PATH INTEGRALS FOR SCALAR FIELDS

3.1 FREE FIELD THEORY

The lagrangian for a free real scalar field ϕ is given by

$$\mathcal{L}_0\left(\phi(x)\right) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)^2\,. \tag{3.1}$$

The Euler-Lagrange equations of motion,

$$\partial_\mu \frac{\partial\mathcal{L}_0}{\partial\left(\partial_\mu\phi(x)\right)} - \frac{\partial\mathcal{L}_0}{\partial\phi(x)}\,, \tag{3.2}$$

yields the Klein-Gordon equation

$$\partial_\mu\partial^\mu\phi(x) + m^2\phi(x) = \left(\partial^2 + m^2\right)\phi(x) = 0\,. \tag{3.3}$$

LORENTZ INVARIANCE Note that $\mathcal{L}_0$ is invariant under Lorentz transformations:

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu\,, \qquad \phi(x) \mapsto \phi'(x) = \phi(\lambda^{-1}x)\,. \tag{3.4}$$

CONJUGATE MOMENTUM The momentum conjugate to $\phi(x)$ can be readily computed:

$$\Pi(x) = \frac{\partial\mathcal{L}_0}{\partial(\partial_0\phi(x))} \tag{3.5}$$

$$= \partial_0\phi(x) = \dot{\phi}(x)\,. \tag{3.6}$$

And therefore the Hamiltonian is

$$\mathcal{H} = \Pi(x)\dot{\phi}(x) - \mathcal{L}_0\left(\phi(x)\right) \tag{3.7}$$

$$= \frac{1}{2}\Pi(x)^2 + \frac{1}{2}\sum_{k=1}^{3}\left(\partial_k\phi(x)\right)^2 + \frac{1}{2}m^2\phi(x)^2\,. \tag{3.8}$$

The quadratic Hamiltonian is the generalization of the harmonic oscillator to the case where we have an infinite number of canonical coordinates, indexed by the continuous spatial coordinate $\mathbf{x}$.

3.2 Path Integral

The vacuum-to-vacuum amplitude in the presence of a source field $J(x)$ is the straightforward generalization of the expression we have derived for the quantum mechanical system. The correspondence between the two systems is as follows:

$$\begin{array}{lcl} q(t) & \longrightarrow & \phi(t,\mathbf{x}) \\ \hat{Q}(t) & \longrightarrow & \hat{\phi}(t,\mathbf{x}) \text{ (operator)} \\ f(t) & \longrightarrow & J(t,\mathbf{x}) \text{ (source)} \end{array}$$

The projection onto the ground state is implemented by the ϵ trick that we introduced in the case of quantum mechanics, *i.e.* by replacing the Hamiltonian with $(1 - i\epsilon)H$. In the case of the scalar field theory, this is conveniently achieved by the substitution $m^2 \mapsto m^2 - i\epsilon$. In all subsequent formulae we will assume that the squared mass has an infinitesimal (negative) imaginary part.

By analogy with the QM computation, we can write the expression for the path integral representation of the vacuum amplitude

$$Z_0[J] = \langle 0|0\rangle_J = \int \mathcal{D}\phi \, \exp\left\{ i \Big[S_0[\phi] + J \cdot \phi \Big] \right\}, \tag{3.9}$$

where

$$S_0[\phi] = \int d^D x \, \mathcal{L}_0\left(\phi(x)\right), \quad J \cdot \phi = \int d^D x \, J(x)\phi(x) . \tag{3.10}$$

For a free theory the integral in Eq. (3.9) is a Gaussian integral, similar to the ones we have seen in the first lecture. In order to make the correspondence more explicit, we can write the action as

$$S_0[\phi] = \int d^D x d^D x' \, \phi(x) K(x,x') \phi(x') + \int d^D x \, J(x)\phi(x) , \tag{3.11}$$

where we recognise a quadratic term, with $K(x,x')$ playing the role of A_{ij}, and $J(x)$ playing the role of the lienar term b_i. The explicit expression for the kernel K is

$$K(x,x') = \Big[-\partial^2 - m^2 \Big] \delta(x-x') . \tag{3.12}$$

It is convenient to work in momentum space, where the kernel in the action is diagonal. Introducing the Fourier transforms

$$\phi(x) = \int \frac{d^D p}{(2\pi)^D} \, e^{-ip\cdot x} \, \tilde{\phi}(p) , \tag{3.13}$$

we can rewrite the kinetic term:

$$\int d^D x\, \partial_\mu \phi(x) \partial^\mu \phi(x) = \int d^D x \int_{p,p'} (-ip_\mu) e^{-ip\cdot x} \tilde{\phi}(p) (-ip'^\mu) e^{-ip'\cdot x} \tilde{\phi}(p') \tag{3.14}$$

$$= \int_p p^2 \tilde{\phi}(p) \tilde{\phi}(-p) = \int_p p^2 \left|\tilde{\phi}(p)\right|^2 , \tag{3.15}$$

where we have introduced the notation

$$\int_p = \int \frac{d^D p}{(2\pi)^D} , \tag{3.16}$$

and used the fact that $\phi(x)$ is real, and hence $\tilde{\phi}(-p) = \tilde{\phi}(p)^*$. Hence the action for the free field in momentum space can be written as

$$S_0[\phi] = \frac{1}{2} \int_p \left\{ \tilde{\phi}(-p) \left[p^2 - m^2 + i\epsilon\right] \tilde{\phi}(p) + \tilde{J}(p)\tilde{\phi}(-p) + \tilde{J}(-p)\tilde{\phi}(p) \right\} . \tag{3.17}$$

Note that the contribution from the ϵ term to the exponential is

$$\exp\left\{ -\epsilon \int_p \left|\tilde{\phi}(p)\right|^2 \right\} , \tag{3.18}$$

which clearly is convergent for large values of $|\tilde{\phi}(p)|$. It is also important to note that the action for the free field is quadratic, and the kernel is diagonal in momentum space. Therefore the path integral for the free field is a simple extension of the Gaussian integrals that we have been discussing in previous lectures. We will use these previous results extensively.

The Gaussian integral can be computed by performing the *usual* shift of the integration variables

$$\tilde{\chi}(p) = \tilde{\phi}(p) + \frac{\tilde{J}(p)}{p^2 - m^2 + i\epsilon} ,$$

so that

$$S_0[\chi] = \frac{1}{2} \int_p \left\{ \tilde{\chi}(-p) \left[p^2 - m^2 + i\epsilon\right] \tilde{\chi}(p) + \tilde{J}(-p) \frac{1}{p^2 - m^2 + i\epsilon} \tilde{J}(p) \right\} . \tag{3.19}$$

Up to a normalization factor

$$Z_0[J] \propto \exp \frac{i}{2} \int_p \tilde{J}(-p) \frac{1}{p^2 - m^2 + i\epsilon} \tilde{J}(p) \tag{3.20}$$

$$= \exp \frac{i}{2} \int d^D x d^D x'\, J(x) \Delta(x, x') J(x') , \tag{3.21}$$

where

$$\Delta(x, x') = \Delta(x - x') = \int_p e^{-ip\cdot(x - x')} \frac{1}{p^2 - m^2 + i\epsilon} . \tag{3.22}$$

You can easily verify that $\Delta = K^{-1}$, *i.e.*

$$\int d^D z\, K(x,z)\Delta(z,x') = \delta(x-x')\,. \tag{3.23}$$

Δ is called the *Feynman propagator*.

Following our previous derivations for Gaussian integrals and QM, you can show that

$$\langle 0|\, T\phi(x_1)\phi(x_2)\,|0\rangle_0 = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) Z_0[J]\Big|_{J=0} \tag{3.24}$$

$$= \frac{1}{i}\Delta(x_1-x_2)\,. \tag{3.25}$$

Further correlators are obtained by taking further more derivatives

$$\langle 0|\, T\phi(x_1)\ldots\phi(x_n)\,|0\rangle_0 = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\ldots\left(\frac{1}{i}\frac{\delta}{\delta J(x_n)}\right) Z_0[J]\Big|_{J=0}\,. \tag{3.26}$$

They can be computed in the free theory using Wick's theorem, again following the arguments we used for Gaussian integrals:

$$\begin{aligned}\langle 0|\, T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,|0\rangle_0 = \frac{1}{i^2}\,[&\Delta(x_1-x_2)\Delta(x_3-x_4)+\\ &+\Delta(x_1-x_2)\Delta(x_3-x_4)+\Delta(x_1-x_2)\Delta(x_3-x_4)]\,.\end{aligned} \tag{3.27}$$

3.3 Interacting theory

Let us now add an interaction term in the lagrangian:

$$\mathcal{L}\left(\phi(x)\right) = \mathcal{L}_0\left(\phi(x)\right) + V\left(\phi(x)\right)\,. \tag{3.28}$$

We are going to consider several examples, *e.g.*

$$V\left(\phi(x)\right) = \frac{1}{3!} g\phi(x)^3\,. \tag{3.29}$$

What is the dimension of the coupling g as a function of D?

Denoting by $S_0[\phi]$ the action of the free theory, we can write the path integral for the interacting theory

$$Z[J] = \langle 0|0\rangle_J = \int \mathcal{D}\phi\, \exp\left\{ i\left(S_0[\phi] + \int d^D x\, V\left(\phi(x)\right) + J\cdot\phi\right)\right\}. \tag{3.30}$$

By performing the same manipulations that we discusssed for gaussian integrals we obtain

$$Z[J] = \exp\left\{ i\int d^D x\, V\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\} \int \mathcal{D}\phi\, \exp\left\{ i\left(S_0[\phi] + J\cdot\phi\right)\right\} \tag{3.31}$$

$$= \exp\left\{ i\int d^D x\, V\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\} Z_0[J]\,. \tag{3.32}$$

A useful expression is obtained by expanding both exponentials

$$Z[J] \propto \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{ig}{3!} \int d^D x \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 \right]^V \times \tag{3.33}$$

$$\sum_{P=0}^{\infty} \frac{1}{P!} \left[\frac{i}{2} \int d^D y \, d^D z \, J(y) \Delta(y-z) J(z) \right]^P \tag{3.34}$$

Consider now the contribution for fixed values of P and V, we are left with $E = 2P - 3V$ sources. Let us look at the details that enter in this contribution.

- the overall factor of 'i':

$$i^V \left(\frac{1}{i} \right)^{3V} i^P = i^{P-2V} = i^{E-P+V} \, . \tag{3.35}$$

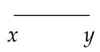
- derivatives acting on sources:

$$\frac{(2P)!}{E!} \text{ combinations} \, . \tag{3.36}$$

Many contractions yield the same result, which we will represent again using a diagrammatic representation.

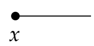
- Propagators, $\Delta(x-y)$, are represented by a line connecting the points x and y.

$$\frac{1}{i} \Delta(x-y) = \tag{3.37}$$



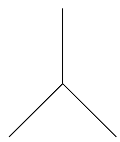
  External sources are represented with a solid dot at the end of a line. Note that the solid dot includes the integration over x.

$$i \int d^D x \, J(x) = \tag{3.38}$$



- Interactions are represented as three-prong vertices, again including the integration over x.

$$ig \int d^D x = \tag{3.39}$$



We can then count the number of contractions that yield a particular diagram, *i.e.* . a particular contribution. Assuming that there is no symmetry in the structure of the diagram, we have the following possibilities.

- permutations of the functional derivatives: $(3!)^V$;
- permutations of the vertices: $V!$;

- permutations of the sources at the end of propagators: 2^P;
- permutations of propagators: $P!$.

These factors match *exactly* the ones that appear in the expansions of the exponentials above. So in the absence of any symmetry in the diagram, each diagram contributes to Eq. (3.34) multiplied by a factor of 1. However, this procedure results in a double counting of the possible contributions if there are symmetries in the structure of the diagram, *i.e.* if a permutation of derivatives results in the same operations as a permutation of the sources.

Example: let us consider the case $P = 2$, $V = 1$, and hence $E = 4 - 3 = 1$. The term in the double expansion is

$$\begin{aligned} \frac{ig}{3!} \int d^D x \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \frac{1}{2!} \frac{i}{2} \int d^D y_1 \, d^D z_1 \, J(y_1) \Delta(y_1 - z_1) J(z_1) \\ \times \int d^D y_2 \, d^D z_2 \, J(y_2) \Delta(y_2 - z_2) J(z_2) \, . \end{aligned} \tag{3.40}$$

It can be rewritten as

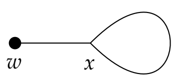
$$\begin{aligned} g \frac{1}{3! \times 2! \times 2 \times 2} \int d^D x \, d^D y_1 \, d^D z_1 \, d^D y_2 \, d^D z_2 \, \Delta(y_1 - z_1) \Delta(y_2 - z_2) \times \\ \times \left(\frac{\delta}{\delta J(x)} \right) \left(\frac{\delta}{\delta J(x)} \right) \left(\frac{\delta}{\delta J(x)} \right) J(y_1) J(z_1) J(y_2) J(z_2) \, . \end{aligned} \tag{3.41}$$

We are going to count the possible contractions in two different ways.

1. Just counting: there are 4 possible choice to decide which J is not paired with a derivative. Then there are $3!$ possible ways of pairing the derivatives with the three Js. The derivatives acting on the Js produce Dirac deltas, which we use to evaluate some of the integrals. The final result is

$$\begin{aligned} & g \frac{1}{3! \times 2! \times 2 \times 2} 4 \times 3! \int d^D x \, d^D w \, J(w) \Delta(w - x) \Delta(x - x) = \\ & = g \frac{1}{2} \int d^D x \, d^D w \, J(w) \Delta(w - x) \Delta(0) \, . \end{aligned} \tag{3.42}$$

The integral can be represented diagrammatically as



(3.43)

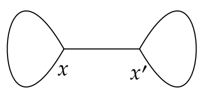
Note that there is one source left in the integral, corresponding to $E = 1$, and hence one external 'dot' in the diagram.

2. Let us now try to figure out the symmetry factor by working out the double counting in the general argument spelled out above. In that counting, swapping derivatives and swapping the ends of a propagator have been counted as distinct operations. However you can check that in this particular example swapping *e.g.* $J(y_1)$ and $J(z_1)$ is equivalent to swapping the two derivatives that act on the currents. Hence, in order to take into account this double counting, we need to divide the integral that is represented by the diagram by a symmetry factor $S = 2$, which precisely the factor of $1/2$ that appears in front of the integral in Eq. (3.42).

Another example: let us now take $P = 3$ and $V = 2$. The number of external legs is $E = 2 \times 3 - 3 \times 2 = 0$. The overall factor of i is $i^{P-2V} = -i$. The terms appearing in the expansion are:

$$\frac{1}{2!} g^2 \frac{1}{(3!)^2} \frac{1}{2^3} \int d^D x_1\, d^D x_2\, d^D y_1\, d^D z_1\, d^D y_2\, d^D z_2\, d^D y_3\, d^D z_3\, \Delta(y_1 - z_1)\Delta(y_2 - z_2)\Delta(y_3 - z_3) \times$$
$$\times \left(\frac{\delta}{\delta J(x_1)}\right)^3 \left(\frac{\delta}{\delta J(x_2)}\right)^3 J(y_1)J(z_1)J(y_2)J(z_2)J(y_3)J(z_3)\,. \tag{3.44}$$

Let us focus on the contractions that yield the following diagrammatic representation



(3.45)

and let us try to work out the combinatorial factor in two ways.

1. Just counting: we need to pair each of the sources in the integrand with one of the derivatives. Starting from y_1 we have 2 ways of choosing which group of derivatives to use, then 3 choices for picking one derivative. The source in z_1 then needs to couple with the one the remaining derivatives in the same group, so there are 2 choices. The three remaining sources need then to be paired with the three remaining derivatives, for which there are 3×2 choices. Hence we obtain
$$g^2 \frac{1}{2^3} \int d^D x d^D x' \Delta(0)^2 \Delta(x - x')\,. \tag{3.46}$$

2. Symmetry factor: swapping the ends of the propagator that appears as a loop closing at x is equivalent to swapping two derivatives at x, which yields a factor of 2; swapping the ends of the propagator that appears as a loop closing at x' is equivalent to swapping two derivatives at x', which yields another factor of 2; and finally swapping the ends of the propagator from x to x' is equivalent to swapping the two groups of three derivatives, which yields yet another factor of 2. Hence in total we have $S = 2^3$, which is consistent with the result above.

The only way to get these things right is by practicing – a lot. There are plenty of examples in Srednicki's book! We should try them together, maybe in a tutorial.

3.4 Disconnected diagrams

A disconnected diagram is made of the product of several connected pieces. We denote by C_I the contribution of a given connected region, including its symmetry factor, and D the total contribution of the diagram. Then

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I} \ , \tag{3.47}$$

where n_I is the number of times that the subdiagram C_I appears. In order to evaluate properly the contribution of the total disconnected diagram we need to evaluate S_D, *i.e.* find out the possible double counting that is left after the symmetry factors of each connected subdiagram has been worked out. The only residual symmetry that is left in the total diagram comes from the exchange of *all* propagators and vertices amongst different but identical connected subdiagrams. The symmetry factor can be readily evaluated

$$S_D = \prod_I n_I! \,. \tag{3.48}$$

We now have all the ingredients to write the sum of diagrams that contribute to the generating functional:

$$\begin{aligned} Z[J] &= \sum_{\{n_I\}} D(\{n_I\}) \propto \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} && (3.49) \\ &= \prod_I \left\{ \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} \right\} && (3.50) \\ &= \exp\left\{ \sum_I C_I \right\} \ , && (3.51) \end{aligned}$$

which shows that $\log Z[j]$ is the sum of all connected diagrams.

We can now address the issue of the normalization of $Z[J]$. We can request that $Z[0] = 1$, or equivalently define

$$\frac{Z[J]}{Z[0]} = \exp\left\{iW[J]\right\} \ , \tag{3.52}$$

where

$$iW[J] = \sum_I{}' C_I \,, \tag{3.53}$$

and the prime in the sum indicates that we do not include the vacuum diagrams in the sum, *i.e.* the diagrams with $E = 0$.