

# Chapter 1

## Linear Equations in Linear Algebra

### 1.1 Systems of Linear Equations

**Definition 1.1.1.** A Linear Equation is the variables  $x_1, x_2, \dots, x_n$  is an equation that can be written in the form  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$  where  $a_1, a_2, \dots, a_n$  are real coefficient and  $b$  is a real number (and known)

**Definition 1.1.2.** A System of Linear Equations 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$
  
m number of equations, n number of unknowns (standard form) (first index row number, second index col number)

**Definition 1.1.3.** A solution of the system is a list  $(s_1, s_2, \dots, s_n)$  of numbers that makes each equation a true statement when the values are substituted for  $x_1, x_2, \dots, x_n$

**Definition 1.1.4.** Solution Set is the set of all possible solutions

Geometric Interpretations Example) Find the Solution set of the system

(a)  $\begin{cases} x_1 - x_2 = 5 \\ 2x_1 + x_2 = 7 \end{cases}$

(b)  $\begin{cases} x_1 - 2x_2 = 4 \\ -2x_1 + 4x_2 = -8 \end{cases}$

(c)  $\begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 + 6x_2 = 5 \end{cases}$

**Definition 1.1.5.** A linear system is consistent if it has either one solution or infinitely many solutions

**Definition 1.1.6.** Matrix of Coefficients 
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

**Definition 1.1.7.** Augmented Matrix of the System

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

## 1.2 Row Reduction and Echelon Forms

**Definition 1.2.1.** A leading of a row in a matrix is the left most non-zero entry

Example) 
$$\left[ \begin{array}{cccccc} 0 & 0 & \textcircled{7} & 3 & 4 & 1 \\ \textcircled{2} & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{-2} & 0 \end{array} \right]$$

**Definition 1.2.2.** A rectangular matrix is in echelon form if it has the following three properties:

1. All non-zero rows are above any zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry above it.
3. All entries in a column below a leading entry are zero.

## 1.3 Vector Equations

**Definition 1.3.1.** Vectors

In  $R^2$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ , in  $R^3$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ , in  $R^n$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

**Definition 1.3.2.** Alebraic Operations of Vectors.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Addition:  $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$

Multiply by Scaler:  $c \in R \quad c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$

**Definition 1.3.3.** Linear Combination of Vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  vectors in  $R^n$  $c_1, c_2, \dots, c_n$  scalarsLinear Combination:  $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ **Definition 1.3.4.** Vector Form of a System of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \quad \quad \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\vec{a}_1 \quad \quad \quad \vec{a}_2 \quad \quad \quad \vec{a}_n \quad \quad \quad \vec{b}_n$$

Short Vector Form:  $\vec{a}_1x_1 + \vec{a}_2x_2 + \dots + \vec{a}_nx_n = \vec{b}_n$ 

$$\text{Long Vector Form: } \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \dots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

**Example 1.3.1.**

$$\text{Standard Form: } \begin{cases} 2x_1 + 3x_2 - 4x_3 = 5 \\ x_1 \quad \quad \quad + 2x_3 = 1 \\ \quad \quad \quad x_2 - x_3 = 4 \end{cases}$$

$$\text{Augmented Matrix: } \left[ \begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 4 \end{array} \right]$$

$$\text{Vector Form: } \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

**Definition 1.3.5.**

If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  are vectors in  $R^N$  then the set of all linear combinations of  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$  is denoted by  $\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$  and is called a subset of  $R^N$  spanned (or generated) by  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ .

**Example 1.3.2.**

$$\text{for } R^3, \text{ describe } \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

All Linear Combinations we get the  $x_1x_3$ -plane

Remark: A system of linear equations is consistent if  $\vec{b}$  is in  $\text{Span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$

**Example 1.3.3.**

$$\text{Determine if } \vec{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix} \text{ is in the } \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} x_3 = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 1 & 7 & -5 \\ 1 & 2 & 5 & 9 \end{array} \right] \text{RowOperations} \rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & -17 & -18 \end{array} \right]$$

Yes it is in Span because it is consistent!

Remark:

- 1) If the question is determine wheter the system is consistent or not. Then usually it is enough to get Echelon Form of the Augmented Matrix.
- 2) If the question is to solve the system, then we need Reduced Echelon Form of the Augmented Matrix

**Example 1.3.4.**

$$\begin{array}{rcl} x_1 + x_2 - 2x_3 & = & 5 \\ x_1 - x_2 + x_3 & = & 7 \\ 5x_1 - x_2 - x_3 & = & 31 \end{array} = \left[ \begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 1 & -1 & 1 & 7 \\ 5 & -1 & -1 & 31 \end{array} \right] \text{RowOperations} \rightarrow$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 1 & -\frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = -x_2 + 2x_3 + 5 \\ x_2 = \frac{3}{2}x_3 - 1 \\ x_3 = \text{Parameter} \end{array}$$

Wrong Because  $-x_2$  is not a parameter. If it's a pivot column, it can't be a parameter.

$$\text{RowOperations} \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 6 \\ 0 & 1 & -\frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = \frac{1}{2}x_3 + 6 \\ x_2 = \frac{3}{2}x_3 - 1 \\ x_3 = \text{Parameter} \end{array}$$

Remember: Echelon Form of a matrix is not unique. Reduced Echelon Form IS unique.

## 1.4 The Matrix Equation $A\vec{x} = \vec{b}$

$$\begin{array}{rcl} a_{11}x_1 & + & a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

Standard Form:

$$\text{Matrix Form: } \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

Theorem: The system  $A\vec{x} = \vec{b}$  has a solution Iff  $\vec{b}$  is a linear combination of  $A, \vec{b} \in \text{Span}\{\text{column vectors of } A\}$

**Example 1.4.1.**  $A = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix} \vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$

$$\text{Standard Form: } \begin{cases} 3x_1 + 5x_2 - 1x_3 = 4 \\ 2x_1 + \phantom{5x_2} + 4x_3 = 2 \\ \phantom{2x_1} 1x_2 + 2x_3 = -1 \end{cases}$$

$$\text{Matrix Form: } \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$\text{Vector Form: } \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

**Example 1.4.2.** How many rows have pivot positions?

$$A = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & -1 & 5 \\ -1 & -2 & 1 & 7 \\ 1 & 1 & 0 & -6 \end{bmatrix} \xrightarrow{\text{RowOperations}} \begin{bmatrix} \textcircled{1} & 3 & -2 & -2 \\ 0 & \textcircled{1} & -1 & 5 \\ 0 & 0 & 0 & \textcircled{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A as above

$A\vec{x} = \vec{b}$  Assume system is consistent

$Q_1$ : On how many parameters does the solution depend?

Answer: One ( $x_3$ )

$Q_2$ : Is it true that  $A\vec{x} = \vec{b}$  has a solution for any  $\vec{b} \in R^4$ ?

Answer: Only if there is a pivot position in each row. - So it's False.

**Example 1.4.3.** Do the vectors  $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 0 \\ 7 \\ 5 \\ -1 \end{bmatrix}$ ,  $\vec{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 2 \\ 1 \end{bmatrix}$  Span  $R^4$ ?

Only 3 vectors, need at least 4 vectors to span  $R^4$  (Still it is not enough, in general)

Theorem: Let  $A$  be an  $m$  row by  $n$  column matrix then the following statements are equivalent.

a) For each  $\vec{b}$  in  $R^m$ , the system  $A\vec{x} = \vec{b}$  has a solution.

b) The columns of  $A$  span  $R^m$ .

c)  $A$  has a pivot position in every row.

## 1.5

## 1.6

## 1.7

## 1.8 Introduction to Linear Transformations

**Example 1.8.1.**

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

**Definition 1.8.1.** Transformation

A transformation (or function or mapping)  $T$  from  $R^n$  to  $R^m$  is a rule that assigns to each vector  $\vec{x}$  in  $R^n$  a vector  $T(\vec{x})$  in  $R^m$ .

The set  $R^n$  is called the Domain of  $T$

The set  $R^m$  is called the Co-Domain of  $T$

$T(\vec{x})$  is called the image of  $\vec{x}$  (under  $T$ )

$\{T(\vec{x}), \vec{x} \in R^n\}$  is called the range of  $T$  (the set of all images)

**Definition 1.8.2.** Linear Transformation

A transformation  $T : R^n \rightarrow R^m$  is linear if:

- (i)  $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$  for any  $\vec{u}, \vec{v}$  from the domain of  $T$
- (ii)  $T(c\vec{u}) = cT(\vec{u})$  for any  $\vec{u}$  and any scalar  $c \in R$

**Example 1.8.2.** .