C.1 Summations and Products

When performing calculations, we'll often end up writing sums of terms, where each term follows a pattern. For example:

$$\frac{1+1^2}{3+1} + \frac{2+2^2}{3+2} + \frac{3+3^2}{3+3} + \cdots + \frac{100+100^2}{3+100}$$

We will often use *summation notation* to express such sums concisely. We could rewrite the previous example simply as:

$$\sum_{i=1}^{100} \frac{i+i^2}{3+i}.$$

In this example, i is called the *index of summation*, and 1 and 100 are the *lower* and *upper bounds* of the summation, respectively. A bit more generally, for any pair of integers j and k, and any function $f : \mathbb{Z} \to \mathbb{R}$, we can use summation notation in the following way:

$$\sum_{i=j}^{k} f(i) = f(j) + f(j+1) + f(j+2) + \cdots + f(k).$$

We can similarly use *product notation* to abbreviate multiplication:[1]

[1] Fun fact: the Greek letter Σ (sigma) corresponds to the first letter of "sum," and the Greek letter Π (pi) corresponds to the first letter of "product."

$$\prod_{i=j}^{k} f(i) = f(j) \times f(j+1) \times f(j+2) \times \cdots \times f(k).$$

It is sometimes useful (e.g., in certain formulas) to allow a summation or product's lower bound to be greater than its upper bound. In this case, we say the summation or product is *empty*, and define their values as follows:[2]

[2] These particular values are chosen so that adding an empty summation and multiplying by an empty product do not change the value of an expression.

- When $j > k$, $\sum_{i=j}^{k} f(i) = 0$.
- When $j > k$, $\prod_{i=j}^{k} f(i) = 1$.

Finally, we'll end off this section with a few formulas for common summation formulas, and a few laws governing how expressions using summation and product notation can be simplified.

Theorem. For all $n \in \mathbb{N}$, the following formulas
hold:

1. For all $c \in \mathbb{R}$, $\sum_{i=1}^{n} c = c \cdot n$ (sum with constant terms).
2. $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ (sum of consecutive numbers).
3. $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of consecutive squares).
4. For all $r \in \mathbb{R}$, if $r \neq 1$ then $\sum_{i=0}^{n-1} r^i = \frac{r^n-1}{r-1}$ (sum of powers).
5. For all $r \in \mathbb{R}$, if $r \neq 1$ then $\sum_{i=0}^{n-1} i \cdot r^i = \frac{n \cdot r^n}{r-1} - \frac{r(r^n-1)}{(r-1)^2}$ (arithmetico-geometric series).

Theorem. For all $m, n \in \mathbb{Z}$, the following formulas hold:

1. $\sum_{i=m}^{n}(a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right)$ (separating sums)
2. $\prod_{i=m}^{n}(a_i \cdot b_i) = \left(\prod_{i=m}^{n} a_i\right) \cdot \left(\prod_{i=m}^{n} b_i\right)$ (separating products)
3. $\sum_{i=m}^{n} c \cdot a_i = c \cdot \left(\sum_{i=m}^{n} a_i\right)$ (factoring out constants, sums)
4. $\prod_{i=m}^{n} c \cdot a_i = c^{n-m+1} \cdot \left(\prod_{i=m}^{n} a_i\right)$ (factoring out constants, products)
5. $\sum_{i=m}^{n} a_i = \sum_{i'=0}^{n-m} a_{i'+m}$ (change of index $i' = i - m$)
6. $\prod_{i=m}^{n} a_i = \prod_{i'=0}^{n-m} a_{i'+m}$ (change of index $i' = i - m$)