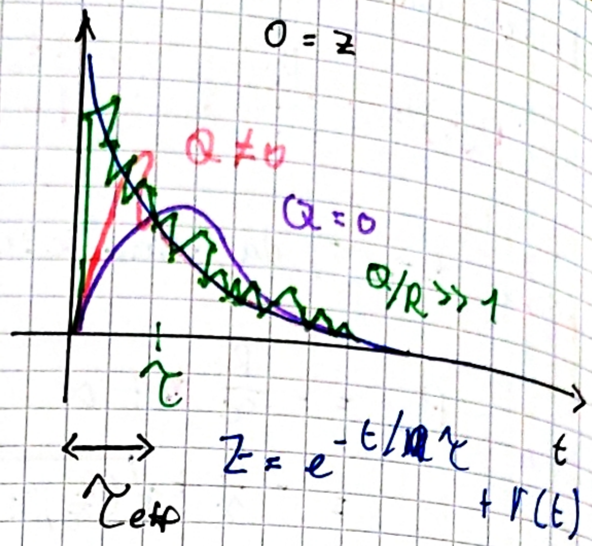
Temu sledimo diskretno

$$\dot{X} = Ax + c$$

$$-\frac{1}{\tau}\dot{X} = Ax + c$$

$$-\frac{1}{\tau} = A$$

$$\left(\Theta = 1 + AT = 1 - \frac{T}{\tau}\right)$$



i) $Q = 0$, $K_\infty = \frac{Q}{R}$

ii) $Q \neq 0$; $\tau_{eff} < \tau$

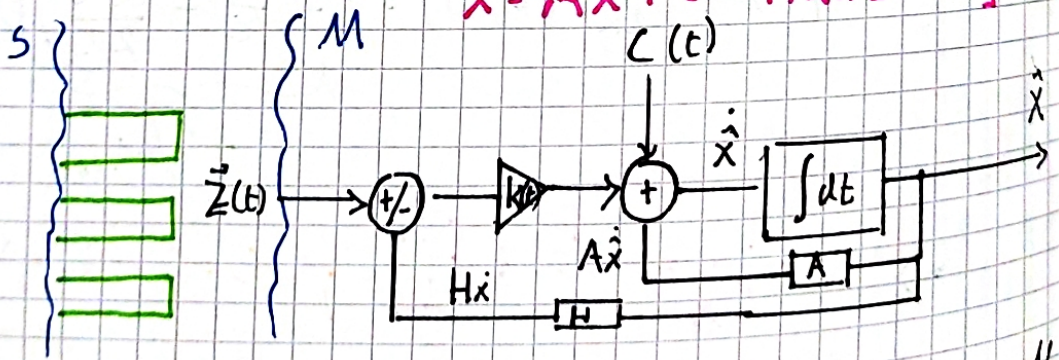
iii) $Q/R \gg 1$; $Q \neq 0$

Izkaže se, da je K_∞ dober za to če hočemo končen rezultat in nas sledenje začetnim tranzientom ne zanima.

(Spomni se temp. vode ko potopiš termometer in rabi nekaj časa, da se ustali. Samo ustaljeno nas zanima)

Poenostavitve kalmanove sheme in povratna zanka

$$\dot{\hat{x}} = A\hat{x} + c(t) + K(t)[z - H\hat{x}]$$



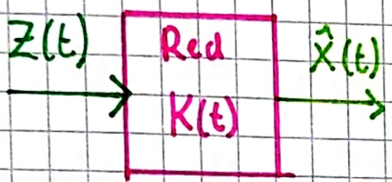
~~Kvant~~ $K(t)[Z - H\hat{x}]$ → stopnja sinhronizacije med S in M

Kadar sta S in M usklajena je lahko K karkoli

=> Tudi K_∞ bo dober

Poglejmo si senzor kot univerzalen merilni sistem. Želimo si:

i) Na izhodu senzorja naj bo napetost $\hat{x} = U(t)$
ii) Odvisnost samo od ene količine (x)
iii) Senzor naj odpravi sam čim več merilnega šuma
iv) Senzor naj čim manj vpliva nazaj na opazovani sistem
v) $\hat{x}(t) = U(t)$; naj bo to berljiva količina



Senzor te dve povezuje preko diferencialne enačbe

· Red senzorja;

Def = Red diferencialne enačbe, ki povezuje $Z(t)$ in $\hat{x}(t)$

Lair. Def = υ-ti red senzorja $(\upsilon > 0)$ obravnavamo kot idealen = optimalen sledilni sistem za spremenljivke $\vec{x}(t)$ v sistemu s katerih dinamike se spreminja kvečjemu:

$$\frac{d^{(\upsilon)}}{dt^{\upsilon}} x(t) = 0 + W(t)$$

Spomni se termometra pod pazduho in grejemo / ima vedno zamik. Če bi bila temp. non lin. odvisna recimo x^2, x^3 bi lahko T čisto pobegnila senzorju

R

f

Senzor 1. reda

V S: $\dot{X} = W(t)$ $\langle W^2 \rangle = Q$

$Z = X + r(t)$ $\langle r^2 \rangle = R$

Kalman za optimalno pravi:

$$\dot{X} = 0 - W$$

$$A = 0$$

$$C = 0$$

$$\Gamma = 1$$

$$H = 1$$

V sistemu M pa:

Ocena na izhodu senzorja

$$\dot{\hat{X}} = K(z - \hat{X})$$

$$\dot{P} = -P^2/R + Q$$

$$K = P/R$$

Če je ojačevalni faktor konstanten $K(t) \to K_\infty = \frac{P_\infty}{R}$

$$\dot{P} = 0, \quad \frac{P_\infty^2}{R} = Q$$

$$\Rightarrow P_\infty = \sqrt{QR} \quad K_\infty = \sqrt{Q/R}$$

Vpeljemo še

$$\tau = \frac{1}{K_\infty} = \sqrt{\frac{R}{Q}}$$

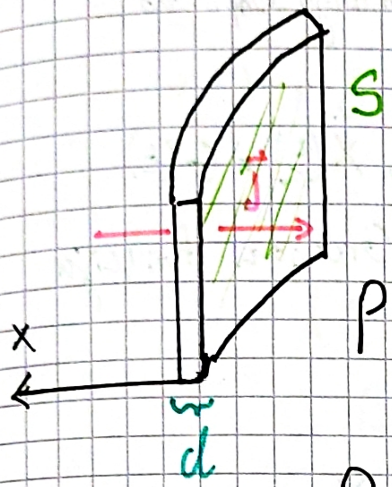
$$\frac{1}{K_\infty}\dot{\hat{X}} + \hat{X} = Z(t)$$

$$\boxed{\tau \dot{\hat{X}} + \hat{X} = Z}$$

Dif. en. 1. reda za Senzor 1. reda

Je optimalni indikator (metoda za sledenje konstante)

Primer: [Termometer]



$$P = S_j = \frac{\lambda S (T_z - T)}{d}$$

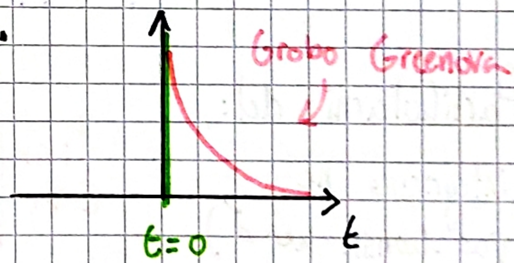
$$P = \frac{dQ}{dt} = m c_p \frac{dT}{dt}$$

$$\frac{d m c_p}{\lambda S} \frac{dT}{dt} = -\frac{\lambda S}{d}(T_z - T)$$

$$\Rightarrow \quad K\dot{T} + T = T_z(t)$$

Enačba senzorja 1. reda!

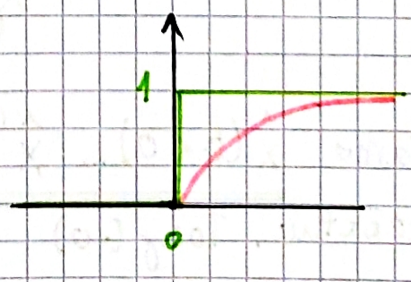
Zanima nas obnašanje senzorja 1. reda, ko $Z(t) \neq$ konst (in sistemske napake, prehodna obdobja...).



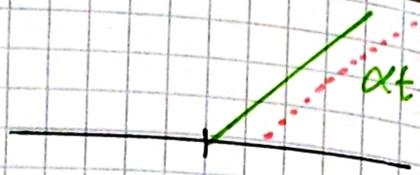
Tipični vhodi $Z(t)$: i) $Z(t) = \delta(t)$

Greenova funkcija nam pove vse od odziva senzorja in tipa ipd.

ii) $Z(t) = H_0(t)$



iii) $z(t) = t^n$



iv) $z(t) = \cos \omega t$ harmoničen ~~vh~~ vhod

Pričakujemo isto frekvenco in fazni zamik in drugo amplitudo

__1. red__ $\tau \dot{\hat{x}} + \hat{x} = z(t)$ $\cancel{\delta(t)}$,,, $\hat{x}(t) = G(t)$

Greenova funkcija

i) $z(t) = \delta(t)$

Homogeni del: $\tau \dot{\hat{x}} + \hat{x} = 0 \qquad \tau \cdot \frac{d\hat{x}}{dt} = -x$

$$\hat{x} = C e^{-\lambda t}$$

$$\tau(-\lambda) C e^{-\lambda t} + C e^{-\lambda t} = 0$$

$$(-\lambda \tau + 1) e^{-\lambda t} = 0$$

$$\Rightarrow \lambda = 1/\tau$$

Homogeni del: $\hat{x}_H = C e^{-t/\tau}$

Partikularni del:
(integriramo ne var. konst. ker δ)

$$\lim_{\varepsilon \to 0} \Bigg[\underbrace{\int_{-\varepsilon}^{\varepsilon} \tau \frac{d\hat{x}}{dt} dt}_{\downarrow} + \underbrace{\int_{-\varepsilon}^{\varepsilon} \hat{x}\, dt}_{\to 0} = \underbrace{\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt}_{1} \Bigg]$$

$$\tau [\underbrace{\hat{x}(\varepsilon)}_{+0} - \underbrace{\hat{x}(-\tau)}_{-0}] = 1$$

Zahtevamo $\hat{x}(t<0) \dots \hat{x}^{(n)}(t<0) = 0$, da je "števec" pri miru na začetku. Torej (-0) zanemarimo

$$\Rightarrow \hat{x}_p(0) = 1/\tau$$

iii) $Z(t) = \alpha t$

homogeni del: Je isti seveda $\hat{X}_H(t) = Ce^{-t/\tau}$

Partikularni del: $\tau\dot{\hat{x}} + \hat{x} = \alpha t$ Nastavek: $\hat{X}_p = At + B$

$$\tau A + At + B = \alpha t$$

$$A = \alpha$$

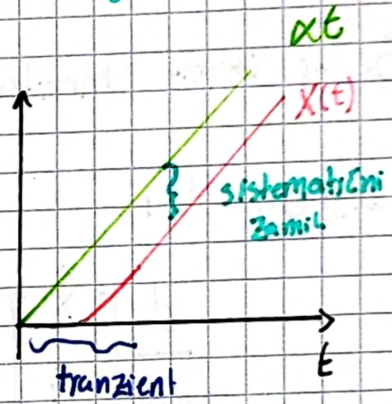
$$A\tau + B = 0$$

$$B = -\alpha\tau \Rightarrow X_p = \alpha t - \alpha\tau$$

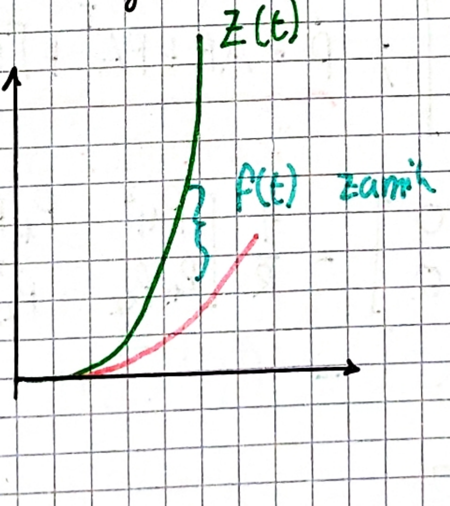
$$\Rightarrow X(t) = \alpha t - \alpha\tau + Ce^{-t/\tau}$$

Zahtevamo $X(0) = 0 \Rightarrow C = \alpha\tau$

$$\Rightarrow X(t) = \alpha(t-\tau) + \alpha\tau e^{-t/\tau}$$



Kaj ce bi s tem senzorjem sledili $Z(t) = \beta t^2$



$\Rightarrow$ Ne sledi