

# Semester 2 Notes

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# 1 Analysis II

## 1.1 Propositional Logic

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- $\neg(p \cup q) = (\neg p) \cap (\neg q)$
- $\neg(p \cap q) = (\neg p) \cup (\neg q)$
- $\neg(\exists p) = \forall p$
- $p \Rightarrow q = (p \cap q) \cup \neg q$
- so  $\neg(p \Rightarrow q) = (\neg p \cup \neg q) \cap \neg \neg q = \neg p \cup q$

## 1.2 Continuous Functions

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### Continuity

- For  $f : D \rightarrow \mathbb{R}$ ,  $f$  is continuous at a point  $a \in D$  if:  
 $\forall \epsilon > 0, \exists \delta > 0 : \forall x \in D, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$
- $f$  is uniformly continuous if continuous  $\forall a$ , or  
 $\forall \epsilon > 0, \exists \delta > 0 : \forall x, y \in D, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
- In general, continuity  $\nRightarrow$  uniform continuity
- A function  $f : D \rightarrow \mathbb{R}$  is continuous at  $a \in \mathbb{R}$  iff:

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$$\forall (x_n)_{n \in \mathbb{N}}, x_n \in D, \lim_{n \rightarrow \infty} x_n = a, \lim_{n \rightarrow \infty} f(x_n) = f(a)$$

- An alternative statement of this is:

$$\lim f(x_n) = f(\lim(x_n))$$

if  $f$  not continuous at  $a$ :

$$\exists \epsilon > 0 : \forall \delta > 0, \exists x \in D :$$

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| \geq \epsilon$$

**Theorem 1 (Intermediate Value Theorem)** for  $f : D \rightarrow C \subseteq \mathbb{R}$   
then  $u \in C \Rightarrow \exists x \in D : u = f(x)$

- A function is also continuous if:
- $\delta > 0 : \forall x \in D \setminus a : |x - a| > \delta$
- $\forall (x_n)_{n \in \mathbb{N}}$

## 2 waves and fields

### 2.1 Periodic Motion

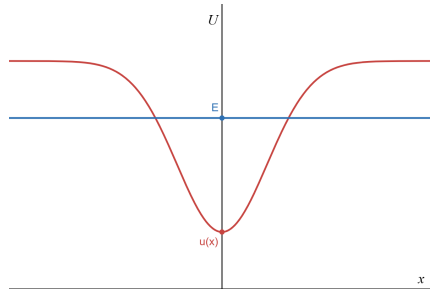
#### Periodic Motion In 1D

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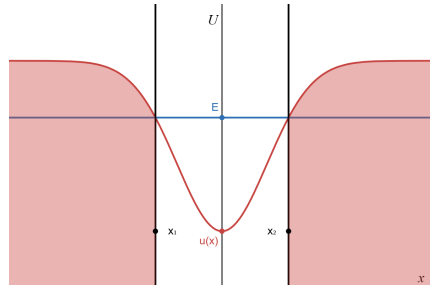
- Periodic motion is one in which the position repeats values regularly
- It can be characterised like so:

$$r(t + nT) = r(t), \forall t \in \mathbb{R}, n \in \mathbb{N}, T = \text{period}$$

- If we consider a situation in a conservative field, with a potential  $u(x)$ , and a system with a total energy  $E$ :



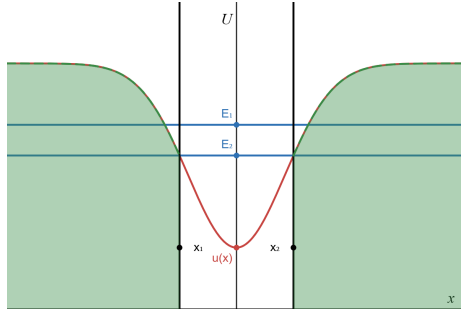
- We can see that the system would be bound within the region  $U < E$ :



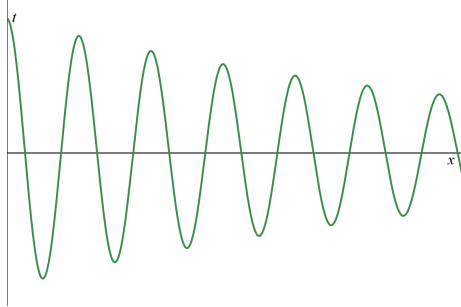
- This naturally leads to periodic motion, as at  $x = x_1, v = 0$
- At this point, the energy is not minimised, and so the system tends towards  $x = 0$ , where  $v \neq 0$ , causing periodic motion.
- One way to achieve this is through simple harmonic motion (SHM)
- SHM is any system where  $\ddot{x} = -x$
- We can see here that  $u(x) = -\int f dx = \int \frac{\ddot{x}}{m} dx$ , or  $f = -\frac{du}{dx} \Rightarrow m\ddot{x} = -\frac{du}{dx}$
- We can then add another term to take energy out of the system:
- $m\ddot{x} = -\frac{du}{dx} - \gamma\dot{x}$  in general.

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- This describes a frictional force, causing damped harmonic motion (DHM)
- This will lead to the energy gradually decreasing:



- Which will result in the following motion:



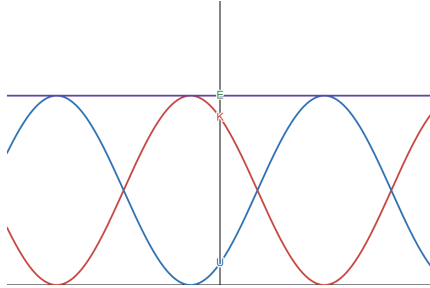
### Periodic Motion In 3D

- Gravity is a conservative force so  $u = u(\vec{r})$ , and angular momentum is conserved
- Evidently, from  $\vec{L} = \vec{r} \times \vec{p}$ , the position  $\vec{r}$  is perpendicular to the constant angular momentum  $\vec{L}$ , and thus the system is confined to a single plane.
- Let  $v_\phi$  = the component of  $v$  perpendicular to  $\vec{r}$ , and  $v_r$  = the component of  $v$  parallel to  $\vec{r}$
- Then  $L = m\vec{r} \times (r\dot{\phi})$
- Since  $K = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{\phi}^2 + \dot{r}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$
- $\dot{\phi} = \frac{L}{mr^2} \Rightarrow K = \frac{1}{2}m(\dot{r}^2 + r^2(\frac{L}{mr^2})^2)$
- So  $E = K + u = \frac{1}{2}m(\dot{r}^2 + (L/mr)^2) + u(\vec{r})$

## Simple Harmonic Motion

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- Simple harmonic motion is any motion of the form  $x(t) = A\cos(\omega t + \phi)$
- From this, we can see that  $\ddot{x}(t) = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x$
- This differential equation,  $\ddot{x} \propto x$ , characterises simple harmonic oscillation (SHO)
- Then,  $f = m\ddot{x} = -m\omega^2 x$ , is only dependant on position
- Then, the potential energy:  $U(x) = -\int f dx = m\omega^2 \int x dx = \frac{1}{2}m\omega^2 x^2$
- Then,  $E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}mA^2\omega^2$
- E, K, and U will look like this over time, with  $\overline{U} = \overline{K}$  over one period:



## Simple Pendulums and Small oscillators

- For a simple pendulum,  $|\tau| = I\ddot{\theta} = -f_{\perp} \cdot r = -f_g \sin(\theta) \cdot l$
- Then, using the small angle approximation  $\sin(\theta) = \theta$ :  

$$I\ddot{\theta} = -f_g l \theta$$
- Another example of simple harmonic motion is the torsion pendulum
- This is a mass on a string that can twist around its axis:  

$$\tau = -c\theta \Rightarrow I\ddot{\theta} = -c\theta$$

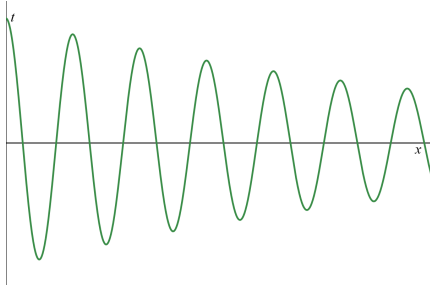
## Damped Harmonic Motion

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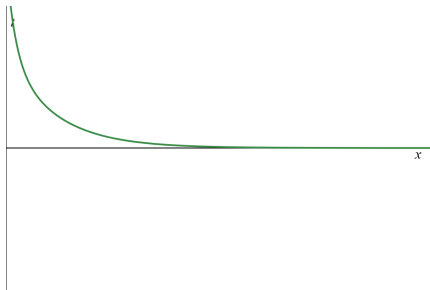
- We expect the amplitude A of a damped harmonic oscillator to decrease over time.
- For DHM, The 2nd law of motion looks like this:
- $m\ddot{x} = -kx - b\dot{x}$
- Setting  $b=0$ , we recover the SHM characteristic equation
- For  $b^2 - 4km < 0$ , This is called underdamping

- Underdamping will have the following form of motion:

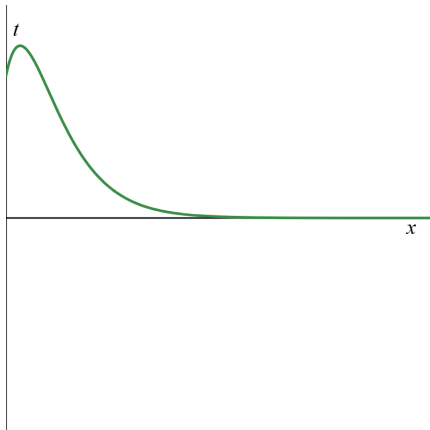
$$x(t) = Ae^{-\frac{b}{2m}t} \cos(\omega t + \varphi) \quad (2.1)$$



- Then considering the quantity  $\frac{2m}{b}$ , we see  $|x(t + \frac{2m}{b})| = e^{-1}|x(t)|$
- If  $b^2 > 4mk$ , we get overdamped motion:
- $x(t) = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t}$



- and if  $b^2 = 4mk$ , critically damped Motion:
- $x(t) = (A_1 + A_2 t)e^{-\frac{b}{2m}t}$



## Forced Harmonic Oscillation & Resonance

7/2/25

- A forced harmonic oscillator (FHO) is one subject to an external driving force  $F_d$
- $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F_d$
- If the driving force is also periodic, the motion will oscillate at the driving frequency  $f_d$

$$\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \quad (2.2)$$

- Where A is maximised at  $\omega_d = \omega_r$

## Coupled Harmonic Oscillators

- for coupled springs with masses  $m_{1,2}$  at positions  $x_{1,2}$
- Clearly,  $m\ddot{x}_1 = -kx_1 - k'(x_1 - x_2)$
- And a similar result for  $x_2$
- If we then consider the case  $x_1 = x_2$ , the middle spring's force vanishes and we get 2 uncoupled harmonic motions with frequency  $\omega_1 = \sqrt{\frac{k}{m}}$
- This is called a normal mode of the system, as the  $x_1(t) \perp x_2(t)$ : They are independant
- Considering the alternate normal mode:  $x_1(t) = -x_2(t)$
- And the motion  $\omega_2 = \sqrt{\frac{k+2k'}{m}}$
- In n dimensions, there are n normal modes, and all normal frequencies are normal frequencies
- All modes of a system are superpositions of the normal modes



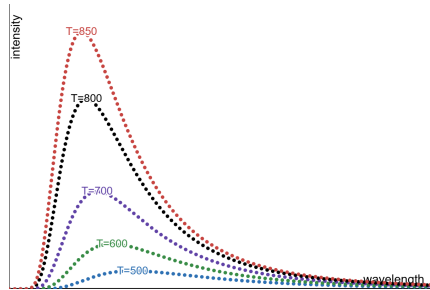
## 3 Modern Physics

### 3.1 Black Body Radiation

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#### Observations

- Black bodies satisfy 3 criteria:
  - absorbs all incident radiation
  - emits as much radiation as possible at any wavelength
  - emits radiation isotropically
- By studying the spectra of black bodies at different temperatures:



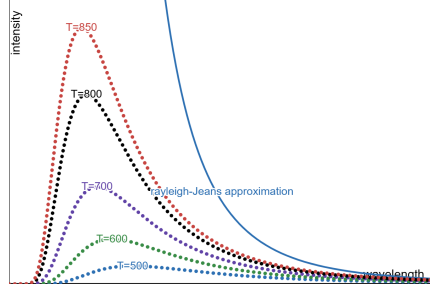
- We can observe 2 main laws:
  - Wein's displacement law:  $\lambda_{max} = \frac{b}{T}$
  - Stephans law:  $P(T) = \sigma AT^4$

#### Rayleigh-Jeans Law

- In June 1900, John Rayleigh discovered that the spectral radiance  $\propto \lambda^{-4}$
- In 1905, James Jean derived the full Rayleigh-Jeans law including constants
- The Rayleigh-Jeans law is a classical approximation of the radiance of a black body
- Rayleigh-Jeans law can be expressed as so:

$$B(\lambda, T) = \frac{2ck_bT}{\lambda^4} \quad (3.1)$$

- This produced the following prediction for spectral radiance at T=800:



- This presents many problems, mainly that it doesn't match for  $\lambda < \lambda_p$ , and  $I \rightarrow \infty$  as  $\lambda \rightarrow 0$

### Planck's Law

- Rayleigh-Jeans law, as seen in the derivation, can be expressed as:
- $R(\lambda) \propto \rho(f) \cdot E_{avg}$
- Where  $\rho$  and  $E$  are the number of waves and average energy of each wave
- In the Rayleigh-Jeans law,  $E_{avg} = kT$
- This was due to the equipartition theorem
- Max Planck's great insight was to abandon the equipartition theorem, instead writing:
- $E = n\epsilon \Rightarrow \overline{E} = \frac{hf}{e^{\frac{hf}{kT}} - 1}$
- And so:

$$R(\lambda) = \frac{2hc}{\lambda^5} \cdot \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \quad (3.2)$$

## 3.2 The Photoelectric Effect

6/2/25

- In 1887, Heinrich Hertz discovered that light rays incident on a metal plate produces an emf across the material
- This is due to electrons being liberated by the light
- This works with metals due to the delocalised electron bonds
- The electrons would be liberated with a range of kinetic energies up to a maximum
- There was also an observed threshold frequency, below which a ray can't liberate electrons

### classical Predictions

- $V_s \propto I$
- $K_{max} \propto A$
- $\frac{dN}{dt} \propto f$

### Einstein's Explanation

- In 1905, Einstein published his explanation of the photoelectric effect
- He explained that the light interacts in a 1:1 manner, with  $E = hf$
- This is insightful as Max Planck had only quantised matter oscillation
- Einstein quantised light
- This explains the intensity profile, as the particle light must interact with the electrons 1:1
- It also explained the threshold frequency phenomenon, as light with insufficient energy just reflects off of the metal
- $E_\gamma = hf = \varphi + \frac{1}{2}m_e v_{max}^2$  or  $hf = \varphi + K_{max}$

## 4 Intro to Quantum Physics

### 4.1 A Brief History

#### Ultraviolet Catastrophe

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- In 1860, Gustav Kirchhoff discovered what is now called black body radiation
- This is where an object with internal energy radiates electromagnetic waves correlated with its thermal properties.
- Classical predictions, using namely the Rayleigh-Jeans law, predicted an infinite amount of low-frequency light waves:
- In 1900, Max Planck produced a solution to this problem which correctly predicted the spectral radiance of black body radiation
- This theory worked by quantising light, which he expected to disappear in the final solution
- He still did not believe that light was actually quantised
- He earned the Nobel Prize in 1918

#### Photoelectric Effect

- In 1887, Heinrich Hertz discovered the photoelectric effect
- This is when you expose a metal plate to high intensity light and observe a voltage over the plate
- The contemporary theories predicted that the energy transferred should be proportional to the frequency
- Instead, there is an observed threshold frequency, below which no energy is transferred
- In 1905, Albert Einstein explained this by employing Max Planck's quantum light
- He won the Nobel Prize in 1921

#### Atomic Models

- In 1911, Ernst Rutherford discovered positively charged atomic nuclei
- The presence of the electron in the atom, then, predicted that atoms were unstable, decaying in  $10^{-11}s$  as the electron would fall into the nuclei
- In 1913, Niels Bohr predicted that angular momentum was quantised, restricting electrons to specific "orbits" and stopping atomic decay

- Neils Bohr could not come up with a mechanism for angular momentum quantisation, however
- He won the Nobel Prize in 1913

### DeBroglie's Electron Strings

- In 1924, Louis DeBroglie suggested wave particle duality
- This suggests that sufficient restrictions upon an electrons position force it to exhibit wave like behaviour
- When employed on an electron in orbit, the electron waves superpose, producing a standing wave
- Standing waves can only exist with an integer multiplier on the fundamental frequency
- This multiplier corresponds to the energy the electron can have, quantising orbital angular momentum
- He won the Nobel Prize in 1929

## 4.2 Rutherford And Compton Scattering

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### Rutherford Scattering

- In 1897, Thompson devised the Plum Pudding model.
- In 1906 in Manchester, Ernst Rutherford devised the  $\alpha$  scattering experiment
- He shot  $\alpha$  particles at a thin sheet of gold and observe the distribution of scattering angles
- The Plum Pudding model predicts the following distribution:
- It predicted little to no scattering due to the diffusivity of the positive charge
- Whereas the distribution Rutherford found was the following:
- In response to this, rutherford devised the nuclear model
- This explained that the nucleus repels the  $\alpha$  particles due to the like charges, sometimes up to a full  $\pi^c$

## Compton Scattering

- In 1923, Arthur Compton discovered Compton scattering
- This is where x-rays incident on an atom reflect at a different wavelength
- The classical model, with  $p_\gamma = 0$ , predicted no shift:
- But in reality, there was a shift:
- This showed that light does carry momentum, and therefore supported the photon model
- Using conservation of momentum and conservation of energy, you can arrive at the following equation:

$$\lambda - \lambda' = \frac{h}{m_0 \cdot c} [1 - \cos(\theta)] \quad (4.1)$$

## 4.3 Milikan's Oil Drop

5/2/25

- In 1897, Robert Milikan devised an experiment to measure the charge of the electron:
- The 2 plates are connected to a power source, producing an emf
- A nozzle containing oil is small to be sprayed
- There is a camera or microscope to observe the drops
- The drops become charged due to friction with the nozzle
- Oil drops that move up are large charge or low mass, and vice versa
- There is also the bouyant and viscous forces
- $\vec{f}_b = \frac{4}{3}\pi gr^3 \rho_{air} \hat{e}_z$  since the oil is displacing the air
- from Stoke's Law,  $\vec{f}_v = -6\pi\eta_{air} \cdot rv$
- $\vec{f}_v = -6\pi\eta_{air}$
- Since it is hard to measure the properties of the drops explicitly, we can instead formulate expressions for them:

$$r = \sqrt{\frac{6\eta_{air}v_{off}}{2(\rho_{oil} - \rho_{air})g}} \quad (4.2)$$

$$q = \frac{g\sqrt{2}(v_{on} + v_{off})d}{V_0} \cdot \sqrt{\frac{\eta_{air}^3 v_{off}}{(\rho_{oil} - \rho_{air})g}} \quad (4.3)$$

- You then repeat this multiple times and record the charge of each drop 6/2/25
- There will be a random distribution of charges, however the smallest difference in charge will be  $e = 1.60 \cdot 10^{-19}$

## 5 Mathematical Methods II

### 5.1 Non Cartesian Co-Ordinates

#### Polar Co-Ordinates

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- for  $x\hat{i} + y\hat{j}$ , the equivalent polar basis vectors are:  
 $\hat{r} = (\cos(\theta), \sin(\theta))$ ,  $\hat{\theta} = (-\sin(\theta), \cos(\theta))$  where  $\tan(\theta) = \frac{y}{x}$
- Then,  $\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta} = \hat{i}$  and  $\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta} = \hat{j}$

#### 3D Polar Co-Ordinates

- There are 2 main 3D Polar Coordinate Bases:
- Spherical  $(\hat{r}, \hat{\theta}, \hat{\phi})$
- And Cylindrical  $(\hat{\rho}, \hat{\theta}, \hat{z})$



## 6 Derivations

### 6.1 Analysis II

### 6.2 Waves and Fields

#### eqn 2.1 Underdamping Motion Form

- $m\ddot{x} = -kx - b\dot{x}$
  - $\Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$
  - This has a characteristic equation  $\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}$
  - With solutions  $\lambda = -\frac{b}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}$
  - For case  $b^2 < 4mk$ ,  $\lambda = -\frac{b}{2m} \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$
- $$\Rightarrow x(t) = \text{Re}[Ae^{-\frac{b}{2m}t}e^{\pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}]$$
- $\Rightarrow \boxed{x(t) = Ae^{-\frac{b}{2m}t} \cos(\omega t + \varphi)}$

#### eqn 2.2 Resonance Frequency

- For  $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f_d$ ,  $f_d = d \cos(\omega_d t + \varphi)$  DE
- We know that for FHM,  $f = f_d$  so:
- Assume  $x(t) = \text{Re}[Ae^{i\omega_d t}]$ ,  $F_d = \text{Re}[De^{i\omega_d t}]$  where  $\mathcal{D} = De^{i\varphi_d}$
- Into DE:  $-\omega_d^2 Ae^{i\omega_d t} + \frac{b}{m}i\omega_d Ae^{i\omega_d t} + \frac{k}{m}Ae^{i\omega_d t} = \frac{1}{m}De^{i\omega_d t}$
- $\Rightarrow \mathcal{A}(\omega_d^2 - \frac{b}{m}i\omega_d - \frac{k}{m}) = -\frac{1}{m}\mathcal{D}$
- $\Rightarrow \mathcal{A} = \frac{\mathcal{D}/m}{\omega_d^2 - \frac{b}{m}i\omega_d - \frac{k}{m}}$  where  $\omega_0 = \sqrt{\frac{k}{m}}$
- Then, amplitude  $A = |\mathcal{A}| = \frac{D/m}{\sqrt{(\omega_d^2 - \omega_0^2)^2 + \frac{b^2}{m^2}\omega_d^2}}$
- It is obvious that  $A_{max}$  occurs at the minimum of the denominator
- $\frac{d\mathcal{W}}{d\omega_d} = \frac{d}{d\omega_0}[\sqrt{(\omega_d^2 - \omega_0^2)^2 + \frac{b^2}{m^2}\omega_d^2}] = \frac{\frac{1}{2}[4(\omega_d^3 - \omega_d\omega_0^2) + 2\frac{b^2}{m^2}\omega_d]}{\mathcal{W}}$
- $\frac{d\mathcal{W}}{d\omega_d}|_{\omega_r} \Rightarrow 2\omega_r[2(\omega_r^2 - \omega_0^2) + \frac{b^2}{m^2}] = 0$
- $\omega_d \neq 0 \Rightarrow \omega_r^2 = \omega_0^2 - \frac{b^2}{2m^2}$
- $\Rightarrow \boxed{\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}}$

### eqn 2.3 Coupled Harmonic Oscillator

- $m\ddot{x}_1 = -kx_1 - k^{prime}(x_1 - x_2)$
- $m\ddot{x}_2 = -kx_2 - k^{prime}(x_2 - x_1)$

## 6.3 Modern Physics

### eqn 3.1 Rayleigh-Jeans Law

- Firstly, we begin by noting that  $R(\lambda) = \rho(f) \cdot \frac{c}{4\pi}$  where  $\rho(f)$  is the energy density inside a Jeans cube
- Then,  $\rho(f) = N(f) \cdot E_{avg}$
- $N(f) = \frac{8\pi f^2}{c^3}$ , and  $E_{avg} = kT$
- Then  $\rho(f) = \frac{8\pi f^2 kT}{c^3}$
- $= \frac{8\pi kT}{\lambda^2}$
- $\Rightarrow \boxed{\frac{2ckT}{\lambda^4}}$

### eqn 3.2 Planck's Law

## 6.4 Intro to Quantum Physics

### eqn 4.1 Compton Wavelength

#### Milikan's Experiment

For plates with p.d.  $V_0$ , separation  $d$ , and droplet of charge  $q$  and radius  $r$  moving at velocity  $v$ :

- $\vec{f}_e = q \cdot \vec{E} = q \cdot \frac{V_0}{d} \cdot \hat{e}_z$
- $\vec{f}_g = mg \cdot \hat{e}_z$  and  $m = V \cdot \rho_{oil} = \frac{4}{3}\pi r^3 \rho_{oil} \Rightarrow \vec{f}_g = \frac{4}{3}\pi r^3 \rho_{oil} g \hat{e}_z$
- oil displacing air  $\Rightarrow \vec{f}_b = w_{displaced} = v_{air} \rho_{air}$
- $v_{air} = v_{oil} \Rightarrow \vec{f}_b = \frac{4}{3}\pi g r^3 \rho_{air} \hat{e}_z$
- from Stoke's Law,  $\vec{f}_v = -6\pi\eta_{air} \cdot rv$

#### eqn 4.2 Milikan's Oil Drop Radius

If we set  $V_0 = 0$  and consider the terminal velocity of the drop  $v_{off}$

- $\vec{f}_g + \vec{f}_b + \vec{f}_v = 0$
- Note,  $\vec{f}_g + \vec{f}_b + \vec{f}_v = 0 \Rightarrow f_v = f_g - f_b$
- $\frac{4}{3}\pi\rho_{oil}gr^3\hat{e}_z - \frac{4}{3}\pi\rho_{air}gr^3\hat{e}_z - 6\pi\eta_{air}v_{off}\hat{e}_z = 0$
- $\frac{4}{3}\pi(\rho_{oil} - \rho_{air})r^3 = 6\eta_{air}v_{off}r$
- $$r = \sqrt{\frac{6\eta_{air}v_{off}}{2(\rho_{oil} - \rho_{air})g}}$$

#### eqn 4.3 Milikan's Oil Drop Charge

apply voltage  $V_0$  such that drop moves upwards to a terminal velocity  $v_{on}$

- $\vec{f}_e + \vec{f}'_v + \vec{f}_b + \vec{f}_e = 0$
- $f_g\hat{e}_z + f'_v\hat{e}_z - f_b\hat{e}_z - f_e\hat{e}_z = 0$
- $f_e = f'_v + (f_g - f_b) = f'_v + f_v$ , where  $f_v$  = viscous force from off case
- $\frac{qV_0}{d} = 6\pi\eta_{air}rV_{on} + 6\pi\eta_{air}v_{off} = 6\pi\eta_{air}r(V_{on} + V_{off})$
- Leading to:

- $$q = \frac{g\sqrt{2}(V_{on} + V_{off}d)}{V_0} \cdot \sqrt{\frac{\eta_{air}^3 V_{off}}{(\rho_{oil} - \rho_{air})g}}$$

## 6.5 Maths Methods II