Chapter 1

Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Definition 1.1.1. A Linear Equations is the variables $x_1, x_2 ... x_n$ is an equation that can be written in the form $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ where $a_1, a_2, ..., a_n$ are real coefficient and b is a real number(and known)

Definition 1.1.2. A System of Linear Equations $\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$
m number of equations, n number of unknowns (standard form) (first index row number, second index col number)

Definition 1.1.3. A solution of the system is a list $(s_1, s_2, ..., s_n)$ of numbers that makes each equation a true statment when the values are substituted for $x_1, x_2, ..., x_n$

Definition 1.1.4. Solution Set is the set of all possible solutions

Geometric Interpretations Example) Find the Solution set of the system

(a) $\begin{cases} x_1 - x_2 & = & 5 \\ 2x_1 + x_2 & = & 7 \end{cases}$

(b) $\begin{cases} x_1 - 2x_2 & = & 4 \\ -2x_1 + 4x_2 & = & -8 \end{cases}$

(c) $\begin{cases} x_1 + 3x_2 & = & 1 \\ 2x_1 + 6x_2 & = & 5 \end{cases}$

Definition 1.1.5. A linear system is consistant if it has either one solution or infinitely many solutions

Definition 1.1.6. Matrix of Coefficients $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

Definition 1.1.7. Augmented Matrix of the System $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$

1.2 Row Reduction and Echelon Forms

Definition 1.2.1. A leading of a row in a matrix is the left most non-zero entry

$$\text{Example)} \begin{bmatrix} 0 & 0 & 7 & 3 & 4 & 1 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{-2} & 0 \end{bmatrix}$$

Definition 1.2.2. A rectangular matrix is in echelon form if it has the following three properties:

1. All non-zero rows are above any zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry above it.
3. All entries in a column below a leading entry are zero.

1.3 Vector Equations

Definition 1.3.1. Vectors

In $R^2, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, in $R^3, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, in $R^n, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Definition 1.3.2. Alebraic Operations of Vectors.

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Addition: $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$

Multipy by Scaler: $c \in R \quad c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$

Definition 1.3.3. Linear Combination of Vectors
$\vec{v_1}, \vec{v_2}, ..., \vec{v_p}$ vectors in R^n
$c_1, c_2, ..., c_n$ scalers
Linear Combination: $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_n\vec{v_n}$

Definition 1.3.4. Vector Form of a System of Linear Equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & = & b_m \\ \vec{a_1} & & \vec{a_2} & & & & \vec{a_n} & & \vec{b_n} \end{array}$$

Short Vector Form: $\vec{a_1}x_1 + \vec{a_2}x_2 + ... + \vec{a_n}x_n = \vec{b_n}$

Long Vector Form: $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + ... + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Example 1.3.1.

Standard Form: $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 5 \\ x_1 \qquad\qquad + 2x_3 = 1 \\ \qquad\quad x_2 - x_3 = 4 \end{cases}$

Augmented Matrix: $\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 4 \end{array}\right]$

Vector Form: $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

Definition 1.3.5.
If $\vec{v_1}, \vec{v_2}, ..., \vec{v_p}$ are vectors in R^N then the set of all linear combonations of $\vec{v_1}, \vec{v_2}, ..., \vec{v_p}$ is denoted by Span$\{\vec{v_1}, \vec{v_2}, ..., \vec{v_p}\}$ and is called a subset of R^N spanned (or generated) by $\vec{v_1}, \vec{v_2}, ..., \vec{v_p}$.

Example 1.3.2.
for R^3, describe Span$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$
All Linear Combontations we get the x_1x_3-plane

Remark: A system of linear equations is consistant if $\vec{b}$ is in Span$\{\vec{a_1}, \vec{a_2}, ..., \vec{a_n}\}$

Example 1.3.3.
Determine if $\vec{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$ is in the Span$\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}\}$

1.4 The Matrix Equation $A\vec{x} = \vec{b}$

Standard Form:
$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & = & b_m \end{array}$$

Matrix Form:
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Theorem: The system $A\vec{x} = \vec{b}$ has a solution Iff $\vec{b}$ is a linear combination of $A, \vec{b} \in$ Span{column vectors of A}