Math 136 - Stochastic Processes
Homework Set 5, Autumn 2013, Due: October 30

1. Exercise 3.2.21. Consider the random variables $\widehat{S}_k$ of Example 1.4.13.

   (a) Applying Proposition 3.2.6 verify that the corresponding characteristic functions are

   $$\Phi_{\widehat{S}_k}(\theta) = [\cos(\theta/\sqrt{k})]^k .$$

   ANS: Let X_i for $i = 1 \dots k$ be i.i.d. RVs with $\mathbf{P}(X_i = -1) = \mathbf{P}(X_i = 1) = 1/2$. Then using Proposition 3.2.6 for the first equality we have

   $$\Phi_{\widehat{S}_k}(\theta) = \prod_{i=1}^{k} \Phi_{X_i/\sqrt{k}}(\theta) = \{\Phi_{X_1/\sqrt{k}}(\theta)\}^k = \{\mathbf{E}(e^{i\theta X_1/\sqrt{k}})\}^k = \{(e^{-\theta/\sqrt{k}} + e^{\theta/\sqrt{k}})/2\}^k = \{\cos(\theta/\sqrt{k})\}^k$$

   (b) Recalling that $\delta^{-2}\log(\cos\delta) \to -0.5$ as $\delta \to 0$, find the limit of $\Phi_{\widehat{S}_k}(\theta)$ as $k \to \infty$ while $\theta \in \mathbb{R}$ is fixed.
   ANS: Note that $\Phi_{\widehat{S}_k}(\theta) = \exp\{k \log[\cos(\theta/\sqrt{k})]\}$. Taking $\delta = \theta/\sqrt{k}$ and exploiting the continuity of the exponential function we get $\Phi_{\widehat{S}_k}(\theta) \to e^{-\theta^2/2}$.

   (c) Suppose random vectors $\underline{X}^{(k)}$ and $\underline{X}$ in $\mathbb{R}^n$ are such that $\Phi_{\underline{X}^{(k)}}(\underline{\theta}) \to \Phi_{\underline{X}}(\underline{\theta})$ as $k \to \infty$, for any fixed $\underline{\theta}$. It can be shown that then the laws of $\underline{X}^{(k)}$, as probability measures on $\mathbb{R}^n$, must converge weakly in the sense of Definition 1.4.20 to the law of $\underline{X}$. Explain how this fact allows you to verify the C.L.T. statement $\widehat{S}_n \xrightarrow{\mathcal{L}} G$ of Example 1.4.13.
   ANS: From the previous part we see that $\Phi_{\widehat{S}_k}(\theta) \to \Phi_G(\theta)$ for all θ, where G is a standard normal random variable. Then what has been stated above implies that $\widehat{S}_k \xrightarrow{\mathcal{L}} G$.

2. Exercise 3.2.22. Consider the random vectors $\underline{X}^{(k)} = (\frac{1}{\sqrt{k}}S_{k/2}, \frac{1}{\sqrt{k}}S_k)$ in $\mathbb{R}^2$, where $k = 2, 4, 6, \dots$ is even, and S_k is the *simple random walk* of Definition 3.1.2, with $\mathbf{P}(\xi_1 = -1) = \mathbf{P}(\xi_1 = 1) = 0.5$.

   (a) Verify that $\Phi_{\underline{X}^{(k)}}(\underline{\theta}) = [\cos((\theta_1 + \theta_2)/\sqrt{k})[\cos(\theta_2/\sqrt{k})]^{k/2}]^{k/2}$, where $\underline{\theta} = (\theta_1, \theta_2)$.
   ANS: Here $\Phi_{\underline{X}^{(k)}}(\underline{\theta}) = \mathbf{E}\exp(\theta_1 S_{k/2}/\sqrt{k} + \theta_2 S_k/\sqrt{k})$ and since $S_k = S_{k/2} + \tilde{S}_{k/2}$ where $\tilde{S}_{k/2}$ is independent, identically distributed copy of $S_{k/2}$, we have

   $$\mathbf{E}\exp(\theta_1 S_{k/2}/\sqrt{k} + \theta_2 S_k/\sqrt{k}) = \mathbf{E}\exp[(\theta_1 + \theta_2)S_{k/2}/\sqrt{k}]\mathbf{E}\exp[\theta_2 S_{k/2}/\sqrt{k}]$$

   The required result now follows by noting that $S_k/\sqrt{k}$ has the same distribution as $\widehat{S}_k$ from Exercise 3.2.21, so their characteristic functions are equal.

   (b) Find the mean vector $\underline{\mu}$ and the covariance matrix Σ of a Gaussian random vector $\underline{X}$ for which $\Phi_{\underline{X}^{(k)}}(\underline{\theta})$ converges to $\Phi_{\underline{X}}(\underline{\theta})$ as $k \to \infty$.
   ANS: Same approach as in part (b) of the Exercise 3.2.21 gives $\Phi_{\underline{X}^{(k)}}(\underline{\theta}) \to e^{-(\theta_1+\theta_2)^2/4}e^{-\theta_2^2/4}$. We now need $\underline{\mu}$ and Σ such that

   $$\exp[-(\underline{\theta}, \Sigma\underline{\theta})/2 + i(\underline{\theta}, \underline{\mu})] = \exp[(-\theta_1^2/2 - \theta_1\theta_2 - \theta_2^2)/2]$$

   which gives $\underline{\mu} = (0, 0)$, $\Sigma_{11} = \Sigma_{12} = \Sigma_{21} = 1/2$ and $\Sigma_{22} = 1$.

   (c) Upon appropriately generalizing what you did in part (b), I claim that the *Brownian motion* of Theorem 3.1.3 must be a Gaussian stochastic process. Explain why, and guess what is the mean $\mu(t)$ and auto-covariance function $\rho(t, s)$ of this process (if needed take a look at Chapter 5).
   ANS: The Brownian motion of Theorem 3.1.3 arises as a weak limit as $k \to \infty$ of linear interpolations of a scaled random walk, which at $t = \frac{i}{k}$ with $i = 0 \dots k$ has values $X_t^{(k)} := \frac{1}{\sqrt{k}}S_{kt}$ (where S_n as in part (b) above). Now, for any $0 \le t_1 < t_2 < \cdots < t_n \le 1$ the random vector $(X_{t_1}^{(k)}, \dots, X_{t_n}^{(k)})$ will be a generalization of the random vector $\underline{X}^{(k)}$ from part (b) above (there $n = 2$, $t_1 = 1/2$ and $t_2 = 1$) and it will converge weakly to a Gaussian RV. But this weak limit is also a f.d.d. of the process obtained as a weak limit in Theorem 3.1.3, i.e. the Brownian motion. So all its f.d.d.'s must be Gaussian, which means that the Brownian motion is a Gaussian process. Guided by part (b) we guess $\mu(t) = 0$ and $\rho(t, s) = \min(t, s)$.

3. Exercise 3.3.5. Suppose that the stochastic process X_t is such that $\mathbf{E}[X_t] = 0$ and $\mathbf{E}[X_t^2] = 1$ for all $t \in [0, T]$.

   (a) Show that $|\mathbf{E}[X_t X_{t+h}]| \le 1$ for any $h > 0$ and $t \in [0, T - h]$.
   ANS: By Jensen's inequality for $g(x) = |x|$ and Proposition 1.2.41:

   $$|\mathbf{E}[X_t X_{t+h}]| \le \mathbf{E}[|X_t X_{t+h}|] \le \sqrt{\mathbf{E}X_t^2}\sqrt{\mathbf{E}X_{t+h}^2} = 1.$$

   (b) Suppose that for some $\lambda < \infty$, $p > 1$, and $h_0 > 0$,

   $$\mathbf{E}[X_t X_{t+h}] \ge 1 - \lambda h^p$$

   for all $0 < h \le h_0$. Using Kolmogorov's continuity theorem show that then X_t has a continuous modification.
   ANS: $\mathbf{E}(|X_{t+h} - X_t|^2) = \mathbf{E}(X_{t+h}^2) + \mathbf{E}(X_t^2) - 2\mathbf{E}(X_{t+h}X_t) = 2(1 - \mathbf{E}[X_{t+h}X_t)] \le (2\lambda)h^p$. By Kolmogorov's theorem with $\alpha = 2$, $c = 2\lambda$ and $\beta = p - 1 > 0$ the process X_t has a continuous modification.

   (c) Suppose that X_t is a Gaussian stochastic process such that $\mathbf{E}[X_t] = 0$ and $\mathbf{E}[X_t^2] = 1$ for all $t \in [0, T]$. Show that if X_t satisfies the inequality (3.3.2) for some $\lambda < \infty$, $p > 0$, and $h_0 > 0$, then for any $0 < \gamma < p/2$, the process X_t has a modification which is locally Hölder continuous with exponent γ. (Hint: see Section 5.1 for the moments of Gaussian R.V.).
   ANS: Since $\{X_t\}$ is a zero-mean Gaussian stochastic process, $X_{t+h} - X_t$ is a zero mean Gaussian random variable, so by (3.3.2),

   $$\mathbf{E}[|X_{t+h} - X_t|^{2n}] = \frac{(2n)!}{2^n n!}\left[\mathbf{E}(X_{t+h} - X_t)^2\right]^n \le [(2n)!/n!]\lambda^n h^{pn}$$

   for any integer n, $0 < h \le h_0$ and $t \in [0, T - h]$. Fix an integer n large enough so $\gamma < \beta/\alpha$ when $\alpha = 2n$ and $\beta = pn - 1$ (i.e. $\gamma < p/2 - 1/(2n)$), and set $c = [(2n)!\lambda^n]/n!$ finite. By the preceding, Kolmogorov's continuity theorem applies for these parameters and yields the existence of a modification of X_t that is locally Hölder continuous with exponent γ.

4. Exercise 3.3.8

   (a) Let $\{X_n\}, \{Y_n\}$ be discrete time S.P.s that are modifications of each other. Show that $\mathbf{P}(X_n = Y_n \text{ for all } n \ge 0) = 1$.
   ANS: For each n let $A_n = \{\omega : X_n(\omega) = Y_n(\omega)\}$. Since $\{X_n\}, \{Y_n\}$ are modifications of each other we know that $\mathbf{P}(A_n) = 1$. Hence $\mathbf{P}(\cap_{n=1}^{\infty} A_n) = 1$ since a *countable* intersection of sets that occur with probability one also occurs with probability one. Noting that $\cap_{n=1}^{\infty} A_n = \{\omega : X_n(\omega) = Y_n(\omega) \text{ for all } n \ge 0\}$ gives the desired result.

   (b) Let $\{X_t\}, \{Y_t\}$ be continuous time S.P.s that are modifications of each other. Suppose that both processes have right-continuous sample paths a.s. Show that $\mathbf{P}(X_t = Y_t \text{ for all } t \ge 0) = 1$.
   ANS: Without loss of generality we assume that the sample paths of $\{X_t\}$ and $\{Y_t\}$ are right-continuous *for all* ω. For each $t \ge 0$, let $A_t = \{\omega : X_t(\omega) = Y_t(\omega)\}$. Since $\{X_t\}, \{Y_t\}$ are modifications of each other we know that $\mathbf{P}(A_t) = 1$. The set $\mathbb{Q}$ of rational numbers is countable, so $A = \cap_{r \in \mathbb{Q}, r \ge 0} A_r$ is a countable intersection of sets A_r such that $\mathbf{P}(A_r) = 1$ and consequently $\mathbf{P}(A) = 1$ as well. It thus suffices to show that $A_t \supseteq \cap_{r \in \mathbb{Q}, r \ge t} A_r$ for all $t \ge 0$ since then $B = \cap_{t \ge 0} A_t \supseteq A$ so $\mathbf{P}(B) \ge \mathbf{P}(A)$ yielding that $\mathbf{P}(B) = 1$ as is claimed. Thus, it suffices to show that if $\omega \in \cap_{r \in \mathbb{Q}, r \ge t} A_r$ for $t \ge 0$ irrational, then $\omega \in A_t$ as well. Indeed, by right continuity of the sample path of both processes,

   $$X_t(\omega) = \lim_{r \in \mathbb{Q}, r \downarrow t} X_r(\omega) = \lim_{r \in \mathbb{Q}, r \downarrow t} Y_r(\omega) = Y_t(\omega),$$

   which gives the desired result.

   (c) Provide an example of two S.P.-s which are modifications of one another but which are not indistinguishable.
   ANS: The underlying probability space is $(\mathbb{R}, \mathcal{B}, U)$ with U the uniform measure on $(0, 1)$. Let $X_t = 0$ be a constant stochastic process and $Y_t(\omega) = 0$ if $t \ne \omega$ and $Y_t(\omega) = 1$ if $t = \omega$, for $t \in [0, 1]$. Then,

   $$\mathbf{P}(X_t = Y_t) = U(\{\omega \in (0, 1) : \omega \ne t\}) = 1$$

   but

   $$\mathbf{P}(\{X_t = Y_t \text{ for all } t \in [0, 1]\}) = 0$$

   since for every $t \in [0, 1]$, $X_t(t) \ne Y_t(t)$.

5. Exercise 5.1.4. Suppose W_t is a Brownian motion and $\alpha, s, T > 0$ are non-random constants. Show the following.

   (a) (Symmetry) $\{-W_t, t \ge 0\}$ is a Brownian motion.
   ANS: Obviously $-W_t$ remains Gaussian, continuous, and has the same mean function and auto-covariance functions as W_t. Indeed,

   $$\mathbf{E}(-W_t) = -\mathbf{E}W_t = 0$$

   and

   $$\mathbf{E}(-W_t)(-W_s) = \mathbf{E}W_t W_s = \min(t, s).$$

   (b) (Time homogeneity) $\{W_{s+t} - W_s, t \ge 0\}$ is a Brownian motion.
   ANS: Again, it is clear that $W_{s+t} - W_s$ is a continuous Gaussian process. Its mean and auto-covariance functions are,

   $$\mathbf{E}(W_{s+t} - W_s) = \mathbf{E}W_{s+t} - \mathbf{E}W_s = 0,$$

   and

   $$\begin{aligned}\mathbf{E}(W_{s+t} - W_s)(W_{s+\sigma} - W_s) &= \mathbf{E}(W_{s+t}W_{s+\sigma} - W_{s+t}W_s - W_{s+\sigma}W_s + W_s^2) \\ &= \min(s + \sigma, s + t) - 2s + s = \min(\sigma, t).\end{aligned}$$

   These agree with that of Brownian motion which gives the desired conclusion.

   (c) (Time reversal) $\{W_T - W_{T-t}, 0 \le t \le T\}$ is a Brownian motion.
   ANS: Clearly, $W_T - W_{T-t}$ is continuous and Gaussian. We compute,

   $$\mathbf{E}(W_T - W_{T-t}) = 0$$

   and

   $$\begin{aligned}\mathbf{E}(W_T - W_{T-t})(W_T - W_{T-s}) &= \mathbf{E}(W_T^2 - W_T W_{T-t} - W_T W_{T-s} + W_{T-t}W_{T-s}) \\ &= T - (T - t) - (T - s) + \min(T - t, T - s) \\ &= t + s - T + \min(T - t, T - s) \\ &= \min(s, t),\end{aligned}$$

   giving the desired result.

   (d) (Scaling, or self-similarity) $\{\sqrt{\alpha}W_{t/\alpha}, t \ge 0\}$ is a Brownian motion.
   ANS: Since both spatial and time scaling are continuous and spatial scaling preserves the Gaussian distribution, $\sqrt{\alpha}W_{t/\alpha}$ is a continuous Gaussian process. Its mean and auto-covariance functions are,

   $$\mathbf{E}\sqrt{\alpha}W_{t/\alpha} = \sqrt{\alpha}\mathbf{E}W_{t/\alpha} = 0$$

   and

   $$\mathbf{E}(\sqrt{\alpha}W_{t/\alpha}\sqrt{\alpha}W_{s/\alpha}) = \alpha\mathbf{E}W_{t/\alpha}W_{s/\alpha} = \alpha\min(t/\alpha, s/\alpha) = \min(t, s).$$

   Hence $\sqrt{\alpha}W_{t/\alpha}$ is a Brownian motion.

   (e) (Time inversion) If $\widetilde{W}_0 = 0$ and $\widetilde{W}_t = tW_{1/t}$, then $\{\widetilde{W}_t, t \ge 0\}$ is a Brownian motion.
   ANS: Again, it is clear that $\widetilde{W}_t$ is a Gaussian process with mean function $\mu(t) \equiv 0$. Fixing $t > 0$ we note that $\mathbf{E}\widetilde{W}_0\widetilde{W}_t = 0 = \min(0, t)$. Further, for any $s > 0$,

   $$\mathbf{E}\widetilde{W}_s\widetilde{W}_t = (st)\min(1/s, 1/t) = \min(s, t).$$

   Hence, $\widetilde{W}_t$ has the same auto-covariance function as Brownian motion. At this point we know that $\widetilde{W}_t$ has the same f.d.d. as Brownian motion and that there exists an event Γ with $\mathbf{P}(\Gamma) = 0$ such that $t \mapsto \widetilde{W}_t(\omega)$ is continuous at any $t > 0$ provided $\omega \notin \Gamma$. So, $\widetilde{W}_t$ is a Brownian motion if almost surely $\widetilde{W}_t \to 0$ when $t \downarrow 0$. The most direct way to show this is to recall that as $\widetilde{W}_t$ has the f.d.d. of a Brownian motion, by Kolmogorov's continuity theorem $\widetilde{W}_t$ has a continuous modification V_t. With both $t \mapsto \widetilde{W}_t(\omega)$ and $t \mapsto V_t(\omega)$ continuous at any $t > 0$ and all $\omega \notin \Gamma'$ such that $\mathbf{P}(\Gamma') = 0$, necessarily $\mathbf{P}(V_t = \widetilde{W}_t \text{ for all } t > 0) = 1$ (by the same argument you used in solving part (b) of Exercise 3.3.8). Since almost surely both $V_t \to V_0 = 0$ when $t \to 0$ and $\widetilde{W}_t = V_t$ for all $t > 0$, it follows that also $\widetilde{W}_t \to 0$ a.s. An alternative proof of the a.s. convergence to 0 of $\widetilde{W}_t$ is by invoking the strong law of large numbers to have that $\widetilde{W}_{1/n} = n^{-1}W_n \to 0$ as $n \to \infty$ (since W_n is a sum of n i.i.d. $N(0, 1)$ random variables) then arguing that the Gaussian process $t^{-1}W_t$ does not fluctuate much on $t \in [n, n + 1]$ (via standard bounds for the tail of $N(0, 1)$ random variables).

6. Exercise 5.1.12 Fix $H \in (0, 1)$. A Gaussian stochastic process $\{X_t, t \ge 0\}$ is called a fractional Brownian motion (or in short, fBM), of Hurst parameter H if $\mathbf{E}(X_t) = 0$ and

   $$\mathbf{E}(X_t X_s) = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t - s|^{2H}], \ s, t \ge 0.$$

   (a) Show that an fBM of Hurst parameter H has a continuous modification that is also locally Hölder continuous with exponent γ for any $0 < \gamma < H$.
   ANS: Fix $0 < \gamma < H$. We have for all t, s and any positive integer n that

   $$\mathbf{E}|X_t - X_s|^{2n} = C_n(\mathbf{E}|X_t - X_s|^2)^n = C_n(\mathbf{E}X_t^2 + \mathbf{E}X_s^2 - 2\mathbf{E}X_t X_s)^n = C_n|t - s|^{2Hn},$$

   where C_n are some non-random finite constants (c.f. the explicit formula for moments of a normal random variable, immediately after the proof of Proposition 5.1.3). So from Kolmogorov's continuity theorem (with $\alpha = 2n$ and $\beta = 2Hn - 1$) we see that X_t possesses a continuous modification with any Hölder exponent in $(0, (2Hn - 1)/2n)$. With $(2Hn - 1)/2n = H - 1/(2n)$ we get the desired result by taking n large enough so that $H - \frac{1}{2n} > \gamma$.

   (b) Verify that in case $H = 1/2$ such a modification yields the (standard) Brownian motion.
   ANS: Since such a modification is a continuous Gaussian process, we just need to show that for $H = 1/2$ the process has the same mean and auto-covariance as the standard Brownian motion. The former is obvious and for the latter, we compute,

   $$\mathbf{E}(X_t X_s) = \frac{1}{2}[t + s - |t - s|] = \min(t, s).$$

   (c) Show the self-similarity property, whereby for any non-random $\alpha > 0$ the process $\{\alpha^H X_{t/\alpha}\}$ is an fBM of the same Hurst parameter H.
   ANS: With $\{X_t\}$ a Gaussian S.P. visibly so is $\{\alpha^H X_{t/\alpha}\}$. It thus suffices to show that the rescaled

   process has the same mean and auto-covariance functions as $\{X_t\}$. The former is obvious and for the latter, we compute,

   $$\mathbf{E}\alpha^H X_{t/\alpha}\alpha^H X_{s/\alpha} = \alpha^{2H}(\frac{1}{2}[(t/\alpha)^{2H} + (s/\alpha)^{2H} - |t/\alpha - s/\alpha|^{2H}]) = \frac{1}{2}[t^{2H} + s^{2H} - |t - s|^{2H}].$$

   (d) For which values of H is the fBM a process of stationary increments and for which values of H is it a process of independent increments?
   ANS: Recall that we have already seen in part (a) that $\mathbf{E}(X_{t+h} - X_t)^2 = h^{2H}$. Since the distributional properties of Gaussian random variables are determined entirely by their mean and variance we thus conclude that the fBM process has stationary increments for all H. As $\{X_t\}$ is Gaussian, it has independent increments if and only if

   $$\mathbf{E}(X_t - X_s)(X_{t'} - X_{s'}) = 0$$

   for all $t > s \ge t' > s'$. We compute,

   $$\begin{aligned}\mathbf{E}(X_t - X_s)(X_{t'} - X_{s'}) &= \mathbf{E}(X_t X_{t'} + X_s X_{s'} - X_t X_{s'} - X_s X_{t'}) \\ &= \frac{1}{2}((t - s')^{2H} + (s - t')^{2H} - (t - t')^{2H} - (s - s')^{2H})\end{aligned}$$

   If $H = 1/2$ it is easy to see that the above vanishes[1]. Suppose $H \ne 1/2$. Then setting $t = 2, s = t' = 1, s' = 0$, the above becomes

   $$\frac{1}{2}(2^{2H} - 2) \ne 0.$$

   Hence $\{X_t\}$ has independent increments if and only if $H = 1/2$.

[1] Note that we could have immediately conclude the independence of increments when $H = 1/2$ from part (b) since the standard Brownian motion has this property; however, we went through the above computation since it useful for the next part.

Math 136 - Stochastic Processes
Homework Set 6, Autumn 2013, Due: November 6

1. Exercise 4.1.6 Provide an example of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a filtration $\{\mathcal{F}_n\}$ and a stochastic process $\{X_n\}$ adapted to $\{\mathcal{F}_n\}$ such that:

   (a) $\{X_n\}$ is a martingale with respect to its canonical filtration but $(X_n, \mathcal{F}_n)$ is not a martingale.
   ANS: Take $\Omega = \{a, b\}$, $\mathcal{F}_0 = \mathcal{F} = 2^\Omega$, $X_0 = 0$, $X_1 = \pm 1$ with probability 1/2 and $X_n = X_1$ for all $n \ge 2$. Then $\{X_n\}$ is a martingale with respect to its canonical filtration since:

   $$X_0 = 0 = \mathbf{E}(X_1) = \mathbf{E}(X_1|X_0)$$

   and

   $$X_n = \mathbf{E}(X_n|X_n) = \mathbf{E}(X_{n+1}|X_n) = \mathbf{E}(X_{n+1}|\sigma(X_0, \dots, X_n))$$

   for all $n \ge 1$. Now consider the filtration $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = 2^\Omega$ for all n. Then,

   $$X_0 = 0 \ne X_1 = \mathbf{E}(X_1|\mathcal{F}_0),$$

   so that $(X_n, \mathcal{F}_n)$ is not a martingale.

   (b) Provide a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ under which $\{X_n\}$ is not a martingale even with respect to its canonical filtration.
   ANS: Let $\mathbf{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $X_1 = 1$ with probability $p > 1/2$ and $X_1 = -1$ with probability $1 - p < 1/2$. Then

   $$\mathbf{E}X_1 = (2p - 1) > 0 \ne 0 = X_0$$

   so that $\{X_n\}$ is not a martingale with respect to its canonical filtration.

2. Exercise 4.1.23 Let $\xi_1, \xi_2, \dots$ be independent with $\mathbf{E}\xi_i = 0$ and $\mathbf{E}\xi_i^2 = \sigma_i^2$.

   (a) Let $S_n = \sum_{i=1}^n \xi_i$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Show that $\{S_n^2\}$ is a sub-martingale and $\{S_n^2 - s_n^2\}$ is a martingale.
   ANS: Using the same argument of Example 4.1.8 we know that $\{S_n\}$ is a martingale with respect to its canonical filtration. Moreover, from the fact that $S_n^2 = \sum_{i=1}^n \xi_i^2 + 2\sum_{1 \le i < j \le n} \xi_i\xi_j$ it is clear that $\mathbf{E}|S_n^2| < \infty$ for all n. Thus since $x \mapsto x^2$ is a convex function it follows from the conditional Jensen inequality that S_n^2 is a sub-martingale. Letting $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$ and using that ξ_{n+1} is independent of $\mathcal{F}_n$, we have

   $$\mathbf{E}[S_{n+1}^2|\mathcal{F}_n] = \mathbf{E}[(S_n + \xi_{n+1})^2|\mathcal{F}_n] = \mathbf{E}[S_n^2 + 2\xi_{n+1}S_n + \xi_{n+1}^2|\mathcal{F}_n] = S_n^2 + \sigma_{n+1}^2.$$

   Hence,

   $$\mathbf{E}[S_{n+1}^2 - s_{n+1}^2|\mathcal{F}_n] = S_n^2 - s_{n+1}^2 + \sigma_{n+1}^2 = S_n^2 - s_n^2.$$

   Thus $\{S_n^2 - s_n^2\}$ is a martingale as desired.

   (b) Suppose also that $m_n = \prod_{i=1}^n \mathbf{E}[e^{\xi_i}] < \infty$. Show that $\{e^{S_n}\}$ is a sub-martingale and $M_n = e^{S_n}/m_n$ is a martingale.
   ANS: By assumption $m_n < \infty$ giving us that $\{e^{S_n}\}$ is an integrable SP. Since $\{S_n\}$ is a martingale and $x \mapsto e^x$ is convex it follows from the conditional Jensen inequality that $\{e^{S_n}\}$ is a sub-martingale. Letting $\mathcal{F}_n = \sigma(M_1, \dots, M_n)$, the independence of ξ_{n+1} and $\mathcal{F}_n$ gives us that

   $$\mathbf{E}[M_{n+1}|\mathcal{F}_n] = \frac{1}{m_{n+1}}\mathbf{E}[e^{S_n}e^{\xi_{n+1}}|\mathcal{F}_n] = \frac{e^{S_n}}{m_{n+1}}\mathbf{E}[e^{\xi_{n+1}}] = \frac{e^{S_n}}{m_n} = M_n.$$

   Therefore $\{M_n\}$ is a martingale.

3. Exercise 4.2.5. Let $\mathcal{G}_t$ denote the canonical filtration of a Brownian motion W_t.

   (a) Show that for any $\lambda \in \mathbb{R}$, the S.P. $M_t(\lambda) = \exp(\lambda W_t - \lambda^2 t/2)$, is a continuous time martingale with respect to $\mathcal{G}_t$.
   ANS: Note that $\mathbf{E}|M_t(\lambda)| = e^{-\lambda^2(t/2)}\mathbf{E}[e^{\lambda W_t}]$ which since W_t is a Gaussian random variable, we know to be finite. Further, $\mathbf{E}e^{\lambda(W_{t+h} - W_t)} = e^{\lambda^2 h/2}$ yielding the identity $\mathbf{E}[M_{t+h}(\lambda)|\mathcal{G}_t] = e^{-\lambda^2(t/2) + \lambda W_t} = M_t(\lambda)$, so $M_t(\lambda)$ is a martingale.

   (b) Explain why $\frac{d^k}{d\lambda^k}M_t(\lambda)$ are also martingales with respect to $\mathcal{G}_t$.
   ANS: Fixing $\lambda \in \mathbb{R}$, let $\lambda_m = \lambda + m^{-1}$ and recall that $M_t(\lambda_m)$ is a MG with respect to $\mathcal{G}_t$. The adapted process $X_t(m, \lambda) := m(M_t(\lambda_m) - M_t(\lambda))$ is then also a MG with respect to $\mathcal{G}_t$. That is, $\mathbf{E}[X_{t+h}(m, \lambda) - X_t(m, \lambda)|\mathcal{G}_t] = 0$ for any non-random $h, t \ge 0$. Further, considering $m \to \infty$ we get by definition of the derivative that $X_t(m, \lambda)$ converges a.s. to the adapted S.P. $Z_t(\lambda) := \frac{d}{d\lambda}M_t(\lambda)$. Now, by the mean value theorem

   $$\sup_m\{|X_t(m, \lambda)|\} \le Y_t := \sup\{|Z_t(\lambda + u)| : 0 \le u \le 1\}.$$

   Computing explicitly $Z_t(\lambda + u)$, it is not hard to check that $Y_t \le (|W_t| + (|\lambda| + 1)t)e^{(|\lambda|+1)|W_t|}$ is integrable (per fixed $t \ge 0$). From the preceding we thus deduce by dominated convergence for C.E. that a.s. $\mathbf{E}[Z_{t+h}(\lambda) - Z_t(\lambda)|\mathcal{G}_t] = 0$. Consequently, given $\lambda \in \mathbb{R}$ non-random, the process $Z_t(\lambda)$ is a MG with respect to $\mathcal{G}_t$. Applying the same reasoning with $M_t(\lambda)$ replaced by $Z_t(\lambda)$ extends our claim from $k = 1$ to $k = 2$, and arguing inductively in k, the same applies for $k = 3, 4, \dots$.

   (c) Compute the first three derivatives in λ of $M_t(\lambda)$ at $\lambda = 0$ and deduce that the S.P. $W_t^2 - t$ and $W_t^3 - 3tW_t$ are also MGs.

ANS: Fixing $x, t \in \mathbb{R}$, the derivative of $M(\lambda) := e^{\lambda x - \lambda^2 t/2}$ is $M'(\lambda) = (x - \lambda t)M(\lambda)$, yielding that $M''(\lambda) = [(x-\lambda t)^2 - t]M(\lambda)$ and $M'''(\lambda) = (x - \lambda t)[(x-\lambda t)^2 - 3t]M(\lambda)$. In case $\lambda = 0$ we have $M(0) = 1$ resulting with $M'(0) = x$, $M''(0) = x^2 - t$ and $M'''(0) = x^3 - 3tx$. Setting $x = W_t$ we deduce by the preceding that $W_t^2 - t$ and $W_t^3 - 3tW_t$ are also MGs.

4. Exercise 4.2.10 Given a positive MG $(Z_t, \mathcal{F}_t)$ with $\mathbf{E}Z_0 = 1$ consider for each $t \geq 0$ the probability measure $\widetilde{\mathbf{P}}_t \colon \mathcal{F}_t \to \mathbb{R}$ given by $\widetilde{\mathbf{P}}_t(A) = \mathbf{E}[Z_t I_A]$.

   (a) Show that $\widetilde{\mathbf{P}}_t(A) = \widetilde{\mathbf{P}}_s(A)$ for any $A \in \mathcal{F}_s$ and $0 \leq s \leq t$.
   ANS: Since Z_t is a martingale and I_A is $\mathcal{F}_s$-measurable, we have by the tower property and taking out what is known that

$$\widetilde{\mathbf{P}}_t(A) = \mathbf{E}[Z_t I_A] = \mathbf{E}[\mathbf{E}[Z_t I_A \,|\, \mathcal{F}_s]] = \mathbf{E}[\mathbf{E}[Z_t \,|\, \mathcal{F}_s] I_A] = \mathbf{E}[Z_s I_A] = \widetilde{\mathbf{P}}_s(A).$$

   (b) Fixing $0 \leq u \leq s \leq t$ and $Y \in L^1(\Omega, \mathcal{F}_s, \widetilde{\mathbf{P}}_t)$, set $X_{s,u} = \mathbf{E}(YZ_s|\mathcal{F}_u)/Z_u$. With $\widetilde{\mathbf{E}}_t$ denoting the expectation under $\widetilde{\mathbf{P}}_t$, deduce that $\widetilde{\mathbf{E}}_t(Y|\mathcal{F}_u) = X_{s,u}$ almost surely under $\widetilde{\mathbf{P}}_t$ (hence also under $\mathbf{P}$, by Exercise 1.4.32).
   ANS: First note that $YZ_s \in L^1(\Omega, \mathcal{F}_s, \mathbf{P})$ since

$$\mathbf{E}(|Y|Z_s) = \mathbf{E}(|Y|\mathbf{E}(Z_t \,|\, \mathcal{F}_s)) = \mathbf{E}(\mathbf{E}(|Y|Z_t \,|\, \mathcal{F}_s)) = \mathbf{E}(|Y|Z_t) = \widetilde{\mathbf{E}}_t(|Y|) < \infty.$$

   Consequently, the $\mathcal{F}_u$ measurable random variable $X_{s,u} = \mathbf{E}(YZ_s \,|\, \mathcal{F}_u)/Z_u$ is well defined. Further, fixing $A \in \mathcal{F}_u$, recall that YI_A is $\mathcal{F}_s$ measurable and $(Z_t, \mathcal{F}_t)$ a martingale. Hence, using the tower property, taking out what is known and applying part (a) for the $\mathcal{F}_u$ measurable $X_{s,u}I_A$ we get that

$$\begin{aligned}\widetilde{\mathbf{E}}_t[YI_A] &= \mathbf{E}[Z_t Y I_A] = \mathbf{E}[\mathbf{E}(Z_t Y I_A \,|\, \mathcal{F}_s)] = \mathbf{E}[YI_A \mathbf{E}(Z_t \,|\, \mathcal{F}_s)] = \mathbf{E}[YZ_s I_A] \\ &= \mathbf{E}[\mathbf{E}(YZ_s I_A \,|\, \mathcal{F}_u)] = \mathbf{E}[\mathbf{E}(YZ_s \,|\, \mathcal{F}_u) I_A] = \mathbf{E}[Z_u X_{s,u} I_A] = \widetilde{\mathbf{E}}_u[X_{s,u}I_A] = \widetilde{\mathbf{E}}_t[X_{s,u}I_A].\end{aligned}$$

   Since this applies for any $A \in \mathcal{F}_u$, we have by definition of the conditional expectation in the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbf{P}}_t)$ that $X_{s,u} = \widetilde{\mathbf{E}}_t(Y|\mathcal{F}_u)$ up to a set $N \in \mathcal{F}$ such that $\widetilde{\mathbf{P}}_t(N) = 0$. Recall Exercise 1.4.32 that $\widetilde{\mathbf{P}}_t(N) = 0$ if and only if $\mathbf{P}(N) = 0$, so the identity $X_{s,u} = \widetilde{\mathbf{E}}_t(Y|\mathcal{F}_u)$ holds for $\mathbf{P}$ almost every ω, as claimed.

5. Exercise 5.1.8. Compute the mean and the auto-covariance functions of the processes B_t, Y_t, U_t, and X_t.

ANS: We compute,

$$\begin{aligned}&\mathbf{E}(B_t) = 0, \\ &\mathbf{E}(B_t B_s) = s(1-t) \text{ when } 0 \leq s \leq t \leq 1, \\ &\mathbf{E}(B_t B_s) = s - 1 \text{ when } 0 \leq 1 \leq s \leq t \text{ and} \\ &\mathbf{E}(B_t B_s) = 0 \text{ when } 0 \leq s \leq 1 \leq t; \\ &\mathbf{E}(Y_t) = e^{t/2} \text{ and } \mathbf{E}[(Y_t - e^{t/2})(Y_s - e^{s/2})] = e^{(t+s)/2}(e^{\min(t,s)} - 1); \\ &\mathbf{E}(U_t) = 0 \text{ and } \mathbf{E}(U_t U_s) = e^{-|t-s|/2}. \\ &\mathbf{E}X_t = x + \mu t \text{ and,} \\ &\mathbf{E}[(X_t - x - \mu t)(X_s - x - \mu s)] = \sigma^2 \mathbf{E}(W_t W_s) = \sigma^2 \min(t,s).\end{aligned}$$

Justify your answers to:

(a) Which of the processes W_t, B_t, Y_t, U_t, X_t is Gaussian?
ANS: We know that W_t is a Gaussian process. The f.d.d. of the S.P. B_t and U_t correspond to deterministic linear combinations of the joint Gaussian r.v. W_{t_i}, hence both B_t and U_t are Gaussian processes. Since $Y_1 = e^{W_1}$ is strictly positive and not almost surely a constant, it can not be a Gaussian r.v, hence Y_t is not a Gaussian process. Finally, X_t is just an affine (time-dependent) translate of a Gaussian process and hence Gaussian.

(b) Which of these processes is stationary?
ANS: Stationarity implies the process has constant mean and its auto-covariance $\rho(t,s)$ is a function only of $|t-s|$. The S.P. W_t, B_t, Y_t and X_t fail to have this property so are non-stationary. The S.P. U_t satisfies these conditions and being also Gaussian, this suffices for U_t being a stationary process.

(c) Which of these processes has continuous sample paths?
ANS: W_t has continuous sample paths by the definition of Brownian motion so B_t, Y_t, U_t, X_t are finite compositions of functions continuous in t. Therefore, all five processes have continuous sample paths.

(d) Which of these processes is adapted to the filtration $\sigma(W_s, s \leq t)$ and which is also a sub-martingale for this filtration?
ANS: Recall that W_t is adapted and is a martingale for its canonical filtration. The processes B_t and U_t depend on values of W_s for $s > t$ so they are not adapted to this filtration. The S.P. Y_t is the composition of the convex function e^x and a martingale and hence a submartingale. Finally, as X_t is an affine translate of W_t, it is visibly adapted to the filtration and is a submartingale provided that $\mu \geq 0$:

$$\mathbf{E}[X_t|\sigma(W_s : s \leq t)] = x + \mu t + \sigma W_s \geq x + \mu s + \sigma W_s = X_s.$$

Note that if $\mu < 0$ we get the reverse inequality.

6. Exercise 5.1.11. Suppose W_t is a Brownian motion.

   (a) Compute the probability density function of the random vector (W_s, W_t). Then compute $\mathbf{E}(W_s|W_t)$ and $\mathrm{Var}(W_s|W_t)$, first for $s > t$, then for $s < t$.
   Hint: Consider Example 2.4.5.
   ANS: Suppose first that $t < s$. Then, $W_s - W_t$ is independent of W_t, having a Gaussian distribution of zero mean and variance $s - t$ Therefore, $\mathbf{E}(W_s|W_t) = W_t$ and $\mathrm{Var}(W_s|W_t) = \mathbf{E}((W_s - W_t)^2|W_t) = s - t$. Moving to deal with $t > s$, note that (W_s, W_t) is a Gaussian random vector, of zero mean and covariance matrix Σ whose entries are $\Sigma_{11} = \Sigma_{12} = \Sigma_{21} = s$, $\Sigma_{22} = t$. Upon finding that Σ is invertible and computing its inverse, we get that (W_s, W_t) has the (joint) probability density function $f_{W_s,W_t}(x,y) = \exp(-x^2/(2s) - (y-x)^2/(2(t-s)))/(2\pi\sqrt{s(t-s)})$. With the density of W_t being $g_{W_t}(y) = \exp(-y^2/2t)/\sqrt{2\pi t}$, we have by Example 2.4.5 that the conditional density of W_s given W_t is $f_{W_s|W_t}(x|W_t)$ for

$$f_{W_s|W_t}(x|y) = f_{W_s,W_t}(x,y)/g_{W_t}(y) = \exp(-(x - sy/t)^2/(2\sigma^2))/(\sqrt{2\pi}\sigma)$$

   where $\sigma^2 = s(t-s)/t$. The latter is the density of a Gaussian random variable of mean sy/t and variance σ^2, so as explained in Example 2.4.5 we have that $\mathbf{E}(W_s|W_t) = (s/t)W_t$ and $\mathrm{Var}(W_s|W_t) = s - s^2/t$.

   (b) Explain why the Brownian Bridge B_t, $0 \leq t \leq 1$ has the same distribution as $\{W_t, 0 \leq t \leq 1$, conditioned upon $W_1 = 0\}$ (which is the reason for naming B_t a Brownian bridge).
   Hint: Both Exercise 2.4.6 and parts of Exercise 5.1.8 may help here.
   ANS: For $s \leq t \leq 1$ we know that $X = W_1 - W_t$ is independent of the random vector $(Y, Z) = (W_s, W_t)$. Consequently, combining part (a) with Exercise 2.4.6 we have that $\mathbf{E}(W_s|W_t, W_1 - W_t) = \mathbf{E}(W_s|W_t) = (s/t)W_t$. Further, $\sigma(W_t, W_1) = \sigma(W_t, W_1 - W_t)$, so also $\mathbf{E}(W_s|W_t, W_1) = (s/t)W_t$. Thus, applying the tower property for $\sigma(W_1) \subseteq \sigma(W_t, W_1)$ and taking out what is known, we see that

$$\mathbf{E}[W_s W_t|W_1] = \mathbf{E}[W_t \mathbf{E}(W_s|W_t, W_1)|W_1] = (s/t)\mathbf{E}(W_t^2|W_1).$$

   Recall that by part (a), $\mathbf{E}(W_t|W_1) = tW_1$ and

$$\mathbf{E}(W_t^2|W_1) = \mathrm{Var}(W_t|W_1) + [\mathbf{E}(W_t|W_1)]^2 = t - t^2 + t^2 W_1^2,$$

implying that

$$\mathbf{E}(W_s W_t|W_1) = s(1-t) + stW_1^2.$$

Though we shall not do so in detail, fixing $0 < s_1 < \ldots < s_n < 1$ one can compute the density of $(W_{s_1}, \ldots, W_{s_n})$ conditional on $\{W_1 = 0\}$, per Example 2.4.5, and verify that it is the density of a (zero-mean) non-degenerate Gaussian random vector. Consequently, $\{W_t, 0 \leq t \leq 1\}$ conditional on the event $\{W_1 = 0\}$ is a Gaussian S.P. Recall Exercise 5.1.8, that $\mathbf{E}(B_t) = 0$ and $\mathbf{E}(B_s B_t) = s(1-t)$ for all $0 \leq s \leq t \leq 1$. In conclusion, we have established that the Gaussian S.P. $\{W_t, 0 \leq t \leq 1\}$ conditional on the event $\{W_1 = 0\}$, has the same mean and auto-covariance functions as the Gaussian S.P. B_t. Therefore, these two S.P. have the same distribution (i.e. the same f.d.d.).

Math 136 - Stochastic Processes
Homework Set 7, Autumn 2013, Due: November 13

1. Exercise 4.3.4. Show that the *first hitting time* $\tau(\omega) = \min\{k \geq 0 : X_k(\omega) \in B\}$ of a Borel set $B \subseteq \mathbb{R}$ by a sequence $\{X_k\}$, is a stopping time for the canonical filtration $\mathcal{F}_n = \sigma(X_k, k \leq n)$. Provide an example where the *last hitting time* $\theta = \sup\{k \geq 0 : X_k \in B\}$ of a set B by the sequence, is not a stopping time (not surprising, since we need to know the whole sequence $\{X_k\}$ in order to verify that there are no visits to B after a given time n).
ANS: We have, $\{\tau \leq n\} = \bigcup_{k=0}^n \{X_k \in B\} \in \mathcal{F}_n$ since it is a finite union of elements in $\mathcal{F}_n$. This verifies that τ is a stopping time for the filtration $\mathcal{F}_n$. Consider the stochastic process corresponding to two coin flips: $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F}_1 = \{\phi, \{HH, HT\}, \{TH, TT\}, \Omega\}$, $\mathcal{F}_2 = 2^{\Omega}$ and $X_k(\omega) = I_{\{\omega_k = H\}}$, $k = 1, 2$ for any $\omega = \omega_1\omega_2 \in \Omega$. Let $B = \{1\}$. Then $\{\theta \leq 1\} = \{HT, TT\} \notin \mathcal{F}_1$. So θ is not a stopping time.

2. Exercise 4.3.15 Let $\mathcal{G}_t$ denote the canonical filtration of the S.P. $\{X_t\}$.

   (a) Verify that $\mathcal{G}_{t^+} = \bigcap_{u>0} \mathcal{G}_{t+u}$ is a right-continuous filtration.
   ANS: Simply note that

$$\bigcap_{h>0} \mathcal{G}_{(t+h)^+} = \bigcap_{h>0} \Big[\bigcap_{u>0} \mathcal{G}_{t+h+u}\Big] = \bigcap_{v>0} \mathcal{G}_{t+v} = \mathcal{G}_{t^+},$$

   so by definition $\{\mathcal{G}_{t^+}\}$ is a right-continuous filtration.

   (b) Considering part (a) of Proposition 4.3.13 for filtration $\mathcal{G}_{t^+}$, deduce that for any fixed $b > 0$ and $\delta > 0$ the random variable $\tau_b^{(\delta)} = \inf\{t \geq \delta : X_{t-\delta} > b\}$ is a stopping time for $\{\mathcal{G}_t\}$, provided $\{X_t\}$ has right-continuous sample path.
   ANS: Note that $\mathcal{G}_t \subseteq \mathcal{G}_{t^+}$ implying that the S.P. $Z_t = X_{t-\delta}$, $t \geq \delta$, is adapted to $\{\mathcal{G}_{(t-\delta)^+}\}$ for any fixed $\delta > 0$. Hence, $\{\tau_b^{(\delta)} \leq t\} \in \mathcal{G}_{(t-\delta)^+}$ for any $t \geq \delta$, by part (a) of Proposition 4.3.13. Further, with $\delta > 0$, it follows that $\mathcal{G}_{(t-\delta)^+} \subseteq \mathcal{G}_t$ for any $t \geq \delta$, hence $\{\tau_b^{(\delta)} \leq t\} \in \mathcal{G}_t$ for all t (the case $t < \delta$ is trivial, for then the relevant event is empty). We conclude that $\tau_b^{(\delta)}$ is a stopping time for $\{\mathcal{G}_t\}$, as claimed.

   (c) With $Y_t = \int_0^t X_s^2 ds$ use part (b) of Proposition 4.3.13 to show that $\theta_1 = \inf\{t \geq 0 : Y_t = b\}$ is another stopping time for $\{\mathcal{G}_t\}$. Then explain why $\theta_2 = \inf\{t \geq 0 : Y_{2t} = b\}$, is in general not a stopping time for this filtration.
   ANS: That θ_1 is a stopping time for $\{\mathcal{G}_t\}$ is immediate from the continuity of the sample path $t \mapsto Y_t$ and the fact that the singleton $\{b\}$ is a closed set (where part (b) of Proposition 4.3.13

does not require right-continuity of the filtration). There are many counterexamples to θ_2 not being a stopping time with respect to $\{\mathcal{G}_t\}$. For example, consider the right continuous process $X_t = 2\xi \mathbf{1}_{\{t \geq 1\}}$ where $\mathbf{P}(\xi = 0) = \mathbf{P}(\xi = 1) = 1/2$. For this process and $b = 1$ it is easy to verify that the event $\{\theta_2 \leq 3/4\}$ is merely $\{\xi = 1\}$. Since in this case $\mathcal{G}_t = \{\emptyset, \Omega\}$ when $t < 1$, it follows that $\{\xi = 1\} \notin \mathcal{G}_{3/4}$.

3. Exercise 4.3.18. Let W_t be a Brownian motion. Fixing $a > 0$ and $b > 0$ let $\tau_{a,b} = \inf\{t \geq 0 : W_t \notin (-a, b)\}$. We will see in Section 5.2 that $\tau_{a,b}$ is finite with probability one.

   (a) Check that $\tau_{a,b}$ is a stopping time and that $W_{t \wedge \tau_{a,b}}$ is uniformly integrable.
   ANS: Since W_t has continuous sample paths $|W_{t \wedge \tau_{a,b}}| \leq \max(a, b)$ is uniformly (in t and ω) bounded, hence U.I. Further, $(-a, b)^c$ is a closed set so $\tau_{a,b}$ is a stopping time by part (b) of Proposition 4.3.13.

   (b) Applying Doob's optional stopping theorem for this stopped martingale, compute the probability that W_t reaches level b before it reaches level $-a$.
   ANS: Since $W_{\tau_{a,b}} \in \{-a, b\}$, applying the optional stopping theorem (we can do this because of part (a) and the assumption $\tau_{a,b} < \infty, a.s.$), we have that $0 = \mathbf{E}[W_0] = \mathbf{E}[W_{\tau_{a,b}}] = -a\mathbf{P}[W_{\tau_{a,b}} = -a] + b\mathbf{P}[W_{\tau_{a,b}} = b]$. Consequently, $\mathbf{P}[W_{\tau_{a,b}} = b] = a/(b+a)$.

   (c) Justify using the optional stopping theorem for $\tau_{b,b}$ and the martingales $M_t(\lambda)$ of Exercise 4.2.5. Deduce from it the value of $\mathbf{E}(e^{-\theta\tau_{b,b}})$ for $\theta > 0$.
   Hint: In part (c) you may use the fact that the S.P. $\{-W_t\}$ has the law as $\{W_t\}$.
   ANS: Let $X = e^{-\lambda^2 \tau_{b,b}/2}$ and $A = \{W_{\tau_{b,b}} = b\}$. Noting that the non-negative $M_{t \wedge \tau_{b,b}}(\lambda) \leq e^{|\lambda| b}$ is a U.I. process and $W_{\tau_{b,b}} \in \{-b, b\}$, it follows by Doob's optional stopping theorem that

$$1 = \mathbf{E}M_0(\lambda) = \mathbf{E}(M_{\tau_{b,b}}(\lambda)) = e^{\lambda b}\mathbf{E}[XI_A] + e^{-\lambda b}\mathbf{E}[XI_{A^c}]. \tag{1}$$

   Suppose we change the sign of the Brownian sample path, from $W_t(\omega)$ to $-W_t(\omega)$. The value of $\tau_{b,b}$, and hence that of X, is invariant under such a change, while the events A and A^c are exchanged by it. With the S.P. $\{-W_t\}$ having the same distribution (i.e. f.d.d.) as $\{W_t\}$, we thus deduce that $\mathbf{E}[XI_A] = \mathbf{E}[XI_{A^c}]$, and hence both are equal to $\mathbf{E}[X]/2$. Plugging this into (1) we get that $1 = \cosh(\lambda b)\mathbf{E}[e^{-\lambda^2 \tau_{b,b}/2}]$. Setting $\lambda = \sqrt{2\theta}$ we conclude that $\mathbf{E}[e^{-\theta\tau_{b,b}}] = 1/\cosh(\sqrt{2\theta}b)$ from which the law of $\tau_{b,b}$ can be computed.

4. Exercise 4.3.20. Consider $M_t = \exp(\lambda Z_t)$ for non-random constants λ and r, where $Z_t = W_t + rt, t \geq 0$, and W_t is a Brownian motion.

   (a) Compute the conditional expectation $\mathbf{E}(M_{t+h}|\mathcal{G}_t)$ for $\mathcal{G}_t = \sigma(Z_u, u \leq t)$ and $t, h \geq 0$.
   ANS: Noting that $\mathcal{G}_t = \sigma(W_u, u \leq t)$, we have that $W_{t+h} - W_t$ is independent of $\mathcal{G}_t$ and hence

$$\begin{aligned}\mathbf{E}(M_{t+h}|\mathcal{G}_t) &= \mathbf{E}(\exp(\lambda Z_{t+h})|\mathcal{G}_t) \\ &= \exp(\lambda Z_t)\mathbf{E}(\exp[\lambda(Z_{t+h} - Z_t)]|\mathcal{G}_t) \\ &= M_t\mathbf{E}(\exp[\lambda(W_{t+h} - W_t + rh)]|\mathcal{G}_t) \\ &= e^{\lambda rh}M_t\mathbf{E}(\exp[\lambda(W_{t+h} - W_t)]) \\ &= e^{\lambda rh + \lambda^2 h/2}M_t\end{aligned}$$

   (b) Find the value of $\lambda \neq 0$ for which $(M_t, \mathcal{G}_t)$ is a martingale.
   ANS: By part (a), $(M_t, \mathcal{G}_t)$ is a martingale if and only if $e^{\lambda rh + \lambda^2 h/2} = 1$ for any $h \geq 0$, which gives $\lambda = -2r$ (when $r \neq 0$).

   (c) Fixing $a, b > 0$, apply Doob's optional stopping theorem to find the law of $Z_{\tau_{a,b}}$ for $\tau_{a,b} = \inf\{t \geq 0 : Z_t \notin (-a, b)\}$.
   ANS: As the case $r = 0$ has been discussed in Exercise 4.3.18, we assume hereafter that $r \neq 0$ and let $\tau_c = \inf\{t \geq 0 : W_t = c\}$ for any $c \in \mathbb{R}$. We show in Section 5.2 that almost surely $\tau_c < \infty$ for each fixed $c \in \mathbb{R}$. When $r > 0$, we have $Z_t \geq W_t$ resulting with $\tau_{a,b} \leq \tau_b$; when $r < 0$, we have $Z_t \leq W_t$ resulting with $\tau_{a,b} \leq \tau_{-a}$. Therefore, $\tau_{a,b} < \infty$ almost surely. By continuity of W and hence of Z, $Z_{\tau_{a,b}} \in \{-a, b\}$. Part (b) tells us that $(M_t, \mathcal{G}_t)$ is a martingale for $M_t = \exp(-2rZ_t)$. Since $M_{t \wedge \tau_{a,b}}$ is uniformly bounded (by $e^{2|r|\max(a,b)}$), hence U.I., we can apply Doob's optional stopping theorem and get

$$1 = \mathbf{E}(M_0) = \mathbf{E}(M_{\tau_{a,b}}) = e^{2ra}\mathbf{P}(Z_{\tau_{a,b}} = -a) + e^{-2rb}\mathbf{P}(Z_{\tau_{a,b}} = b).$$

   Consequently, with $1 = \mathbf{P}(Z_{\tau_{a,b}} = -a) + \mathbf{P}(Z_{\tau_{a,b}} = b)$ we get that

$$\mathbf{P}(Z_{\tau_{a,b}} = b) = \frac{e^{2ra} - 1}{e^{2ra} - e^{-2rb}} \quad \text{and} \quad \mathbf{P}(Z_{\tau_{a,b}} = -a) = \frac{1 - e^{-2rb}}{e^{2ra} - e^{-2rb}}.$$

5. Exercise 5.2.4. Let W_t be a Brownian motion.

   (a) Show that $-\min_{0 \leq t \leq T} W_t$ and $\max_{0 \leq t \leq T} W_t$ have the same distribution which is also the distribution of $|W_T|$.
   ANS: Recall that W_t is a Gaussian process of zero mean. Since its f.d.d. have densities which are symmetric around the origin, it follows that the S.P. W_t and $-W_t$ have the same law. With $-\min_{t \in [0,T]} W_t = \max_{t \in [0,T]}(-W_t)$, we see that the latter two R.V. have the same distribution. We know that $\mathbf{P}(\max_{t \in [0,T]} W_t \geq \alpha) = 2\mathbf{P}(W_T \geq \alpha) = \mathbf{P}(|W_T| \geq \alpha)$ for all $\alpha \geq 0$. So, the three R.V. $|W_T|$, $\max_{0 \leq t \leq T} W_t$ and $-\min_{0 \leq t \leq T} W_t$ have the same distribution.

   (b) Show that the probability α that the Brownian motion W_u attains the value zero at some $u \in (s, s+t)$ is given by $\alpha = \int_{-\infty}^{\infty} p_t(|x|)\phi_s(x)dx$, where $p_t(x) = \mathbf{P}(|W_t| \geq x)$ for $x, t > 0$ and $\phi_s(x)$ denotes the probability density of the R.V. W_s for $s > 0$.
   Remark: The explicit formula $\alpha = (2/\pi)\arccos(\sqrt{s/(s+t)})$ is obtained in [**KT75**, page 348] by computing this integral.
   ANS: Let $\mathcal{H}_s = \sigma(W_s)$ and A denote the event $\{\exists u \in (s, s+t) : W_u = 0\}$. Then, by the tower property $\alpha = \mathbf{P}(A) = \mathbf{E}(\mathbf{P}(A|\mathcal{H}_s))$. Since $\mathbf{P}(W_{s+t} = 0|\mathcal{H}_s) = 0$ and the Brownian path is continuous, we have that

$$I_{\{W_s<0\}}\mathbf{P}(A|\mathcal{H}_s) = I_{\{W_s<0\}}\mathbf{P}(\max_{u \in [0,t]}(W_{s+u} - W_s) \geq -W_s|\mathcal{H}_s).$$

   We know that conditional on $\mathcal{H}_s$, the S.P. $\{W_{s+u} - W_s : u \geq 0\}$ has the original Brownian law (for example, see Proposition 5.2.3). Applying part (a), we deduce that $I_{\{W_s<0\}}\mathbf{P}(A|\mathcal{H}_s) = I_{\{W_s<0\}}p_t(-W_s)$. The same considerations yield in case $W_s > 0$ that

$$I_{\{W_s>0\}}\mathbf{P}(A|\mathcal{H}_s) = I_{\{W_s>0\}}\mathbf{P}(-\min_{u \in [0,t]}(W_{s+u} - W_s) \geq W_s|\mathcal{H}_s).$$

   It follows by part (a) then that $I_{\{W_s>0\}}\mathbf{P}(A|\mathcal{H}_s) = I_{\{W_s>0\}}p_t(W_s)$. With $W_s \neq 0$ almost surely, combining these two formulas we have that

$$\alpha = \mathbf{E}[\mathbf{P}(A|\mathcal{H}_s)] = \mathbf{E}[p_t(|W_s|)] = \int_{-\infty}^{\infty} p_t(|x|)\phi_s(x)dx$$

   as stated.

6. Exercise 5.2.5. Show that $\mathbf{E}(\tau_{\beta,\alpha}) = \alpha\beta$ by applying Doob's optional stopping theorem for the uniformly integrable stopped martingale $W_{t \wedge \tau_{\beta,\alpha}}^2 - t \wedge \tau_{\beta,\alpha}$.
ANS: We have seen en-route to (5.2.2) that $\tau_{\beta,\alpha} \leq \tau_\alpha < \infty$ almost surely. Considering the martingale $X_t = W_t^2 - t$ of continuous sample path we have further assumed in the statement of the exercise that $X_{t \wedge \tau_{\beta,\alpha}}$ is U.I. Thus, Doob's optional stopping theorem (Theorem 4.3.16) applies here, leading to the identity $\mathbf{E}(W_{\tau_{\beta,\alpha}}^2 - \tau_{\beta,\alpha}) = \mathbf{E}(W_0^2 - 0) = 0$. That is,

$$\mathbf{E}\tau_{\beta,\alpha} = \mathbf{E}W_{\tau_{\beta,\alpha}}^2 = \alpha^2\mathbf{P}(W_{\tau_{\beta,\alpha}} = \alpha) + \beta^2\mathbf{P}(W_{\tau_{\beta,\alpha}} = -\beta) = \frac{\alpha^2\beta}{\alpha+\beta} + \frac{\beta^2\alpha}{\alpha+\beta} = \alpha\beta.$$

Math 136 - Stochastic Processes
Homework Set 8, Autumn 2013, Due: November 20

1. Exercise 4.4.10. Find a non-random $f(t)$ such that $X_t = e^{W_t - f(t)}$ is a martingale, and for this value of $f(t)$ find the increasing process associated with the martingale X_t via the Doob-Meyer decomposition. Hint: Try an increasing process $A_t = \int_0^t e^{2W_s - h(s)}ds$ and use Fubini's theorem to find the non-random $h(s)$ for which $M_t = X_t^2 - A_t$ is a martingale with respect to the filtration $\mathcal{G}_t = \sigma(W_s, s \leq t)$.
ANS: By Exercise 4.2.5 we know that $e^{W_t - t/2}$ is a martingale, hence we take $f(t) = t/2$. We assume that the increasing process in the Doob-Meyer decomposition has the form $A_t = \int_0^t e^{2W_u - h(u)}du$. Clearly, $A_0 = 0$. Also, A_t has continuous sample paths, since W_t does; A_t depends only on the values of W_u for $0 \leq u \leq t$ so it is $\{\mathcal{G}_t\}$-adapted; and A_t is nondecreasing since $e^x > 0$. Further, $\mathbf{E}[e^{2W_u}|\mathcal{G}_s] = e^{2W_s + 2(u-s)}$ for all $u \geq s$, hence $\mathbf{E}[X_t^2|\mathcal{G}_s] = e^{2W_s - 2s + t}$ and by Fubini's theorem also,

$$\mathbf{E}[A_t - A_s|\mathcal{G}_s] = \int_s^t \mathbf{E}[e^{2W_u - h(u)}|\mathcal{G}_s]du = e^{2W_s - 2s}\int_s^t e^{2u - h(u)}du,$$

when $t \geq s$. The remaining condition of $(X_t^2 - A_t, \mathcal{G}_t)$ a martingale thus amounts to

$$\mathbf{E}[X_t^2 - (A_t - A_s)|\mathcal{G}_s] - X_s^2 = e^{2W_s - 2s}(e^t - e^s - \int_s^t e^{2u - h(u)}du) = 0,$$

which evidently holds for $h(u) = u$. In conclusion, the increasing part associated with the MG $(X_t, \mathcal{G}_t)$ is $A_t = \int_0^t e^{2W_s - s}ds$.

2. Exercise 4.5.4. Consider an urn that at stage 0 contains one red ball and one blue ball. At each stage a ball is drawn at random from the urn, with all possible choices being equally likely, and it and one more ball of the same color are then returned to the urn. Let R_n denote the number of red balls at stage n and $M_n = R_n/(n+2)$ the corresponding fraction of red balls.

   (a) Find the law of R_{n+1} conditioned on $R_n = k$ and use it to compute $\mathbf{E}(R_{n+1}|R_n)$.
   ANS: At time n, there are k red balls and $(n+2-k)$ blue balls if $R_n = k$. So we have that R_{n+1} can only take the values k and $k+1$ with non-zero probabilities $(n+2-k)/(n+2)$ and $k/(n+2)$, respectively. Thus, $\mathbf{E}(R_{n+1}|R_n) = (n+2-R_n)R_n/(n+2) + R_n(R_n+1)/(n+2) = \frac{n+3}{n+2}R_n$.

   (b) Check that M_n is a martingale with respect to its canonical filtration.
   ANS: We have that M_n is bounded so it is integrable. Note that the canonical filtration $\mathcal{G}_n$ for $\{M_n\}$ is the same as that of $\{R_n\}$. Further, per fixed given value of R_n, the value of R_{n+1} is independent of $(R_0, R_1, \ldots, R_{n-1})$. Hence,

$$\mathbf{E}(M_{n+1}|\mathcal{G}_n) = \frac{1}{n+3}\mathbf{E}(R_{n+1}|\mathcal{G}_n) = \frac{1}{n+3}\mathbf{E}(R_{n+1}|R_n) = \frac{1}{n+2}R_n = M_n,$$

   so M_n is indeed a martingale with respect to its canonical filtration.

(c) Applying Proposition 4.5.3 conclude that $M_n \to M_\infty$ in L^2 and that $\mathbf{E}(M_\infty) = \mathbf{E}(M_0) = 1/2$.

ANS: Since $0 \le M_n \le 1$, we have $\mathbf{E}M_n^2 \le 1, n = 1, 2, \ldots$. By Proposition 4.5.3 there exists a R.V. M_∞ such that $M_n \to M_\infty$ a.s. and in L^2. Consequently, as shown for example in Exercise 1.3.21 (or by the bounded convergence of Corollary 1.4.29), $\mathbf{E}(M_\infty) = \lim_{n\to\infty} \mathbf{E}(M_n) = \mathbf{E}(M_0) = 1/2$.

(d) Using Doob's (maximal) inequality show that $\mathbf{P}(\max_{k\ge1} M_k > 3/4) \le 2/3$.

ANS: By part (c) and Doob's inequality, we have that $\mathbf{P}(\max_{k\ge1} M_k > 3/4) \le (4/3)\mathbf{E}(M_\infty) = 2/3$.

3. Exercise 4.6.8. Suppose $\{Z_n\}$ is a branching process with $\mathbf{P}(N = 1) < 1$ and $Z_0 = 1$. Show that

$$\mathbf{P}(\lim_{n\to\infty} Z_n = \infty) = 1 - p_{ex},$$

first in case $m \le 1$, then in case $\mathbf{P}(N = 0) = 0$ and finally using the preceding exercise, for $m > 1$ and $\mathbf{P}(N = 0) > 0$.

ANS: Since $\mathbf{P}(N = 1) < 1$ we have by Propositions 4.6.3 and 4.6.5 that $p_{ex} = 1$ when $m \le 1$. That is, in this case w.p.1. $Z_n = 0$ for all n large enough, yielding the stated claim.

In contrast, if $\mathbf{P}(N = 0) = 0$ then Z_n is non-decreasing, so $p_{ex} = 0$. Further, in this case Z_n is bounded only if $N_1^{(k)} = 1$ for all k large enough, which with $\mathbf{P}(N = 1) < 1$ occurs with zero probability, again resulting with the stated claim.

Finally, for $m > 1$ and $\mathbf{P}(N = 0) > 0$ we have from Exercise 4.6.7 that $p_{ex} = \rho \in (0, 1)$ and further,

$$1 - p_{ex} = 1 - \mathbf{P}(M_\infty = 1) = \mathbf{P}(M_\infty = 0) = \mathbf{P}(\lim_{n\to\infty} \rho^{Z_n} = 0) = \mathbf{P}(\lim_{n\to\infty} Z_n = \infty),$$

as claimed.

4. Exercise 4.6.9. Let $\{Z_n\}$ be a branching process with $Z_0 = 1$. Compute p_{ex} in each of the following situations and specify for which values of the various parameters the extinction is certain.

(a) The offspring distribution satisfies, for some $0 < p < 1$,

$$\mathbf{P}(N = 0) = p,\ \mathbf{P}(N = 2) = 1 - p.$$

ANS: We have that $m = \mathbf{E}N = 2(1 - p)$ with $m \le 1$ if and only if $p \ge 1/2$. Thus, $p_{ex} = 1$ when $p \ge 1/2$ by Proposition 4.6.3 (for $p > 1/2$) and Proposition 4.6.5 (for $p = 1/2$, applicable since here $\mathbf{P}(N = 1) = 0 < 1$). Finally, if $p < 1/2$ then $m > 1$ so $\{Z_n\}$ is super-critical with $\mathbf{P}(N = 0) = p > 0$. We have shown in Exercise 4.6.7 that in this case p_{ex} is the unique solution in $(0, 1)$ of

$$0 = x - \phi(x) = x - \mathbf{P}(N = 0) - \mathbf{P}(N = 2)x^2 = x - p - (1 - p)x^2$$

(taking the function $\phi(x)$ per equation (4.6.2) that corresponds to our law of N). As

$$x - p - (1 - p)x^2 = (1 - p)(1 - x)(x - p/(1 - p)),$$

we conclude that $p_{ex} = p/(1 - p) < 1$ when $p < 1/2$.

(b) The offspring distribution is (shifted) Geometric, i.e. for some $0 < p < 1$,

$$\mathbf{P}(N = k) = p(1 - p)^k,\ k = 0, 1, 2, \ldots$$

ANS: We have now that $m = \mathbf{E}N = \sum_{k=1}^\infty kp(1 - p)^k = (1 - p)/p$ (where to get the last identity differentiate in p the identity $\sum_{k=0}^\infty (1 - p)^k = 1/p$ and multiply both sides by $-p(1 - p)$). As in part (a), if $p \ge 1/2$ then $m \le 1$ and consequently $p_{ex} = 1$ (for here too $\mathbf{P}(N = 1) = p(1 - p) < 1$). In contrast, $p < 1/2$ yields a super-critical branching process with $\mathbf{P}(N = 0) = p > 0$, so again from Exercise 4.6.7 we have that p_{ex} is the unique solution in $(0, 1)$ of

$$0 = x - \phi(x) = x - \sum_{k=0}^\infty \mathbf{P}(N = k)x^k = x - p\sum_{k=0}^\infty (1 - p)^k x^k = x - \frac{p}{1 - (1 - p)x}.$$

Thus, p_{ex} is the unique root in $(0, 1)$ of the quadratic equation

$$0 = x(1 - (1 - p)x) - p = x - p - (1 - p)x^2,$$

and as you have seen in part (a), it follows that $p_{ex} = p/(1-p)$. Thus, though the law of N in part (b) is different from its law in part (a), both result with same values of p_{ex} (for all choices of p).

5. Exercise 5.3.10. Suppose $(W_t, \mathcal{F}_t)$ satisfies Lévy's characterization of the Brownian motion. Namely, it is a square-integrable martingale of right-continuous filtration and continuous sample path such that $(W_t^2 - t, \mathcal{F}_t)$ is also a martingale. Suppose X_t is a bounded $\mathcal{F}_t$-adapted simple process. That is,

$$X_t = \eta_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^\infty \eta_i \mathbf{1}_{(t_i, t_{i+1}]}(t),$$

where the non-random sequence $t_k > t_0 = 0$ is strictly increasing and unbounded (in k), while the (discrete time) S.P. $\{\eta_n\}$ is uniformly (in n and ω) bounded and adapted to $\mathcal{F}_{t_n}$. Provide an explicit formula for $A_t = \int_0^t X_u^2 du$, then show that both

$$I_t = \sum_{j=0}^{k-1} \eta_j(W_{t_{j+1}} - W_{t_j}) + \eta_k(W_t - W_{t_k}),\ \text{when } t \in [t_k, t_{k+1}),$$

and $I_t^2 - A_t$ are martingales with respect to $\mathcal{F}_t$ and explain why this implies that $\mathbf{E}I_t^2 = \mathbf{E}A_t$ and $V_t^{(2)}(I) = A_t$.

ANS: Since the intervals $(t_i, t_{i+1}]$ are pairwise disjoint,

$$X_t^2 = \eta_0^2 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^\infty \eta_i^2 \mathbf{1}_{(t_i, t_{i+1}]}(t).$$

Consequently,

$$A_t = \int_0^t X_u^2 du = \sum_{j=0}^{k-1} \eta_j^2 (t_{j+1} - t_j) + \eta_k^2 (t - t_k),\ \text{when } t \in [t_k, t_{k+1}),$$

Next note that I_t is adapted to $\mathcal{F}_t$ (on account of the adaptedness of $\{\eta_n\}$ to $\mathcal{F}_{t_n}$ and that of $\{W_t\}$ to $\{\mathcal{F}_t\}$), and is integrable (for each summand is integrable due to boundedness of η_n and the integrability of W_t). Further, by the tower property, if $(I_t - I_{t_k}, \mathcal{F}_t)$ satisfies the martingale property for $t \in [t_k, t_{k+1}]$ and each fixed k then also $(I_t, \mathcal{F}_t)$ is a martingale (for all $t \ge 0$). Fixing k and $t_k \le s < t \le t_{k+1}$, note that taking out η_k which is measurable on $\mathcal{F}_{t_k} \subseteq \mathcal{F}_s$, we get by the martingale property of $(W_t, \mathcal{F}_t)$ that

$$\mathbf{E}[(I_t - I_{t_k}) - (I_s - I_{t_k})|\mathcal{F}_s] = \mathbf{E}[\eta_k(W_t - W_s)|\mathcal{F}_s] = \eta_k(\mathbf{E}[W_t|\mathcal{F}_s] - W_s) = 0,$$

as needed for proving that $(I_t, \mathcal{F}_t)$ is a martingale.

Similarly, note that $J_t = I_t^2 - A_t$ is $\mathcal{F}_t$-adapted and integrable (on account of square integrability of $\{W_t\}$ and boundedness of η_n). As before, to show that $(J_t, \mathcal{F}_t)$ is a martingale it suffices to verify the martingale property for $(J_t - J_{t_k}, \mathcal{F}_t)$ with $t \in [t_k, t_{k+1}]$ and k fixed. To this end, note that

$$J_t - J_{t_k} = 2I_{t_k}(I_t - I_{t_k}) + \eta_k^2[(W_t - W_{t_k})^2 - (t - t_k)],$$

and recall that we have shown this property already for $(I_t - I_{t_k}, \mathcal{F}_t)$. Since I_{t_k} is measurable on $\mathcal{F}_{t_k} \subseteq \mathcal{F}_t$, the same applies for $(I_{t_k}(I_t - I_{t_k}), \mathcal{F}_t)$. Further, η_k^2 is also measurable on $\mathcal{F}_{t_k} \subseteq \mathcal{F}_t$ and by the preceding, linearity of the C.E. and taking out what is known, we only need to verify that $(\widehat{W}_u^2 - u, \mathcal{F}_{t_k+u})$ has the martingale property for $\widehat{W}_u = W_{t_k+u} - W_{t_k}$ and $0 \le u \le t_{k+1} - t_k$. This in turn follows from our assumption that W_t is Brownian motion (with respect to $\mathcal{F}_t$), hence by Proposition 5.2.3 so is $\widehat{W}_u$ (now with respect to $\mathcal{F}_{t_k+u}$).

Clearly, the martingale property of J_t implies that $\mathbf{E}J_t = \mathbf{E}J_0 = 0$, that is $\mathbf{E}I_t^2 = \mathbf{E}A_t$. We have proved that both $(I_t, \mathcal{F}_t)$ and $(I_t^2 - A_t, \mathcal{F}_t)$ are martingales of continuous sample path and right-continuous filtration, with $A_0 = 0$ and $t \mapsto A_t$ non-decreasing. Thus, A_t is the increasing process associated with $(I_t, \mathcal{F}_t)$ via the Doob-Meyer decomposition. As stated in Corollary 5.3.5, A_t must then be also the quadratic variation $V_t^{(2)}(I)$ of the "stochastic integral" I_t.

6. Exercise 5.3.14. Consider the stochastic process $Y(t) = W(t)^2$, for $0 \le t \le 1$, with $W(t)$ a Brownian motion.

(a) Show that for any $\gamma < 1/2$ the sample path of $Y(t)$ is locally Hölder continuous of exponent γ with probability one.

ANS: Suppose that f is a function on $[0, 1]$ that is locally Hölder continuous of exponent $\gamma > 0$. Then, the same holds for the function f^2. Indeed, here $M = \sup_{x\in[0,1]} f(x)$ is finite (since f is continuous on the bounded interval $[0, 1]$) and we have for all $s, t \in [0, 1]$ that

$$\frac{|f^2(t) - f^2(s)|}{|t - s|^\gamma} = |f(t) + f(s)|\frac{|f(t) - f(s)|}{|t - s|^\gamma} \le 2M\frac{|f(t) - f(s)|}{|t - s|^\gamma}.$$

Our claim follows from this fact in view of part (a) of Exercise 5.1.12 (in case $H = 1/2$ there; see also Exercise 5.3.7).

Alternatively, noting that $Y(s + h) - Y(s) = (2Z + X)X$ for the independent Gaussian $X = W(s + h) - W(s)$ and $Z = W(s)$ of zero mean and variances h and s, respectively, it is not hard to show that $\mathbf{E}[(Y(s + h) - Y(s))^{2p}] \le C(p)h^p$ for any positive integer p, some finite $C(p)$ and all $h, s \in [0, 1]$. The claim then follows by an application of Kolmogorov's continuity theorem (where $\gamma < \beta/\alpha = (p - 1)/2p$ once p is large enough).

(b) Compute $\mathbf{E}V_{(\pi)}^{(2)}(Y)$ for a finite partition π of $[0, t]$ to k intervals, and find its limit as $\|\pi\| \to 0$.

ANS: Using notations of part (a) it is not hard to check that for all $h > 0$, $s \ge 0$,

$$(Y(s + h) - Y(s))^2 = 4Z^2X^2 + 4ZX^3 + X^4,$$

where by independence of $X \sim N(0, h)$ and $Z \sim N(0, s)$ it follows that

$$\mathbf{E}[|Y(s + h) - Y(s)|^2] = 4\mathbf{E}Z^2\mathbf{E}X^2 + 4\mathbf{E}Z\mathbf{E}X^3 + \mathbf{E}X^4 = 4sh + 3h^2.$$

With $\Delta t_i = t_{i+1} - t_i$ for the partition $\pi = \{0 = t_0, t_1, \ldots, t_k = t\}$, using this identity and the linearity of the expectation we have

$$\mathbf{E}V_{(\pi)}^{(2)}(Y) = \mathbf{E}[\sum_{i=0}^{k-1}(Y(t_{i+1}) - Y(t_i))^2] = 4\sum_{i=0}^{k-1} t_i\Delta t_i + 3\sum_{i=0}^{k-1}\Delta t_i^2$$

Note that $\sum_{i=0}^{k-1} t_i\Delta t_i$ is a Riemann sum approximation for the integral $\int_0^t u du = t^2/2$ that thus converges to $t^2/2$ as $||\pi|| \to 0$. Further, with

$$\sum_{i=0}^{k-1}\Delta t_i^2 \le ||\pi||\sum_{i=0}^{k-1}\Delta t_i = ||\pi||t \to 0$$

as $||\pi|| \to 0$, we conclude that $\mathbf{E}V_{(\pi)}^{(2)}(Y)$ converges to $2t^2$ in the limit $||\pi|| \to 0$.

Indeed, we note in passing that for the partition π of $[0, t]$ to k intervals of equal length $\Delta t_i = t/k$, i.e. taking $t_i = i(t/k)$, the expectation we consider simplifies to $2t^2 + t^2/k$ that evidently converges to $2t^2$ as $k \to \infty$.

(c) Show that the total variation of $Y(t)$ on the interval $[0, 1]$ is infinite.

ANS: In Proposition 5.3.12 we saw that the Brownian motion has infinite total variation in any fixed interval $[a, b]$, $b > a$. Inside any open interval (a, b) there is a sub-interval $[r, q]$ with $q > r$ rational numbers. As there are only countably many such sub-intervals, we deduce that there exists an event A of probability one such that if $\omega \in A$ then $t \mapsto W(t, \omega)$ is continuous and has an infinite total variation in every open interval in $(0, 1)$. In particular, fixing $\omega \in A$ this implies that there exists $t \in (0, 1)$ such that $W(t, \omega) \ne 0$ (since otherwise $W(t, \omega) = 0$ for all t and such a path would have finite total variation). Fixing such $t = t(\omega)$ we assume with out loss of generality that $W(t, \omega) = 2\delta > 0$ and note that by continuity of the sample path there exists $\epsilon > 0$ such that $|W(s, \omega) - W(t, \omega)| \le \delta$ for all $s \in (t - \epsilon, t + \epsilon)$. This implies that for any s_1 and s_2 in $(t - \epsilon, t + \epsilon)$ we have $W(s_1, \omega) + W(s_2, \omega) \ge 2\delta$ and hence

$$|Y(s_1, \omega) - Y(s_2, \omega)| = |W(s_1, \omega) + W(s_2, \omega)||W(s_1, \omega) - W(s_2, \omega)| \ge 2\delta|W(s_1) - W(s_2)|.$$

It follows that the total variation of $Y(s, \omega)$ on the interval $(t - \epsilon, t + \epsilon)$ is bounded below by 2δ times the total variation of $W(s, \omega)$ on the same interval. Our claim follows since we already know that the latter quantity is infinite for $\omega \in A$.