13.5 Introduction to Binary Search Trees

In the next part of this chapter, we're going to learn about one particular application of trees for storing data called the *binary search tree*. This new data structure forms the basis of more advanced tree-based data structures you'll learn about in future courses like CSC263/265.

The Multiset abstract data type

To start, let's introduce one new abstract data type that is an extension of the Set ADT that allows for duplicates:

- **Multiset**
    - Data: an unordered collection of values, allowing duplicates
    - Operations: get size, insert a value, remove one occurrence of a specified value, check membership in the multiset.

You might be imagining a few different possible implementations for the Multiset ADT already using familiar Python data types like `list` and `dict`. Let's briefly consider two possible implementations the Multiset ADT.

Implementing Multiset using `List` and `Tree`

Suppose we use a `list` to implement the Multiset ADT, where we simply append new items to the end of the list. As we discussed all the way back in Chapter 8, this implementation would make *searching* for a particular item a $\Theta(n)$ operation, proportional to the size of the collection.

If we used a `Tree` instead of a `list`, we'd get the same behaviour: if the item is not in the tree, every item in the tree must be checked. So just switching from lists to trees isn't enough to do better!

However, one of the great insights in computer science is that adding some additional structure to the input data can enable new, more efficient algorithms. You have seen a simple form of this called *augmentation* in previous tutorials, and in this section we'll look at more complex "structures".

In the case of Python lists, if we assume that the list is *sorted*, then we can use the **binary search** algorithm to improve the efficiency of searching to $\Theta(\log n)$.[1] But though ensuring the list is sorted does make searching faster, the same limitations of array-based lists apply:

[1] We haven't yet formally covered binary search, though we'll see it later in the course. If you're curious we've put in some reference videos from CSC108 at the bottom of this section.

insertion and deletion into the front of the list takes $\Theta(n)$. So the question is: can we achieve efficient (faster than linear) search, insertion, and deletion implementations all at once? *Yes we can!*[2]

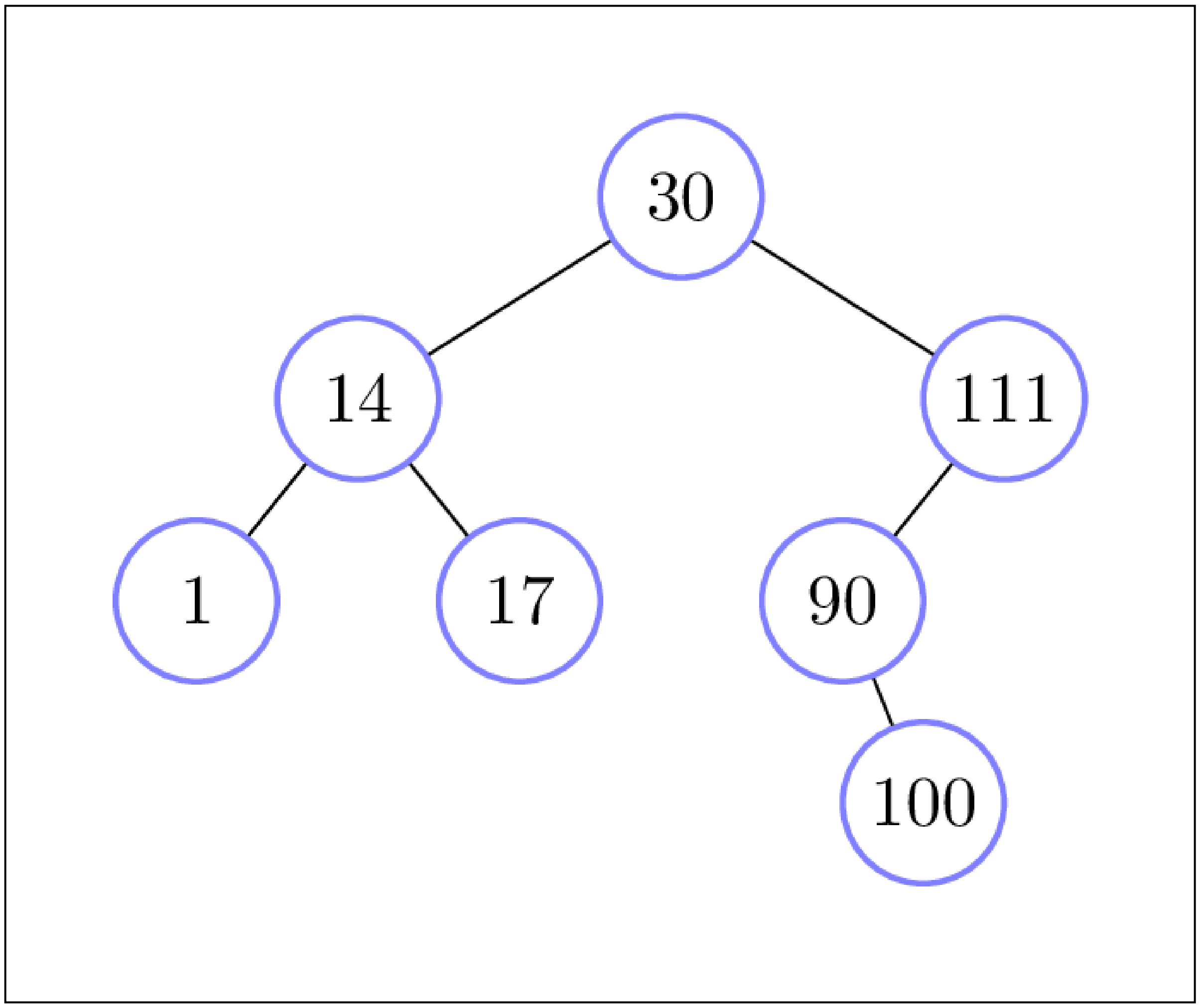
[2] Now, you might be recalling that Python's `set` and `dict` already have constant-time search, insertion, and deletion! It is indeed possible to implement the Multiset ADT efficienty using a `dict` (keeping track of duplicates in a "smart" way). But we're going to explore a different approach this chapter that allows for other efficient operations as well.

Binary search trees: definitions

To implement the Multiset ADT efficiently, we will combine the branching structure of trees with the idea of binary search to develop a notion of a "sorted tree".

First, some definitions. A **binary tree** is a tree in which every item has two (possibly empty) subtrees, which are labelled its *left* and *right* subtrees. An item in a binary tree satisfies the **binary search tree property** when its value is greater than or equal to all items in its left subtree, and less than or equal to all items in its right subtree.[3]

[3] Note that duplicates of the root are allowed in *either* subtree in this version.



A binary tree is a **binary search tree (BST)** when *every* item in the tree satisfies the binary search tree property (the "every" is important: for an arbitrary binary tree, it's possible that some items satisfy this property but others don't). An example of a binary search tree is shown on the right.

We can think of binary search trees as a "sorted tree": even if the data isn't inserted in sorted order, the BST keeps track of it in a sorted fashion. This makes BSTs extremely efficient in doing operations like searching for an item; but unlike sorted Python lists, they can be much more efficient at insertion and deletion while maintaining the sortedness of the data!

Representing a binary search tree in Python

We'll define a new class `BinarySearchTree` to represent this form of tree. This class is based on the `Tree` class we studied, but with a few important differences. First, because we know there are only two subtrees, and the left/right ordering matters, we use explicit attributes to refer to the left and right subtrees:[4]

[4] So we've replaced the `Tree` `_subtrees` attribute with two new attributes, `_left` and `_right`.

```
class BinarySearchTree:
    """Binary Search Tree class.
    """
    # Private Instance Attributes:
    #   - _root:
    #       The item stored at the root of this tree, or None if this tree is
    #     empty.
    #   - _left:
    #       The left subtree, or None if this tree is empty.
    #   - _right:
    #       The right subtree, or None if this tree is empty.
    _root: Optional[Any]
    _left: Optional[BinarySearchTree]
    _right: Optional[BinarySearchTree]
```

Another difference between `BinarySearchTree` and `Tree` how we distinguish between empty and non-empty trees. In the `Tree` class, an empty tree has a `_root` value of `None`, and an *empty list* `[]` for its list of subtrees. In the `BinarySearchTree` class, an empty tree also has a `_root` value of `None`, but its `_left` and `_right` attributes are set to `None` as well. Moreover, for `BinarySearchTree`, an empty tree is the *only* case where any of the attributes can be `None`; when we represent a non-empty tree, we do so by storing the root item (which isn't `None`) at the root, and storing `BinarySearchTree` objects in the `_left` and `_right` attributes. The attributes `_left` and `_right` might refer to *empty* binary search trees, but this is different from them being `None`![5] We document these as representation invariants for

[5] This is also what we did for `RecursiveList` back in Section 12.5.

BinarySearchTree, which are Python translations of "self._root is None if and only if self._left is None" and "self._root is None if and only if self._right is None".

```python
class BinarySearchTree:
    """Binary Search Tree class.

    Representation Invariants:
      - (self._root is None) == (self._left is None)
      - (self._root is None) == (self._right is None)
    """
```

And finally, we add one more English representation invariant for the BST property itself. Here is our full BinarySearchTree class header, docstring, and instance attributes:

```python
class BinarySearchTree:
    """Binary Search Tree class.

    Representation Invariants:
      - (self._root is None) == (self._left is None)
      - (self._root is None) == (self._right is None)
      - (BST Property) if self._root is not None, then
          all items in self._left are <= self._root, and
          all items in self._right are >= self._root
    """
    # Private Instance Attributes:
    #   - _root:
    #       The item stored at the root of this tree, or None if this tree is
         empty.
    #   - _left:
    #       The left subtree, or None if this tree is empty.
    #   - _right:
    #       The right subtree, or None if this tree is empty.
    _root: Optional[Any]
    _left: Optional[BinarySearchTree]
    _right: Optional[BinarySearchTree]
```

And here are the initializer and is_empty methods for this class, which are based on the corresponding methods for the Tree class:

```python
class BinarySearchTree:

    def __init__(self, root: Optional[Any]) -> None:
        """Initialize a new BST containing only the given root value.

        If <root> is None, initialize an empty BST.
        """
        if root is None:
            self._root = None
            self._left = None
            self._right = None
        else:
            self._root = root
            self._left = BinarySearchTree(None)   # self._left is an empty
 BST
            self._right = BinarySearchTree(None)  # self._right is an empty
 BST

    def is_empty(self) -> bool:
        """Return whether this BST is empty.
        """
        return self._root is None
```

Note that we do not allow client code to pass in left and right subtrees as parameters to the initializer. This is because binary search trees have a much stronger restriction on where values can be located in the tree, and so a separate method is used to insert new values into the tree that will ensure the BST property is always satisfied.

But before we get to the BST mutating methods (inserting and deleting items), we'll finish this section by studying the most important non-mutating BST method: searching for an item.

Searching a binary search tree

For general trees, the standard search algorithm is to compare the `item` against the root, and then recursively search in each of the subtrees until either the item is found, or all the subtrees have been searched. When `item` is not in the tree, every item must be searched.

In stark contrast, for BSTs *the initial comparison to the root tells you which subtree you need to check*. For example, suppose we're searching for the number 111 in a BST. We check the root of the BST, which is 50. Since 111 is greater than 50, we know that if it does appear in the BST, it must appear in the *right* subtree, and so that's the only subtree we need to recurse on.[6]

[6] On the other hand, if the root of the BST were 9000, when we would need to recurse on the left subtree, since 111 is less than 9000.

That is, in the recursive step for `BinarySearchTree.__contains__`, only one recursive call needs to be made, rather than two!

```
class BinarySearchTree:
    def __contains__(self, item: Any) -> bool:
        """Return whether <item> is in this BST.
        """
        if self.is_empty():
            return False
        else:
            if item == self._root:
                return True
            elif item < self._root:
                return self._left.__contains__(item)
            else:
                return self._right.__contains__(item)
```

While this code structure closely matches the recursive code template for the general `Tree` class, we can also combine the two levels of nested ifs to get a slightly more concise version:

```
class BinarySearchTree:
    def __contains__(self, item: Any) -> bool:
        """Return whether <item> is in this BST.
        """
        if self.is_empty():
            return False
        elif item == self._root:
            return True
        elif item < self._root:
            return self._left.__contains__(item)
        else:
            return self._right.__contains__(item)
```

References

- CSC108 videos: Binary Search (part 1, part 2, part 3, part 4)