Semester 2 Notes

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Contents

1 Analysis II

1.1 Propositional Logic

- $\neg(p \cup q) = (\neg p) \cap (\neg q)$

- $\neg(p \cap q) = (\neg p) \cup (\neg q)$

- $\neg(\exists p) = \forall p$

- $p \Rightarrow q = (p \cap q) \cup \neg q$

- so $\neg(p \Rightarrow q) = (\neg p \cup \neg q) \cap \neg\neg q = \neg p \cup q$

1.2 Continuous Functions

Continuity

- For $f : D \to \mathbb{R}$, f is continuous at a point $a \in D$ if:
  $\forall \epsilon > 0, \exists \delta > 0 : \forall x \in D, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

- f is uniformly coninuous if continuous $\forall a$, or

  $\forall \epsilon > 0, \exists \delta > 0 : \forall x, y \in D, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

- In general, continuity $\not\Rightarrow$ uniform continuity

- A function $f : D \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff:
  $\forall (x_n)_{n \in \mathbb{N}}, x_n \in D, \lim_{n \to \infty} x_n = a, \lim_{n \to \infty} f(x_n) = f(a)$

Theorem 1 (Intermediate Value Theorem) *for $f : D \to C \subseteq \mathbb{R}$
    then $u \in C \Rightarrow \exists x \in D : u = f(x)$*

2 waves and fields

2.1 Periodic Motion

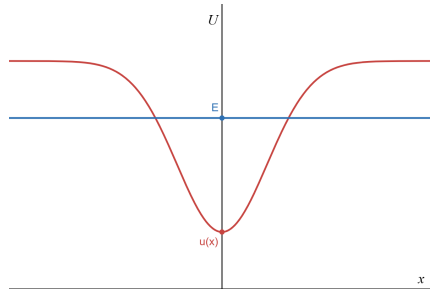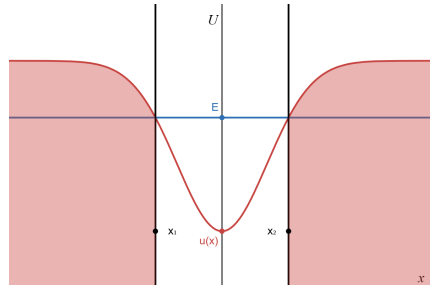**Periodic Motion In 1D**

- Periodic motion is one in which the position repeats values regularly

- It can be characterised like so:

$$r(t + nT) = r(t), \forall t \in \mathbb{R}, n \in \mathbb{N}, T = period$$

- If we consider a situation in a consevative field, with a potential $u(x)$, and a system with a total energy E:

- We can see that the system would be bound within the region $U < E$:

- This naturally leads to periodic motion, as at $x = x_1, v = 0$

- At this point, the energy is not minimised, and so the system tends towards $x = 0$, where $v \neq 0$, causing periodic motion.

- One way to achieve this is through simple harmonic motion (SHM)

- SHM is any system where $\ddot{x} = -x$

- We can see here that $u(x) = -\int f dx = \int \frac{\ddot{x}}{m} dx$, or $f = -\frac{du}{dx} \Rightarrow m\ddot{x} = -\frac{du}{dx}$

- We can then add another term to take energy out of the system:

- $m\ddot{x} = -\frac{du}{dx} - \gamma\dot{x}$ in general.

- This describes a frictional force, causing damped harmonic motion (DHM)

- This will lead to the energy gradually decreasing:
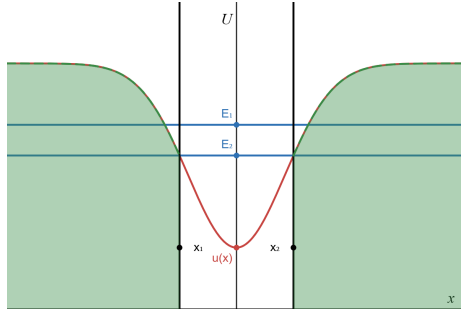
- Which will result in the following motion:
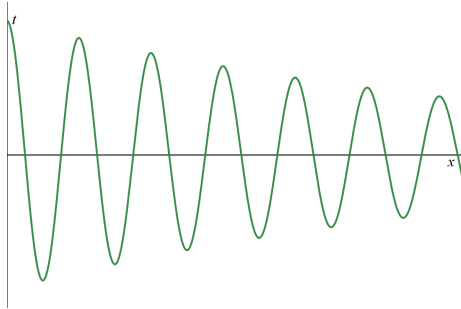
Periodic Motion In 3D

- Gravity is a conservative force so $u = u(\vec{r})$, and angular momentum is conserved

- Evidently, from $\vec{L} = \vec{r} \times \vec{p}$, the position $\vec{r}$ is perpendicular to the constant angular momentum $\vec{L}$, and thus the system is confined to a single plane.

- Let $v_\phi =$ the component of v perpendicular to $\vec{r}$, and $v_r =$ the component of v parallel to $\vec{r}$

- Then $L = m\vec{r} \times (r\dot{\phi})$

- Since $K = \frac{1}{2}m\dot{\vec{r}} = \frac{1}{2}m(\vec{v_\phi}^2 + \vec{v_r}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$

- $\dot{\phi} = \frac{L}{mr^2} \Rightarrow K = \frac{1}{2}m(\dot{r}^2 + r^2(\frac{L}{mr^2})^2)$

- So $E = K + u = \frac{1}{2}m(\dot{r}^2 + (L/mr)^2 + u(\vec{r}))$

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Simple Harmonic Motion

- Simple harmonic motion is any motion of the form $x(t) = A cos(\omega t + \phi)$

- From this, we can see that $\ddot{x}(t) = -A\omega^2 \cos(\omega t + \varphi) = -\omega^2 x$

- This differential equation, $\ddot{x} \propto x$, characterises simple harmonic oscillation(SHO)

- Then, $f = m\ddot{x} = -m\omega^2 x$, is only dependant on position

- Then, the potential energy: $U(x) = -\int f dx = m\omega^2 \int x dx = \frac{1}{2}m\omega^2 x^2$

- Then, $E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}mA^2\omega^2$

- E, K, and U will look like this over time, with $\overline{U} = \overline{K}$ over one period:
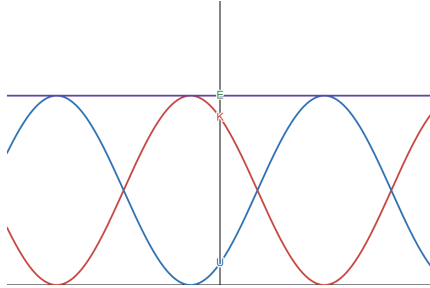
Simple Pendulums and Small oscillators

- For a simple pendulum, $|\tau| = I\ddot{\theta} = -f_\perp \cdot r = -f_g \sin(\theta) \cdot l$

- Then, using the small angle approximation $\sin(\theta) = \theta$:
$I\ddot{\theta} = -f_g l\theta$

- Another example of simphe harmonic motion is the torsion pendulum

- This is a mass on a string that can twist around its axis:
$\tau = -c\theta \Rightarrow I\ddot{\theta} = -c\theta$

Damped Harmonic Motion

- We expect the amplitude A of a damped harmonic oscillator to decrease over time.

- For DHM, The 2nd law of motion looks like this:

- $m\ddot{x} = -kx - b\dot{x}$

- Setting b=0, we recover the SHM characteristic equation

- For $b^2 - 4km < 0$, This is called underdamping

- Underdamping will have the following form of motion:

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \varphi) \tag{2.1}$$
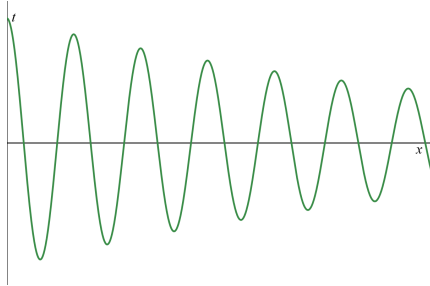
- Then considering the quantity $\frac{2m}{b}$, we see $|x(t + \frac{2m}{b})| = e^{-1}|x(t)|$

- If $b^2 > 4mk$, we get overdamped motion:

- $x(t) = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t}$
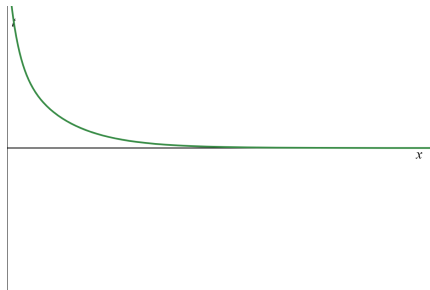
- and if $b^2 = 4mk$, critically damped Motion:

- $x(t) = (A_1 + A_2 t)e^{-\frac{b}{2m}t}$

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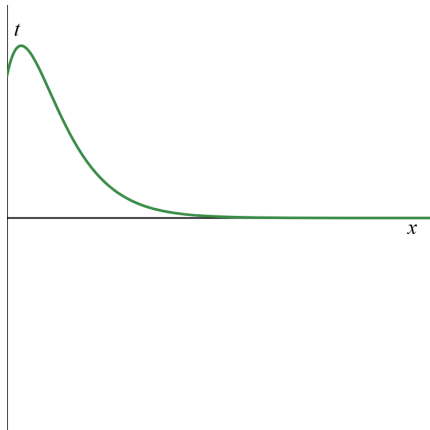
Forced Harmonic Oscillation & Resonance

- A forced harmonic oscillator (FHO) is one subject to an external driving force F_d

- $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m} = \frac{1}{m}F_d$

- If the driving force is also periodic, the motion will oscillate at the driving frequency f_d

$$\omega_r = \omega_0\sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \tag{2.2}$$

- Where A is maximised at $\omega_d = \omega_r$

3 Modern Physics

3.1 Black Body Radiation

Observations

- Black bodies satisfy 3 criteria:

    - absorbs all incident radiation

    - emits as much radiation as possible at any wavelength

    - emits radiation isotropically

- By studying the spectra of black bodies at different temperatures:
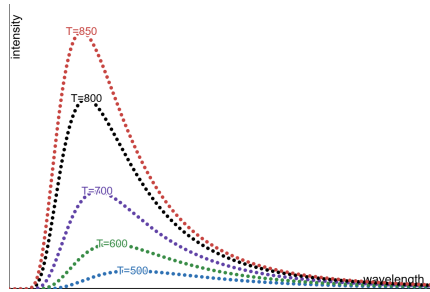
- We can observe 2 main laws:

    - Wein's displacement law: $\lambda_{max} = \frac{b}{T}$
    - Stephans law: $P(T) = \sigma A T^4$

Rayleigh-Jeans Law

- In June 1900, John Rayleigh discovered that the spectral raiance $\propto \lambda^{-4}$

- In 1905, James Jean derived the full Rayleigh-Jeans law including constants

- The Rayleigh-Jeans law is a classical approximation of the radiance of a black body

- Rayleigh-Jeans law can be expressed as so:

$$B(\lambda, T) = \frac{2ck_bT}{\lambda^4} \tag{3.1}$$

- This produced the following prediction for spectral radiance at T=800:
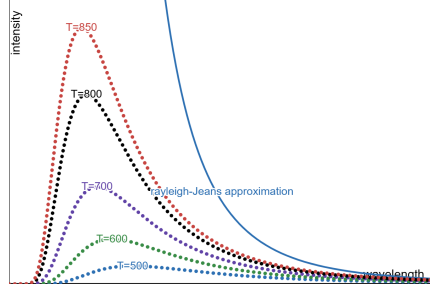
- This presents many problems, mainly that it doesnt match for $\lambda < \lambda_p$, and $I \to \infty$ as $\lambda \to 0$

Planck's Law

- Rayleigh-Jeans law, as seen in the derivation, can be expressed as:

- $R(\lambda) \propto \rho(f) \cdot E_{avg}$

- Where $\rho \, and \, E$ are the number of waves and average energy of each wave

- In the Rayleigh-Jeans law, $E_{avg} = kT$

- This was due to the equipartition theorem

- Max Planck's great insight was to abandon the equipartition theorem, instead writing:

- $E = n\epsilon \Rightarrow \overline{E} = \dfrac{hf}{e^{\frac{hf}{kT}} - 1}$

- And so:

$$R(\lambda) = \frac{2hc}{\lambda^5} \cdot \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \tag{3.2}$$

3.2 The Photoelectric Effect

6/2/25

- In 1887, Heinrich Hertz discovered that light rays incident on a metal plate produces an emf across the material

- This is due to electrons being liberated by the light

- This works with metals due to the delocalised electron bonds

- The electrons would be liberated with a range of kinetic energies up to a maximum

- There was also an observed threshold frequency, below which a ray can't liberate electrons

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classical Predictions

- $V_s \propto I$

- $K_{max} \propto A$

- $\frac{dN}{dt} \propto f$

Einstein's Explanation

- In 1905, Einstein published his explanation of the photoelectric effect

- He explained that the light interacts in a 1:1 mannor, with $E = hf$

- This is insightful as Max Planck had only quantised matter oscillation

- Einstein qunatised light

- This explains the intensity profile, as the particle light must interact with the electrons 1:1

- It also explained the threshold frequency phenomenon, as light with insufficient energy just reflects off of the metal

- $E_\gamma = hf = \varphi + \frac{1}{2}m_e v_{max}^2$ or $hf = \varphi + K_{max}$

4 Intro to Quantum Physics

4.1 A Brief History

Ultraviolet Catastrophe

- In 1860, Gustav Kirchhoff discovered what is now called black body radiation

- This is where an object with internal energy radiates electromagnetic waves correlated with its thermal properties.

- Classical predictions, using namely the Rayleigh-Jeans law, predicted an infinite ammount of low-frequency light waves:

- In 1990, Max Planck produced a soloution to this problem which correctly predicted the spectral radiance of black body radiation

- This theory worked by quantising light, which he expected to disapear in the final solution

- He still did not believe that light was acually quantised

- He earned the Nobel Prize in 1918

Photoelectric Effect

- In 1887, Heinrich Hertz discovered the photoelectric effect

- This is when you expose a metal plate to high intensity light and observe a voltage over the plate

- The contemporary theories predicted that the energy transfered should be proportional to the frequency

- Instead, there is an observed threshold frequency, below which no energy is transfered

- In 1905, Albert Einstein explained this by employing Max Plancks quantum light

- He won the Nobel Prize in 1921

Atomic Models

- In 1911, Ernst Rutherford discovered positively charged atomic nuclei

- The presence of the electron in the atom, then, predicted that atoms were unstable, decaying in 10^-11s as the electron would fall into the nuclei

- In 1913, Neils Bohr predicted that angular momentum was quantised, restricting electrons to specific "orbits" and stopping atomic decay

- Neils Bohr could not come up with a mechanism for angular momentum quantisation, however

- He won the Nobel Prize in 1913

DeBroglie's Electron Strings

- In 1924, Louis DeBroglie suggested wave particle duality

- This suggests that sufficient restrictions upon an electrons position force it to exhibit wave like behaviour

- When employed on an electron in orbit, the electron waves superpose, producing a standing wave

- Standing waves can only exist with an integer multiplier on the fundamental frequency

- This multiplier corresponds to the energy the electron can have, quantising orbital angular momentum

- He won the Nobel Prize in 1929

4.2   Rutherford And Compton Scattering

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Rutherford Scattering

- In 1987, Thompson devised the Plum Pudding model.

- In 1906 in Manchester, Ernst Rutherford devised the α scattering experiment

- He shot α particles at a thin sheet of gold and observe the distribution of scattering angles

- The Plum Pudding model predicts the following distribution:

- It predicted little to no scattering due to the diffusity of the positive charge

- Whereas the distribution Rutherford found was the following:

- In response to this, rutherford devised the nuclear model

- This explained that the nucleus repels the α particles due to the like charges, sometimes up to a full π^c

Compton Scattering

- In 1923, Arthur Compton discovered Compton scattering

- This is where x-rays incident on an atom reflect at a different wavelength

- The classical model, with $p_\gamma = 0$, predicted no shift:

- But in reality, there was a shift:

- This showed that light does carry momentum, and therefore supported the photon model

- Using conservation of momentum and conservation of energy, you can arrive at the following equation:

$$\lambda - \lambda' = \frac{h}{m_0 \cdot c}[1 - \cos(\theta)] \tag{4.1}$$

4.3   Milikan's Oil Drop

5/2/25

- In 1897, Robert Milikan devised an experiment to measure the charge of the electron:

- The 2 plates are connected to a power source, producing an emf

- A nozzle containing oil is small to be sprayed

- There is a camera or microscope to observe the drops

- The drops become charged due to friction with the nozzle

- Oil drops that move up are large charge or low mass, and vice versa

- There is also the bouyant and viscous forces

- $\vec{f_b} = \frac{4}{3}\pi g r^3 \rho_{air} \hat{e}_z$ since the oil is displacing the air

- from Stoke's Law, $\vec{f_v} = -6\pi\eta_{air} \cdot rv$

- $\vec{f_v} = -6\pi\eta_{air}$

- Since it is hard to measure the properties of the drops explicitly, we can instead formulate expressions for them:

$$r = \sqrt{\frac{6\eta_{air}v_{off}}{2(\rho_{oil} - \rho_{air})g}} \tag{4.2}$$

$$q = \frac{g\sqrt{2}(v_{on} + v_{off})d}{V_0} \cdot \sqrt{\frac{\eta_{air}^3 v_{off}}{(\rho_{oil} - \rho_{air})g}} \tag{4.3}$$

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- You then repeat this multiple times and record the charge of each drop   6/2/25

- There will be a random distribution of charges, however the smallest difference in charge will be $e = 1.60 \cdot 10^{-19}$

5   Mathematical Methods II

5.1   Non Cartesian Co-Ordinates

Polar Co-Ordinates

- for $x\hat{\imath} + y\hat{\jmath}$, the equivalent polar basis vectors are:

  $\hat{r} = (\cos(\theta), \sin(\theta))$, $\hat{\theta} = (-\sin(\theta), \cos(\theta))$ where $\tan(\theta) = \frac{y}{x}$

- Then, $\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta} = \hat{\imath}$ and $\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta} = \hat{\jmath}$

3D Polar Co-Ordinates

- There are 2 main 3D Polar Coordinate Bases:

- Spherical $(\hat{r}, \hat{\theta}, \hat{\varphi})$

- And Cylindrical $(\hat{\rho}, \hat{\theta}, \hat{z})$

6 Derivations

6.1 Analysis II

6.2 Waves and Fields

eqn 2.1 Underdamping Motion Form

- $m\ddot{x} = -kx - b\dot{x}$

- $\Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

- This has a characteristic equation $\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}$

- With soloutions $\lambda = -\frac{b}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{b}{m}^2\right) - 4\frac{k}{m}}$

- For case $b^2 < 4mk, \lambda = -\frac{b}{2m} \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

$$\Rightarrow x(t) = Re[Ae^{-\frac{b}{2m}t}e^{\pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}]$$

- $\Rightarrow \boxed{x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \varphi)}$

eqn 2.2 Resonance Frequency

DE

- For $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f_d, f_d = d\cos(\omega_d t + \varphi)$

- We know that for FHM, $f = f_d$ so:

- Assume $x(t) = Re[\mathcal{A}e^{i\omega_d t}], F_d = Re[\mathcal{D}e^{i\omega_d t}]$

- Where $\mathcal{D} = De^{i\varphi_d}$

- Into DE:

- $-\omega_d^2 \mathcal{A}e^{i\omega_d t} + \frac{b}{m}i\omega_d \mathcal{A}e^{i\omega_d t} + \frac{k}{m}\mathcal{A}e^{i\omega_d t} = \frac{1}{m}\mathcal{D}e^{i\omega_d t}$

- $\Rightarrow \mathcal{A}(\omega_d^2 - \frac{b}{m}i\omega_d - \frac{k}{m}) = -\frac{1}{m}\mathcal{D}$

- $\Rightarrow \mathcal{A} = \frac{\mathcal{D}/m}{\omega_d^2 - \frac{b}{m}i\omega_d - \omega_0^2}$ where $\omega_0 = \sqrt{\frac{k}{m}}$

- Then, amplitude $A = |\mathcal{A}| = \frac{D/m}{\sqrt{(\omega_d^2 - \omega_0^2)^2 + \frac{b^2}{m^2}\omega_d^2}}$

- It is obvious that A_{max} occurs at the minimum of the denominator

- $\frac{d\mathcal{W}}{d\omega_0} = \frac{d}{d\omega_0}[\sqrt{(\omega_d^2 - \omega_0^2)^2 + \frac{b^2}{m^2}\omega_d^2}] = \frac{\frac{1}{2}[4(\omega_d^3 - \omega_d\omega_0^2) + 2\frac{b}{m}^2\omega_d]}{\mathcal{W}}$

- $\frac{d\mathcal{W}}{d\omega_0}|_{\omega_r} \Rightarrow 2\omega_r[(\omega_r^2 - \omega_0^2) + \frac{b}{m}^2] = 0$

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- $\omega_d \neq 0 \Rightarrow \omega_r^2 = \omega_0^2 - \frac{b^2}{2m^2}$

- $\Rightarrow \boxed{\omega_r = \sqrt{\dfrac{k}{m} - \dfrac{b^2}{2m^2}}}$

6.3   Modern Physics

eqn 3.1 Rayleigh-Jeans Law

- Firstly, we begin by noting that $R(\lambda) = \rho(f) \cdot \frac{c}{4\pi}$ where $\rho(f)$ is the energy density inside a Jeans cube

- Then, $\rho(f) = N(f) \cdot E_{avg}$

- $N(f) = \frac{8\pi f^2}{c^3}$, and $E_{avg} = kT$

- Then $\rho(f) = \frac{8\pi f^2 kT}{c^3}$

- $= \frac{8\pi kT}{\lambda^2}$

- $\Rightarrow \boxed{\dfrac{2ckT}{\lambda^4}}$

eqn 3.2 Planck's Law

6.4   Intro to Quantum Physics

eqn 4.1 Compton Wavlength

Milikan's Experiment

For plates with p.d. V_0, separation d, and droplet of charge q and radius r moving at velocity v:

- $\vec{f_e} = q \cdot \vec{E} = q \cdot \frac{V_0}{q} \cdot \hat{e}_z$

- $\vec{f_g} = mg \cdot \hat{e}_z$ and $m = V \cdot \rho_{oil} = \frac{4}{3}\pi r^3 \rho_{oil} \Rightarrow \vec{f_g} = \frac{4}{3}\pi r^3 \rho_{oil} g \hat{e}_z$

- oil displacing air $\Rightarrow \vec{f_b} = w_{displaced} = v_{air}\rho_{air}$

- $v_{air} = v_{oil} \Rightarrow \vec{f_b} = \frac{4}{3}\pi g r^3 \rho_{air} \hat{e}_z$

- from Stoke's Law, $\vec{f_v} = -6\pi\eta_{air} \cdot rv$

eqn 4.2 Milikan's Oil Drop Radius

If we set $V_0 = 0$ and consider the terminal velocity of the drop v_{off}

- $\vec{f_g} + \vec{f_b} + \vec{f_v} = 0$

- Note, $\vec{f_g} + \vec{f_b} + \vec{f_v} = 0 \Rightarrow f_v = f_g - f_b$

- $\frac{4}{3}\pi\rho_{oil}gr^3\hat{e}_z - \frac{4}{3}\pi\rho_{air}gr^3\hat{e}_z - 6\pi\eta_{air}v_{off}\hat{e}_z = 0$

- $\frac{4}{3}\pi(\rho_{oil} - \rho_{air}g)r^3 = 6\eta_{air}v_{off}r$

- $$\boxed{r = \sqrt{\frac{6\eta_{air}v_{off}}{2(\rho_{oil} - \rho_{air})g}}}$$

eqn 4.3 Milikan's Oil Drop Charge

apply voltage V_0 such that drop moves upwards to a terminal velocity v_{on}

- $\vec{f_e} + \vec{f_v'} + \vec{f_b} + \vec{f_e} = 0$

- $f_g\hat{e}_z + f_v'\hat{e}_z - f_b\hat{e}_z - f_e\hat{e}_z = 0$

- $f_e = f_v' + (f_g - f_b) = f_v' + f_v$, where f_v = viscous force from off case

- $\frac{qV_0}{d} = 6\pi\eta_{air}rV_{on} + 6\pi\eta_{air}v_{off} = 6\pi\eta_{air}r(V_{on} + V_{off})$

- Leading to:

- $$\boxed{q = \frac{g\sqrt{2}(V_{on} + V_{off}d)}{V_0} \cdot \sqrt{\frac{\eta_{air}^3 V_{off}}{(\rho_{oil} - \rho_{air})g}}}$$

6.5   Maths Methods II