

Discussion on two kinds of discrete time crystal

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The spontaneous breaking of time-translation symmetry in periodically driven system leads to discrete time crystal (DTC). The discrete time crystal can be divided into two cases, quantum time crystal and classical time crystal. In this work, we explore the existence and properties of these two kinds of time crystal. In quantum case, we work out a time crystal phase and give critical exponents $\nu = 1.53$ and $\beta = 0.28$ by numerical simulation. In classical case, the true long-range order discrete time crystal phase only exists in the 3D system while thermal noise will melt the crystalline order in 1D and 2D system. We obtain that these two kinds of discrete time crystal belong to Ising class.

I. INTRODUCTION

Landau symmetry theory has always been the traditional paradigm of condensed matter physics where the spontaneous symmetry breaking is the vital topic. Under this frame, the translation and rotation symmetry breaking of ground states lead to the general crystal. F. Wilczek raised the issue whether there is a kind of crystal whose ground state has time translation symmetry breaking (TTSB) in 2012.[1] H. Watanabe and M. Oshikawa, however, established the no-go theorem and proved that it is impossible to construct such a crystal under Wilczek frame.[1] In 2016, D. Else put forward the criteria for the TTSB in a floquet system: TTSB occurs for every short ranged correlation state $|\psi(t)\rangle$, and there is an operator Φ such that $\langle\psi(t+T)|\Phi|\psi(t+T)\rangle \neq \langle\psi(t)|\Phi|\psi(t)\rangle$, where $|\psi(t+T)\rangle$ is a state evolves a period T . Namely, the eigenstates of the Floquet operator cannot be short-ranged correlated. In a following study, D. Else constructed the discrete time crystal by the Floquet Hamiltonian.[2] Then N. Yao came up with a proposal to construct time crystal[3], which was realized in laboratory by C. Monroe's team and M. Lukin's team.[4][5]

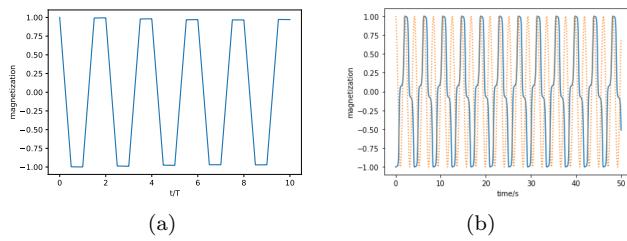


FIG. 1. (a). The quantum discrete time crystal exhibits the $2T$ periodic oscillation. The x axis stands for period and y axis stands for magnetization. (b). The classical discrete time crystal shows $2T$ periodic oscillation. The yellow line is external field with T period while the blue line is the magnetization of classical time crystal.

The beingness of time crystal has been settled yet an-

other question arose: Whether the time crystal phase exists in the classical Hamiltonian system? It is sophisticated since the time crystal system is not in equilibrium. N. Yao attempted to answer the question and found that the activated classical discrete time crystal could survive in 1D system with subharmonic response.[6] Particularly, such a system couples to a finite temperature bath so that it is possible to introduce temperature in an nonequilibrium system. Besides, based on N. Yao's work, Z. Cai found another kind classical discrete time crystal.[7]

This paper discusses the discrete time crystal in its quantum case and classical case. In part two, we discuss quantum case, floquet time crystal, by introducing Floquet Hamiltonian. The numeric result of its rigidity and critical exponents is shown in this part. In part three, we introduce the classical time crystal governed by the Landau-Lifshitz-Gilbert equation. We study the behavior of the system with zero temperature and finite temperature. We also try to characterize its nonequilibrium phase transition and figure out the corresponding universality class.

II. QUANTUM CASE

A. Model and method

Here let's consider Floquet time crystal model, with Floquet Hamiltonian $H(t) = H(t + nT)$:

$$H_f(t) = \begin{cases} H_1 \equiv (g - \epsilon) \sum_i \sigma_i^x, & 0 < t < T_1 \\ H_2 \equiv \sum_i J_i^z \sigma_i^z \sigma_{i+1}^z, & T_1 < t < T, \end{cases}$$

where $J_i^z \in [J_z/2, 3J_z/2]$. Time evolution of the system after a period can be presented by $|\psi(t+T)\rangle = U_f |\psi(t)\rangle$, where $U_f = e^{-iH_2(T-T_1)} e^{-iH_1T_1}$ is the time evolution operator.[2]

For a special case, $g = \pi/2$, $\epsilon = 0$, the system's period is $2T$. For example, the initial state is $|\psi(t)\rangle = |\uparrow\downarrow\downarrow\uparrow\downarrow\cdots\rangle$. In our case, H_1 only flips every site for $e^{-iH_1T_1} = \exp(i(\pi/2) \sum_i \sigma_i^x) = \prod_i i\sigma_i^x$ and H_2 only changes the phase since σ_i^z and H_2 are commutative. Thus, once the evolution operator U_f acts on the state $|\psi(t)\rangle$, the state flips and the phase changes $|\psi(t)\rangle = e^{i\phi} |\downarrow\uparrow\uparrow\downarrow\cdots\rangle$.

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And the operator acts on a period again, the state flips back yet phase alters. If we consider the autocorrelation function $Z(t) = \langle \sigma_i^z(t)\sigma_i^z(0) \rangle$, the system has $2T$ period indeed.

Generally, we can prove that the result is by no means coincidence. Even if there is perturbation and randomness in the system, it could have $2T$ period under the thermodynamics limit. Numeric results can verify it.

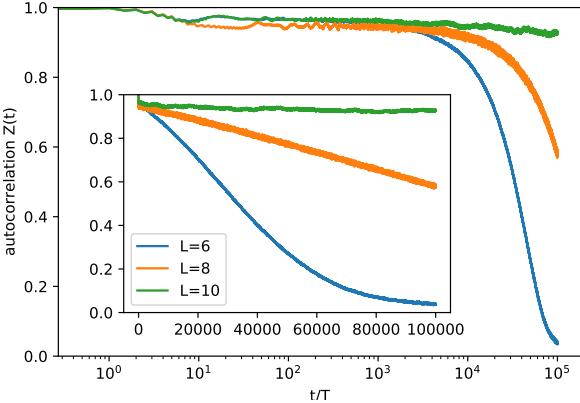


FIG. 2. The decay of 50 times disorder-averaged magnetization $Z(t)$ with $\epsilon = 0.04$ and $J_z = 0.1$ for different size of the system. We find that with the lifetime of the system prolong as the system size grows.

B. result and discussion

The rigidity of floquet time crystal The numeric results in Fig. 2 show that the decay of the system grows exponentially as system's size grows. This result is consistent with ergodicity breaking condition where the lifetime of ergodicity breaking system is diverge as system size growing. Hence, the system is stable under thermodynamics limit.

In Fig. 3, we try to give a phase diagram of our system. The phase diagram shows that perturbation and randomness can not destroy the time crystal phase. Hence, the time crystal phase exists.

The critical behavior To acquire more information about the system, we analyze it from theoretical and numerical simulation aspects. The Hamiltonian tells that the model is random transverse field Ising model. [8] An approximate renormalization group method can be applied to this model. Since the randomness is strong, typical scaling behavior and mean scaling behavior are governed by two length scales with exponents $\nu_{typical} = 1$ and $\nu_{average} = 2$ respectively in $\xi \sim |\epsilon - \epsilon_c|^{-\nu}$.

The critical exponents $\nu_{average}$ can be achieved by numerical methods. From the phase diagram, we know that the matter of phase is up to ϵ and J_z , which are similar to the temperature and pressure in ordinary cases. Besides, the further reason for why ϵ and J_z affect the phase lies

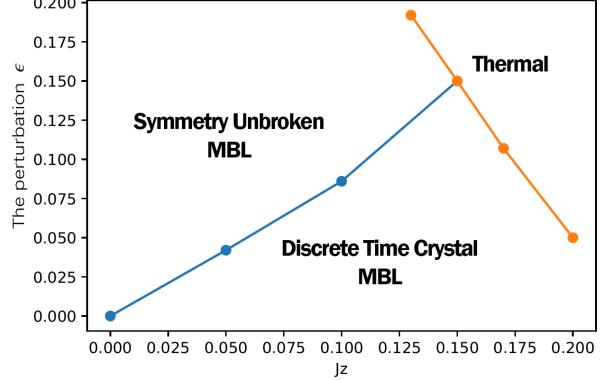


FIG. 3. The phase diagram of quantum case. Data originates from $L = 10$ under 20 disorder averages. We care about the bottom area where the discrete time crystal exists.

in TTSB criteria: the system should have long-ranged correlation. It is the ϵ and J_z that change the correlation length of the system. Intuitively, we choose mutual information $I(A, B) = S_A + S_B - S_{AB}$ as the order parameter since mutual information reflects the relevance of a system. The mutual information obeys the scaling form $I \sim f(L/\xi)$ where $\xi \sim |\epsilon - \epsilon_c|^{-\nu}$. Besides, the system is finite size scaling so that it satisfies $I \sim L^{-\beta}$. Thus, the mutual information has scaling behavior $I \sim f(L/\xi)/L^\beta$, which describes the critical behavior.

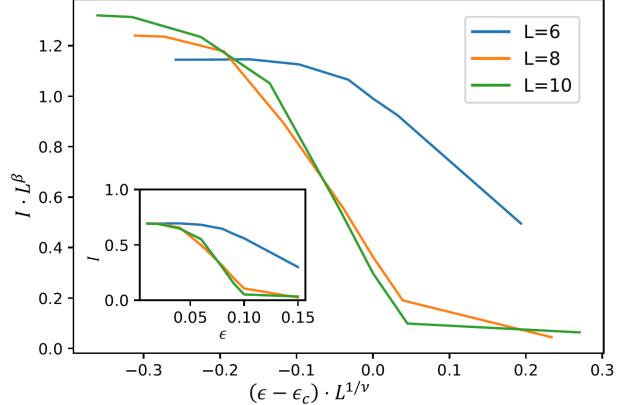


FIG. 4. The mutual information between the 2 left- and right-most sites. We perform 20 times disorder average. The result is $\nu = 1.53$, $\beta = 0.28$ and $\epsilon = 0.09$.

In Fig. 4, we perform a two parameter scaling collapse. We obtain the critical exponent $\nu = 1.53$ and $\beta = 0.28$. But we should question the result. For $L = 6$ case, the model's size is too small to consider correlation length. The right two part has correlation with the left two part significantly. Besides, the errors come from the disorder. We only make 20 times disorder averages so that the specimen is too small to get the right answer.

III. CLASSICAL CASE

A. Model and method

First, it is a transverse Ising model that is discussion with time-varied coupling strength. Its Hamiltonian writes:

$$H_s(t) = -V(t) \sum_{\langle ij \rangle} s_i^x s_j^x + \sum_i h_z s_i^z$$

Where the dynamical variable \hat{s}_i is a classical vector and $\langle ij \rangle$ represents summing over the neighboring spin of \hat{s}_i . The interaction strength (t) periodically oscillates as $V(t) = V_0 \cos \omega_0 t$ since the formation of discrete time crystal needs the periodic-driven. h_z represents for the strength of unified magnetic field along the z axis and we fix h_z as J in this paper. In the absence of thermal bath, the equation of motion is described as $\dot{\mathbf{s}}_i = \mathbf{h}_i^0 \times \mathbf{s}_i$, where \mathbf{h}_i^0 represents the effective magnetic field: $\mathbf{h}_i^0 = [V(t)\bar{s}_i^x, 0, h_z]$, and \bar{s}_i^x means averaging over the x component of the neighbouring spins of \hat{s}_i : $\bar{s}_i^x = \frac{1}{2} \sum_{\langle j \rangle} s_j^x$.

To describe the phase transition in the system, we define its autocorrelation function $C(t) = \langle \frac{1}{L^2} \sum_i s_i^x(0) s_i^x(t) \rangle_\xi$ as the order parameter. we fix the initial state $s_i(0)$ as $s_i(0) = [-1, 0, 0]$ for any i . Therefore, the evolution of the average magnetization along the x direction $M(t) = \langle \frac{1}{L^3} \sum_i s_i^x(t) \rangle_\xi$ is proportional to $C(t)$. For convenience, $C(t)$ will be replaced by $M(t)$ in following calculation.

Besides, we consider three geometrical configurations with periodic boundary conditions shown in Fig.5, which correspond with 1 to 3 dimension lattice.

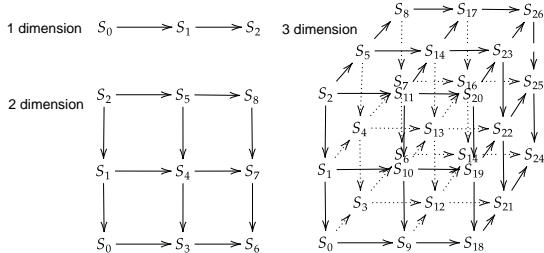


FIG. 5. The geometrical configurations of 1 to 3 dimension lattice models with $L=3$.

Before we further our discussion, the temperature should be defined first. Intuitively, our periodic driven spin system is a nonequilibrium state where the temperature is ill-defined. However, Yao et al. solved the problem by coupling a classical Hamiltonian dynamics to a finite temperature bath. The bath's degree of freedom is way more larger than that of the spin system, so that the impact of the spin system has on the bath can be

neglected. That's to say, the thermal bath have a well-defined temperature (T), and the temperature of the spin system can be defined as the thermal bath's temperature. The impact of the thermal bath can be described by Brownian motion and stochastic process. To specify, the equation of motion can be described by stochastic Landau-Lifshitz-Gilbert (s-LLG) equation:

$$\dot{\mathbf{s}}_i = \mathbf{h}_i \times \mathbf{s}_i - \lambda \mathbf{s}_i \times (\mathbf{s}_i \times \mathbf{h}_i) \quad (1)$$

where λ is a damping parameter and here we fix λ as J . $\mathbf{h}_i = \mathbf{h}_i^0 + \xi_i(t)$ is the effective magnetic field, where $\xi_i(t)$ is a three dimension stochastic magnetic field, representing for the thermal noise. We further assume that the thermal noise is independent so that the every component of every spin at every moment satisfies the Gaussian distribution with mean 0 and variance D :

$$\begin{aligned} \langle \xi_i^\alpha(t) \rangle_\xi &= 0 \\ \langle \xi_i^\alpha(t) \xi_j^\beta(t') \rangle_\xi &= D^2 \delta_{\alpha\beta} \delta_{ij} \delta(t - t') \end{aligned}$$

D represents for the strength of thermal noise and the fluctuation-dissipation theorem indicates that the strengths of the friction and noise satisfy $D^2 = 2T\lambda$.

Here the HEUN algorithm has been used to solve the s-LLG equation.

B. result and discussion

The evolution of 3D system First we investigate the behavior of 3D systems at zero temperature.

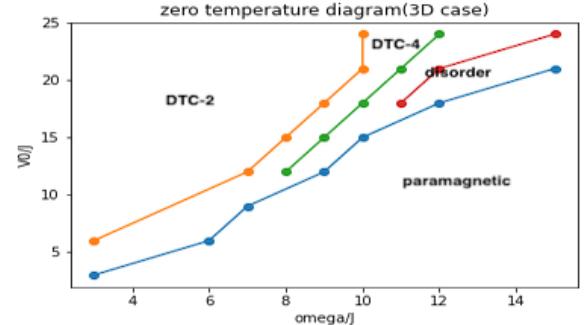


FIG. 6. The zero temperature phase diagram of 3D system when $L = 8$. DTC-n means that the ferromagnetic order parameter $M(t)$ oscillates with a period n times that of the driving $(\frac{2\pi}{\omega_0})$.

As played in Fig. 6, the system shows abundant dynamical behaviors after long-time evolution. For example, the system would be in DTC-2 phase (Fig. 7(b)) when V_0 is large enough yet ω small enough. In addition, DTC-3 phase (Fig. 7(d)) and DTC-4 phase are also observed. Paramagnetic phase refers to the situation when order parameter decay to 0 (Fig. 7(a)). This

could happen when ω is relatively large and V_0 is relatively small. The last phase observed in our simulation is "chaotic phase" (Fig. 7(c)), when $M(t)$ would oscillate erratically instead of decaying to zero.

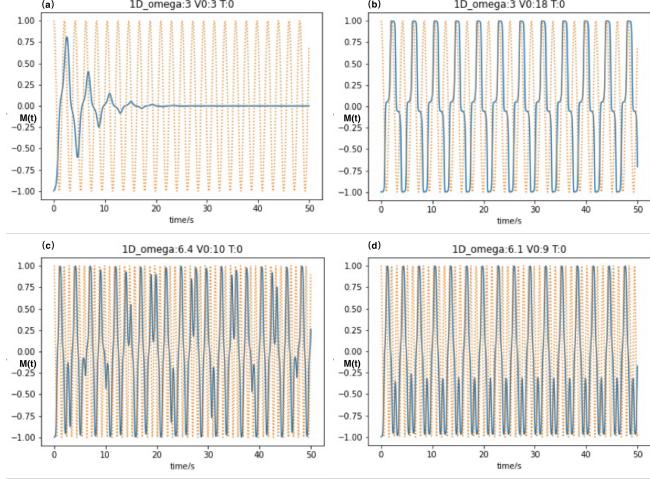


FIG. 7. The dynamics of the ferromagnetic order parameter $M(t)$ at zero temperature

After studying the behavior of the system under zero temperature, a question naturally raised: Will thermal fluctuation melt the crystalline order, just as it does for conventional crystals? According to Fig. 6, we choose $V_0 = 21, \omega = 7$ when the system is deep time crystal phase at zero temperature with 10 times disordered average. As shown in Fig. 8, the system will decay. The higher the temperature is, the faster it will decay.

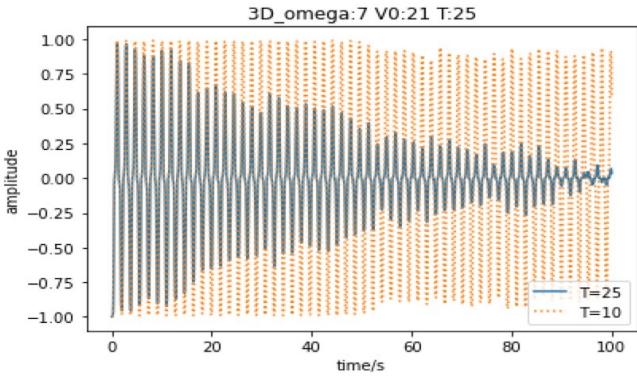


FIG. 8. The dynamics of the ferromagnetic order parameter $M(t)$ at temperatures $T=10$ and $T=25$

In Fig. 9, we fit the relationship between the amplitude of each oscillation and time. According to the phase transition theory, at critical point, the We find $M(t)$ would decay exponentially: $M(t) \sim e^{-0.02t}$.

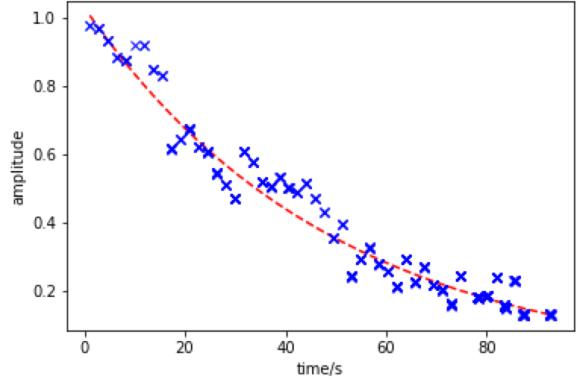


FIG. 9. The amplitude of order parameter when $V_0 = 21, \omega = 7, T = 25$.

This exponentially decay can be interpreted as a 'kink' activated by thermal fluctuation: the thermal noise can induce tunnelings from one of the degenerate DTC phases to the other by a π -phase shifting as shown in Fig. 10. This topological activation would inevitably destroy long-range crystalline order. This could also explain why temporal long-range order does not exist in 1D finite temperature system. Under low temperature($T = 10$), the thermal fluctuation is not strong enough to induce a π -phase shift. Instead, it can only have subtle impact on amplitude. Then we can conclude that the persistent oscillation when $T = 10$ indicates a temporal long-range DTC phase.

We can further ask how to characterize such a nonequilibrium phase transition and what is the corresponding universality class.

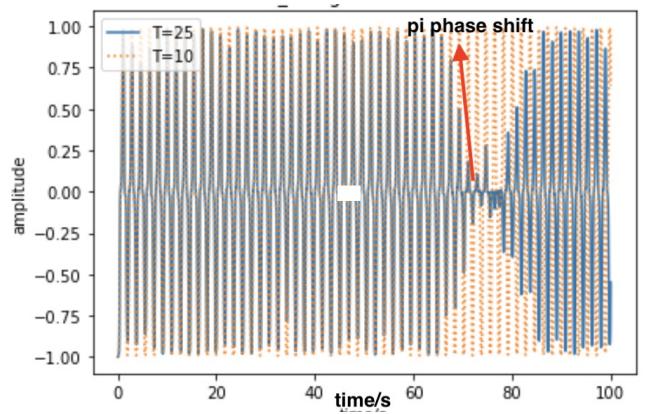


FIG. 10. In the evolution under a single noise trajectory, a π -phase shifting was activated by noise when $V_0 = 21, \omega = 7, T = 25$.

However, the critical behavior can only be observed when the system is close to the thermodynamic limit. Namely, only when the scale of the system is large enough, can critical behavior be observed. For example,

in the recent work of Cai et al. they take $L = 20 \sim 28$, and average thousands of evolution as an approximation to the ensemble average. However, due to the limitation of computing power, it took us two days to average the evolution of ten systems with $L = 16$. That's to say, instead of calculating the critical temperature and critical index in our system with numerical method, we could only point out how to calculate them theoretically here.

We know that near the critical temperature, the order parameter $M(t)$ would decay exponentially: $M(t) \sim e^{-\Delta_c t}$, where Δ_c is a function of L : $\Delta_c = \Delta_c(L)$. We can define relaxation time τ as $\tau_c = \Delta_c^{-1}$, which is expected to be proportional to L^z . Based on that, we can define a renormalized relaxation time $\tilde{\tau} = \frac{\tau_c}{L^z}$. According to the phase transition theory, we have $\tau \sim |T - T_c|^{-z\nu}$ near the crucial point. Therefore, $\tilde{\tau}$ is a function of T and L and at the temperature of T_c , $\tilde{\tau}$ is independent of L . We can choose different L and draw $\tilde{\tau} - T$ curves respectively. Theoretically these curves would intersect at a point and the corresponding abscissa is the critical temperature T_c . We find out the critical exponent ν by adjusting ν until all $\tilde{\tau} - |T - T_c|^* L^{1/\nu}$ curves corresponding to different L overlap.

The evolution of 1D and 2D systems We found it impossible to find stable temporal crystalline order in dimension lower than 3D, even though we can observe similar phase diagrams at zero temperature.

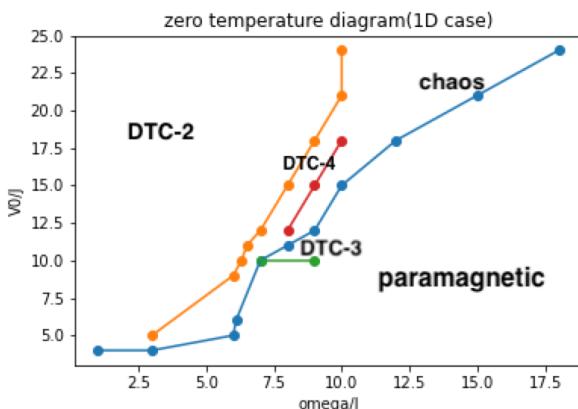


FIG. 11. The zero temperature phase diagram of 1D system when $L = 8$.

The Fig. 12(a) shows the evolution of 1D system under zero temperature. Here we exhibit the evolution between 400-500s. It shows that the order parameter $M(t)$ oscillates with a period is twice that of the driving without decaying. Fig. 12(b) shows the evolution when we increase the temperature to $T = 0.0001$ in the 1D system. Fig. 12(c) exhibits the evolution of a 2D system under zero temperature, which is similar to 1D case. Fig. 12(d) shows the evolution when we increase the temperature of the 2D system to $T = 0.01$.

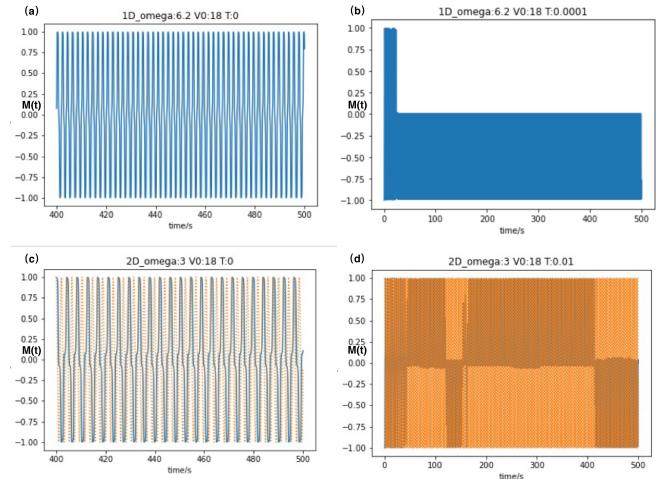


FIG. 12. The evolution of 1D and 2D system under finite temperature.

Due to the introduction of thermal noise, we can not have stable DTC phase in 1D or 2D. The situation would not change with the increase of L .

We can use Mermin–Wagner theorem to explain it, which states that continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions in dimensions $d \leq 2$. In our case, the spin interact with its neighbour only. Thus classical 1D or 2D case unable to exist when the system has noise.

IV. CONCLUSION

In conclusion, we discuss the quantum time crystal and the classical time crystal in this paper. The quantum crystal spontaneously breaks the discrete time translational symmetry from \mathbb{Z} to \mathbb{Z}_2 . Although time crystal is in nonequilibrium state, it is a phase of matter for its rigidity against perturbation. The numeric result shows that the lifetime of the time crystal exponentially grows as the size grows. We attempt to obtain the critical exponents in our system yet some issue on them. In our classical case, only in 3D systems exists DTC phases with true long-range crystalline order persisting forever, while in 1D and 2D system, thermal noise would melt the crystalline order.

For future work, we try to find a 1D system that has phase transition, like N. Yao's work. It will help us understand nonequilibrium physics and finite temperature phase transition.

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