CS/ECE 374 P29

Jiawei Tang, Pengxu Zheng, Abhay Varmaraja

TOTAL POINTS

100 / 100

QUESTION 1

1 29.A. **20 / 20**

- ✓ **- 0 pts** Correct
- **- 15 pts** IDK
- **- 5 pts** Minor error
- **- 15 pts** Major error
- **- 20 pts** Completely wrong (e.g. assuming algorithm terminates after k iterations) or blank

QUESTION 2

2 29.B. **20 / 20**

- ✓ **- 0 pts** Correct
- **- 15 pts** IDK
- **- 5 pts** No English description
- **- 5 pts** Minor error
- **- 10 pts** Major error
- **- 20 pts** Completely wrong

QUESTION 3

3 29.C. **20 / 20**

- ✓ **- 0 pts** Correct
- **- 15 pts** IDK
- **- 5 pts** No English description
- **- 5 pts** Minor error but made use of (b)
- **- 10 pts** Major error
- **- 20 pts** Completely wrong

QUESTION 4

4 29.D. **40 / 40**

- ✓ **- 0 pts** Correct
- **- 30 pts** IDK
- **- 5 pts** Minor notation error
- **- 10 pts** Minor reasoning error
- **- 5 pts** No English description
- **- 20 pts** Major reasoning error
- **- 40 pts** Completely wrong

HW 0 Solution

CS/ECE 374: Algorithms & Models of Computation, Spring 2019 Version: **1.0**

Submitted by:

- **≪Abhay Varmaraja≫**: **≪abhaymv2≫**
- **≪JiaWei Tang≫**: **≪jiaweit2≫**
- **≪Eric Zheng≫**: **≪pzheng5≫**

29. (100 PTS.) Spies.

Given a graph G with two sets of vertices S and C and edges E, where C are citizens of the nation and an edge represents a friendship and S are citizens who are willing to spy on other citizens. The government is attempting find a set $S' \subseteq S$ of minimal size such that they can spy on all citizens.

Let us define a utilize greedy algorithm. Let us define $S_0 = S$ and $C_0 = C$. On the i^{th} iteration the algorithm chooses a s from S such that it connects to as many elements of C_{i-1}. Let us define $S_i = S_{i-1} \cup s_i$ and $C_i = C_{i-1}/\Gamma(s_i)$ where $\Gamma(s_i$ is the set of all citizens that the chosen spy is connected to. We stop the algorithm as soon as C is empty, meaning we are spying on everyone.

(A)

Solution:

Given that the optimal solution is of size k. Prove that for any i in our greedy algorithm, we are selecting at least $|C_{i-1}|/k$ citizens to spy on.

To begin with when we order the optimal solution by "magnitude" we can easily prove that the j^{th} element in the optimal solution is going to be connected to at least $|C_{j-1}|/(k-j)$ elements. This is because the solution has $|C_{j-1}$ citizens to spy on remaining and only $k-j$ elements are left in the solution meaning the average case is $|C_{j-1}|/(k-j)$ and since this first element is the greatest as we are sorting since order is irrelevant, it is always greater than or equal to the average case, since if it is less, then all following elements are also less and the whole set of citizens cannot possibly be spied on. This gives a bound for the number of connections $|\Gamma(s_j) \cap C_{j-1}| \geq |C_{j-1}|/(k-j) \geq |C_{j-1}|/k$.

We can use this to assist in our proof for the greedy algorithm. On the i^{th} iteration, our solution contains some subset of as few 0 elements of the optimal solution, let us say j elements. Then on said iteration the set of C_{i-1} citizens is spanned by the remaining $k-j$ spies in the ordered optimal solution. Hence by our above proof for the optimal solution sizes, on any given iteration of our greedy algorithm, the next step in the optimal solution would choose a spy with connections $|\Gamma(s_j) \cap C_{j-1}| \geq |C_{i-1}|/(k-j) \geq |C_{i-1}|/k$. Since our algorithm is greedy, it will choose a node with number of connections $|\Gamma(s_i) \cap C_{i-1}| \geq |\Gamma(s_j) \cap C_{i-1}| \geq$. Consequently, it follows that $|\Gamma(s_i) \cap C_{i-1}| \geq |C_{i-1}|/k$.

(B)

Solution:

Prove that $|C_i| \leq (1 - 1/k)|C_{i-1}|$.

We can assume from a that any iteration choose s_i with a number of connected citizens $con(s_i) \geq |C_{i-1}|/k$, we also know that $|C_i| = |C_{i-1}| - con(s_i)$. Therefore we have: $|C_{i-1}| - |C_i| \geq |C_{i-1}|/k$

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(B)

Solution:

Prove that $|C_i| \leq (1 - 1/k)|C_{i-1}|$.

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$-|C_i| \geq |C_{i-1}|/k - |C_{i-1}|$
$|C_i| \leq |C_{i-1}| - |C_{i-1}|/k$
$|C_i| \leq |C_{i-1}|(1 - 1/k)$

(C)

Solution:

Using that $(1 - 1/k)^k \leq 1/e$, prove that for all i, we have that $|C_{i+k}| \leq |Ci|/e$

Let us utilize the property proved in B $|C_i| \leq (1 - 1/k)|C_{i-1}|$. Let us reformat this such that $|C_{i+1}| \leq (1-1/k)|C_i|$ it becomes clear that we can repeatedly apply this equation let us say n times and go from $C_{i+n} \leq (1-1/k)^n|C_i|$. The case we are interested in is $n = k$ so $C_{i+k} \leq (1-1/k)^k|C_i|$. Since we are given that $(1 - 1/k)^k \leq 1/e$. This equation becomes $C_{i+k} \leq (1 - 1/k)^k|C_i| \leq |C_i|/e$ or simply $|C_{i+k}| \leq |Ci|/e$. Thus proving our property.

(D)

Solution:

Let us prove, that the greedy algorithm outputs a set of at most $k(\lceil ln(n) \rceil + 1)$ spies, such that all the citizens of C are spied on by these spies.

Let us start of by identifying two features, one that we begin with C of size n and second that C tells us performing k iterations results in C decreasing by a factor of e or more, more formally $|C_{i+k}| \leq |Ci|/e$. Therefore we need to determine how many sets of k iterations must be completed in order to completely reduce C. This mathematically translates to $ln(n)$. What we are claiming is that $|C|/e^{ln(n)} = 1$. This simplifies to $n/e^{ln(n)} = 1$ and further to $n/n = 1$. Leaving 1 more iteration for our algorithm to provide a solution. Therefore we need to add k elements to our set of spies $ln(n)$ times and then perform one more iteration to eliminate the constant. Thus providing us the upper bound of $k(\lceil ln(n) \rceil + 1)$ on our output as it is possible for C to decrease by more than e on some iterations.

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