# 16.2 Selection Sort

We'll begin our study of sorting in earnest with two different *iterative*, or loop-based, sorting algorithms. In this section we'll study *selection sort*, and in the following section we'll study *insertion sort*. Even though these two algorithms solve the same problem (take a list and sort it), they use different approaches that we will be able to compare and learn from.

## *The algorithm idea*

The **selection sort** algorithm has at its core a fairly intuitive idea. Given a collection of unsorted elements, we repeatedly extract the smallest element from the collection, building up a sorted list from these elements. Selection sort gets its name from the fact that at each step, we *select* the smallest element from the ones remaining.

For example, suppose we start with the list `[3, 7, 2, 5]`.

- The smallest element is `2`, so that becomes the first element of our sorted list: `[2]`.
- The remaining elements are `[3, 7, 5]`. The smallest element remaining is `3`, and so that is the next element of our sorted list: `[2, 3]`.
- The remaining elements are `[7, 5]`. The smallest element remaining is `5`, and our sorted list becomes `[2, 3, 5]`.
- The remaining element is `[7]`. The smallest element is `7`, and our sorted list becomes `[2, 3, 5, 7]`.

We can summarize this behaviour in the table below:

| Items to be sorted | Smallest element | Sorted list |
|---|---|---|
| `[3, 7, 2, 5]` | `2` | `[2]` |
| `[3, 7, 5]` | `3` | `[2, 3]` |
| `[7, 5]` | `5` | `[2, 3, 5]` |
| `[7]` | `7` | `[2, 3, 5, 7]` |
| `[]` | | `[2, 3, 5, 7]` |

This table looks an awful lot like a loop accumulation table, and indeed we can implement this version of selection sort using a loop.

```
def selection_sort_simple(lst: list) -> list:
    """Return a sorted version of lst."""
```

```
    sorted_so_far = []

    while lst != []:
        smallest = min(lst)
        lst.remove(smallest)
        sorted_so_far.append(smallest)

    return sorted_so_far
```

This implementation does correctly return a list, but has a significant flaw: it mutates its input `lst`, which is not documented in its specification.[1] We could fix this by, for example,

[1] For example, the built-in `sorted` function also returns a new list, but it makes sure not to mutate its input.

making a copy of `lst` and operating on that copy instead. However, for the rest of this section we'll introduce a new implementation of selection sort that instead mutates its argument, without creating any additional lists.

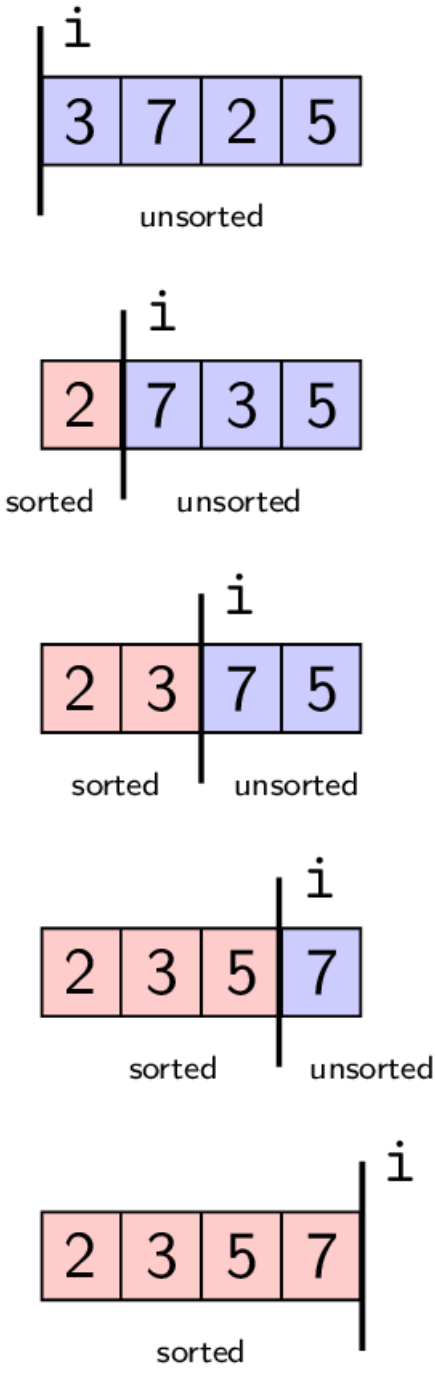
## *In-place selection sort*

We say that a sorting algorithm is **in-place** when it sorts a list by mutating its input list, and without using any additional list objects (e.g., by creating a copy of the list). In-place algorithms may still use new computer memory to store primitive values like integers, this additional amount of memory is constant with respect to the size of the input list.[2]

[2] We can say that the amount of additional memory used is $\Theta(1)$, ignoring the amount of memory used to store the input itself.

So to implement an in-place version of selection sort, we cannot use a new list to accumulate the sorted values. Instead, our approach will move elements around in the input list, so that at iteration `i` in our loop, the first `i` elements of the list are the sorted part, and the remaining parts are unsorted.

Here is our above example of sorting the list `[3, 7, 2, 5]`, now using this in-place approach.




So in this example, `i` starts at 0.

1. At iteration `i = 0`, the entire list is unsorted. We find the smallest element in the list (the `2`), and *swap it* with the item at index `0`, obtaining the list `[2, 7, 3, 5]`.
2. Then at iteration `i = 1`, only the elements `[7, 3, 5]` are unsorted. We again do a swap, moving the `3` to index `1`, obtaining the list `[2, 3, 7, 5]`.

3. This repeats again for `i = 2`, and the `7` and `5` get swapped, so that we get `[2, 3, 5, 7]`.
4. When `i = 3`, the "unsorted" part consists of just a single number, and so no more swaps occur. The list is sorted!

### *Some loop invariants*

In the above diagram, we used the variable `i` to represent the boundary between the sorted and unsorted parts of the list. We can represent this formally as a *loop invariant*. Here is a partial implementation of this in-place selection sort algorithm, with this loop invariant:

```
def selection_sort(lst: list) -> None:
    """Sort the given list using the selection sort algorithm.

    Note that this is a *mutating* function.
    """
    for i in range(0, len(lst)):
        assert is_sorted(lst[:i])

        ...


def is_sorted(lst: list) -> bool:
    """Return whether this list is sorted."""
    # Implementation omitted
```

There is another loop invariant that is useful for our implementation of selection sort. At iteration $i$, the first $i$ items aren't just sorted; because we've always chosen the smallest item at each iteration, the first $i$ items must be smaller than all other items in the list. We can express this as a second loop invariant:

```
def selection_sort(lst: list) -> None:
    """Sort the given list using the selection sort algorithm.

    Note that this is a *mutating* function.
    """
    for i in range(0, len(lst)):
        assert is_sorted(lst[:i])
        assert i == 0 or all(lst[i - 1] <= lst[j] for j in range(i,
         len(lst)))

        ...
```

### *Implementing the loop*

Now that we have our loop invariants in place, let's get to implementing the loop. Remember that at iteration `i`, we need to find the smallest item in the "unsorted" section of the list, which is `lst[i:]`, and swap it with `lst[i]`.

Here is our full implementation of this function. We've pulled out the code to find the smallest item in the unsorted portion into a helper.

```
def selection_sort(lst: list) -> None:
    """Sort the given list using the selection sort algorithm.

    Note that this is a *mutating* function.

    >>> lst = [3, 7, 2, 5]
    >>> selection_sort(lst)
    >>> lst
    [2, 3, 5, 7]
    """
    for i in range(0, len(lst)):
        # Loop invariants
        assert is_sorted(lst[:i])
        assert i == 0 or all(lst[i - 1] <= lst[j] for j in range(i,
         len(lst)))

        # Find the index of the smallest item in lst[i:] and swap that
        # item with the item at index i.
        index_of_smallest = _min_index(lst, i)
        lst[index_of_smallest], lst[i] = lst[i], lst[index_of_smallest]


def _min_index(lst: list, i: int) -> int:
    """Return the index of the smallest item in lst[i:].

    In the case of ties, return the smaller index (i.e., the index that
         appears first).

    Preconditions:
        - 0 <= i <= len(lst) - 1
    """
    index_of_smallest_so_far = i

    for j in range(i + 1, len(lst)):
        if lst[j] < lst[index_of_smallest_so_far]:
            index_of_smallest_so_far = j

    return index_of_smallest_so_far
```

## *Running-time analysis for selection sort*

To wrap up, let's analyze the running time of our in-place selection sort algorithm, *ignoring* the assert statements. We need to first analyze the helper function `_min_index`. Let $n$ be the length of the input list `lst`.

- The statements outside of the loop take constant time. We'll treat them as just a single step.
- The loop iterates $n - i - 1$ times, for $j = i + 1, \ldots, n - 1$, and the body takes constant time (1 step). So the running time of the loop is $n - i - 1$ steps.
- So the total running time of `_min_index` is $(n - i - 1) + 1$, which is $\Theta(n - i)$.

Now let's look at `selection_sort`. Once again, let $n$ be the length of the input list `lst`.

Inside the body of the loop, there are two statements (ignoring the assertions). The first statement is the call to `_min_index`, which takes $n - i$ steps, where $i$ is the value of the for loop variable.[3] The second statement (swapping the `lst` values) takes constant time, so

[3] Note that we've translated the $\Theta(n - i)$ running time for `_min_index` into an exact $n - i$ number of steps in the loop body.

we'll count that as 1 step. So the running time of one iteration of the for loop is $n - i + 1$, and the for loop iterations once for each $i$ between 0 and $n - 1$, inclusive.

This gives us a total running time of:

$$\begin{aligned}\sum_{i=0}^{n-1} n - i + 1 &= n(n+1) - \sum_{i=0}^{n-1} i \\ &= n(n+1) - \frac{n(n-1)}{2} \\ &= \frac{n(n+3)}{2}\end{aligned}$$

Therefore the running time of `selection_sort` is $\Theta(n^2)$.

# *References*

- CSC108 videos: Selection Sort (Part 1, Part 2)