C.2 Inequalities

In this course we will deal heavily with the manipulation of *inequalities*. While many of these operations are very similar to manipulating equalities, there are enough differences to warrant a comprehensive list.

Theorem. (*Arithmetic manipulations*) For all real numbers a, b, and c, the following are true:

a. If $a \leq b$ and $b \leq c$, then $a \leq c$.
b. If $a \leq b$, then $a + c \leq b + c$.
c. If $a \leq b$ and $c > 0$, then $ac \leq bc$.
d. If $a \leq b$ and $c < 0$, then $ac \geq bc$.
e. If $0 < a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.
f. If $a \leq b < 0$, then $\frac{1}{a} \geq \frac{1}{b}$.

Moreover, if we replace any of the "if" inequalities with a strict inequality (i.e., change $\leq$ to $<$), then the corresponding "then" inequality is also strict.[1]

[1] For example, the following is true: "If $a < b$, then $a + c < b + c$."

The previous theorem tells us that basic operations like adding a number or multiplying by a positive number preserves inequalities. However, other operations like multiplying by a negative number or taking reciprocals *reverses* the direction of the inequality, which is something we didn't have to worry about when dealing with equalities. But it turns out that, at least for non-negative numbers, most of our familiar functions preserve inequalities.

Definition. Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$. We say that f is when for all $x, y \in \mathbb{R}^{\geq 0}$, if $x < y$ then $f(x) < f(y)$.

Most common functions are strictly increasing:

- Raising to a positive power, e.g., $f(x) = x^2$ or $f(x) = x^{3.14}$.[2]

  [2] Remember that we're restricting ourselves to the $\mathbb{R}^{\geq 0}$ for the domain of these functions! $f(x) = x^2$ is not increasing on the domain $\mathbb{R}$, for example.

- Logarithms with a base greater than one, e.g., $f(x) = \log_3(x + 1)$.

- Exponential functions with a base greater than one, e.g., $f(x) = 2^x$.

Moreover, adding two strictly increasing functions, or multiplying a strictly increasing function by a positive constant or another always-positive strictly increasing function, results in another strictly increasing function. So for example, we know that $f(x) = 300x^2 + x \log_3 x + 2^{x+100}$ is also strictly increasing.

It should be clear from this definition that the following property holds, which enables us to manipulate inequalities using a host of common functions.

Theorem. For all non-negative real numbers a and b, and all strictly increasing functions $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, if $a \leq b$, then $f(a) \leq f(b)$.

Moreover, if $a < b$, then $f(a) < f(b)$.

It is this theorem that allows us to perform several common operations on inequalities as a "step" in a computation. For example, if we know $0 < a \leq b$, then we can conclude that $a^2 \leq b^2$, or $\log_2(a) \leq \log_2(b)$, because both of the functions x^2 and $\log_2(x)$ are strictly increasing functions.