PATH INTEGRALS FOR GAUGE FIELDS

6.1 PATH INTEGRAL FOR GAUGE FIELDS

The action for the gauge field, $A_\mu(x)$, can be written as

$$S[A] = \int d^D x \left(-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\right), \tag{6.1}$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. Computing the variation of the action yields the classical equations of motion, *i.e.* Maxwell's equations in vacuum:

$$\partial_\mu F^{\mu\nu}(x) = 0\,. \tag{6.2}$$

Note that the action only depends on the field strength $F_{\mu\nu}$, and therefore is invariant under local *gauge transformations*,

$$A_\mu(x) \mapsto A_\mu^\Lambda(x) = A_\mu(x) + \partial_\mu\Lambda(x)\,. \tag{6.3}$$

Other symmetries, like translation invariance and invariance under Lorentz transformations are also encoded in Eq. (6.1). The action is quadratic in the fields, and can be recast as

$$S[A] = \frac{1}{2}\int d^D x\, A_\mu(x)\left[\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right]A_\nu(x)\,. \tag{6.4}$$

It would be tempting at this stage to define the path integral by analogy to the case of the scalar field:

$$Z[J] = \int \mathcal{D}A \exp\left\{i\int d^D x\left[-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + J_\mu(x)A^\mu(x)\,.\right]\right\} \tag{6.5}$$

Unfortunately, as discussed previously for the case of Gaussian integrals and scalar fields, the integral above can be performed only if the kernel is invertible. Writing the action in momentum space yields the kernel in its diagonalised form,

$$S[A] = \int \frac{d^D k}{(2\pi)^D}\tilde{A}_\mu(k)\left[K^{\mu\nu}(k)\right]\tilde{A}_\nu(-k)\,, \tag{6.6}$$

where the kernel is given by

$$K^{\mu\nu}(k) = -k^2 g^{\mu\nu} + k^\mu k^\nu\,. \tag{6.7}$$

It is clear from Eq. (6.6) that any longitudinal component of the gauge field,

$$\tilde{A}_\mu(k) = k_\mu\tilde{\Lambda}(k)\,,$$

is an eigenfunction of the action kernel with vanishing eigenvalue. Equivalently one could notice that the kernel is proportional to the projector $\Pi^{\mu\nu}$ on the transverse components of the gauge field:

$$K^{\mu\nu}(p) = -k^2 \Pi^{\mu\nu}(k)\,. \tag{6.8}$$

The action does not depend at all on the longitudinal components. They are 'projected out' of the action, leaving a divergent integral over a non compact domain. This is a direct consequence of the redundancy in the usage of a four-vector to describe photons. There are indeed four degrees of freedom in a real vector field, which is used to represent photon with only *two* physical, transverse polarizations. Using a four-vector allows an elegant implementation of Lorentz covariance, but the price we pay is that we have unphysical degrees of freedom in the action. The redundancy is at the origin of the gauge symmetry of the action. Going back to position space, it is easy to see that the longitudinal modes are *pure gauge* ones, *i.e.* $A_\mu(x) = \partial_\mu \Lambda(x)$. The solution to this problem is to identify the redundant degrees of freedom, and factor out the integration over these modes.

6.2 Faddeev-Popov procedure

A gauge *orbit* is a set of gauge configurations that are related by gauge transformations:

$$\Omega_A = \left\{ A^\Lambda_\mu(x), \text{for all } \Lambda(x) \right\}\,. \tag{6.9}$$

As discussed above, for a given $A_\mu(x)$, all the field configurations in Ω_A represent the same physical state, and therefore a single representative should be included in the path integral for each gauge orbit. We can select such representative by requiring it to be the solution of a *gauge fixing* condition:

$$G\left(A_\mu(x)\right) = 0\,. \tag{6.10}$$

A useful identity In order to insert a gauge fixing condition in the path intgral we are going to make use of the following identity:

$$1 = \int \mathcal{D}G\, \delta(G) = \int \prod_x \left[dG(x)\, \delta\left(G(x)\right) \right] \tag{6.11}$$

$$= \int \mathcal{D}\Lambda\, \delta\left(G\left(A^\Lambda_\mu\right)\right) \det\left(\frac{\delta G\left(A^\lambda_\mu\right)}{\delta\Lambda} \right)\,. \tag{6.12}$$

Lorentz gauge As an example, the Lorentz gauge corresponds to the choice

$$G\left(A_\mu(x)\right) = \partial_\mu A^\mu(x)\,; \tag{6.13}$$

and therefore

$$G\left(A_\mu^\Lambda(x)\right) = \partial_\mu\left(A^\mu(x) + \partial^\mu\Lambda(x)\right) \tag{6.14}$$

$$= \partial_\mu A^\mu(x) + \partial^2\Lambda(x)\,. \tag{6.15}$$

In order to computer the Jacobian of the change of variables in Eq. (6.12), we need to consider $G\left(A_\mu^\Lambda(x)\right)$ has a function of Λ, so that

$$\frac{\delta G\left(A_\mu^\Lambda(x)\right)}{\delta\Lambda(y)} = \delta(x-y)\partial^2\,. \tag{6.16}$$

In this simple case, we note that the functional derivative does not depend on the gauge field A_μ, and therefore we do not need to work out the determinant in full detail.

Gauge-fixed path integral We can now use Eq. (6.12), and rewrite the functional integral for a gauge theory as

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\delta G\left(A_\mu^\lambda\right)}{\delta\Lambda}\right)\int \mathcal{D}\Lambda \int \mathcal{D}A\, e^{iS[A]}\,\delta\left(G\left(A_\mu^\Lambda\right)\right) \tag{6.17}$$

$$\propto \int \mathcal{D}\Lambda \int \mathcal{D}A^\Lambda\, e^{iS[A^\Lambda]}\,\delta\left(G\left(A_\mu^\Lambda\right)\right)\,. \tag{6.18}$$

The integration measure is invariant under the transformation $A_\mu \mapsto A_\mu + \partial_\mu\Lambda$, which is only a shift oof the integration variables. The gauge invariance of the action means that $S[A] = S[A^\Lambda]$. Finally we can rename the integration variable, and rewrite Eq. (6.18)

$$\int \mathcal{D}A\, e^{iS[A]} \propto \int \mathcal{D}\Lambda \int \mathcal{D}A\, e^{iS[A]}\,\delta\left(G\left(A_\mu\right)\right)\,. \tag{6.19}$$

We have therefore achieved our goal, namely to separate the integration over the gauge copies, which is now factored out in front of the path integral. The remaining integration is over the gauge fields but includes the gauge-fixing delta function, which selects one representative for each gauge orbit. This procedure is called *Faddeev-Popov* method, and turns out to be particular simple for the U(1) theory, where the Jacobian turns out to be independent of the gauge field, and drops out of the integral. The procedure yields a more interesting result for the case of non-Abelian gauge symmetry.

Generalised gauge We can generalise the gauge condition considering

$$G\left(A_\mu(x)\right) = \partial_\mu A^\mu(x) - \omega(x)\,, \tag{6.20}$$

where ω is a generic function. Using the Faddeev-Popov trick, we can write the path integral as

$$\int \mathcal{D}A\, e^{iS[A]} \propto \int \mathcal{D}\Lambda \int \mathcal{D}A\, e^{iS[A]}\,\delta\Big(\partial_\mu A^\mu(x) - \omega(x)\Big)\,. \tag{6.21}$$

Eq. (6.21) can be integrated over ω with a Gaussian weight:

$$\int \mathcal{D}A\, e^{iS[A]} \propto \int \mathcal{D}\omega\, \exp\left[-i \int d^D x\, \frac{\omega(x)^2}{2\xi}\right] \int \mathcal{D}\Lambda \int \mathcal{D}A\, e^{iS[A]}\, \delta\Big(\partial_\mu A^\mu(x) - \omega(x)\Big)$$

$$= \left(\int \mathcal{D}\Lambda\right) \int \mathcal{D}A\, e^{iS[A]}\, \exp\left[-i \int d^D x\, \frac{1}{2\xi} \left(\partial_\mu A^\mu(x)\right)^2\right]. \tag{6.22}$$

Once again we have factored out the volume of the gauge orbit, but now instead of a gauge fixing delta function in the path integral, we have a non-trivial weight for the longitudinal modes in the exponential. The modified action can be written as:

$$S[A] - \frac{1}{2\xi} \int d^D x\, \left(\partial_\mu A^\mu(x)\right)^2 =$$

$$= \frac{1}{2} \int d^D x\, A_\mu(x) \left[\partial^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu\right] A_\nu(x)\,. \tag{6.23}$$

The new kernel in momentum space is

$$K_\xi^{\mu\nu}(k) = -k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu\,, \tag{6.24}$$

and it can be readily checked that the longitudinal modes are no longer zero modes of K_ξ, which is now invertible. Its inverse, which we denote $\tilde{D}_F^{\mu\nu}(k)$ is the Feynman propagator for the photon field:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon} \left[g^{\mu\nu} - (1 - \xi)\, \frac{k^\mu k^\nu}{k^2}\right]. \tag{6.25}$$

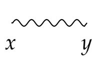
The choices $\xi = 0$ and $\xi = 1$ are called respectively the Landau and Feynman gauge propagators.

The Gaussian integral for the free theory can now be performed,

$$Z_0[J] = \exp\left[\frac{1}{2} \int \frac{d^D k}{(2\pi)^D}\, \tilde{J}_\mu(k) \tilde{D}_F^{\mu\nu}(k) J_\nu(-k)\right]. \tag{6.26}$$

The propagator in momentum space is obtained by Fourier transforming the expresssion in momentum space,

$$D_f^{\mu\nu}(x - y) = \int \frac{d^D k}{(2\pi)^D}\, e^{-ik\cdot(x-y)} \tilde{D}_F^{\mu\nu}(k) \tag{6.27}$$

$=$  . (6.28)

The photon propagator is usually represented by a wavy line, as shown in the second line above.