Which Random Matching Markets Exhibit a Stark Effect of Competition?

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Abstract

We revisit the popular random matching market model introduced by Knuth (1976) and Pittel (1989), and shown by Ashlagi, Kanoria and Leshno (2013) to exhibit a "stark effect of competition", i.e., with any difference in the number of agents on the two sides, the short side agents obtain substantially better outcomes. We generalize the model to allow "partially connected" markets with each agent having an average degree d in a random (undirected) graph. Each agent has a (uniformly random) preference ranking over only their neighbors in the graph. We characterize stable matchings in large markets and find that the short side enjoys a significant advantage only for d exceeding $\log^2 n$ where n is the number of agents on one side: For moderately connected markets with $d = o(\log^2 n)$, we find that *there is no stark effect of competition*, with agents on both sides getting a $\sqrt{d}$-ranked partner on average. Notably, this regime extends far beyond the connectivity threshold of $d = \Theta(\log n)$. In contrast, for densely connected markets with $d = \omega(\log^2 n)$, we find that the short side agents get $\log n$-ranked partner on average, while the long side agents get a partner of (much larger) rank $d/\log n$ on average. Numerical simulations of our model confirm and sharpen our theoretical predictions. Since preference list lengths in most real-world matching markets are much below $\log^2 n$, our findings may help explain why available datasets do not exhibit a strong effect of competition.

Keywords: random matching markets, incomplete preference lists, stable matching, competition.

1 Introduction

Ashlagi, Kanoria and Leshno (Ashlagi et al. 2017) found that in a matching market with n men and $n + 1$ women, and uniformly randomly complete preference lists, independent across agents,

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there is a nearly unique stable matching, where the average rank of men for their wives is just $\log n(1 + o(1))$ (the same as it would have been under random serial dictatorship, where each man in turn selects their favorite remaining woman), whereas the average rank of women for their husbands is $\frac{n}{\log n}(1 + o(1))$. For example, with $n = 1,000$, men get matched to their seventh most desired woman, whereas women are matched to only their 145th most preferred man. Of course the situation is completely reversed if, instead, there are 999 women, while the number of men is still $1,000$. This led Ashlagi et al. (2017) to conclude "... *we find that matching markets are extremely competitive, with even the slightest imbalance greatly benefiting the short side.*"

Meanwhile, over the past two decades, a large number of real world matching market datasets from deferred acceptance (DA) based clearinghouses have become available to different researchers in the field, e.g., from centralized labor markets like the National Residency Matching Program (NRMP) (Roth and Peranson 1999) and the Israel Psychology Masters Match (Hassidim et al. 2017b), college admissions (e.g., Rios et al. 2019), and school choice (e.g., Abdulkadiroglu et al. 2005). Since these (and other) clearinghouses run the incentive-compatible deferred acceptance (DA) algorithm, the preference rankings collected may be assumed to reflect the true underlying preferences.[1] Notably, *none* of these real world data sets exhibit a "stark effect of competition" phenomenon, i.e., we do not see an abrupt change in the stable matching for a small change in market composition. As a representative example, we provide numerical counterfactuals for high school admissions data collected in one of the major cities in the U.S.: The data includes the preference lists provided by nearly 75,000 applicants, and 700 programs with a total capacity of 73,000. To study the effect of competition, we vary the market "imbalance" across a wide range by dropping up to 20,000 students from the data (uniformly at random) at one extreme, and duplicating up to 20,000 students (uniformly at random) at the other extreme, while holding the set of programs and their capacities fixed. We numerically evaluate the effect of thus varying the number of students on the resulting allocation of programs to students under the student-proposing DA algorithm. As per the usual practice, we summarize the allocation in terms of the fraction of students who are allotted to one of their top-k most preferred programs (for $k = 1, 3$) and the fraction who are unassigned; see the solid lines in Figure 1. Observe that the summary statistics

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[1]Incentive compatibility of DA for the proposing side was established by Dubins and Freedman (1981). For the receiving side, approximate IC is strongly suggested by the findings of Kojima and Pathak (2009) and Ashlagi et al. (2017), among others, and the mechanism further seems very hard to manipulate in most practical settings. However, it is worth noting Hassidim et al. (2017a) has found empirical evidence of incorrect preference reporting in certain situations, while Echenique et al. (2020) suggests that participants may not report options they like if those options are infeasible. Since the extent and nature of misreporting in our data (if any) is unclear, we simply assume the preference reports to be truthful.

vary extremely smoothly and slowly in the number of students over a wide range. In other words, *we observe no stark effect of competition in real world data*, which is at odds with the aforementioned conclusion of Ashlagi et al. (2017).
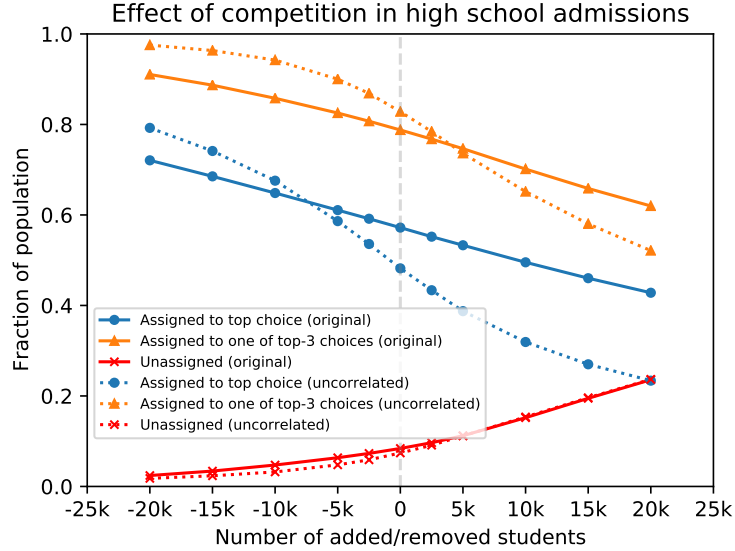

Figure 1: Fraction of students who are assigned to one of their top-k most preferred programs (for $k = 1, 3$) and the fraction of students who are unassigned, as a function of the number of students removed or duplicated uniformly at random. Simulations are based on the actual high school admissions data containing 75k applicants and 73k seats across 700 programs (averaged across 100 realizations). The solid lines use the student preference rankings and program priorities in the original data, and implement a single tie-breaking rule[3]. The dotted lines are based on randomizing preferences and priorities: Each student's preference list has unchanged length but its entries are drawn without replacement with the sampling probability of each program being proportional to the number of students who have applied to it in the original dataset, and each program uses a uniformly random and independent priority ordering over students.

The stochastic model of matching markets considered in Ashlagi et al. (2017) is often called a "random matching market"; one where agents have independent, complete and uniformly random preference lists over the other side. The model was introduced by Knuth (Knuth 1976), heavily studied by Pittel and others (Pittel 1989, Knuth et al. 1990, Pittel 1992) and this model (and variants) remains a workhorse for research in the area (e.g., Lee 2016, Menzel 2015) and even for deriving operational insights, e.g., which tie-breaking rule to use (Ashlagi and Nikzad 2016), making it imperative that we understand how its predictions might depart from reality, and the role played by each of the stylized assumptions in the model.

It is natural to ask whether correlation in preferences in real markets is the reason that they do

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[3]Under single tie-breaking, each student receives a random lottery number at the beginning of matching process, which is used by all programs for breaking ties between applicants with the same priority.

not exhibit a strong effect of competition. Indeed, if preferences on the "men" side of the market are fully correlated (i.e., all agents have the same preference ordering) while the other "women" side has arbitrary preferences, then there is a unique stable matching which can be computed by running serial dictatorship by women (women serially pick their favorite available man), in the order of the womens' universal ranking by men. One would expect this unique stable matching to transform smoothly as the number of agents on one side of the market is varied. To test whether correlation is indeed the reason we see only a weak effect of competition in the actual high school admissions data introduced above, we ran an additional experiment with "randomized" preferences: We took only the student and program "degrees" (i.e., preference list lengths and number of times the program is listed) from the data, generated both student preference lists and program priorities independently at random, and studied the resulting allocation as a function of the number of students; see the dotted lines in Figure 1 (the distribution over preferences is precisely specified in the caption of the figure). While the effect of competition under randomized preferences is somewhat stronger than in the original data,[4] it bears no resemblance to the abrupt phase transition found by Ashlagi et al. (2017). Thus it appears that even without correlation in preferences, and despite being very well connected,[5] realistic markets seem to lack a strong effect of competition. This prompts us to investigate the effect of the level of connectivity in the random matching market model on the effect of competition.

Model. Our model generalizes the random matching market model to allow "partially connected" markets with each agent having an average degree d in a random (undirected) connectivity graph. Each agent has a preference ranking over only their neighbors in the connectivity graph. We assume there are $n + k$ men and n women, where the "imbalance" k may be positive or negative but we restrict to "small" imbalances $|k| = o(n)$. For technical convenience, the random graph model we work with is one where each man is connected to a uniformly random subset of exactly d women, independent of other men.[6]

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[4]Define the elasticity of the fraction of students who get their top choice as the percent change in this fraction for every 1% change in the number of students. Near the status quo number of students, we find that the elasticity of the high school market is nearly -1.0, whereas the elasticity of the randomized high school market is nearly -2.8.

[5]In the high school admissions setting, both without and with randomization of preferences, over 97% of student pairs are within two hops of each other and nearly 100% are within three hops of each other (a pair of students is within one hop of each other if they list a program in common).

[6]As a result, each woman has $\text{Binomial}(n + k, d/n) \xrightarrow[n \to \infty]{\text{d}} \text{Poisson}(d)$ neighbors where $\xrightarrow{\text{d}}$ denotes convergence in distribution. Throughout the paper we will restrict attention to $d = \omega(1)$, as a result of which $\text{Poisson}(d) \xrightarrow{\text{p}} d$, i.e., the degree of each woman is also very close to d, and so the asymmetry between the two sides in the model is mainly technical.

Main findings. We characterize stable matchings as a function of d and the number of women n and find that the short side enjoys a significant advantage only for d exceeding $\log^2 n$: For moderately connected markets, specifically any d such that $d = o(\log^2 n)$ and $d = \omega(1)$ and large n, we find that *there is no stark effect of competition*, namely, the short and long sides of the market are almost equally well off (for $|k| = O(n^{1-\epsilon})$ market imbalance), with agents on both sides getting a $\sqrt{d}(1+o(1))$-ranked partner on average. Notably, this regime extends far beyond the connectivity threshold (above which the connectivity graph is connected with high probability) of $d = \Theta(\log n)$. On the other hand, for densely connected markets, specifically for any $d = \omega(\log^2 n)$ and large n, we find that there is a stark effect of competition: assuming a small imbalance $|k| = o(n)$, the short side agents get a partner of rank $\log n$ on average, while the long side agents get a partner of (much larger) rank $d/\log n$ on average. Numerical simulations of our model confirm the theoretical predictions, and in fact further enhance our understanding: they suggest a *sharp threshold between the two regimes close to $d \approx 1.0 \times \log^2 n$* and that *this holds even for small n down to $n \gtrsim 10$.* Figure 2 provides a schematic depicting our main findings (including the $d \approx 1.0 \times \log^2 n$ threshold between regimes suggested by numerics).

Since preference list lengths in most real markets are much below $\log^2 n$ (the latter is nearly 48 for $n = 1000$ and nearly 117 for $n = 50000$), and correlation in preferences only appears to reduce the effect of competition (see Figure 1 for indicative evidence), our findings may explain why real world matching datasets do not exhibit a strong effect of competition.

We highlight that the "no stark effect" regime includes well connected markets for connectivity in the range $d \in (\Theta(\log n), o(\log^2 n))$. This is in sharp contrast to buyer-seller market, where, roughly, connectivity implies a stark effect of competition where the short side of the market captures all the surplus (see Remark 1 in Section 3 for a detailed discussion). In particular, our results imply that the informal claim in Ashlagi et al. (2017) of strong *similarity* between the two kinds of markets is incorrect for moderately connected markets (parallel to the fact that most real world matching markets are well connected but do not exhibit a stark effect of competition).

Intuition for our findings. We now provide the high level intuition behind our main results. Ashlagi et al. (2017) showed a stark effect of competition for fully connected markets $d = n$. We find that as d is decreased, this phenomenon remains intact if all short side agents are matched: if women are on the long side ($k < 0$), though the average rank of women for their husbands $R_{\text{WOMEN}} \approx d/\log n$ decreases as d decreases, it remains true that men are significantly better off than women $R_{\text{WOMEN}}/R_{\text{MEN}} \geq 1 + \Omega(1)$. As d falls below a certain threshold, a positive number of
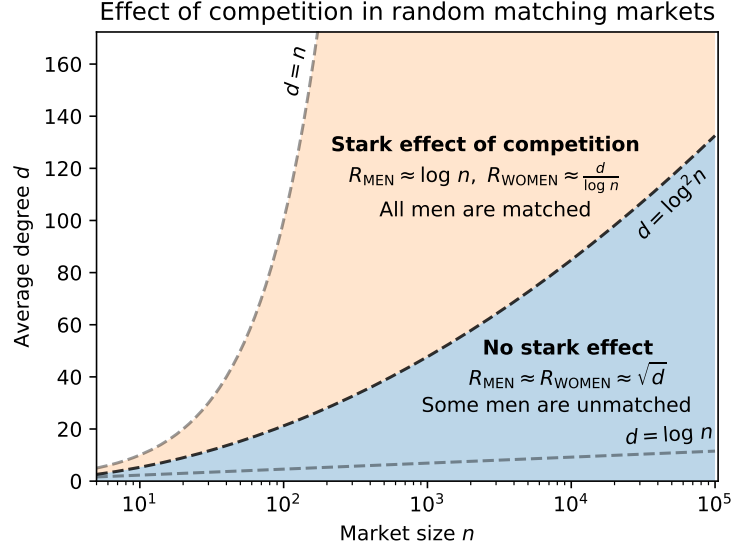
Figure 2: Schematic showing the two "competitiveness" regimes for partially connected random matching markets with small imbalance, n women, men being on the short side, and connectivity (average preference list length) d. R_{MEN} denotes the average rank of men for their wives, and R_{WOMEN} the average rank of women for their husbands.

men remains unmatched with high probability. The threshold turns out to be $\log^2 n$, corresponding to the fact that the maximum number of proposals made by any man in the fully connected random market is $\Theta(\log^2 n)$; see Pittel (2019a).

But does a few agents remaining unmatched have any bearing on the stark effect of competition phenomenon? A priori it is unclear that this should be the case. After all, it is easy to construct matching markets such that some short side agents remain unmatched, but where the short side is nevertheless significantly better off than the long side.[7] Remarkably, we find that in *random matching markets there is no stark effect of competition if some short side agents remain unmatched.* We now provide some informal intuition why random matching markets have this property: Clearly, due to the matching constraint the number of unmatched men must be exactly k plus the number of unmatched women. Hence, assuming a small imbalance k (the balanced market with $k = 0$ being a special case), the number of unmatched agents on the two sides must be nearly the same. But the number of unmatched men should grow with R_{MEN} (the more proposals men need to make in men-proposing DA, the larger the number of men that will reach the end of their preference list), whereas the number of unmatched women should similarly grow with R_{WOMEN} (e.g., one can

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[7]For instance, consider a densely connected random matching market with the modification that a few agents on the short side have empty (or short) preference lists. The latter agents will remain unmatched, but the short side agents will nevertheless have a much smaller average rank for the their partners than the long side agents.

consider women-proposing DA, and assume — or prove separately — that, as usual, the WOSM is close to the MOSM). We deduce that we should have $R_{\text{MEN}} \approx R_{\text{WOMEN}}$ in the $d \ll \log^2 n$ regime, i.e., there is almost no advantage from being on the short side of the market.

Next, we provide more detailed quantitative intuition leading informally to the sharp estimates of R_{MEN} and R_{WOMEN} in our characterization of moderately connected markets. This intuition is based on a detailed heuristic picture of the stable outcome in a random matching market (we do not formalize the full detailed picture in this paper, and instead prove our main theorem via a "shortcut" described below). Intuitively, both R_{MEN} and the number of unmatched men δ^m should be governed by the (endogenous) probability p_{MEN} that a neighboring woman j (independently of other women) is "interested" in given man i (the woman j is said to be interested if she receives no proposal which she prefers to i): in particular, the rank of man i for his wife (his most preferred woman who accepts his proposal) should be distributed as Geometric(p_{MEN}) truncated at d, leading to $R_{\text{MEN}} \approx 1/p_{\text{MEN}}$ (assuming $1/p_{\text{MEN}} \ll d$) and $\delta^m \approx n\mathbb{P}(\text{Geometric}(p_{\text{MEN}}) > d) = n(1-p_{\text{MEN}})^{-d} \approx n\exp(-dp_{\text{MEN}})$. Analogously for women, letting p_{WOMEN} denote the (endogeneous) probability that woman j receives a proposal from each neighboring man i, we expect $R_{\text{WOMEN}} \approx 1/p_{\text{WOMEN}}$ and $\delta^w \approx n\exp(-dp_{\text{WOMEN}})$. For k small, we have that both sides must have nearly the same number of unmatched agents $\delta^w \approx \delta^m$ and hence $p_{\text{WOMEN}} \approx p_{\text{MEN}}$ and $R_{\text{MEN}} \approx R_{\text{WOMEN}}$. But we can further get quantitative estimates: the average number of proposals received by women is nearly the same as the average number of proposals made by men $(n+k)R_{\text{MEN}}/n \approx R_{\text{MEN}} \approx 1/p_{\text{MEN}}$, and since $p_{\text{WOMEN}} \approx$ average number of proposals received/(typical length of preference list) $\approx 1/(dp_{\text{MEN}})$. We deduce that $p_{\text{MEN}} \approx p_{\text{WOMEN}} \approx \frac{1}{\sqrt{d}}$ and so $R_{\text{MEN}} \approx R_{\text{WOMEN}} \approx \sqrt{d}$ and $\delta^m \approx \delta^w \approx ne^{-\sqrt{d}}$.

Technical contributions. Our characterization of the stable matching in partially connected random matching markets (as a function of connectivity d) is novel: stable matchings have not been previously characterized either in balanced or in unbalanced random markets under partial connectivity $d < n$ and $d = \omega(1)$. (A few papers have studied the extreme case of sparsely connected markets $d = \Theta(1)$ under various preference models; see Section 1.1.)

Our characterization showing a stark effect of competition in densely connected markets $d = \omega(\log^2 n)$ is proved via an analysis similar to Ashlagi et al. (2017). In contrast, our characterization showing no stark effect of competition in moderately connected markets $d = o(\log^2 n)$ and $d = \omega(1)$ overcomes significant technical difficulties via a novel approach as we now describe.

The main challenge we face relative to previous works studying fully connected markets (e.g., Ashlagi et al. 2017, Knuth 1976) is the complexity in the way that DA terminates when there is a

positive (but vanishing) fraction of unmatched agents on both sides of the market. Note that DA terminates when the number of women who have received at least one proposal equals the number of men who have not exhausted their lists. In the large d regime (including fully connected markets as studied in the prior literature), it is likely, assuming that men are on the short side ($k < 0$), that the event of $n + k$ distinct women each receiving at least one proposal happens before any man reaches the bottom of his list. As a result, the total number of proposals in DA can be well approximated by the solution of the coupon collector's problem where $n + k$ distinct coupons must be drawn, and key properties of the MOSM can be deduced from there. In the small to medium d regime, however, it is likely that a positive number of men have reached the end of their lists during the run of DA. As a result, to estimate the number of proposals of DA, it is necessary to get a handle on the number of men who have been rejected d times, which is considerably more complicated to analyze, especially for $d = \omega(1)$ as we consider, where the fraction of unmatched men is positive but vanishing (previous works, especially Immorlica and Mahdian (2005), have developed a machinery to handle the case of $d = \Theta(1)$ which leads to a $\Theta(1)$ fraction of unmatched agents). One of our technical contributions is resolving this difficulty. Instead of proving the detailed heuristic picture given in the previous paragraph, we control two main quantities: (i) the total number of proposals before DA terminates (this quantity is the one tracked in the related literature), and (ii) the number of unmatched men and women when DA terminates. The matching constraint tells us that the number of unmatched men is exactly k plus the number of unmatched women. Thus, to control (ii) it suffices to control the number of unmatched men. We estimate (bound) this quantity by constructing a "fake" process where a man who is accepted and then later rejected is allowed to make d additional proposals. It turns out this process is much easier to analyze and it yields a sufficiently good estimate of the number of men who end up unmatched under the assumption $d = o(\log^2 n)$.

1.1 Related work

The closest papers to our work are the ones studying random matching markets Knuth (1976), Pittel (1989), Knuth et al. (1990), Pittel (1992), Ashlagi et al. (2017), Pittel (2019a). All of these papers assume *complete* preference lists. Whereas the early papers focused on balanced random markets and found that the proposing side (in DA) has a substantial advantage, Ashlagi et al. (2017) and follow up papers found that in unbalanced markets, the short side has a substantial advantage. The main technical difficulty we face relative to these papers is that a positive number of agents remain

unmatched on *both* sides of the market in moderately connected markets $d = o(\log^2 n)$, preventing us from directly leveraging the analogy with the coupon collector problem as in the previous works.

Notable papers by Immorlica and Mahdian and others (Immorlica and Mahdian 2005, Kojima and Pathak 2009) show a small core (i.e., a small set of stable matchings) while working with short (constant-sized) preference lists, leading to a linear fraction of unmatched agents. Arnosti (Arnosti 2015) and Menzel (Menzel 2015) characterize the (nearly unique) stable outcome in settings with constant-sized preference lists, and in particular, we expect their characterizations can be used to show that the outcome changes "smoothly" as a function of the market imbalance under short lists. In contrast to the aforementioned papers, our work restricts attention to the case $d = \omega(1)$ and indeed identifies the existence of a threshold at $d \sim \log^2 n$, as a result of which the fraction of unmatched agents in our setting is vanishing. Technically, the consequence of this phenomenon is that "rejection chains" in the progress of DA are $\omega(1)$ in length in our work, making them harder to analyze, and the (approximate) system "state" no longer has bounded dimension as in Arnosti (2015).

Our work belongs to a vast theoretical literature on matching markets, which began with the work of Gale and Shapley (Gale and Shapley 1962) introducing stable matching and the deferred acceptance algorithm, and has developed over the last six decades with major contributions by Roth, Sotomayor, and a large number of other prominent researchers (see, e.g., Roth and Sotomayor 1990, David 2013). Key combinatorial properties of stable matchings are extremely well understood for multiple decades now, and more recently, it has been generally accepted that in typical matching markets, the man optimal stable matching is nearly the same as the woman optimal stable matching Immorlica and Mahdian (2005), Kojima and Pathak (2009), Ashlagi et al. (2017), allowing one to talk about *the* stable matching in typical settings.

What still remains troublingly mysterious is the *nature* of the stable matching as a function of market primitives, especially in settings where there is a significant idiosyncratic/horizontal component to preferences and preference lists are not short (when there is a strong vertical component to preferences, the outcome is known to be approximately assortative, e.g., see Legros and Newman (2007)). Ashlagi et al. (2017) suggested that the outcome depends heavily on which side is the short side of the market, but innumerable datasets and the present theoretical work indicate that this is *not* the case in typical markets. The present paper aims to explain the relative lack of competitiveness of typical matching markets, and overall to take a small step towards a better understanding of how the stable matching depends on market primitives. Reasoning based on Ashlagi et al. (2017) has the potential to lead theorists (and perhaps practitioners) astray, given that we often want to

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derive operational insights, e.g., which tie breaking rule to use Ashlagi and Nikzad (2016), based on the analysis of models resembling the random matching market model.

There is a robust and growing body of practical work on designing real world matching markets, especially in the contexts of school and college admissions (e.g., Rios et al. 2019, Abdulkadiroglu et al. 2005, Dur et al. 2018), and various labor markets (e.g., Roth and Peranson 1999, Hassidim et al. 2017b). Stability, namely, that no pair of agents should prefer to match with each other, has been found to be crucial in the design of centralized clearinghouses (Roth 1991) and predictive of outcomes in decentralized matching markets (Kagel and Roth 2000, Hitsch et al. 2010). We are not aware of any real world matching dataset in which the short side of the market is vastly better off even if the imbalance is small. It further appears that most practitioners are aware that a platform operator cannot make one side of the market vastly better off by slightly tilting the market imbalance in favor of that side.

Organization of the paper. In Section 2, we introduce our model of partially connected random matching markets. In Section 3, we state our main theorems (Theorem 1 and 2) and discuss them. An overview of our proof of our characterization of moderately connected markets (Theorem 1) is provided in Section 4. In Section 5, we provide the simulation results that confirm and sharpen our theoretical predictions. Formal proofs are relegated to the appendix.

2   Model

We consider a two-sided market that consists of a set of men $\mathcal{M} = \{1, \ldots, n + k\}$ and a set of women $\mathcal{W} = \{1, \ldots, n\}$. Here k is a positive or negative integer, which we call the *imbalance*.

We fix a positive integer $d \leq n$ which we call the *connectivity* (or *average degree*) of the market. Each man i has a strict preference list $\succ_i$ over a uniformly random subset $\mathcal{W}_i \subset \mathcal{W}$ of $|\mathcal{W}_i| = d$ women (from among the $\binom{n}{d}$ possibilities), where the subsets $\mathcal{W}_i$ are drawn independently across men. Each woman j has strict preferences $\succ_j$ over only the men who include her in their preference list[8]

$$\mathcal{M}_j = \{i \in \mathcal{M} : j \in \mathcal{W}_i\}.$$

A *matching* is a mapping μ from $\mathcal{M} \cup \mathcal{W}$ to itself such that for every $i \in \mathcal{M}$, $\mu(i) \in \mathcal{W} \cup \{i\}$, and

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[8]Equivalently, we sample an undirected bipartite random graph G connecting men $\mathcal{M}$ to women $\mathcal{W}$, where each man has degree exactly d and the d neighboring women of each man are selected uniformly at random and independently across men. Given G, for each agent has a strict preference ranking over all his/her neighbors in G and does not rank any other agents.

for every $j \in \mathcal{W}$, $\mu(j) \in \mathcal{M} \cup \{j\}$, and for every $i, j \in \mathcal{M} \cup \mathcal{W}$, $\mu(i) = j$ implies $\mu(j) = i$. We use $\mu(j) = j$ to denote that agent j is unmatched under μ.

A matching μ is *unstable* if there are a man i and a woman j such that $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. A matching is *stable* if it is not unstable.

A *random matching market* is generated by drawing, for each man i, a uniformly random preference list over $\mathcal{W}_i$ (from among the $|\mathcal{W}_i|!$ possibilities), and for each woman j, a uniformly random preference list over $\mathcal{M}_j$, independently across agents.

A stable matching always exists, and can be found using the Deferred Acceptance (DA) algorithm by Gale and Shapley (Gale and Shapley 1962). They show that the men-proposing DA finds the *men-optimal stable matching* (MOSM), in which every man is matched with his most preferred stable woman. The MOSM matches every woman with her least preferred stable man. Likewise, the women-proposing DA produces the women-optimal stable matching (WOSM) with symmetric properties. All of our results will characterize the MOSM. Given the strong evidence from Immorlica and Mahdian (2005), Kojima and Pathak (2009), Ashlagi et al. (2017) and other works that the MOSM and WOSM are nearly the same in typical matching markets (with the exception of balanced and densely connected random markets, which we avoid by assuming $k < 0$ in Theorem 2), we omit to formally show this fact for our setting in the current version of the paper though we believe it can be done, e.g., using the method developed in Cai and Thomas (2019) (the property MOSM $\approx$ WOSM is found to hold consistently in our numerical simulations of our model).

We are interested in how matched agents rank their assigned partners under stable matching, and in the number of agents who remain unmatched. Denote the rank of woman j in the preference list $\succ_i$ of man i by $\mathrm{Rank}_i(j) \equiv |\{j' : j' \succeq_i j\}|$. A smaller rank is better, and i's most preferred woman has a rank of 1. Symmetrically, denote the rank of i in the preference list of j by $\mathrm{Rank}_j(i)$.

Definition 1. *Given a matching μ, the* men's average rank of wives *is given by*

$$R_{\mathrm{MEN}}(\mu) = \frac{1}{n+k} \left(|\bar{\mathcal{M}}(\mu)|(d+1) \; + \sum_{i \in \mathcal{M} \setminus \bar{\mathcal{M}}(\mu)} \mathrm{Rank}_i(\mu(i)) \right),$$

where $\bar{\mathcal{M}}(\mu)$ is the set of men who are unmatched under μ, and the number of unmatched men *is denoted by $\delta^m(\mu)$, i.e., $\delta^m(\mu) = |\bar{\mathcal{M}}(\mu)|$.*

Similarly, the women's average rank of husbands *is given by*

$$R_{\mathrm{WOMEN}}(\mu) = \frac{1}{n} \left(\sum_{j \in \bar{\mathcal{W}}(\mu)} (|\mathcal{M}_j| + 1) \; + \sum_{j \in \mathcal{W} \setminus \bar{\mathcal{W}}(\mu)} \mathrm{Rank}_j(\mu(j)) \right)$$

where $\bar{\mathcal{W}}(\mu)$ is the set of women who are unmatched under μ, and the number of unmatched women *is denoted by $\delta^w(\mu)$, i.e., $\delta^w(\mu) = |\bar{\mathcal{W}}(\mu)|$.*

(Note here that if an agent is unmatched, we take the rank for the agent to be one more than the length of the agent's preference list.) By the rural hospital theorem Roth (1986), the set of unmatched agents ($\bar{\mathcal{M}}(\mu)$ and $\bar{\mathcal{W}}(\mu)$) is the same in every stable matching μ, and therefore we simply represent the number of unmatched men and women under stable matching by δ^m and δ^w respectively throughout the remainder of paper.

We remark that the only asymmetry in our model is that the lengths of men's preference lists are deterministically d, whereas each woman has $\text{Binomial}(n+k, d/n) \xrightarrow[n\to\infty]{\text{d}} \text{Poisson}(d)$ neighbors where $\xrightarrow{\text{d}}$ denotes convergence in distribution. Since our theoretical analysis will assume $d = \omega(1)$, we have $\text{Poisson}(d) \xrightarrow{\text{p}} d$, i.e., the degree of each woman is also very close to d, and so the asymmetry between the two sides in the model is mainly technical.

3   Results

In this section we state and discuss our main results.

Before stating our results, we restate a main finding of Ashlagi et al. (2017) (Theorem 2 in that paper) on the structure of stable matchings in fully connected random markets. (The statement has been modified —and slightly weakened in the process— with the aim of allowing easy comparison with our main theorems.)

Theorem (Ashlagi et al. (2017), Fully connected markets). *Consider a sequence of random matching markets indexed by n, with $n + k$ men and n women, for $k = k(n) \in [-n/2, -1]$, and complete preference lists on both sides of the market (i.e., connectivity $d = n$). For fixed $\epsilon > 0$, with high probability the following hold for every stable matching[9] μ:*

$$R_{\text{MEN}}(\mu) \le (1 + \varepsilon)\left(\tfrac{n}{n+k}\right)\log\left(\tfrac{n}{|k|}\right),$$

$$R_{\text{WOMEN}}(\mu) \ge \frac{n + k}{1 + (1 + \varepsilon)\left(\tfrac{n}{n+k}\right)\log\left(\tfrac{n}{|k|}\right)},$$

and all men are matched.

The theorem shows that even a slight imbalance in the number of agents on the two sides of the market results in a stark effect on stable outcomes that strongly favors the agents on the short side

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[9]Though our definition of average rank is slightly different from that of Ashlagi et al. (2017), the bounds stated are nevertheless valid for our definition.

of the market: agents on the short side are essentially able to freely *choose* their partners (as Ashlagi et al. (2017) explain, R_{MEN} is nearly the same as it would be under random serial dictatorship by the men), whereas agents on the long side do only a little better than being matched with a random partner. In particular, even with $k = -1$, it holds that $R_{\text{MEN}}(\mu) \leq 1.01 \log n$ and $R_{\text{WOMEN}}(\mu) \geq \frac{0.99n}{\log n}$ in every stable matching, w.h.p. In the present paper, we investigate stable matchings in random *partially connected* matching markets, and compare with the above finding of Ashlagi et al. (2017).

Moderately and sparsely connected markets. In our first main result, we show that the short-side advantage disappears in partially connected markets (with small or zero imbalance) whose connectivity parameter d is below $\log^2 n$.

Theorem 1 (Moderately Connected Markets). *Consider a sequence of random matching markets indexed by n, with $n + k$ men and n women ($k = k(n)$ can be positive or negative or zero), and connectivity (average degree) $d = d(n)$, with $d = \omega(1)$ and $d = o(\log^2 n)$, and[10] $|k| = O(ne^{-\sqrt{d}})$. Then with high probability,[11] we have*

$$\left| R_{\text{MEN}}(\text{MOSM}) - \sqrt{d} \right| \leq d^{0.3},$$

$$\left| R_{\text{WOMEN}}(\text{MOSM}) - \sqrt{d} \right| \leq d^{0.3},$$

$$\left| \log \delta^m - \log \left(ne^{-\sqrt{d}} \right) \right| \leq d^{0.3},$$

$$\left| \log \delta^w - \log \left(ne^{-\sqrt{d}} \right) \right| \leq d^{0.3}.$$

Informally, in large random matching markets with average degree $d = o(\log^2 n)$ and a small imbalance $k = O(n^{1-\epsilon})$, under stable matching we have $R_{\text{MEN}} \approx R_{\text{WOMEN}} \approx \sqrt{d}$ irrespective of which side is the short side, and there are approximately $ne^{-\sqrt{d}} = \omega(1)$ unmatched agents on both sides of the market. Thus there is no short-side advantage and agents on both sides are matched to their $\sqrt{d}$-th ranked partner on average. A significant number of agents are left unmatched even on the short side, in contrast to a fully connected unbalanced matching market where all agents on the short side are matched. Though we only characterize the MOSM in the present version of the paper, we believe the same characterization extends to the WOSM as well. We give an overview of the proof of Theorem 1 in Section 4 and the formal proof in Appendix B.

The main intuition for Theorem 1 is that for $d = o(\log^2 n)$, a positive number of men remain unmatched with high probability, because they reach the end of their preference lists in men-proposing DA (Pittel (2019a) showed that some men need to go $\log^2 n$ deep in their preference lists

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[10]In particular, for arbitrary fixed $\epsilon > 0$, the result holds for any $k = k(n)$ that satisfies $|k(n)| = O(n^{1-\epsilon})$.
[11]Specifically, our characterization holds with probability at least $1 - O(\exp(-d^{1/4})) = 1 - o(1)$.

in the fully connected market). Clearly, the number of unmatched men must be exactly k plus the number of unmatched women. Then, assuming a small imbalance k, the number of unmatched agents on the two sides must be nearly the same. But the number of unmatched men should grow with R_{MEN} (the more men need to propose, the larger the number that will reach the end of their preference lists), whereas the number of unmatched women should similarly grow with R_{WOMEN} (e.g., one can consider women proposing DA, and assume that, as usual, the WOSM is close to the MOSM). We deduce that we should have $R_{\text{MEN}} \approx R_{\text{WOMEN}}$ in the $d \ll \log^2 n$ regime. (Informal quantitative intuition leading to the precise estimates of R_{MEN} and δ^m is provided in the introduction; we avoid reproducing it here.)

We highlight that Theorem 1 encompasses a wide range of connectivity parameters $d = o(\log^2 n)$, which extends far beyond the connectivity threshold $d \approx \log n$ (this is the connectivity threshold in our model, the same as for Erdős-Rényi random graphs). Thus our "no stark effect of competition" result does not require a disconnected or fragmented market. Rather, the result applies even to very well connected markets.[12] This is in sharp contrast to buyer-seller markets, where, roughly, connectivity implies a stark effect of competition, as captured in the following remark.

Remark 1 (Connected buyer-seller markets exhibit a stark effect of competition)**.** *Consider a buyer-seller market where each of $n+k$ sellers is selling one unit of the same commodity, and each of n buyers wants to buy one unit and has value 1 for a unit. A bipartite graph G with sellers on one side and buyers on the other captures which trades are feasible. (This is a special case of the Shapley-Shubik assignment model (Shapley and Shubik 1971).) We say that an unbalanced market with $k > 0$ (or $k < 0$) exhibits a stark effect of competition if, in any equilibrium, all trades occur at price 0 (or 1), i.e., the agents on the short side, namely buyers (sellers), capture all the surplus. Then we know (Shapley and Shubik 1971) that for $k \neq 0$ the market exhibits a stark effect of competition if the following requirement is satisfied:*

$$\mathcal{E} \equiv \{ \text{For each agent } j \text{ on the long side, there exists a matching in } G$$
$$\text{where all short side agents are matched but agent } j \text{ is unmatched} \}.$$

Requirement $\mathcal{E}$ is only slightly stronger than connectivity of G: Suppose, as in our model in Section 2, that each seller is connected to a uniformly random subset of d buyers. Under this stochastic model for G, for any sequence of k such that $1 \leq |k| = O(1)$, event $\mathcal{E}$ occurs (i.e., there is a stark

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[12]For example, with $n = 1,000$, $\log^2 n \approx 48$. Taking $d = 10$ (much less than 48), numerics tell us that 9.6% of pairs of men are within 1 hop of each other (i.e., there is woman who is ranked by both men), and 99.98% of pairs of men are within 2 hops of each other.

effect of competition) for all d exceeding the connectivity threshold at $d = \log n$:

(i) For any $\epsilon > 0$ and $d \geq (1 + \epsilon) \log n$, with high probability, G is connected and moreover, event $\mathcal{E}$ occurs, i.e., there is a stark effect of competition.

(ii) For any $\epsilon > 0$ and $d \leq (1 - \epsilon) \log n$, with high probability, the connectivity graph G is disconnected (in fact a positive number of buyers have degree zero).

Numerical simulations in the Section 5 show that the finding in Theorem 1 holds up extremely well for all $d \lesssim 1.0 \log^2 n$ for realistic values of n (not just asymptotically in n for $d = o(\log^2 n)$). Now $\log^2 n$ is quite large for realistic market sizes (see Figure 2 in the introduction), far in excess of preference list lengths in many real markets: we have $\log^2 n \approx 48$ for $n = 1000$, 85 for $n = 10000$ and 132 for $n = 100000$. In contrast, we have $n \approx 80,000$ for the high school admissions data introduced in the Section 1 and preference lists have length no more than 12 (the average length is only around 6.9), $n \approx 30,000$ for the National Residency Matching Program and preference lists have length only about 11 on average. Thus, real preference list lengths are typically much smaller than $\log^2 n$. Moreover, correlation in preferences should only reduce the effect of competition (e.g., see the evidence in Figure 1), leading us to contend that *the vast majority of real matching markets live in the "no stark effect of competition" regime covered by Theorem 1.* This may explain why, in simulation experiments on real data like the one shown in Figure 1, only a relatively weak effect of competition is observed.

Densely connected markets. Our next main result shows that $d \sim \log^2 n$ is the threshold level of connectivity above which the finding of Ashlagi et al. (2017) holds true, i.e., the short side is markedly better off even in (large) markets with a small imbalance. Moreover, this benefit of being on the short side arises in conjunction with the key property that *all agents on the short side of the market are matched* (an implausible occurrence in real world markets).

Theorem 2 (Densely Connected Markets)**.** *Consider a sequence of random matching markets indexed by n, with $n + k$ men and n women, and connectivity (average degree) $d = d(n)$, with $k = k(n) < 0$ and $|k| = o(n)$, $d = \omega(\log^2 n)$ and $d = o(n)$. Then, with high probability, all men are matched under stable matching, and we have*

$$R_{\text{MEN}}(\text{MOSM}) \leq (1 + o(1)) \log n \,,$$

$$R_{\text{WOMEN}}(\text{MOSM}) \geq (1 + o(1)) \frac{d}{\log n} \,.$$

This result shows that the short-side advantage emerges in densely connected markets even when the imbalance is small (including for an imbalance of one, i.e., $k = -1$). More specifically, when $d = \omega(\log^2 n)$, it predicts that the agents on the short side are matched to their $\log n$-th ranked partner on average whereas the agents on the long side are matched to their $\left(\frac{d}{\log n}\right)$-th ranked partner on average. Theorem 2 smoothly interpolates between the result in AKL (Ashlagi et al. 2017) and our Theorem 1 (though the extremes $d = \Omega(n)$ and $d = \Theta(\log^2 n)$ are not covered by the formal statement in present form): as connectedness d increases, a phase transition happens at $d = \Theta(\log^2 n)$, and the short side advantage starts to emerge for $d = \omega(\log^2 n)$. The magnitude of the advantage increases as the market becomes denser. Combining Theorems 1 and 2, we conclude that, assuming a small imbalance, a short-side advantage exists if and only if a matching market is connected densely enough, and the threshold level of connectivity $d \sim \log^2 n$.

The analysis leading to Theorem 2 is similar to that leading to (Ashlagi et al. 2017, Theorem 2). The number of proposals in men-proposing DA remains unaffected; the only change is that women now have rank lists of approximate length d (instead of length $n + k$), and so, receiving about $\log n$ proposals leads to an average rank of husband of about $d/\log n$. The proof is in Appendix C.

4 Overview of the proof of Theorem 1

This section provides an overview of the proof of Theorem 1, which is our characterization of moderately connected random matching markets. Our proof uses the well-known analogy between DA and the coupon collector problem to bound women's average rank of their husbands, but also encounters and tackles the challenge of tracking the (strictly positive) number of men who have reached the bottom of their preference lists by constructing a novel bound using a tractable stochastic process. The latter challenge did not arise in the setting of Ashlagi et al. (2017) where *all* short side agents are matched under stable matching, and similarly doesn't arise in our "densely connected markets" setting (Theorem 2). Following Ashlagi et al. (2017) and the majority of other theoretical papers on matching markets, we prove our characterizations for large n (and then use numerics to demonstrate that they extend to small n; see Section 5). Alongside an overview of the proof this section provides parenthetical pointers to the relevant formal lemmas; their statements and proofs can be found in Appendix B.

Our analysis tracks the progress of the following McVitie-Wilson (McVitie and Wilson 1971) (sequential proposals) version of the men-proposing Deferred Acceptance algorithm that outputs MOSM (the final outcome is known to be the MOSM, independent of the sequence in which

proposals are made). Under this algorithm, only one man proposes at a time, and "rejection chains" are run to completion before the next man is allowed to make his first proposal. The algorithm takes the preference rankings of the agents as its input.

Algorithm 1 (Man-proposing Deferred Acceptance). *Initialize "men who have entered" $\hat{\mathcal{M}} \leftarrow \phi$, unmatched women $\bar{\mathcal{W}} \leftarrow \mathcal{W}$, the number of proposals $t \leftarrow 0$, the number of unmatched men $\delta^m \leftarrow 0$.*

1. *If $\mathcal{M} \backslash \hat{\mathcal{M}}$ is empty then terminate. Else, let i be the man with the smallest index in $\mathcal{M} \backslash \hat{\mathcal{M}}$. Add i to $\hat{\mathcal{M}}$.*

2. *If man i has not reached the end of his preference list, do $t \leftarrow t + 1$ and man i proposes to his most preferred woman j whom he has not yet proposed. If he is at the end of his list, do $\delta^m \leftarrow \delta^m + 1$ go to Step 1.*

3. *Decision of j:*

   (a) *If $j \in \bar{\mathcal{W}}$, i.e., j is currently unmatched, then she accepts i. Remove j from $\bar{\mathcal{W}}$. Go to Step 1.*

   (b) *If j is currently matched, she accepts the better of her current partner and i, and rejects the other. Set i to be the rejected man and continue at Step 2.*

Principle of deferred decisions. As we are interested in the behavior of Algorithm 1 on a random matching market, we think of the deterministic algorithm on a random input as a randomized algorithm, which is easier to analyze. The randomized, or coin flipping, version of the algorithm does not receive preferences as input, but draws them through the process of the algorithm. This is often called the *principle of deferred decisions*. The algorithm reads the next woman in the preference of a man in step 2 and whether a woman prefers a man over her current proposal in step 3b. No man applies twice to the same woman during the algorithm, and therefore the algorithm never reads previously revealed preferences. In step 2 the randomized algorithm selects the woman j uniformly at random from those to whom man i has not yet proposed. In step 3b, the probability that j prefers i over her current match is $1/(\nu(j) + 1)$ where $\nu(j)$ is the number of proposals previously received by woman j.

Stopping time. Algorithm 1 defines that "time" t ticks whenever a man makes a proposal. First observe that the current number of unmatched men $\delta^m[t] = \delta^m$ at time t, i.e., men who have reached the bottom of their lists and are still unmatched, is *non-decreasing* over time, whereas the current number of unmatched women $\delta^w[t] = |\bar{\mathcal{W}}|$ at time t, i.e., women who have yet to receive their first

proposal, is *non-increasing* over time. The MOSM is found when the number of unmatched men exactly equals the number of unmatched women plus k. We view this total number of proposals τ when DA terminates as a stopping time:

$$\tau = \min\{t \geq 1 : \delta^m[t] = \delta^w[t] + k\}. \tag{1}$$

This total number of proposals τ serves as a key quantity enabling our formal characterization of the MOSM (see Figure 3 for an illustration). On the men's side, the sum of men's rank of wives is
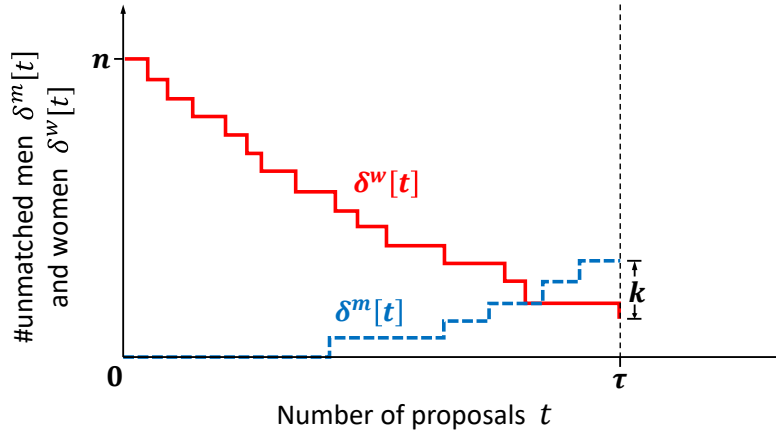

Figure 3: Illustration of a sample path of the current number of unmatched men $\delta^m[t]$ and unmatched women $\delta^w[t]$ under Man-proposing Deferred Acceptance (Algorithm 1). The algorithm terminates at $t = \tau$, the first time $\delta^m[t] = \delta^w[t] + k$. (In this illustration $k > 0$).

approximately the total number of proposals τ (more precisely, this sum is $\tau + \delta^m[\tau]$ given that the rank for an unmatched agent is defined as one more than the length of the agent's preference list, but $\tau \gg \delta^m[\tau]$ is the dominant term). On women's side, since each proposal goes approximately to a uniformly random woman, as a function of the total number of proposals we can tightly control the distribution of the number of proposals received by individual women (this distribution is close to Poisson and tightly concentrates around its average) and therefore their average rank of husbands (Propositions 5 and 6), as well as the number of unmatched women (Propositions 2 and[13] 4).

Therefore, the bulk of the proof of Theorem 1 is dedicated to bounding the total number of proposals τ. Because of the aforementioned technical challenge that a positive number of agents remain unmatched on both sides, a direct application of the coupon collector analogy is not enough. Instead, we control the two stochastic processes that track the current number of unmatched

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[13]In Proposition 4, we first upper bound the number of unmatched women, and then use the aforementioned observation to lower bound the number of proposals.

men $\delta^m[t]$ and unmatched women $\delta^w[t]$ at each time t and make use of the identity (1) that $\delta^m[\tau] = \delta^w[\tau] + k$. (Upon termination, the number of unmatched men must be k plus the number of unmatched women.) For technical purposes, we extend the definition of $\delta^m[t]$ and $\delta^w[t]$ to $t > \tau$ as follows: if there are no men waiting to propose (i.e., a stable matching has been found), we introduce a fake man who is connected to d women (uniformly and independently drawn) with a uniformly random preference ranking over them, and keep running Algorithm 1.

Upper bound on the total number of proposals. We show (in Proposition 1) that the total number of proposals cannot be too large, i.e., $\tau \leq (1 + \epsilon)n\sqrt{d}$ with high probability for $\epsilon = d^{-1/4} = o(1)$. We establish this bound by showing that after a large enough number of proposals have been made, i.e., at time $t = (1 + \epsilon)n\sqrt{d}$, the current number of unmatched women $\delta^w[t]$ has (with high probability) dropped below $ne^{-\sqrt{d}}$ whereas the current number of unmatched men $\delta^m[t]$ has (with high probability) increased above some level which is $\omega(ne^{-\sqrt{d}})$ and hence, since $k = O(ne^{-\sqrt{d}})$, the stopping event $(\delta^m[\tau] = \delta^w[\tau] + k)$ must have happened earlier, i.e., $\tau \leq (1+\epsilon)n\sqrt{d}$. The upper bound on $\delta^w[(1+\epsilon)n\sqrt{d}]$ (see Lemma 7) is derived using a standard approach that utilizes the analogy to the coupon collector problem. The lower bound on $\delta^m[(1+\epsilon)n\sqrt{d}]$ (see Lemma 10) is obtained by counting the number of occurrences of d-rejections-in-a-row during the men-proposing DA procedure (whenever rejections take place d times in a row, at least one man becomes unmatched). Thus, our lower bound on $\delta^m[(1+\epsilon)n\sqrt{d}]$ ignores that some men are first accepted, and then later rejected causing them to reach the end of their preference lists via less than d consecutive rejections. Our conservative approach provides tractability and saves us from needing to track how far down their preference lists the currently matched men are. Nevertheless, the slack in this step necessitates our stronger assumption $d = o(\log^2 n)$, despite our conjecture that the characterization extends for all $d < 0.99 \log^2 n$.

Lower bound on the total number of proposals. We prove (in Proposition 4) that the total number of proposals cannot be too small, i.e., $\tau \geq (1 - \epsilon)n\sqrt{d}$ with high probability for some $\epsilon = o(1)$. We start with upper bounding (in Lemma 13) the expected number of unmatched men in the stable matching, $\mathbb{E}[\delta^m]$, by showing that the probability of the last proposing man being rejected cannot be too large given that each woman has received at most $(1 + \epsilon)\sqrt{d}$ proposals on average (recall that $\tau \leq (1 + \epsilon)n\sqrt{d}$ w.h.p.). We then use Markov's inequality to derive an upper bound on δ^m which holds with high probability, and deduce (in Proposition 3) an upper bound on δ^w using the identity $\delta^m = \delta^w + k$. Then we again use the coupon collector analogy to bound τ from below: the process cannot stop too early since the current number of unmatched women $\delta^w[t]$

does not decay fast enough to satisfy the upper bound on $\delta^w[\tau]$ $(= \delta^w)$ if τ is too small.

5 Numerical Simulations

This section provides simulation results that confirm and sharpen the theoretical predictions made in Section 3. Our simulations reveal (i) a sharp threshold at connectivity $d \approx 1.0 \log^2 n$ with no stark effect of competition observed for d below this threshold, and (ii) that our findings hold even for small values of n. We also investigate the role of imbalance k. Finally, we observe that the connectivity in the actual high school admissions data resembles that in a market with $n = 500$ and $d = 7 \ll \log^2 500 \approx 40$, providing some explanation for why that dataset does not exhibit a stark effect of competition.

We first examine the effect of connectivity on stable matchings in a random matching market of a fixed size. Specifically, we consider a market with 1,000 men and 1,001 women ($n = 1001$, $k = -1$) where the length of each man's preference list d varies from 5 to 150. For each degree d we generate 500 random realizations of matching markets according to the generative model described in Section 2, and for each realization we compute the MOSM via the men-proposing DA algorithm. Figure 4 reports the men's average rank of wives and the women's average rank of husbands (left) and the number of unmatched men and women (right) at each d. While not reported here to avoid cluttering the figures, we observe almost identical results for the WOSM. Observe that when $d < \log^2 n$ both men's average rank and women's average rank are highly concentrated at $\sqrt{d}$ and both the number of unmatched men and the number of unmatched women are close to $ne^{-\sqrt{d}}$, which confirms the estimates in Theorem 1. As d grows beyond $\log^2 n$, the average rank of men and women start to deviate from each other, and specifically, the average rank of short side (men) stops increasing whereas the average rank of long side (women) increases linearly: i.e., $R_{\text{MEN}} \approx \log n$ and $R_{\text{WOMEN}} \approx \frac{d}{\log n}$ when $d > \log^2 n$, confirming Theorem 2. We also remark that the number of unmatched men quickly vanishes as d increases beyond $\log^2 n$ (note that the y-axis of the plot has a log-scale).

The above observation extends to a wide range of market size n (even for small $n \leq 50$). To better illustrate, we investigate three kinds of threshold degree levels $d^*_{\text{rank}}(n)$, $d^*_\delta(n)$, and $d^*_{\text{conn}}(n)$ that sharply characterize the phase transitions that occur when degree d varies in random matching markets of size n. We define these thresholds as follows: given that $k = -1$ as above,

$$d^*_{\text{rank}}(n) = \min_d \left\{ \mathbb{E}_{n,d}[R_{\text{WOMEN}}(\text{MOSM})]/\mathbb{E}_{n,d}[R_{\text{MEN}}(\text{MOSM})] \geq 1.15 \right\}, \tag{2}$$
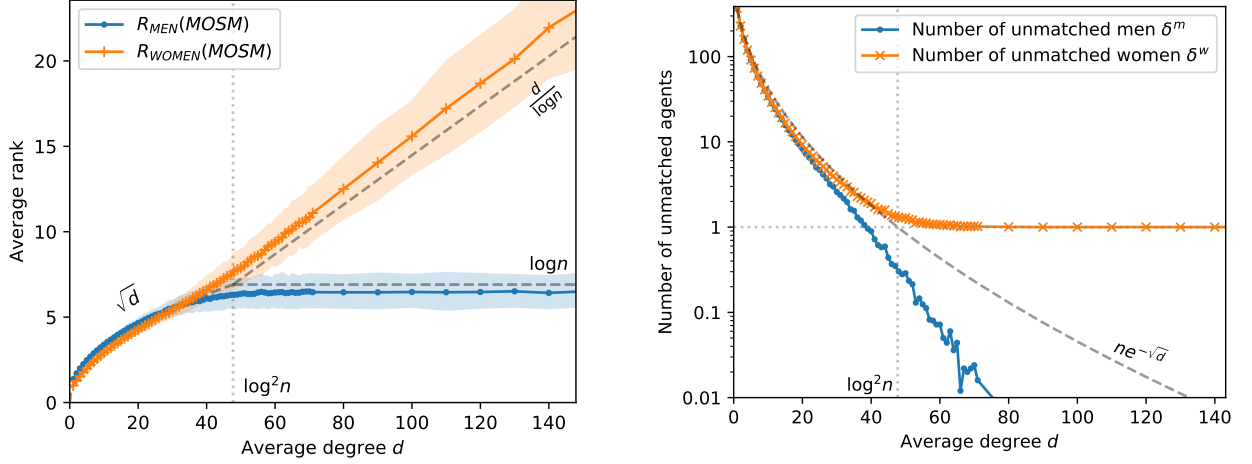
Figure 4: Men's average rank of wives R_{MEN} and women's average rank of husbands R_{WOMEN} (left) and the number of unmatched men δ^m and the number of unmatched women δ^w (right) under MOSM in random matching markets with 1,000 men and 1,001 women ($n = 1001$, $k = -1$), and a varying length of men's preference list d. In both figures, solid lines indicate the average value across 500 random realizations, and gray dashed lines indicate our theoretical predictions (Theorem 1 and 2) annotated with their expressions. In the left figure, the shaded areas surrounding solid lines represent the range between the top and bottom 10th percentiles of 500 realizations of men's and women's average rank.

$$d_\delta^*(n) = \min_d \left\{ \mathbb{E}_{n,d}[\delta^m] \leq 0.5 \right\}, \tag{3}$$

$$d_{\mathrm{conn}}^*(n) = \min_d \left\{ \mathbb{E}_{n,d}[\text{the number of connected components}] \leq 2 \right\}, \tag{4}$$

where $\mathbb{E}_{n,d}[\cdot]$ represents the expected value of some random variable in a random matching market with $n-1$ men each of whose degree is d and n women. The rank-gap threshold $d_{\mathrm{rank}}^*(n)$ indicates the degree value beyond which men's average rank and women's average rank start to deviate from each other (in particular, we require a 15% or larger difference in the average ranks on the two sides of the market); the unmatched-man threshold $d_\delta^*(n)$ is the degree value beyond which all men are (typically) matched; and the connectivity threshold $d_{\mathrm{conn}}^*(n)$ is the degree value beyond which the entire market is typically connected. We quantify these threshold values based on numerical simulations. More specifically, we vary the number of men n from 10 to 2,500, and for each n we use bisection method with a varying d to find the threshold degrees, where the expected values are approximated with sample averages across 500 random realizations. We find that bisection method is adequate since each of the measures on which the above thresholds are defined is observed to monotonically increase or decrease in d, and further to change rapidly near the threshold value d^* that we want to estimate.

Figure 5 plots the measured threshold degrees. Remarkably, the thresholds $d_{\mathrm{rank}}^*(n)$ and $d_\delta^*(n)$

are very close to $\log^2 n$ for all tested values of n. This suggests that our predicted threshold is fairly sharp: the short-side advantage emerges if and only if $d \gtrsim 1.0 \times \log^2 n$. Also note that this threshold is much larger than the connectivity threshold $d^*_{\text{conn}}(n) \approx \log n$.
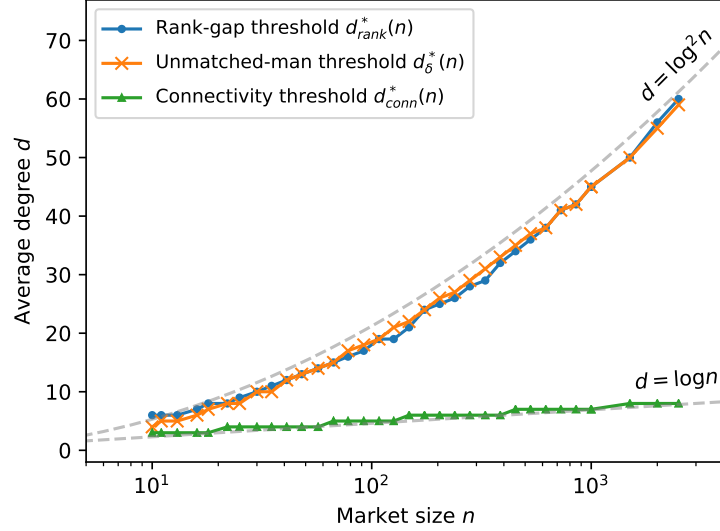

Figure 5: Threshold degrees $d^*_{\text{rank}}(n)$, $d^*_\delta(n)$, and $d^*_{\text{conn}}(n)$, defined in (2)–(4), in random matching markets with $n-1$ men and n women where n ranges from 10 to 2,500. For each n, the threshold values are found using bisection method in which we simulate 500 realizations at each attempted d. The gray dashed lines indicate the theoretical predictions annotated with their expressions.

We next investigate the effect of imbalance k on the stable outcomes and characterize it at the different levels of connectivity d. Analogous to the numerical experiment for the high school admissions discussed in Section 1, we fix the number of women $n = 500$ (so $\log^2 n \approx 40$), and measure men's average rank under MOSM (averaged across 500 realizations) where the number of men varies from 450 to 550. To facilitate easier comparison, we compute the normalized average rank R_{MEN}/d: e.g., $R_{\text{MEN}}/d \approx 0.2$ implies that in average a man is matched to his top-20% most preferred woman out of his preference list. Figure 6 shows how the (normalized) men's average rank changes as we add or remove men in the market, tested with different values of d. Observe that for large $d > \log^2 n$ (e.g., $d = 100, 450$) there is a stark effect when we inject a slight imbalance into the balanced market; compare 500 men vs. 501 men. In contrast, for small $d < \log^2 n$ (e.g., $d = 10, 20$), the stable outcome changes very "smoothly" across a wide range of imbalance, which is consistent with simulation results based on high school admissions data (see Figure 1).

We conclude this section by providing some statistics that illustrate the level of connectivity in the high school admissions example and showing that random matching markets with $n = 500$ and
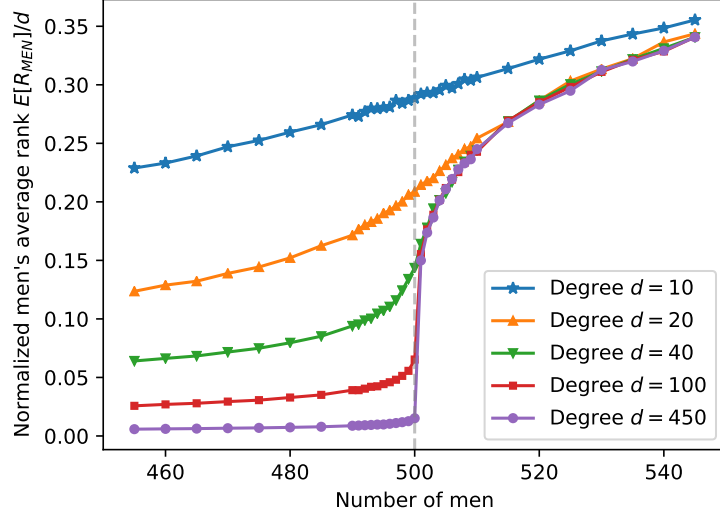
Figure 6: The effect of imbalance k on men's average rank in random matching markets with a fixed number of women $n = 500$ ($\log^2 n \approx 40$). For each $d \in \{10, 20, 40, 100, 450\}$, the corresponding curve reports men's average rank under MOSM normalized by d, i.e., $\mathbb{E}[R_{\text{MEN}}(\text{MOSM})]/d$, where the number of men varies from 450 to 550 (i.e., $k = -50, \ldots, 50$). Each data point reports the average value across 500 realizations.

$d = 7$ exhibit a similar level of connectivity. We focus on the pairwise distance among students as a measure of the connectivity of a matching market: e.g., the distance between two students is one hop if they applied to the same program. On the actual high school admissions data, we sample 1,000 students out of total 75,202 students, and measure the distance from each of selected students to all the other 75,201 students. We observe that 10.1% of student pairs are within 1 hop, 97.8% of pairs are within 2 hops, and 100.0% of pairs are within 3 hops. (Recall that the average preference list length in this high school admissions data was 6.9.) We apply the same analysis on our random matching market model and find that the model with $n = 500$ and $d = 7$ yields a comparable outcome: 9.4% of man pairs are within 1 hop, 98.1% of pairs are within 2 hops, and 100.0% of pairs are within 3 hops. Given that $\log^2 n \approx 40$ for $n = 500$, far in excess of $d = 7$, and since correlation in preferences seems only to reduce the effect of competition (see evidence in Figure 1), we deduce that the high school admissions market seems to lie well within the "no stark effect" regime covered by Theorem 1, which provides an explanation as to why we do not see a stark effect of competition (Figure 1).

6 Discussion

We investigated stable matchings in random matching markets which are partially connected, and asked which random matching markets exhibit a stark effect of competition. In particular, unlike many previous papers which study whether there is a nearly unique stable matching, we focus on the issue of how well (or poorly) agents do under stable matching, as a function of market primitives. The parameter d captured the connectivity (average degree), n captured the market size and k captured the imbalance, whereas preferences were assumed to be uniformly random and independent. We found that, in densely connected markets $d = \omega(\log^2 n)$, the short side of the market enjoys a significant advantage, generalizing the finding of Ashlagi et al. (2017) in fully connected markets. In contrast, in moderately connected markets $d = o(\log^2 n)$, we found that for any $k = o(n)$, the two sides of the market do almost equally well, challenging the claim of Ashlagi et al. (2017) that "*matching markets are extremely competitive*". Notably, this "no stark effect of competition" regime extends far beyond the connectivity threshold of $d = \log n$ and thus includes well connected markets. Numerical simulation results not only support our theory but further indicate that our findings extend to small n and that there is a sharp threshold between the two regimes at $d \approx 1.0 \log^2 n$. We argued informally that most real world matching markets lie in the no stark effect of competition regime, providing some explanation why matching market datasets do not exhibit a stark effect of competition.

Following the theoretical matching literature, we have analyzed a highly stylized model in the interest of tractability and obtaining sharp results. (Even so, we encounter and overcome significant new technical challenges.) We leave as interesting and challenging directions for future work to characterize stable matchings while incorporating various features of real world market such as many-to-one matching, correlation in preferences, and small market sizes.

References

A. Abdulkadiroglu, P. A. Pathak, and A. E. Roth. The New York City high school match. *American Economic Review*, 95(2):364–367, 2005.

N. Arnosti. Short lists in centralized clearinghouses. *EC '15: Proceedings of the Sixteenth ACM Conference on Economics and Computation*, page 751, 2015.

I. Ashlagi and A. Nikzad. What matters in school choice tie-breakings? how competition guides design. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 767–768, 2016.

I. Ashlagi, Y. Kanoria, and J. D. Leshno. Unbalanced random matching markets: The stark effect of competition. *Journal of Political Economy*, 25(1):65–98, 2015.

I. Ashlagi, Y. Kanoria, and J. D. Leshno. Unbalanced random matching markets: The stark effect of competition. *Journal of Political Economy*, 125(1):69–98, 2017.

L. Cai and C. Thomas. The short-side advantage in random matching markets. *arXiv preprint arXiv:1910.04406*, 2019.

M. David. *Algorithmics of matching under preferences*, volume 2. World Scientific, 2013.

D. Dubhashi and D. Ranjan. Balls and bins: A study in negative dependence. *Random Structures & Algorithms*, 13(2):99–124, 1998.

L. E. Dubins and D. A. Freedman. Machiavelli and the Gale-Shapley algorithm. *The American Mathematical Monthly*, 88(7):485–494, 1981.

U. Dur, S. D. Kominers, P. A. Pathak, and T. Sönmez. Reserve design: Unintended consequences and the demise of boston's walk zones. *Journal of Political Economy*, 126(6):2457–2479, 2018.

F. Echenique, R. Gonzalez, A. Wilson, and L. Yariv. Top of the batch: Interviews and the match. *arXiv preprint arXiv:2002.05323*, 2020.

D. Gale and L. S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69(1):9–15, 1962.

A. Hassidim, D. Marciano, A. Romm, and R. I. Shorrer. The mechanism is truthful, why aren't you? *American Economic Review*, 107(5):220–24, 2017a.

A. Hassidim, A. Romm, and R. I. Shorrer. Redesigning the israeli psychology master's match. *American Economic Review*, 107(5):205–09, 2017b.

G. J. Hitsch, A. Hortaçsu, and D. Ariely. Matching and sorting in online dating. *American Economic Review*, 100(1):130–63, 2010.

N. Immorlica and M. Mahdian. Marriage, honesty, and stability. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 53–62. Society for Industrial and Applied Mathematics, 2005.

K. Joan-Dev and F. Proschan. Negative association of random variables with applications. *The Annals of Statistics*, 11(1):286–295, 1983.

J. H. Kagel and A. E. Roth. The dynamics of reorganization in matching markets: A laboratory experiment motivated by a natural experiment. *The Quarterly Journal of Economics*, 115(1):201–235, 2000.

D. E. Knuth. *Mariages stables et leurs relations avec d'autres problemes combinatoires: introduction a l'analysis mathematique des algorithmes-*. Les Presses de l'Université de Montréal, 1976.

D. E. Knuth. Stable marriage and its relation to other combinatorial problems. In *An introduction to the mathematical analysis of algorithms*, volume 10. American Mathematical Society, 1996.

D. E. Knuth, R. Motwani, and B. Pittel. Stable husbands. *Random Structures & Algorithms*, 1(1):1–14, 1990.

F. Kojima and P. A. Pathak. Incentives and stability in large two-sided matching markets. *American Economic Review*, 99(3):608–27, 2009.

S. Lee. Incentive compatibility of large centralized matching markets. *The Review of Economic Studies*, 84(1):444–463, 2016.

P. Legros and A. F. Newman. Beauty is a beast, frog is a prince: Assortative matching with nontransferabilities. *Econometrica*, 75(4):1073–1102, 2007.

D. G. McVitie and L. B. Wilson. The stable marriage problem. *Communications of the ACM*, 14(7):486–490, 1971.

K. Menzel. Large matching markets as two-sided demand systems. *Econometrica*, 83(3):897–941, 2015.

B. Pittel. The average number of stable matchings. *SIAM J. Disc. Math*, 2(4):530–549, 1989.

B. Pittel. On likely solutions of a stable marriage problem. *The Annals of Applied Probability*, pages 358–401, 1992.

B. Pittel. On likely solutions of the stable matching problem with unequal numbers of men and women. *Mathematics of Operations Research*, 44(1):122–146, 2019a.

B. Pittel. On likely solutions of the stable matching problem with unequal numbers of men and women. *Mathematics of Operations Research*, 44(1):122–146, 2019b.

I. Rios, T. Larroucau, G. Parra, and R. Cominetti. Improving the chilean college admissions system. *Oper. Res*, 2019.

A. E. Roth. On the allocations of residents to rural hospitals: A general property of two-sided matching markets. *Econometrica*, 54(2):425–427, 1986.

A. E. Roth. A natural experiment in the organization of entry-level labor markets: regional markets for new physicians and surgeons in the united kingdom. *The American economic review*, pages 415–440, 1991.

A. E. Roth and E. Peranson. The redesign of the matching market for american physicians: Some engineering aspects of economic design. *American Economic Review*, 89(4):748–780, September 1999.

A. E. Roth and M. A. O. Sotomayor. *Two-sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Number 18. Cambridge University Press, Cambridge, UK, 1990.

L. S. Shapley and M. Shubik. The assignment game i: The core. *International Journal of game theory*, 1 (1):111–130, 1971.

Appendix to "Which Random Matching Markets Exhibit a Stark Effect of Competition?"

Organization of the appendix. The technical appendix is organized as follows.

- Appendix A describes several concentration inequalities and auxiliary stochastic processes that will be heavily used in the following theoretical analysis.

- Appendix B establishes Theorem 1, the main result for moderately connected markets. The proof is lengthy and will be further divided into several steps, with an overview provided at the beginning of each step.

- Appendix C establishes Theorem 2, the main result for densely connected markets.

A   Preliminaries

A.1   Basic Inequalities

Lemma 1. *The following inequalities hold:*

- *For any $|x| \leq \frac{1}{2}$, we have $e^{-x-x^2} \leq 1 - x \leq e^{-x}$.*

- *For any $k > 0$ and $\epsilon \in \left(0, \frac{1}{k}\right)$, we have $1 + k\epsilon \leq \frac{1}{1-k\epsilon}$.*

A.2   Negative Association of Random Variables

The concept of negative association provides a stronger notion of negative correlation, which is useful to analyze the concentration of the sum of dependent random variables.

Definition 2 (Negatively Associated Random Variables (Dubhashi and Ranjan 1998))**.** *A set of random variables $X_1, X_2, \ldots, X_n$ are* negatively associated *(NA) if for any two disjoint index sets $I, J \subseteq \{1, \ldots, n\}$,*

$$\mathbb{E}\left[f(X_i : i \in I) \cdot g(X_j : j \in J)\right] \leq \mathbb{E}\left[f(X_i : i \in I)\right] \cdot \mathbb{E}\left[g(X_j : j \in J)\right]$$

for any two functions $f : \mathbb{R}^{|I|} \mapsto \mathbb{R}$ and $g : \mathbb{R}^{|J|} \mapsto \mathbb{R}$ that are both non-decreasing or both non-increasing (in each argument).

The following lemma formalizes that the sum of negatively associated (NA) random variables is as concentrated as the sum of independent random variables:

Lemma 2 (Chernoff-Hoeffding Bound for Negatively Associated Random Variables (Dubhashi and Ranjan 1998))**.** *Let $X_1, X_2, \ldots, X_n$ be NA random variables with $X_i \in [a_i, b_i]$ always. Then, $S \triangleq \sum_{i=1}^{n} X_i$ satisfies the following tail bound:*

$$\mathbb{P}\left(|S - \mathbb{E}[S]| \geq t\right) \leq 2 \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right). \tag{5}$$

We refer to Dubhashi and Ranjan (1998) for the proof.

The following lemma provides sufficient conditions for a set of random variables to be NA. For each sufficient condition, we provide a pointer to a paper where it has been established.

Lemma 3 (Sufficient Conditions for Negative Association)**.** *The followings hold:*

(i) (Permutation distribution (Joan-Dev and Proschan 1983, Theorem 2.11)) *Let $x_1, x_2, \ldots, x_n$ be n real numbers and let $X_1, X_2, \ldots, X_n$ be random variables such that $(X_1, X_2, \ldots, X_n)$ is a uniformly random permutation of $(x_1, x_2, \ldots, x_n)$. Then $X_1, X_2, \ldots, X_n$ are NA.*

(ii) (Union of independent sets of NA random variables (Joan-Dev and Proschan 1983, Property 7)) *If $X_1, X_2, \ldots, X_n$ are NA, $Y_1, Y_2, \ldots, Y_m$ are NA, and $\{X_i\}_i$ are independent of $\{Y_j\}_j$, then $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are NA.*

(iii) (Concordant monotone functions (Joan-Dev and Proschan 1983, Property 6)) *Increasing functions defined on disjoint subsets of a set of NA random variables are NA. More precisely, suppose $f_1, f_2, \ldots, f_k$ are all non-decreasing in each coordinate, or all non-increasing in each coordinate, with each $f_j : \mathbb{R}^{|I_j|} \mapsto \mathbb{R}$ defined on $(X_i)_{i \in I_j}$ for some disjoint index subsets $I_1, \ldots, I_k \subseteq \{1, \ldots, n\}$. If $X_1, X_2, \ldots, X_n$ are NA, then the set of random variables $Y_1 \triangleq f_1(X_i : i \in I_1), Y_2 \triangleq f_2(X_i : i \in I_2), \ldots, Y_k \triangleq f_k(X_i : i \in I_k)$ are NA.*

A.3   Balls-into-bins

A balls-into-bins process with T balls and n bins is defined as follows: at each time $t = 1, \ldots, T$, a ball is placed into one of n bins uniformly at random, independently of the past. Index the bins by $j \in \{1, \ldots, n\}$, and let $I_{j,t}$ be an indicator variable that equals one if the tth ball is placed in the jth bin and equals zero otherwise. Further define $W_j \triangleq \sum_{t=1}^{T} I_{j,t}$ representing the total number of balls placed into the jth bin.

A particular random variable of interest is the number of empty bins at the end of the process, for which we have the following concentration inequality.

Lemma 4 (Number of empty bins)**.** *Let X be the number of empty bins at the end of a balls-into-bins process with T balls and n bins. For any $\epsilon > 0$, we have*

$$\mathbb{P}\left(\frac{1}{n}X - \left(1 - \frac{1}{n}\right)^T \geq \epsilon\right) \leq \exp\left(-2n\epsilon^2\right),$$

$$\mathbb{P}\left(\frac{1}{n}X - \left(1 - \frac{1}{n}\right)^T \leq \epsilon\right) \leq \exp\left(-2n\epsilon^2\right).$$

Proof. Observe that $\{I_{j,t}\}_{j \in \{1,\ldots,n\}, t \in \{1,\ldots,T\}}$ are negatively associated (NA) since $\{I_{j,t}\}_{j \in \{1,\ldots,n\}}$ are NA for each t (by Lemma 3–(i), since $\{I_{j,t}\}_{j \in \{1,\ldots,n\}}$ is a uniformly random permutation of $n-1$ zeros and a single one) and they are independent across t (Lemma 3–(ii)). Consequently, $W_1, \ldots, W_n$ are NA due to Lemma 3–(iii), since $f_j(I_{j,1}, \ldots, I_{j,T}) \triangleq \sum_{t=1}^{T} I_{j,t}$ is non-decreasing in each coordinate.

Define $Y_j \triangleq \mathbb{I}(W_j = 0)$ indicating whether the jth bin is empty at the end. Although Y_j's are not independent, they are NA (again, by Lemma 3–(iii)). Because $Y_j \sim \text{Bernoulli}\left(\left(1 - \frac{1}{n}\right)^T\right)$ and $X = \sum_{j=1}^{n} Y_j$, by applying Hoeffding's bound (Lemma 2), we obtain the desired result. $\square$

Lemma 5. *Let W_j denotes the number of balls in the jth bin at the end of a balls-into-bins process with T balls and n bins. For any $\Delta > 0$, we have*

$$\mathbb{P}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{W_j + 1} \geq \frac{n}{T} + \Delta\right) \leq \exp\left(-2n\Delta^2\right).$$

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Proof. Since $W_j \sim \text{Binomial}\left(T, \frac{1}{n}\right)$, we have

$$\mathbb{E}\left[\frac{1}{W_j+1}\right] = \sum_{k=0}^{T} \frac{1}{k+1} \times \binom{T}{k}\left(\frac{1}{n}\right)^k \left(1-\frac{1}{n}\right)^{T-k}$$

$$= \frac{n}{T+1}\sum_{k=0}^{T}\binom{T+1}{k+1}\left(\frac{1}{n}\right)^{k+1}\left(1-\frac{1}{n}\right)^{(T+1)-(k+1)}$$

$$= \frac{n}{T+1}\times\left(1-\left(1-\frac{1}{n}\right)^{T+1}\right) \leq \frac{n}{T}\,.$$

In the proof of Lemma 4, we have shown that $W_1,\ldots,W_n$ are NA. By Lemma 3–(iii), $\frac{1}{W_1+1},\ldots,\frac{1}{W_n+1}$ are also NA. Therefore, by applying Hoeffding's bound (Lemma 2), we obtain the desired result. □

A.4 Chernoff's Bound on Random Sum

Lemma 6. *Fix any $p \in (0,1)$ and any $p' \in (0,1)$. Define the random sum*

$$S \triangleq \sum_{i=1}^{N} X_i\,,$$

where X_i's are i.i.d. random variables and have distribution[14] Geometric(p), and $N \sim$ Geometric(p') and is independent of X_i's. Let S_i's be i.i.d. random variables and have the same distribution as S, for $\lambda > \mathbb{E}[S] = 1/(pp')$ we have

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}S_i \geq \lambda\right) \leq \exp\left(-\frac{n}{2\lambda^2}\left(\lambda - \mathbb{E}[S]\right)^2\right)\,.$$

Proof. Denote $q \triangleq 1-p$, $q' \triangleq 1-p'$. In the first step, we derive the moment generating function of S, which we denote by $M(t)$. Note that

$$M(t) = \mathbb{E}[e^{tS}] = \mathbb{E}\left[e^{t\sum_{i=1}^{N}X_i}\right] = \mathbb{E}\left[\left[\mathbb{E}e^{tX}\right]^N\right] = \mathbb{E}\left[\gamma^N\right]\,.$$

where

$$\gamma \triangleq \begin{cases} \frac{pe^t}{1-qe^t} & \text{if } qe^t < 1\,, \\ \infty & \text{otherwise}\,. \end{cases} \tag{6}$$

we have

$$\mathbb{E}[\gamma^N] = \sum_{k=1}^{\infty}\gamma^k(1-p')^{k-1}p' = p'\gamma\sum_{k=1}^{\infty}\gamma^{k-1}(q')^{k-1} = \begin{cases} \frac{p'\gamma}{1-\gamma q'} & \text{if } \gamma q' < 1\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

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[14]Here, by Geometric(p) we mean the distribution $\mathbb{P}(X_i = k) = p(1-p)^{k-1}$ for $k \geq 1$, i.e., the support of the distribution is $\{1, 2, \ldots\}$, and its expectation is $1/p > 1$.

By plugging in γ, we obtain

$$M(t) = \begin{cases} \dfrac{p'\frac{pe^t}{1-qe^t}}{1-q'\frac{pe^t}{1-qe^t}} = \dfrac{p'pe^t}{1-e^t(q+q'p)} & \text{if } t < \bar{t} \triangleq \log(1/(q+q'p))\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

Here we used that $q+q'p > q$ to simplify the condition for $M(t)$ to be finite to $e^t(q+q'p) < 1 \Leftrightarrow t < \bar{t}$.

Now we derive the convex conjugate of $\log M(t)$, a.k.a. the large deviation rate function. Note that $\mathbb{E}[S] = 1/(pp')$. Define $\Lambda^* : [1/(pp'), \infty) \to \mathbb{R}$ as

$$\Lambda^*(\lambda) \triangleq \sup_{t \geq 0} (\lambda t - \log M(t)) = \sup_{t \in [0,\bar{t})} (\lambda t - \log M(t))$$

Fix $\lambda \geq 1/(pp')$ and let t^* be the maximizer of the supremum above. The derivative of $\lambda t - \log M(t)$ with respect to t for $t \in [0,\bar{t})$ is

$$\lambda - 1 - \frac{e^t(q+q'p)}{1 - e^t(q+q'p)} = \lambda - \frac{1}{1 - e^t(q+q'p)}\,,$$

and in particular it is decreasing in t, corresponding to the fact that $\lambda t - \log M(t)$ is concave in t (we already knew concavity holds because the log moment generating function is always convex). Note further that the derivative at $t = 0$ is non-negative since

$$\lambda - \frac{1}{1 - (q+q'p)} = \lambda - 1/(pp') \geq 0\,,$$

and that the derivative eventually becomes negative since it tends to $-\infty$ as $t \to \bar{t}^-$. Hence the first order condition will give us the maximizer $t^* \in [0,\bar{t})$ of $\lambda t - \log M(t)$ as follows:

$$\lambda = \frac{1}{1 - e^{t^*}(q+q'p)} \qquad \Rightarrow \qquad e^{t^*} = \frac{1 - \frac{1}{\lambda}}{q+q'p}\,.$$

Therefore, we have

$$\Lambda^*(\lambda) = \lambda \log\left(1 - \frac{1}{\lambda}\right) - \lambda \log\left(q+q'p\right) - \log\left(\frac{p'p\frac{1-\frac{1}{\lambda}}{q+q'p}}{1/\lambda}\right)$$

$$= \lambda \log\left(1 - \frac{1}{\lambda}\right) - \lambda \log\left(q+q'p\right) - \log\left(\lambda - 1\right) + C,$$

where C is a constant. A short calculation tells us that

$$\frac{d\Lambda^*}{d\lambda}(\lambda) = \log\left(1 - \frac{1}{\lambda}\right) - \log\left(q+q'p\right)\,, \qquad \frac{d^2\Lambda^*}{d\lambda^2}(\lambda) = \frac{1}{\lambda(\lambda-1)}\,. \tag{7}$$

Let $S_1, \cdots, S_n$ be i.i.d. random variables with the same distribution as S. Using Chernoff's bound, for $\lambda \geq \mathbb{E}[S]$ we have

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} S_i \geq \lambda\right) \leq \exp\left(-n\Lambda^*(\lambda)\right)\,. \tag{8}$$

Since $\Lambda^*(\cdot)$ is a large deviation rate function, we have that $\Lambda^*(\mathbb{E}[S]) = 0$ and $\frac{d\Lambda^*}{d\lambda}(\mathbb{E}[S]) = 0$. We

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will now use Taylor's theorem taking terms up to second order for $\Lambda^*(\lambda)$ around $\mathbb{E}[S]$ to obtain the desired bound. Note that at any $\lambda' \in (\mathbb{E}[S], \lambda)$, using the explicit form of $\frac{d^2 \Lambda^*}{d\lambda^2}$ in (7) we have

$$\frac{d^2 \Lambda^*}{d\lambda^2}(\lambda') \geq \frac{1}{(\lambda')^2} \geq \frac{1}{\lambda^2},$$

where we used $\mathbb{E}[S] > 1$. Now, using Taylor's theorem, we know that for some $\lambda' \in (0, \lambda)$ we have

$$\Lambda^*(\lambda) = \frac{1}{2} \frac{d^2 \Lambda^*}{d\lambda^2}(\lambda') (\lambda - \mathbb{E}[S])^2 \geq \frac{1}{2\lambda^2} (\lambda - \mathbb{E}[S])^2 .$$

Plugging into (8), we obtain

$$\mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n S_i \geq \lambda \right) \leq \exp \left(-\frac{n}{2\lambda^2} (\lambda - \mathbb{E}[S])^2 \right)$$

as required. $\qquad \square$

A.5   Notations and Preliminary Observations

We here introduce the variables that formally describe the state of a random matching market over the course of the men-proposing deferred-acceptance (MPDA) procedure (Algorithm 1).

The *time t* ticks whenever a man makes a proposal. Let $I_t \in \mathcal{M}$ be the man who proposes at time t, and $J_t \in \mathcal{W}$ be the woman who receives that proposal. We define $M_{i,t} \triangleq \sum_{s=1}^t \mathbb{I}(I_s = i)$ that counts the number of proposals that a man i has made up to time t, and define $W_{j,t} \triangleq \sum_{s=1}^t \mathbb{I}(J_s = j)$ that counts the number of proposals that a woman j has received up to time t. We will often use $\vec{M}_t \triangleq (M_{i,t})_{i \in \mathcal{M}}$ and $\vec{W}_t \triangleq (W_{j,t})_{j \in \mathcal{W}}$ as vectorized notations. By definition, we have

$$\sum_{i \in \mathcal{M}} M_{i,t} = \sum_{j \in \mathcal{W}} W_{j,t} = t,$$

for any $0 \leq t \leq \tau$ where τ is the total number of proposals under MPDA.

Let $\mathcal{H}_t \subseteq \mathcal{W}$ be the set of women that the man I_t had proposed to before time t: i.e., $\mathcal{H}_t \triangleq \{J_s : I_s = i \text{ for some } s \leq t-1\}$ and we have $|\mathcal{H}_t| < d$. According to the principle of deferred decisions, the tth proposal goes to one of women that the man I_t had not proposed to yet: i.e., J_t is sampled from $\mathcal{W} \setminus \mathcal{H}_t$ uniformly at random. And then, the proposal gets accepted by the woman J_t with probability $1/(W_{J_t, t-1} + 1)$.

We denote *the current number of unmatched men* and *women* at time t by $\delta^m[t]$ and $\delta^w[t]$, respectively. More precisely, $\delta^m[t]$ represents the number of men who have exhausted all his preference list but left unmatched[15] at time t: i.e., $\delta^m[t] \triangleq \sum_{i \in \mathcal{M}} \mathbb{I}(M_{i,t} = d, \mu_t(i) = i)$ where μ_t is the current matching at time t. Also note that once a woman receives a proposal, she remains matched until the end of MPDA procedure: i.e., $\delta^w[t] \triangleq \sum_{j \in \mathcal{W}} \mathbb{I}(\mu_t(j) = j) = \sum_{j \in \mathcal{W}} \mathbb{I}(W_{j,t} = 0)$. We observe that $\delta^m[t]$ starts from zero (at $t = 0$) and is *non-decreasing* over time, and $\delta^w[t]$ starts from n and is *non-increasing* over time.

Recall that τ is the *the total number of proposals* that is made until the end of MPDA, i.e., the time at which the men-optimal stable matching (MOSM) is found. MPDA ends when there

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[15]It is important that the definition of $\delta^m[t]$ does not count the men who have not entered the market until time t. In other words, it counts the number of men who are "confirmed" to be unmatched under MOSM, and correspond to the variable δ^m described in Algorithm 1. This quantity is different from the number of unmatched men under the current matching μ_t, which may decrease when a man proposes to a woman who has never received any proposal.

is no more man to make a proposal, i.e., when every unmatched man had already exhausted his preference list. In (1), we expressed τ as a stopping time, namely,

$$\tau = \min\{t \geq 1 : \delta^m[t] = \delta^w[t] + k\}.$$

In particular, we have

$$\delta^m[\tau] = \delta^w[\tau] + k,$$

since the number of matched men equals to the number of matched women under any feasible matching. Furthermore, we have

$$R_{\mathrm{MEN}}(\mathrm{MOSM}) = \frac{\tau + \delta^m[\tau]}{n + k},$$

by the definition of men's rank.

An extended process. We introduce an *extended process* as a natural continuation of the MPDA procedure that continues to evolve even after the MOSM is found (i.e., the extended process continues for $t > \tau$). Recall that the MPDA procedure under the principle of deferred decisions works as follows: As described in Algorithm 1, $n + k$ men in $\mathcal{M}$ sequentially enter the market one by one, and whenever a new man enters, he makes a proposal and the acceptance/rejection process continues until all men who have entered are either matched or have reached the bottom of their preference lists (i.e., until it finds a new MOSM among the men who have entered including the newly entered man).

To define the extended process, we start by defining an extended market, which has the same n women but an infinite supply of men: $n + k$ "real" men $\mathcal{M}$ who are present in the original market, and an infinity of "fake" men $\mathcal{M}_{\mathrm{fake}}$ in addition. The distribution of preferences in the extended market is again as described in Section 2 (in particular, the preference distribution does not distinguish real and fake men). We then define the *extended process* as tracking the progress of Algorithm 1 on the extended market: the $n + k$ real men enter first in Algorithm 1, as before, and we then continue Algorithm 1 after time τ for all $t > \tau$ by continuing to introduce additional (fake) men sequentially after time τ. In particular, the extended process is identical to the original MPDA process until the MOSM is found (i.e., for $t \leq \tau$).

Observe that in this extended process, the MOSM among $\mathcal{M} \cup \mathcal{W}$ can be understood as a stable outcome found after $n + k$ men have entered the market. Therefore, all the aforementioned notations (I_t, J_t, $M_{i,t}$, $W_{j,t}$, $\mathcal{H}_t$, μ_t, $\delta^m[t]$, $\delta^w[t]$) are well-defined for any time $t \geq 0$ while preserving all their properties characterized above, and we similarly denote by $\hat{\mathcal{M}}[t] \subset \mathcal{M} \cup \mathcal{M}_{\mathrm{fake}}$ the set of men who have entered so far (consistent with the notation in Algorithm 1). In the later proofs, we utilize these notations and their properties (e.g., $\delta^m[\tau] \leq \delta^m[t]$ implies that $\tau \leq t$ since $\delta^m[t]$ is non-decreasing over time for $t = 0, 1, \ldots$).

Balls-into-bins process analogy. When we analyze the women side, we heavily utilize the balls-into-bins process as done in Knuth (1976). We make an analogy between the number of proposals that each of n women has received (denoted by $W_{j,t}$) and the number of balls that had been placed into each of n bins. For example, the number of unmatched women at time t corresponds to the number of empty bins after t balls had been placed.

Recall that, according to the principle of deferred decisions, the tth proposal goes to one of women uniformly at random among whom he had not yet proposed to (i.e., $\mathcal{W} \setminus \mathcal{H}_t$), and thus the recipients of proposals, $J_1, J_2, \ldots$, are not independent. In the balls-into-bins process, in contrast, the tth ball is placed into one of n bins uniformly at random, independently of the other balls'

placement. Despite this difference (sampling without replacement v.s. sampling with replacement), the balls-into-bins process provides a good enough approximation as the number of proposals made by an individual man (i.e., $|\mathcal{H}_t|$) is much smaller than the total number of men and women. We will show that (e.g., in Lemma 8 in the next section) that the corresponding error term can be effectively bounded.

B Proof for Small to Medium-Sized d: the case of $d = o(\log^2 n)$, $d = \omega(1)$

In this section, we consider the case such that $d = o(\log^2 n)$ and $d = \omega(1)$. We will prove the following quantitative version of Theorem 1.

Theorem 3 (Quantitative version of Theorem 1). *Consider a sequence of random matching markets indexed by n, with $n + k$ men and n women ($k = k(n)$ can be positive or negative), and the men's degrees are $d = d(n)$. If $|k| = O(ne^{-\sqrt{d}})$, $d = \omega(1)$ and $d = o(\log^2 n)$, then with probability $1 - O(\exp(-d^{\frac{1}{4}}))$ we have*

1. *(Men's average rank of wives)*

$$\left| R_{\mathrm{MEN}}(\mathrm{MOSM}) - \sqrt{d} \right| \leq 6d^{\frac{1}{4}}.$$

2. *(Women's average rank of husbands)*

$$\left| R_{\mathrm{WOMEN}}(\mathrm{MOSM}) - \sqrt{d} \right| \leq 8d^{\frac{1}{4}}.$$

3. *(The number of unmatched men)*

$$\left| \log \delta^m - \log ne^{-\sqrt{d}} \right| \leq 3d^{\frac{1}{4}}.$$

4. *(The number of unmatched women)*

$$\left| \log \delta^w - \log ne^{-\sqrt{d}} \right| \leq 2.5d^{\frac{1}{4}}.$$

The proofs are organized as follows:

- (Section B.1) We first show that with high probability, the stopping time of MPDA (Algorithm 1), namely, τ, is bounded above as $\tau \leq n\left(\sqrt{d} + d^{\frac{1}{4}}\right)$, by utilizing the coupled extended process defined in Section A.5. This yields a high probability upper bound on $R_{\mathrm{MEN}}(\mathrm{MOSM})$ and a lower bound on the number of unmatched men δ^m and unmatched women δ^w.

- (Section B.2) We prove the complementary bounds on $R_{\mathrm{MEN}}(\mathrm{MOSM})$, δ^m, and δ^w: a lower bound on $R_{\mathrm{MEN}}(\mathrm{MOSM})$ and an upper bound on the number of unmatched men δ^m and unmatched women δ^w. To this end, we start by analyzing the rejection chains triggered by the last man to enter in MPDA, and deduce upper bounds on $\mathbb{E}[\delta^m]$ and $\mathbb{E}[\delta^w]$, using the fact that the order in which men enter does not matter. Using Markov's inequality, we then obtain high probability upper bounds on δ^m and δ^w, which lead to lower bounds on τ and $R_{\mathrm{MEN}}(\mathrm{MOSM})$.

- (Section B.3) We construct the concentration bounds on $R_{\mathrm{WOMEN}}(\mathrm{MOSM})$ based upon the concentration results on τ. In this step, we utilizes the balls-into-bins process to analyze the women's side while carefully controlling the difference between the MPDA procedure and the balls-into-bins process. This completes the proof of Theorem 3.

B.1 Step 1: Upper Bound on the Total Number of Proposals τ

We prove the following two propositions.

Proposition 1 (Upper bound on men's average rank)**.** *Consider the setting of Theorem 1. With probability $1 - O(\exp(-\sqrt{n}))$, we have the following upper bounds on the total number of proposals and men's average rank:*

$$\tau \leq n\left(\sqrt{d} + d^{\frac{1}{4}}\right), \qquad R_{\mathrm{MEN}}(\mathrm{MOSM}) \leq \sqrt{d} + 2d^{\frac{1}{4}}.$$

Proposition 2 (Lower bound on the number of unmatched women)**.** *Consider the setting of Theorem 1. With probability $1 - O(\exp(-\sqrt{n}))$, we have the following lower bounds on the number of unmatched men δ^m and unmatched women δ^w:*

$$\delta^m \geq n\exp\left(-\sqrt{d} - 3d^{\frac{1}{4}}\right), \qquad \delta^w \geq n\exp\left(-\sqrt{d} - 2d^{\frac{1}{4}}\right).$$

Throughout the proofs we utilize the extended process defined in Section A.5, which enables us to analyze the state dynamics even after the termination of original DA procedure. Most of the work is in proving Proposition 1, which is done in Sections B.1.1–B.1.4. We then deduce Proposition 2 from Proposition 1 in Section B.1.5. The overall proof structure is as follows:

- (Sections B.1.1 and B.1.2) We first analyze the women side using balls-into-bins process analogy: Given that a sufficient number of proposals have been made (in particular, for $t = (1 + \epsilon)n\sqrt{d}$), we construct a high probability upper bound on the current number of unmatched women $\delta^w[t]$ and the probability p_t of a proposal being accepted.

- (Sections B.1.3 and B.1.4) We then analyze the men side and obtain a lower bound on the current number of unmatched men $\delta^m[t]$ at $t = (1 + \epsilon)n\sqrt{d}$ by utilizing the upper bound on acceptance probability p_t. Since this lower bound exceeds the upper bound on $\delta^w[t]$ (plus k) which holds at the same t, we deduce that, whp, the algorithm has already terminated, $\tau \leq t = (1 + \epsilon)n\sqrt{d}$, since we know that $\delta^m[\tau] = \delta^w[\tau] + k$. See Figure 3 in Section 4 for illustration. Consequently, an upper bound on R_{MEN} follows from the identity $R_{\mathrm{MEN}} = \frac{\tau + \delta^m}{n + k}$, thus completing the proof of Proposition 1.

- (Section B.1.5) Given the upper bound on τ, we obtain a lower bound on δ^w using the balls-into-bins analogy again. This leads to a lower bound on δ^m due to the identity $\delta^m = \delta^w + k$, which completes the proof of Proposition 2.

B.1.1 Upper bound on number of unmatched women after a sufficient number of proposals

The following result formalizes the fact that there cannot be too many unmatched women after a sufficient number of proposals have been made.

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Lemma 7. *Consider the setting of Theorem 1 and the extended process defined in Section A.5. For any $\epsilon \in (0, \frac{1}{2})$ and $n \in \mathbb{Z}_+$, we have*

$$\mathbb{P}\left(\delta^w[(1+\epsilon)n\sqrt{d}] > ne^{-(1+\frac{\epsilon}{2})\sqrt{d}}\right) \leq \exp\left(-\frac{1}{2}nd\epsilon^2 e^{-3\sqrt{d}}\right). \tag{9}$$

In words, after $t = (1+\epsilon)n\sqrt{d}$ proposals have been made, at most $ne^{-(1+\frac{\epsilon}{2})\sqrt{d}}$ women remain unmatched with high probability.

Proof. It is well known that for any $t > 0$, $\delta^w[t]$ is stochastically dominated by the number of empty bins at the end of a balls-into-bins process (defined in Section A.3) with t balls and n bins, which we denote by $X_{t,n}$. (See, e.g., Knuth (1996); the idea is that since men's preference lists sample women without replacement, the actual process has a weakly larger probability of proposing to an unmatched woman at each step relative to picking a uniformly random woman, and hence a stochastically smaller number of unmatched women at any given t.) Therefore we have

$$\mathbb{P}\left(\delta^w[(1+\epsilon)n\sqrt{d}] > ne^{-(1+\frac{\epsilon}{2})\sqrt{d}}\right)$$

$$\leq \mathbb{P}\left(X_{(1+\epsilon)n\sqrt{d},n} > ne^{-(1+\frac{\epsilon}{2})\sqrt{d}}\right)$$

$$= \mathbb{P}\left(\frac{1}{n}X_{(1+\epsilon)n\sqrt{d},n} - \left(1-\frac{1}{n}\right)^{(1+\epsilon)n\sqrt{d}} > e^{-(1+\frac{\epsilon}{2})\sqrt{d}} - \left(1-\frac{1}{n}\right)^{(1+\epsilon)n\sqrt{d}}\right).$$

By Lemma 1 and Lemma 4, we further have

$$\mathbb{P}\left(\delta^w[(1+\epsilon)n\sqrt{d}] > ne^{-(1+\frac{\epsilon}{2})\sqrt{d}}\right)$$

$$\leq \mathbb{P}\left(\frac{1}{n}X_{(1+\epsilon)n\sqrt{d},n} - \left(1-\frac{1}{n}\right)^{(1+\epsilon)n\sqrt{d}} > e^{-(1+\frac{\epsilon}{2})\sqrt{d}} - e^{-(1+\epsilon)\sqrt{d}}\right)$$

$$\leq \exp\left(-2n\left(e^{-(1+\frac{\epsilon}{2})\sqrt{d}} - e^{-(1+\epsilon)\sqrt{d}}\right)^2\right).$$

For $0 < a < b$, using the convexity of function $f(x) = e^{-x}$ we have $e^{-a} - e^{-b} \geq e^{-b}(b-a)$, and therefore for $\epsilon \in (0, \frac{1}{2})$ and any $n \in \mathbb{Z}_+$,

$$\mathbb{P}\left(\delta^w[(1+\epsilon)n\sqrt{d}] > ne^{-(1+\frac{\epsilon}{2})\sqrt{d}}\right) \leq \exp\left(-2nd\left(\epsilon - \frac{\epsilon}{2}\right)^2 e^{-2(1+\epsilon)\sqrt{d}}\right) \leq \exp\left(-\frac{1}{2}nd\epsilon^2 e^{-3\sqrt{d}}\right).$$

This concludes the proof. $\qquad\square$

B.1.2 Upper bound on ex-ante acceptance probability

We define the *ex-ante acceptance probability* as

$$p_t \triangleq \frac{1}{|\mathcal{W} \setminus \mathcal{H}_t|} \sum_{j \in \mathcal{W} \setminus \mathcal{H}_t} \frac{1}{W_{j,t-1}+1}. \tag{10}$$

This is the probability that the tth proposal is accepted after the proposer I_t is declared but the recipient J_t is not yet revealed (recall that I_t is the identity of the man who makes the tth proposal, J_t is the identity of the woman who receives it, and $\mathcal{H}_t$ is the set of women whom I_t has previously proposed to). In the following lemma, we construct a high probability upper bound

on the summation in (10), and the subsequent lemma will use it to obtain an upper bound on $p_{(1+\frac{\epsilon}{2})n\sqrt{d}}$ for small ϵ.

Lemma 8. *For any $\Delta > 0$ and t such that $1 \le t \le nd$, we have*

$$\mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} \ge \frac{n}{t} + \frac{d^2}{n} + \Delta\right) \le 2\exp\left(-\frac{n\Delta^2}{8d}\right).$$

This is also valid for the extended process (i.e., when $t \ge \tau$).

Proof. Consider a balls-into-bins process with t balls and n bins, and let $\tilde{J}_s \in \{1, \dots, n\}$ be the index of bin into which the sth ball is placed, and let $\tilde{W}_{j,t} \triangleq \sum_{s=1}^{t} \mathbb{I}(\tilde{J}_s = j)$ be the total number of balls placed in the jth bin. Recall that $\tilde{J}_s$ is being sampled from $\{1, \dots, n\}$ ($= \mathcal{W}$) uniformly at random.

We make a coupling between the MPDA procedure and the balls-into-bins process as follows: when determining the sth recipient J_s, we take $J_s \leftarrow \tilde{J}_s$ if $\tilde{J}_s \notin \mathcal{H}_s$, or otherwise, sample J_s among $\mathcal{W} \setminus \mathcal{H}_s$ uniformly at random. In other words, the man I_s first picks a woman $\tilde{J}_s$ among the entire $\mathcal{W}$ uniformly at random, and then proposes to her only if he had not proposed to her yet; if he already had proposed before, he proposes to another woman randomly sampled among $\mathcal{W} \setminus \mathcal{H}_s$. It is straightforward that the evolution of the recipient process J_s under this coupling is identical to that under the usual MPDA procedure.

Define $D_t \triangleq \sum_{s=1}^{t} \mathbb{I}(J_s \ne \tilde{J}_s)$ representing the total discrepancy between the MPDA procedure and its coupled balls-into-bins process. Observe that $\mathbb{I}(\tilde{J}_s \ne J_s) = \mathbb{I}\left(\tilde{J}_s \in \mathcal{H}_s\right)$ and thus $\mathbb{P}\left(\tilde{J}_s \ne J_s \middle| \mathcal{F}_{s-1}\right) = \mathbb{P}\left(\tilde{J}_s \in \mathcal{H}_s \middle| \mathcal{F}_{s-1}\right) \le \frac{d}{n}$ where $\mathcal{F}_{s-1}$ represents all information revealed up to time $s-1$. Let $Z_s \triangleq D_s - \frac{d}{n}s$ and observe that $(M_s)_{s \ge 0}$ is a supermartingale with $Z_0 = 0$ and $|Z_{s+1} - Z_s| \le 1$. By Azuma's inequality, we have for any $\Delta_0 > 0$,

$$\mathbb{P}\left(D_t \ge \frac{dt}{n} + \Delta_0\right) \le \mathbb{P}(Z_t - Z_0 \ge \Delta_0) \le \exp\left(-\frac{\Delta_0^2}{2t}\right).$$

On the other hand, since $0 \le \frac{1}{w+1} \le 1$ for any $w \ge 0$, we deduce that

$$\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} - \sum_{j\in\mathcal{W}}\frac{1}{\tilde{W}_{j,t}+1} \le \sum_{j\in\mathcal{W}:W_{j,t}<\tilde{W}_{j,t}}\left(\frac{1}{W_{j,t}+1} - \frac{1}{\tilde{W}_{j,t}+1}\right)$$

$$\le \left|\{j \in \mathcal{W} : W_{j,t} < \tilde{W}_{j,t}\}\right| \le D_t,$$

where the last inequality follows from the fact that in order to observe $W_{j,t} < \tilde{W}_{j,t}$ for some j, at least one mismatch $\{\tilde{J}_s \ne J_s\}$ should take place. Based on the high probability upper bound on D_t obtained above, we have for any $\Delta_1 > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} - \frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{\tilde{W}_{j,t}+1} \ge \frac{dt}{n^2} + \Delta_1\right) \le \mathbb{P}\left(D_t \ge \frac{dt}{n} + n\Delta_1\right) \le \exp\left(-\frac{n^2\Delta_1^2}{2t}\right). \tag{11}$$

We now utilize the result derived for the balls-into-bins process. From Lemma 5, we have for

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any $\Delta_2 > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{\tilde{W}_{j,t}+1} \geq \frac{n}{t} + \Delta_2\right) \leq \exp\left(-2n\Delta_2^2\right).$$

Combined with (11),

$$\mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} \geq \frac{n}{t} + \Delta_2 + \frac{dt}{n^2} + \Delta_1\right)$$

$$\leq \mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} \geq \frac{n}{t} + \Delta_2 + \frac{dt}{n^2} + \Delta_1, \frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{\tilde{W}_{j,t}+1} < \frac{n}{t} + \Delta_2\right)$$

$$+ \mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{\tilde{W}_{j,t}+1} \geq \frac{n}{t} + \Delta_2\right)$$

$$\leq \mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} - \frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{\tilde{W}_{j,t}+1} \geq \frac{dt}{n^2} + \Delta_1\right) + \exp\left(-2n\Delta_2^2\right)$$

$$\leq \exp\left(-\frac{n^2\Delta_1^2}{2t}\right) + \exp\left(-2n\Delta_2^2\right),$$

for any $\Delta_1 > 0$ and $\Delta_2 > 0$.

We are ready to prove the claim. Given any $\Delta > 0$, take $\Delta_1 = \Delta_2 = \Delta/2$. Then,

$$\mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} \geq \frac{n}{t} + \frac{d^2}{n} + \Delta\right)$$

$$\leq \mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} \geq \frac{n}{t} + \frac{dt}{n^2} + \Delta_1 + \Delta_2\right)$$

$$\leq \exp\left(-\frac{n^2\Delta_1^2}{2t}\right) + \exp\left(-2n\Delta_2^2\right)$$

$$= \exp\left(-\frac{n^2\Delta^2}{8t}\right) + \exp\left(-\frac{1}{2}n\Delta^2\right)$$

$$\leq \exp\left(-\frac{n\Delta^2}{8d}\right) + \exp\left(-\frac{1}{2}n\Delta^2\right)$$

$$\leq 2\exp\left(-\frac{n\Delta^2}{8d}\right),$$

where we utilized the fact that $\frac{dt}{n^2} \leq \frac{d^2}{n}$ and $\frac{n}{d} \leq \frac{n^2}{t}$ under the given condition $t \leq nd$. $\qquad\square$

Lemma 9. *Fix any $\alpha \in (0,1)$, $\epsilon < 0.2$ and sequences $(d(n))_{n\in\mathbb{N}}$, and $(\gamma(n))_{n\in\mathbb{N}}$ such that $d = d(n) = \omega(1)$ and $d = o(\log^2 n)$, and $\gamma = \gamma(n) = \Theta\left(n^{-\alpha}\right)$. Define the maximal ex-ante acceptance probability (for any $t \leq nd$) as*

$$\bar{p}_t \triangleq \max_{\mathcal{H}\subset\mathcal{W}:|\mathcal{H}|\leq d}\left\{\frac{1}{|\mathcal{W}\setminus\mathcal{H}|}\sum_{j\in\mathcal{W}\setminus\mathcal{H}}\frac{1}{W_{j,t-1}+1}\right\}. \tag{12}$$

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Then there exists $n_0 < \infty$ such that for all $n > n_0$, we have

$$\mathbb{P}\left(\overline{p}_{(1+\frac{\epsilon}{2})n\sqrt{d}} \geq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\right) \leq 2\exp\left(-\frac{\gamma^2}{32}\frac{n}{d^2}\right).$$

This is also valid for the extended process (i.e., when $(1+\frac{\epsilon}{2})n\sqrt{d} \geq \tau$).

Proof. Let $t \triangleq (1+\frac{\epsilon}{2})n\sqrt{d}$ and $\mathcal{H}^*$ be the maximizer of (12). Observe that $|\mathcal{W} \setminus \mathcal{H}^*| \geq n - d$ and $\sum_{j\in\mathcal{W}\setminus\mathcal{H}^*} \frac{1}{W_{j,t-1}+1} \leq \sum_{j\in\mathcal{W}} \frac{1}{W_{j,t-1}+1}$, and hence

$$\overline{p}_t = \frac{1}{|\mathcal{W} \setminus \mathcal{H}^*|} \sum_{j\in\mathcal{W}\setminus\mathcal{H}^*} \frac{1}{W_{j,t-1}+1} \leq \frac{1}{n-d}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t-1}+1} \leq \frac{1}{n-d}\left(1+\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1}\right).$$

The last inequality uses that at most one of the terms in the summation decreases from $t-1$ to t, and the decrease in that term is less than 1.

Let $r \triangleq \frac{t}{n} = (1+\frac{\epsilon}{2})\sqrt{d}$, and $\Delta \triangleq \frac{\gamma}{2\sqrt{d}}$. Under the specified asymptotic conditions, for n large enough we have

$$r\Delta = (1+\tfrac{\epsilon}{2})\sqrt{d} \cdot \frac{\gamma}{2\sqrt{d}} \leq 0.6\gamma\,, \quad \frac{r}{n} = \frac{(1+\frac{\epsilon}{2})\sqrt{d}}{n} \leq 0.1\gamma\,, \quad \frac{rd^2}{n} \leq \frac{(1+\frac{\epsilon}{2})d^{5/2}}{n} \leq 0.1\gamma\,, \quad \frac{d}{n} \leq 0.1\gamma\,.$$

Consequently, since $\gamma = o(1)$, for large enough n we have

$$\begin{aligned}
\frac{n}{n-d}\left(\frac{1}{r}+\frac{d^2}{n}+\Delta+\frac{1}{n}\right) &= \frac{1}{1-d/n}\cdot\frac{1}{r}\cdot\left(1+\frac{rd^2}{n}+r\Delta+\frac{r}{n}\right) \\
&\leq \frac{1}{r}\cdot\frac{1}{1-0.1\gamma}\cdot(1+0.1\gamma+0.6\gamma+0.1\gamma) \\
&\leq \frac{1}{r}\cdot(1+\gamma) = \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\,.
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbb{P}\left(\overline{p}_{(1+\frac{\epsilon}{2})\tau^*} \geq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\right) &\leq \mathbb{P}\left(\frac{1}{n-d}\left(1+\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1}\right) \geq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\right) \\
&\leq \mathbb{P}\left(\frac{1}{n-d}\left(1+\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1}\right) \geq \frac{n}{n-d}\times\left(\frac{1}{r}+\frac{d^2}{n}+\Delta+\frac{1}{n}\right)\right) \\
&= \mathbb{P}\left(\frac{1}{n}\sum_{j\in\mathcal{W}}\frac{1}{W_{j,t}+1} \geq \frac{n}{t}+\frac{d^2}{n}+\Delta\right) \\
&\leq 2\exp\left(-\frac{n\Delta^2}{8d}\right) = 2\exp\left(-\frac{\gamma^2}{32}\frac{n}{d^2}\right),
\end{aligned}$$

where the last inequality follows from Lemma 8. $\qquad\square$

B.1.3 Lower bound on the number of unmatched men after a sufficient number of proposals

The following result formalizes the fact that there cannot be too few unmatched men after an enough number of proposals have been made.

Lemma 10. *Consider the setting of Theorem 1 and the extended process defined in Section A.5. For any sequence $(\epsilon(n))_{n\in\mathbb{N}}$ such that $\epsilon = \epsilon(n) < 0.2$ and $\epsilon(n) = \omega\left(\frac{1}{n^{0.49}}\right)$, there exists $n_0 < \infty$ such that for all $n > n_0$, we have*

$$\mathbb{P}\left(\delta^m[(1+\epsilon)n\sqrt{d}] \leq \frac{\epsilon}{16}ne^{-(1-\frac{\epsilon}{3})\sqrt{d}}\right) \leq \exp\left(-\sqrt{n}\right). \tag{13}$$

In words, after $(1+\epsilon)n\sqrt{d}$ proposals have been made, at least $\frac{\epsilon}{16}ne^{-(1-\frac{\epsilon}{3})\sqrt{d}}$ men become unmatched with high probability.

Proof. Let $\tau^* \triangleq n\sqrt{d}$. To obtain a lower bound on the number of unmatched men at time $(1+\epsilon)\tau^*$, we count the number of *d-rejection-in-a-row* events that occur during $[(1+\frac{\epsilon}{2})\tau^*, (1+\epsilon)\tau^*]$. This will provide a lower bound since whenever the rejection happens d times in a row the number of unmatched men increases at least by one.

For this purpose, we first utilize the upper bound on the ex-ante acceptance probability. By Lemma 9 we have: given that $\gamma = \gamma(n) = \Theta\left(\frac{1}{n^\alpha}\right)$ for some $\alpha \in (0,1)$, $\epsilon = \epsilon(n) < 0.2$, and that $d = d(n) = \omega(1)$ and $d = o(\log^2 n)$, there exists $n_0 > 0$ such that for all $n > n_0$,

$$\mathbb{P}\left(\overline{p}_{(1+\frac{\epsilon}{2})\tau^*} \geq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\right) \leq 2\exp\left(-\frac{\gamma^2}{32}\frac{n}{d^2}\right). \tag{14}$$

Let $\hat{p} \triangleq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}$ and consider the events where $\overline{p}_{(1+\frac{\epsilon}{2})\tau^*} \leq \hat{p}$ is satisifed. Since $\overline{p}_t$ is non-increasing over time on each sample path, we have $p_t \leq \hat{p}$ for all $t \geq (1+\frac{\epsilon}{2})\tau^*$ on this sample path: i.e., a proposal after time $(1+\frac{\epsilon}{2})\tau^*$ is accepted with probability at most $\hat{p}$.

As an analogy, we imagine a coin tossing process with head probability $\hat{p}$ (which is an exaggeration of the actual acceptance probability, making it underestimate the occurrence of rejections and provides a valid lower bound on the actual number of *d*-rejection-in-a-row events), and count how many times *d*-tail-in-a-row takes place during $\frac{\epsilon}{2}\tau^*$ coin tosses. With $X_i \overset{\text{i.i.d.}}{\sim}$ Geometric$(\hat{p})$ representing the number of coin tosses required to observe one head (acceptance), the total number of coin tosses required to observe one *d*-tail-in-a-row is given by $\sum_{i=1}^{N}\min\{X_i, d\}$ where N is the smallest i such that $X_i > d$. Note that $N \sim$ Geometric$\left((1-\hat{p})^d\right)$. However, N is correlated with X_i's. To upper bound the random sum, observe that conditioned on N, $\{X_1, \cdots, X_{N-1}\}$ are independent *truncated* Geomtric$(\hat{p})$ variables that only take value on $\{1, \cdots, d\}$, which are stochastically dominated by Geomtric$(\hat{p})$ random variables. Since $\min\{X_N, d\} \leq d$, the random sum of interest is stochastically dominated by $d + S$, where $S = \sum_{i=1}^{N'} X_i$, and $N' \sim$ Geometric$\left((1-\hat{p})^d\right)$ independent of X_i's. (Note that by Wald's identity we have $\mathbb{E}[S] = \hat{p}^{-1}(1-\hat{p})^{-d}$.) Consequently, the total number of coin tosses required to observe $\frac{\epsilon}{8}ne^{-d\hat{p}}$ *d*-tail-in-a-row's is stochastically dominated by

$$\sum_{j=1}^{\frac{\epsilon}{8}ne^{-d\hat{p}}}(d + S_j),$$

where $S_1, S_2, \ldots$ are i.i.d. random variables with the same distribution as S defined above.

Let R denote the total number of d-tail-in-a-row events that occur during $[(1 + \frac{\epsilon}{2})\tau^*, (1 + \epsilon)\tau^*]$. From the above argument, we deduce that

$$\mathbb{P}\left(R \leq \frac{\epsilon}{8}ne^{-d\hat{p}}\right) \leq \mathbb{P}\left(\sum_{j=1}^{\frac{\epsilon}{8}ne^{-d\hat{p}}} (d + S_j) \geq \frac{\epsilon}{2}\tau^*\right) = \mathbb{P}\left(\sum_{j=1}^{\frac{\epsilon}{8}ne^{-d\hat{p}}} S_j \geq \frac{\epsilon}{2}\tau^* - \frac{\epsilon}{8}nde^{-d\hat{p}}\right). \quad (15)$$

We now proceed to bound the RHS of (15). Note that

$$\frac{\frac{\epsilon}{2}\tau^* - \frac{\epsilon}{8}nde^{-d\hat{p}}}{\frac{\epsilon}{8}ne^{-d\hat{p}}} = \frac{4n\sqrt{d}}{ne^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}} - d = 4\sqrt{d}e^{\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}} - d.$$

Recall that $\gamma = \Theta\left(\frac{1}{n^\alpha}\right)$, $\epsilon < 0.2$, and $d = \omega(1)$, we have for large enough n, $4\sqrt{d}e^{\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}} - d > 3.9\sqrt{d}e^{\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}$. Plugging $\lambda \triangleq 3.9\sqrt{d}e^{\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}$ into Lemma 6, we obtain

$$\mathbb{P}\left(\sum_{j=1}^{\frac{\epsilon}{8}ne^{-d\hat{p}}} S_j \geq \frac{\epsilon}{2}\tau^* - \frac{\epsilon}{8}nde^{-d\hat{p}}\right) \leq \exp\left(-\frac{\frac{\epsilon}{8}ne^{-d\hat{p}}}{2\lambda^2}(\lambda - \mathbb{E}[S])^2\right)$$

$$\leq \exp\left(-\frac{\epsilon n}{16}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\left(1 - \frac{\mathbb{E}[S]}{\lambda}\right)^2\right). \quad (16)$$

We also have $\hat{p} = \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}} = o(1)$ and thus for large enough n,

$$(1 - \hat{p})^{-d} \leq \left(e^{-\hat{p}-\hat{p}^2}\right)^{-d} = e^{\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}+\left(\frac{1+\gamma}{1+\frac{\epsilon}{2}}\right)^2},$$

where we use the fact that $1 - x \geq e^{-x-x^2}$ for any $|x| \leq 0.5$. Further observe that for large enough n,

$$\frac{1+\frac{\epsilon}{2}}{1+\gamma}e^{\left(\frac{1+\gamma}{1+\frac{\epsilon}{2}}\right)^2} \leq 1.2e < 3.3,$$

and therefore,

$$\mathbb{E}[S] = \hat{p}^{-1}(1-\hat{p})^{-d} \leq \frac{1+\frac{\epsilon}{2}}{1+\gamma}\sqrt{d}e^{\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}+\left(\frac{1+\gamma}{1+\frac{\epsilon}{2}}\right)^2} \leq 3.3\sqrt{d}e^{\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}.$$

For RHS of (16), we deduce that for large enough n,

$$\exp\left(-\frac{\epsilon n}{16}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\left(1 - \frac{\mathbb{E}[S]}{\lambda}\right)^2\right) \leq \exp\left(-\frac{\epsilon n}{16}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\left(1 - \frac{3.3}{3.9}\right)^2\right)$$

$$\leq \exp\left(-\frac{\epsilon n}{800}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right).$$

Combining all these results, for large enough n, we obtain

$$\mathbb{P}\left(R \leq \frac{\epsilon}{8}ne^{-d\hat{p}}\right) \leq \exp\left(-\frac{\epsilon n}{800}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right).$$

As a result, we obtain a high probability lower bound on the number of unmatched men for the

sample paths satisfying $\overline{p}_{(1+\frac{\epsilon}{2})\tau^*} \leq \hat{p}$:

$$\mathbb{P}\left(\delta^m[(1+\epsilon)\tau^*] \leq \frac{\epsilon}{8}ne^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}} \middle| \overline{p}_{(1+\frac{\epsilon}{2})\tau^*} \leq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\right)$$
$$\leq \mathbb{P}\left(R \leq \frac{\epsilon}{8}ne^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right)$$
$$\leq \exp\left(-\frac{\epsilon n}{800}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right).$$

Combining with (14), we obtain

$$\mathbb{P}\left(\delta^m[(1+\epsilon)\tau^*] \leq \frac{\epsilon}{8}ne^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right)$$
$$\leq \mathbb{P}\left(\delta^m[(1+\epsilon)\tau^*] \leq \frac{\epsilon}{8}ne^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}} \middle| \overline{p}_{(1+\frac{\epsilon}{2})\tau^*} \leq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\right) + \mathbb{P}\left(\overline{p}_{(1+\frac{\epsilon}{2})\tau^*} \geq \frac{1+\gamma}{(1+\frac{\epsilon}{2})\sqrt{d}}\right)$$
$$\leq \exp\left(-\frac{\epsilon n}{800}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right) + 2\exp\left(-\frac{\gamma^2}{32}\frac{n}{d^2}\right).$$

Now we take $\gamma = n^{-1/5}$. First observe that, for large enough n, since $d = o(\log^2 n)$, we have

$$2\exp\left(-\frac{\gamma^2}{32}\frac{n}{d^2}\right) = 2\exp\left(-\frac{1}{32}\frac{n^{3/5}}{d^2}\right) \leq \frac{1}{2}\exp\left(-\sqrt{n}\right),$$

and furthermore, since $\epsilon < 0.2$,

$$e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}} \geq e^{-(1+\gamma)(1-\frac{\epsilon}{3})\sqrt{d}} = e^{-(1-\frac{\epsilon}{3})\gamma\sqrt{d}} \cdot e^{-(1-\frac{\epsilon}{3})\sqrt{d}} \geq \frac{1}{2}e^{-(1-\frac{\epsilon}{3})\sqrt{d}}.$$

Therefore, we obtain

$$\mathbb{P}\left(\delta^m[(1+\epsilon)\tau^*] \leq \frac{\epsilon}{16}ne^{-(1-\frac{\epsilon}{3})\sqrt{d}}\right) \leq \mathbb{P}\left(\delta^m[(1+\epsilon)\tau^*] \leq \frac{\epsilon}{8}ne^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right)$$
$$\leq \exp\left(-\frac{\epsilon n}{800}e^{-\frac{1+\gamma}{1+\frac{\epsilon}{2}}\sqrt{d}}\right) + 2\exp\left(-\frac{\gamma^2}{32}\frac{n}{d^2}\right)$$
$$\leq \exp\left(-\frac{\epsilon n}{1600}e^{-(1-\frac{\epsilon}{3})\sqrt{d}}\right) + \frac{1}{2}\exp\left(-\sqrt{n}\right)$$
$$\leq \exp\left(-\frac{\epsilon n}{1600}e^{-\sqrt{d}}\right) + \frac{1}{2}\exp\left(-\sqrt{n}\right).$$

Given that $\epsilon = \omega(\frac{1}{n^{0.49}})$, we further have for large enough n,

$$\frac{\epsilon n}{1600}e^{-\sqrt{d}} \geq \frac{1}{1600}n^{0.51}e^{-\sqrt{d}} \geq \sqrt{n} + \log 2,$$

thus,

$$\exp\left(-\frac{\epsilon}{1600}ne^{-2\sqrt{d}}\right) \leq \frac{1}{2}\exp\left(-\sqrt{n}\right),$$

which concludes the proof. $\qquad\qquad\square$

B.1.4 Upper bound on the total number of proposals τ and men's average rank R_{MEN} (Proposition 1)

With the help of the coupling between the extended process and the men-proposing DA, we are now able to prove Proposition 1.

Proof of Proposition 1. We make use of Lemma 7. Denote $n\sqrt{d}$ by τ^*. Plug $\epsilon = d^{-\frac{1}{4}}$ in (9). For the RHS of (9) we have

$$\exp\left(-\frac{1}{2}nd\epsilon^2 e^{-3\sqrt{d}}\right) = \exp\left(-\frac{1}{2}n\sqrt{d}e^{-3\sqrt{d}}\right) \leq \exp\left(-\frac{1}{2}ne^{-3\sqrt{d}}\right) \leq \exp\left(-\sqrt{n}\right).$$

Here the last inequality holds because $d = o(\log^2 n)$, and it follows that for any $\alpha > 0$, $e^{-3\sqrt{d}} = \omega\left(\frac{1}{n^\alpha}\right)$. Therefore,

$$\mathbb{P}\left(\delta^w[(1+d^{-\frac{1}{4}})\tau^*] > ne^{-\sqrt{d}}\right) \leq \exp\left(-\sqrt{n}\right). \tag{17}$$

We further utilize Lemma 10. Plug $\epsilon = d^{-\frac{1}{4}}$ in (13). For the LHS of (13), because $\frac{1}{16}\frac{1}{x}e^{\frac{1}{3}x} \geq e^{\frac{1}{4}x}$ for large enough x, we have for large enough n,

$$\frac{\epsilon}{16}ne^{-(1-\frac{\epsilon}{3})\sqrt{d}} = \frac{1}{16}ne^{-\sqrt{d}}d^{-\frac{1}{4}}e^{\frac{1}{3}d^{\frac{1}{4}}} \geq ne^{-\sqrt{d}}e^{\frac{1}{4}d^{\frac{1}{4}}},$$

and hence

$$\mathbb{P}\left(\delta^m[(1+\epsilon)\tau^*] \leq ne^{-\sqrt{d}}e^{\frac{1}{4}d^{\frac{1}{4}}}\right) \leq \mathbb{P}\left(\delta^m[(1+\epsilon)\tau^*] \leq \frac{\epsilon}{16}ne^{-(1-\frac{\epsilon}{3})\sqrt{d}}\right) \leq \exp\left(-\sqrt{n}\right). \tag{18}$$

Note that by assumption on the imbalance k, i.e., $|k| = O(ne^{-\sqrt{d}})$, there exists some constant C such that $|k| \leq Cne^{-\sqrt{d}}$ for large enough n. Consequently, since $C + 1 \leq e^{\frac{1}{4}d^{\frac{1}{4}}}$ for large enough d (and hence for large enough n as $d = \omega(1)$), we have for large enough n,

$$|k| \leq Cne^{-\sqrt{d}} \leq ne^{-\sqrt{d}}\left(e^{\frac{1}{4}d^{\frac{1}{4}}} - 1\right).$$

Recall that τ is the smallest t such that

$$\delta^m[t] - \delta^w[t] = k,$$

where the process $\delta^m[t] - \delta^w[t]$ is non-decreasing over time. Therefore, we have

$$\begin{aligned}
\mathbb{P}\left(\tau \geq (1+d^{-\frac{1}{4}})\tau^*\right) &\leq \mathbb{P}\left(\delta^m[(1+d^{-\frac{1}{4}})\tau^*] - \delta^w[(1+d^{-\frac{1}{4}})\tau^*] \leq k\right) \\
&= \mathbb{P}\left(\delta^m[(1+d^{-\frac{1}{4}})\tau^*] - \delta^w[(1+d^{-\frac{1}{4}})\tau^*] \leq k, \ \delta^w[(1+d^{-\frac{1}{4}})\tau^*] \leq ne^{-\sqrt{d}}\right) \\
&\quad + \mathbb{P}\left(\delta^m[(1+d^{-\frac{1}{4}})\tau^*] - \delta^w[(1+d^{-\frac{1}{4}})\tau^*] \leq k, \ \delta^w[(1+d^{-\frac{1}{4}})\tau^*] > ne^{-\sqrt{d}}\right) \\
&\leq \mathbb{P}\left(\delta^m[(1+d^{-\frac{1}{4}})\tau^*] \leq ne^{-\sqrt{d}} + k\right) + \mathbb{P}\left(\delta^w[(1+d^{-\frac{1}{4}})\tau^*] > ne^{-\sqrt{d}}\right) \\
&\leq \mathbb{P}\left(\delta^m[(1+d^{-\frac{1}{4}})\tau^*] \leq ne^{-\sqrt{d}}e^{\frac{1}{4}d^{\frac{1}{4}}}\right) + \mathbb{P}\left(\delta^w[(1+d^{-\frac{1}{4}})\tau^*] > ne^{-\sqrt{d}}\right) \\
&\leq 2\exp\left(-\sqrt{n}\right),
\end{aligned}$$

where we made use of (17) and (18) in the last step.

As a result, when the imbalance satisfies $|k| = O(ne^{-\sqrt{d}})$, with probability $1 - O(\exp(-\sqrt{n}))$, we have

$$\tau \leq n\left(\sqrt{d} + d^{\frac{1}{4}}\right).$$

By definition of $R_{\text{MEN}}(\text{MOSM})$, we have

$$R_{\text{MEN}}(\text{MOSM}) = \frac{\tau + \delta^m}{n + k} \leq \frac{\tau + n}{n + k}.$$

Hence for $\tau \leq n\left(\sqrt{d} + d^{\frac{1}{4}}\right)$, we have for large enough n,

$$R_{\text{MEN}}(\text{MOSM}) \leq \frac{n}{n + k}\left(\sqrt{d} + d^{\frac{1}{4}} + 1\right) \leq \left(1 + 0.5d^{-\frac{1}{4}}\right)\left(\sqrt{d} + d^{\frac{1}{4}} + 1\right) \leq \sqrt{d} + 2d^{\frac{1}{4}},$$

where we utilized the fact that $\frac{n}{n+k} \leq \frac{n}{n-|k|} \leq \frac{1}{1-Ce^{-\sqrt{d}}} \leq 1+2Ce^{-\sqrt{d}} \leq 1+0.5d^{-\frac{1}{4}}$ for large enough d.

$\square$

B.1.5 Lower bounds on the number of unmatched women δ^w and unmatched men δ^m (Proposition 2)

We now derive a lower on the number of unmatched women δ^w. Similar to the proof of Lemma 7, we again make an analogy between balls-into-bins process and DA procedure, but we now consider a variation of balls-into-bins process that exaggerates the effect of "sampling without replacement" as opposed to the original balls-into-bins process that assumes sampling with replacement. The lower bound on the number of empty bins in this process provides a lower bound on the number of unmatched women δ^w, which immediately leads to a lower bound on the number of unmatched men δ^m by the identity $\delta^m = \delta^w + k$.

Lemma 11. *For any $t \geq d$ and $\Delta > 0$, we have*

$$\mathbb{P}\left(\frac{\delta^w[t]}{n - d} - \left(1 - \frac{1}{n - d}\right)^{t-d} \leq -\Delta\right) \leq \exp\left(-2(n - d)\Delta^2\right).$$

This is also valid for the extended process defined in Section A.5.

Proof. Note that the tth proposal goes to a woman chosen uniformly at random after excluding the set of women $\mathcal{H}_t$ that the man has previously proposed to. Therefore,

$$\mathbb{P}\left(t^{\text{th}} \text{ proposal goes to one of unmatched women}\Big| \delta^w[t-1], \mathcal{H}_t\right) = \frac{\delta^w[t-1]}{n - |\mathcal{H}_t|} \leq \frac{\delta^w[t-1]}{n - d},$$

since $|\mathcal{H}_t| \leq d$. Consider a process $\underline{\delta}^w[t]$ defined as

$$\underline{\delta}^w[t] = \underline{\delta}^w[t-1] - X_t \quad \text{where} \quad X_t \sim \text{Bernoulli}\left(\min\left\{\frac{\delta^w[t-1]}{n - d}, 1\right\}\right).$$

Since the process $\underline{\delta}^w[t]$ exaggerates the likelihood of an unmatched woman receiving a proposal and hence exaggerates the likelihood of decrementing by 1 at each level, $\delta^w[t]$ stochastically dominates $\underline{\delta}^w[t]$: i.e., $\mathbb{P}\left(\delta^w[t] \leq x\right) \leq \mathbb{P}\left(\underline{\delta}^w[t] \leq x\right)$ for all $x \in \mathbb{N}$. We also observe that $\underline{\delta}^w[t]$ counts the number

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of empty bins in a process (we refer to it below as the original process) similar to balls-into-bins process where d bins are occupied during the first d periods, and then the regular balls-into-bins process begins with $n - d$ empty bins. Consider Lemma 4 applied to the "modified" balls-into-bins process of putting t' balls into $n - d$ bins, where the bins correspond to those which are not occupied by the first d balls in the original process, and t' is the total number of balls which go into these bins in the original process up to t. Clearly, $t' \leq t - d$, since the first d balls do not go into these bins. We hence deduce from Lemma 4 that

$$\mathbb{P}\left(\frac{\delta^w[t]}{n-d} - \left(1 - \frac{1}{n-d}\right)^{t-d} \leq -\Delta\right) \leq \mathbb{P}\left(\frac{\delta^w[t]}{n-d} - \left(1 - \frac{1}{n-d}\right)^{t'} \leq -\Delta\right)$$

$$\leq \exp\left(-2(n-d)\Delta^2\right).$$

$\square$

Lemma 12. *Consider the setting of Theorem 1 and the extended process defined in Section A.5. Then there exists $n_0 < \infty$ such that for all $n > n_0$, we have the following lower bounds on the number of unmatched women:*

$$\mathbb{P}\left(\delta^w[(1 + d^{-\frac{1}{4}})n\sqrt{d}] \leq ne^{-(1+2d^{-\frac{1}{4}})\sqrt{d}}\right) \leq \exp\left(-\sqrt{n}\right), \tag{19}$$

$$\mathbb{P}\left(\delta^w[(1 - 5d^{-\frac{1}{4}})n\sqrt{d}] \leq ne^{-(1-2.5d^{-\frac{1}{4}})\sqrt{d}}\right) \leq \exp\left(-\sqrt{n}\right). \tag{20}$$

Proof. Let $\tau^* \triangleq n\sqrt{d}$.
Proof of (19). Fix $t = (1 + d^{-\frac{1}{4}})\tau^* = (1 + d^{-\frac{1}{4}})n\sqrt{d}$. For large enough n, we have

$$\frac{d}{n} \leq \frac{d}{e^{\sqrt{d}}} \leq 0.1d^{-\frac{1}{4}}, \quad \frac{t-d}{n-d} \leq \frac{t}{n} \cdot \frac{1}{1 - d/n} \leq \sqrt{d} \cdot \frac{1 + d^{-\frac{1}{4}}}{1 - 0.1d^{-\frac{1}{4}}} \leq \sqrt{d}(1 + 1.2d^{-\frac{1}{4}}).$$

Consequently, with $\Delta \triangleq e^{-(1+2d^{-\frac{1}{4}})\sqrt{d}}$, for large enough n we have

$$\frac{n-d}{n}\left[\left(1 - \frac{1}{n-d}\right)^{t-d} - \Delta\right] \geq \left(1 - \frac{d}{n}\right) \cdot \left[\exp\left(-\frac{t-d}{n-d} - \frac{t-d}{(n-d)^2}\right) - e^{-(1+2d^{-\frac{1}{4}})\sqrt{d}}\right]$$

$$\geq \frac{1}{2} \cdot \left[\exp\left(-\sqrt{d}(1 + 1.2d^{-\frac{1}{4}}) \cdot (1 + 1/(n-d))\right) - e^{-(1+2d^{-\frac{1}{4}})\sqrt{d}}\right]$$

$$\geq \frac{1}{2} \cdot \left[\exp\left(-\sqrt{d}(1 + 1.4d^{-\frac{1}{4}})\right) - \exp\left(-\sqrt{d}(1 + 2d^{-\frac{1}{4}})\right)\right]$$

$$= e^{-\sqrt{d}} \times \frac{1}{2} \cdot \left(\exp\left(-1.4d^{\frac{1}{4}}\right) - \exp\left(-2d^{\frac{1}{4}}\right)\right)$$

$$\geq e^{-\sqrt{d}} \times \frac{1}{2} \cdot e^{-2d^{\frac{1}{4}}} \left(2.0d^{\frac{1}{4}} - 1.4d^{\frac{1}{4}}\right)$$

$$= e^{-\sqrt{d}} \times e^{-2d^{\frac{1}{4}}} \times 0.3d^{\frac{1}{4}} \geq e^{-(1+2d^{-\frac{1}{4}})\sqrt{d}}.$$

In the second last inequality, we utilize the fact that $e^{-a} - e^{-b} \geq e^{-b}(b - a)$ for any $0 < a < b$. Therefore, by Lemma 11,

$$\mathbb{P}\left(\delta^w[t] \leq ne^{-(1+2d^{-\frac{1}{4}})\sqrt{d}}\right) = \mathbb{P}\left(\frac{\delta^w[t]}{n} \leq e^{-(1+2d^{-\frac{1}{4}})\sqrt{d}}\right)$$

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$$\leq \mathbb{P}\left(\frac{\delta^w[t]}{n} \leq \frac{n-d}{n}\left(\left(1-\frac{1}{n-d}\right)^{t-d} - \Delta\right)\right)$$

$$\leq \mathbb{P}\left(\frac{\delta^w[t]}{n-d} - \left(1-\frac{1}{n-d}\right)^{t-d} \leq -\Delta\right)$$

$$\leq \exp\left(-2(n-d)\Delta^2\right) = \exp\left(-2(n-d)e^{-2(1+d^{-\frac{1}{4}})\sqrt{d}}\right).$$

The claim follows from the fact that $2(n-d)e^{-2(1+d^{-\frac{1}{4}})\sqrt{d}} \geq \sqrt{n}$ for large enough n.

Proof of (20). Fix $t = (1 - 5d^{-\frac{1}{4}})\tau^* = (1 - 5d^{-\frac{1}{4}})n\sqrt{d}$. For large enough n, we have

$$\frac{d}{n} \leq \frac{d}{e^{\sqrt{d}}} \leq 0.1d^{-\frac{1}{4}}, \quad \frac{t-d}{n-d} \leq \frac{t}{n} \cdot \frac{1}{1-d/n} \leq \sqrt{d} \cdot \frac{1-5d^{-\frac{1}{4}}}{1-0.1d^{-\frac{1}{4}}} \leq \sqrt{d}(1-4.8d^{-\frac{1}{4}}),$$

Consequently, with $\Delta \triangleq e^{-(1-2.5d^{-\frac{1}{4}})\sqrt{d}}$,

$$\frac{n-d}{n}\left[\left(1-\frac{1}{n-d}\right)^{t-d} - \Delta\right] \geq \left(1-\frac{d}{n}\right) \cdot \left[\exp\left(-\frac{t-d}{n-d} - \frac{t-d}{(n-d)^2}\right) - e^{-(1-2.5d^{-\frac{1}{4}})\sqrt{d}}\right]$$

$$\geq \frac{1}{2} \cdot \left[\exp\left(-\sqrt{d}(1-4.8d^{-\frac{1}{4}}) \cdot (1+1/(n-d))\right) - e^{-(1-2.5d^{-\frac{1}{4}})\sqrt{d}}\right]$$

$$\geq \frac{1}{2} \cdot \left[\exp\left(-\sqrt{d}(1-4.6d^{-\frac{1}{4}})\right) - \exp\left(-\sqrt{d}(1-2.5d^{-\frac{1}{4}})\right)\right]$$

$$= e^{-\sqrt{d}} \times \frac{1}{2} \cdot \left(\exp\left(4.6d^{\frac{1}{4}}\right) - \exp\left(2.5d^{\frac{1}{4}}\right)\right)$$

$$\overset{(a)}{\geq} e^{-\sqrt{d}} \times \frac{1}{2} \cdot e^{2.5d^{\frac{1}{4}}}\left(4.6d^{\frac{1}{4}} - 2.5d^{\frac{1}{4}}\right)$$

$$\geq e^{-\sqrt{d}} \times e^{2.5d^{\frac{1}{4}}} \times d^{\frac{1}{4}} \geq e^{-(1-2.5d^{-\frac{1}{4}})\sqrt{d}},$$

for large enough n. Here (a) follows from the fact that $f(x) = e^x$ is convex hence $f(x_2) - f(x_1) \geq f'(x_1)(x_2 - x_1)$ for $x_2 > x_1$. Therefore, by Lemma 11,

$$\mathbb{P}\left(\delta^w[t] \leq ne^{-(1-2.5d^{-\frac{1}{4}})\sqrt{d}}\right) = \mathbb{P}\left(\frac{\delta^w[t]}{n} \leq e^{-(1-2.5d^{-\frac{1}{4}})\sqrt{d}}\right)$$

$$\leq \mathbb{P}\left(\frac{\delta^w[t]}{n} \leq \frac{n-d}{n}\left(\left(1-\frac{1}{n-d}\right)^{t-d} - \Delta\right)\right)$$

$$\leq \mathbb{P}\left(\frac{\delta^w[t]}{n-d} - \left(1-\frac{1}{n-d}\right)^{t-d} \leq -\Delta\right)$$

$$\leq \exp\left(-2(n-d)\Delta^2\right) = \exp\left(-2(n-d)e^{-2(1-2.5d^{-\frac{1}{4}})\sqrt{d}}\right).$$

The claim follows from the fact that $2(n-d)e^{-2(1-2.5d^{-\frac{1}{4}})\sqrt{d}} \geq \sqrt{n}$ for large enough n. $\qquad\square$

We are now able to prove Proposition 2.

Proof of Proposition 2. By Proposition 1 and the monotonicity of $\delta^w[t]$, we have for large enough

n,

$$\mathbb{P}\left(\delta^w[\tau] \le e^{-2d^{\frac{1}{4}}} n e^{-\sqrt{d}}\right) \le \mathbb{P}\left(\delta^w[\tau] \le e^{-2d^{\frac{1}{4}}} n e^{-\sqrt{d}}, \tau \le (1 + d^{-\frac{1}{4}})\tau^*\right) + \mathbb{P}\left(\tau \ge (1 + d^{-\frac{1}{4}})\tau^*\right)$$

$$\le \mathbb{P}\left(\delta^w[(1 + d^{-\frac{1}{4}})\tau^*] \le e^{-2d^{\frac{1}{4}}} n e^{-\sqrt{d}}\right) + \exp\left(-\sqrt{n}\right). \tag{21}$$

Moreover, by Lemma 12, we have for large enough n,

$$\mathbb{P}\left(\delta^w[(1 + d^{-\frac{1}{4}})\tau^*] \le e^{-2d^{\frac{1}{4}}} n e^{-\sqrt{d}}\right) \le \exp\left(-\sqrt{n}\right).$$

From (21), we conclude that with probability $1 - 2\exp(-\sqrt{n})$,

$$\delta^w \ge n e^{-\sqrt{d} - 2d^{\frac{1}{4}}}.$$

Since $|\delta^m - \delta^w| = |k| \le O(n e^{-\sqrt{d}})$, it follows that with probability $1 - 2\exp(-\sqrt{n})$,

$$\delta^m \ge n e^{-\sqrt{d} - 3d^{\frac{1}{4}}}.$$

$\square$

B.2   Step 2: Lower Bound on the Total Number of Proposals τ

In this section, we prove the following two propositions.

Proposition 3. *Consider the setting in Theorem 1. With probability $1 - O\left(\exp\left(-d^{\frac{1}{4}}\right)\right)$, we have the following upper bounds on the number of unmatched men δ^m and unmatched women δ^w:*

$$\delta^m \le n \exp\left(-\sqrt{d} + 2.5d^{\frac{1}{4}}\right), \qquad \delta^w \le n \exp\left(-\sqrt{d} + 2.5d^{\frac{1}{4}}\right).$$

Proposition 4. *Consider the setting of Theorem 1. With probability $1 - O\left(\exp\left(-d^{\frac{1}{4}}\right)\right)$, we have the following lower bound on the total number of proposals and men's average rank under the men-optimal stable matching:*

$$\tau \ge n\left(\sqrt{d} - 5d^{\frac{1}{4}}\right), \qquad R_{\text{MEN}}(\text{MOSM}) \ge \sqrt{d} - 6d^{\frac{1}{4}}.$$

The proofs of Proposition 3 and 4 have the following structure:

- (Sections B.2.1 and B.2.2) Proof of Proposition 3: We first derive an upper bound on the expected number of unmatched men $\mathbb{E}[\delta^m]$ in Lemma 13, utilizing the fact that the probability of the last proposing man being rejected cannot be too large given that the total number of proposals τ is limited by its upper bound (Proposition 1). We immediately deduce an upper bound $\mathbb{E}[\delta^w]$ by using the identity $\delta^m = \delta^w + k$. The high probability upper bounds on δ^m and δ^w follow by applying Markov's inequality.

- (Section B.2.3) Proof of Proposition 4: We obtain a lower bound on the total number of proposals τ by showing that the current number of unmatched women $\delta^w[t]$ does not decay fast enough (again argued with a balls-into-bins analogy) and hence it will violate the upper bound on $\delta^w[\tau]\ (= \delta^w)$ derived in Proposition 3 if τ is too small. The lower bound on τ immediately translates into the lower bound on $R_{\text{MEN}}(\text{MOSM})$ due to the identity $R_{\text{MEN}}(\text{MOSM}) = \frac{\tau + \delta^m}{n + k}$.

B.2.1 Upper bound on the expected number of unmatched women $\mathbb{E}[\delta^w]$

Using a careful analysis of the rejection chains triggered by the last proposing man's proposal, we are able to derive an upper bound on the expected number of unmatched women.

Lemma 13. *Consider the setting of Theorem 1. There exists $n_0 < \infty$ such that for all $n > n_0$, we have the following upper bounds on the expected number of unmatched men and women under stable matching*

$$\mathbb{E}[\delta^m] \leq n \exp(-\sqrt{d} + 1.4d^{1/4}), \quad \mathbb{E}[\delta^w] \leq n \exp(-\sqrt{d} + 1.5d^{1/4}). \tag{22}$$

Proof. We will track the progress of the man proposing DA algorithm making use of the principle of deferred decisions, and further make use of a particular sequence of proposals: we will specify beforehand an arbitrary man i (before any information whatsoever is revealed), and then run DA to convergence on the other men, before man i makes a single proposal. We will show that the probability that the man i remains unmatched is bounded as

$$\mathbb{P}(\mu(i) = i) \leq \exp(-\sqrt{d} + 1.4d^{1/4}) \tag{23}$$

for large enough n. This will imply that, by symmetry across men, the expected number of unmatched men under stable matching will be bounded above as

$$\mathbb{E}[\delta^m] \leq (n + k) \exp(-\sqrt{d} + 1.4d^{1/4}).$$

Finally the number of unmatched women at the end is exactly $\delta^w = \delta^m - k$, and so

$$\mathbb{E}[\delta^w] = \mathbb{E}[\delta^m] - k \leq (n + k) \exp(-\sqrt{d} + 1.4d^{1/4}) - k \leq n \exp(-\sqrt{d} + 1.5d^{1/4})$$

for large enough n as required, using $k = O(ne^{-\sqrt{d}})$. The rest of proof is devoted to establishing (23).

Using Proposition 1, we have that with probability $1 - O(\exp(-\sqrt{n}))$, at the end of DA, τ is bounded above as

$$\tau \leq n\big(\sqrt{d} + d^{\frac{1}{4}}\big), \tag{24}$$

and using Proposition 2, we have that with probability $1 - O(\exp(-\sqrt{n}))$,

$$\delta^w \geq ne^{-\sqrt{d} - 2d^{\frac{1}{4}}}, \tag{25}$$

at the end of DA. Note that if (24) holds at the end of DA, then the RHS of (24) is an upper bound on t throughout the run of DA. Similarly, since the number of unmatched woman $\delta^w[t]$ is monotone non-increasing in t, if (25) holds at the end of DA, then the RHS of (25) is a lower bound on $\delta^w[t]$ throughout the run of DA. If, at any stage during the run of DA either (24) (with t instead of τ) or (25) (with $\delta^w[t]$ instead of δ^w) is violated, declare a "failure" event $\mathcal{E} \equiv \mathcal{E}_\tau$. By union bound, we know that $\mathbb{P}(\mathcal{E}) = O(\exp(-\sqrt{n}))$. For $t \leq \tau$, let $\mathcal{E}_t$ denote the event that no failure has occurred during the first t proposals of DA. We will prove (23) by showing an upper bound on the likelihood that man i remains unmatched for sample paths where no failure occurs, and assuming the worst (i.e., that i certainly remains unmatched) in the rare cases where there is a failure.

Run DA to convergence on men besides i. Now consider proposals by i. At each such proposal, the recipient woman is drawn uniformly at random from among at least $n - d + 1$ "candidate"

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women (the ones to whom i has not yet proposed). Assuming $\mathcal{E}_t^c$, we know that

$$t \leq n\left(\sqrt{d} + d^{\frac{1}{4}}\right), \tag{26}$$

and hence the total number of proposals received by candidate women is at most $n\left(\sqrt{d} + d^{\frac{1}{4}}\right)$, and hence the average number of proposals received by candidate women is at most $n(\sqrt{d} + d^{\frac{1}{4}})/(n - d + 1) \leq \sqrt{d}(1 + d^{-1/4} + \log^2 n/n) \leq \sqrt{d}(1 + 1.1d^{-\frac{1}{4}}) \leq \sqrt{d} + 1.1d^{\frac{1}{4}}$ for large enough n, using $d = o(\log^2 n)$. If the proposal goes to woman j, the probability of it being accepted is $\frac{1}{w_{j,t}+1}$. Averaging over the candidate women and using Jensen's inequality for the function $f(x) = \frac{1}{x+1}$, the probability of the proposal being accepted is at least $\frac{1}{\sqrt{d}+1.1d^{1/4}+1} \geq \frac{1}{\sqrt{d}+1.2d^{1/4}}$. If the proposal is accepted, say by woman j, this triggers a rejection chain. We show that it is very unlikely that this rejection chain will cause an additional proposal to woman j (which will imply that it is very unlikely that the rejection chain will cause i himself to be rejected): For every additional proposal in the rejection chain, the likelihood that it goes to an unmatched woman far exceeds the likelihood that it goes to woman j: if the current time is t' and $\mathcal{E}_{t'}^c$ holds, then, since all $\delta^w[t']$ unmatched women are certainly candidate recipients of the next proposal, the likelihood of the proposal being to an unmatched woman is at least $\delta^w[t'] \geq ne^{-\sqrt{d}-2d^{\frac{1}{4}}} \geq \sqrt{n}$ times the likelihood of it being to woman j for n large enough, using $d = o(\log^2 n)$. Now if the proposal is to an unmatched woman, this causes the rejection chain to terminate, hence the expected number of proposals to an unmatched woman in the rejection chain is at most 1. We immediately deduce that if a failure does not occur prior to termination of the chain, the expected number of proposals to woman j in the rejection chain is at most $\frac{1}{\sqrt{n}}$. It follows that

$\mathbb{P}(i$ is displaced from j by the rejection chain triggered when j accepts his proposal)

$\leq \mathbb{P}(j$ receives a proposal in the rejection chain triggered)

$\leq \mathbb{E}[$Number of proposals received by j in the rejection chain triggered$]$

$\leq \dfrac{1}{\sqrt{n}}$,

for n large enough. Overall, the probability of the proposal by i being "successful" in that it is both (a) accepted, and then (b) man i is not pushed out by the rejection chain, is at least

$$\frac{1}{\sqrt{d} + 1.2d^{1/4}}\left(1 - \frac{1}{\sqrt{n}}\right) \leq \frac{1}{\sqrt{d} + 1.3d^{1/4}},$$

for large enough n. Hence the probability of an unsuccessful proposal (if there is no failure) is at most

$$1 - \frac{1}{\sqrt{d} + 1.3d^{1/4}} \leq \exp\left\{-\frac{1}{\sqrt{d} + 1.3d^{1/4}}\right\},$$

and so the probability of all d proposals being unsuccessful (if there is no failure) is at most

$$\exp\left\{-\frac{d}{\sqrt{d} + 1.3d^{1/4}}\right\} \leq \exp\left\{-\sqrt{d} + 1.3d^{1/4}\right\}.$$

Formally, what we have obtained is an upper bound on the quantity $\mathbb{E}\left[\mathbb{I}(\mu(i) = i)\mathbb{I}(\mathcal{E}^c)\right]$, namely,

$$\mathbb{E}\left[\mathbb{I}(\mu(i) = i)\mathbb{I}(\mathcal{E}^c)\right] \leq \exp\left\{-\sqrt{d} + 1.3d^{1/4}\right\}.$$

Since the probability of failure is bounded as $\mathbb{P}(\mathcal{E}) \le O(\exp(-\sqrt{n}))$, the overall probability that of man i remaining unmatched is bounded above as

$$
\begin{aligned}
\mathbb{P}(\mu(i) = i) &\le \mathbb{E}\big[\mathbb{I}(\mu(i) = i)\mathbb{I}(\mathcal{E}^c)\big] + \mathbb{P}(\mathcal{E}) \\
&\le \exp\left\{-\sqrt{d} + 1.3d^{1/4}\right\} + O(\exp(-\sqrt{n})) \le \exp\left\{-\sqrt{d} + 1.4d^{1/4}\right\}
\end{aligned}
$$

for large enough n, i.e., the bound (23) which we set out to show. □

B.2.2 Upper bound on the number of unmatched men δ^m and unmatched women δ^w (Proposition 3)

Proof. Proof of Proposition 3. Recall the results in Lemma 13:

$$
\mathbb{E}[\delta^m] \le n \exp(-\sqrt{d} + 1.4d^{1/4}), \quad \mathbb{E}[\delta^w] \le n \exp(-\sqrt{d} + 1.5d^{1/4}). \tag{27}
$$

We use Markov's inequality for each δ^m and δ^w:

$$
\begin{aligned}
\mathbb{P}\left(\delta^m > n \exp(-\sqrt{d} + 2.4d^{1/4})\right) &\le \frac{\mathbb{E}[\delta^m]}{n \exp(-\sqrt{d} + 2.4d^{1/4})} \le \exp(-d^{1/4}), \\
\mathbb{P}\left(\delta^w > n \exp(-\sqrt{d} + 2.5d^{1/4})\right) &\le \frac{\mathbb{E}[\delta^w]}{n \exp(-\sqrt{d} + 2.5d^{1/4})} \le \exp(-d^{1/4}).
\end{aligned}
$$

□

B.2.3 Lower bound on the number of total proposals τ (Proposition 4)

Proof. Proof of Proposition 4. Consider the extended process defined in Appendix A.5, and let $\delta^w[t]$ be the number of unmatched woman at time t of the extended process. Let τ be the time when the men-optimal stable matching is found, i.e., $\delta^w = \delta^w[\tau]$. Let $\epsilon \triangleq d^{-1/4}$. We have

$$
\begin{aligned}
\mathbb{P}\left(\tau < (1 - 5\epsilon)n\sqrt{d}\right) &\le \mathbb{P}\left(\tau < (1 - 5\epsilon)n\sqrt{d}, \ \delta^w[\tau] < ne^{-(1-2.5\epsilon)\sqrt{d}}\right) + \mathbb{P}\left(\delta^w[\tau] \ge ne^{-(1-2.5\epsilon)\sqrt{d}}\right) \\
&\le \mathbb{P}\left(\delta^w[(1 - 5\epsilon)n\sqrt{d}] < ne^{-(1-2.5\epsilon)\sqrt{d}}\right) + \mathbb{P}\left(\delta^w[\tau] \ge ne^{-(1-2.5\epsilon)\sqrt{d}}\right). \tag{28}
\end{aligned}
$$

Here the last inequality holds because $\delta^w[t]$ is non-increasing over t on each sample path. It follows from Proposition 3 that the second term on the RHS of (28) is $O(e^{-d^{1/4}})$.

It remains to bound the first term on the RHS of (28). By Lemma 12, we have

$$
\mathbb{P}\left(\delta^w[(1 - 5\epsilon)n\sqrt{d}] < ne^{-(1-2.5\epsilon)\sqrt{d}}\right) \le \exp(-\sqrt{n}),
$$

for large enough n. By plugging this in the RHS of (28), we obtain

$$
\mathbb{P}\left(\tau < (1 - 5\epsilon)n\sqrt{d}\right) = O\left(\exp(-d^{\frac{1}{4}})\right). \tag{29}
$$

Note that by the definition of $R_{\text{MEN}}(\text{MOSM})$, we have

$$
R_{\text{MEN}}(\text{MOSM}) \ge \frac{\tau}{n + k}.
$$

Since $|k| = O(ne^{-\sqrt{d}})$, using an argument similar to the one at the end of the proof of Proposition

1, we can deduce from (29) that

$$\mathbb{P}\left(R_{\mathrm{MEN}}(\mathrm{MOSM}) < (1 - 6\epsilon)\sqrt{d}\right) = O\left(\exp\left(-d^{\frac{1}{4}}\right)\right).$$

This concludes the proof. $\qquad\square$

B.3 Step 3: Upper and Lower Bounds on Women's Average Rank R_{WOMEN}

In this section, we prove the following two propositions.

Proposition 5 (Lower bound on women's average rank). *Consider the setting of Theorem 1. With probability $1 - \frac{3}{n}$, we have the following lower bound on women's average rank:*

$$R_{\mathrm{WOMEN}}(\mathrm{MOSM}) \geq \sqrt{d} - 3d^{\frac{1}{4}},$$

Proposition 6 (Upper bound on women's average). *Consider the setting of Theorem 1. With probability $1 - O(\exp(-d^{\frac{1}{4}}))$, we have the following upper bound on the women's average rank:*

$$R_{\mathrm{WOMEN}}(\mathrm{MOSM}) \leq \sqrt{d} + 8d^{\frac{1}{4}}.$$

In order to characterize the women side, we introduce a different extended process which we call the *continue-proposing process* that is slightly different from one introduced in Section A.5. Until the MOSM is found (i.e., $t \leq \tau$), the continue-proposing process is identical to the original DA procedure. After the MOSM is found (i.e., $t > \tau$), the proposing man I_t is chosen arbitrarily among the men who have not yet exhausted their preference list (i.e., $\{i \in \mathcal{M} : M_{i,t-1} < d\}$), and we let him propose to his next candidate. We do not care about the matching nor the acceptance/rejection after τ, since we only keep track of the number of proposals that each man has made, $M_{i,t}$, and each woman has received, $W_{j,t}$. The continue-proposing process terminates at time $t = (n+k)d$, when all men exhaust their preference lists.

To analyze the concentration of R_{WOMEN}, we first construct upper and lower bounds on its conditional expectation. More formally, we define

$$\bar{R}[t] \triangleq \frac{1}{n} \sum_{j \in \mathcal{W}} \frac{W_{j,(n+k)d} - W_{j,t}}{W_{j,t} + 1}, \tag{30}$$

where $W_{j,(n+k)d}$ represents the degree of woman j in a random matching market so that $W_{j,(n+k)d} - W_{j,t}$ represents the number of remaining proposals that woman j will receive after time t. In Lemma 15, we prove that $\bar{R}[\tau]$ is concentrated around $\sqrt{d}$ given $\tau \approx n\sqrt{d}$. In Lemma 16, we show that $\bar{R}[\tau]$ (plus 1) is indeed the conditional expectation of R_{WOMEN} given $W_{j,\tau}$'s and $W_{j,(n+k)d}$'s, and further characterize the conditional distribution of R_{WOMEN} given $\bar{R}[\tau]$, which leads to the concentration bounds on R_{WOMEN}. Within the proofs, we also utilize the fact that $\bar{R}[t]$ is decreasing over time on each sample path.

B.3.1 Concentration of expected women's average rank $\bar{R}_t$

We first state a preliminary lemma that will be used to show the concentration of $\bar{R}_t$.

Lemma 14. *Fix any t and T such that $t < T$ and positive numbers $c_1, \ldots, c_n$ such that $c_j \in [0,1]$ for all j, and define*

$$Y_{t,T} \triangleq \sum_{j \in \mathcal{W}} c_j(W_{j,T} - W_{j,t}).$$

With $S \triangleq \sum_{j=1}^n c_j$, we have

$$\mathbb{P}\left(Y_{t,T} \geq (1+\epsilon)\frac{(T-t)S}{n-d} \,\middle|\, \vec{W}_t \right) \leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{(T-t)S}{n} \right) \tag{31}$$

$$\mathbb{P}\left(Y_{t,T} \leq (1-\epsilon)\frac{(T-t)(S-d)}{n-d} \,\middle|\, \vec{W}_t \right) \leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{(T-t)(S-d)}{n-d} \right) \tag{32}$$

for any $\epsilon \in [0,1]$.

Proof. Throughout this proof, we assume that $W_{1,t}, \ldots, W_{n,t}$ are revealed, i.e. we consider the conditional probabilities/expectations given $W_{1,t}, \ldots, W_{n,t}$. In addition, we assume that $c_1 \leq c_2 \leq \ldots \leq c_n$ without loss of generality.

Proof of (31): We first establish an upper bound using a coupling argument. Recall that $W_{j,s}$ counts the number of proposals that a woman j had received up to time s, which is governed by the recipient process J_s. We construct a coupled process $(\overline{W}_{j,s})_{j \in \mathcal{W}, s \geq t}$ that counts based on $\overline{J}_s$ as follows:

(i) Initialize $\overline{W}_{j,t} \leftarrow W_{j,t}$ for all j.

(ii) At each time $s = t+1, t+2, \ldots, T$, after the recipient J_s is revealed (which is uniformly sampled among $\mathcal{W} \setminus \mathcal{H}_s$), determine $\overline{J}_s \in \{d+1, \ldots, n\}$:

 - If $J_s \in \{d+1, \ldots, n\}$, set $\overline{J}_s \leftarrow J_s$.
 - If $J_s \in \{1, \ldots, d\}$, sample $\overline{J}_s$ according to the probability distribution $p_s(\cdot)$ defined as (the motivation for this definition is provided below)

$$p_s(j) = \begin{cases} 0 & \text{if } j \in \{1, \ldots, d\}, \\ \frac{1}{n-d} \big/ \frac{|\{1,\ldots,d\}\setminus\mathcal{H}_s|}{|\mathcal{W}\setminus\mathcal{H}_s|} & \text{if } j \in \{d+1, \ldots, n\} \cap \mathcal{H}_s, \\ \left(\frac{1}{n-d} - \frac{1}{|\mathcal{W}\setminus\mathcal{H}_s|}\right) \big/ \frac{|\{1,\ldots,d\}\setminus\mathcal{H}_s|}{|\mathcal{W}\setminus\mathcal{H}_s|} & \text{if } j \in \{d+1, \ldots, n\} \setminus \mathcal{H}_s. \end{cases}$$

(iii) Increase the counter of $\overline{J}_s$ instead of J_s: i.e., $\overline{W}_{j,s} \leftarrow \overline{W}_{j,s-1} + \mathbb{I}\{\overline{J}_s = j\}$ for all j.

In words, whenever a proposal goes to one of d women who have smallest c_j values (i.e., when $J_s \in \{1, \ldots, d\}$), we randomly pick one among the other $n - d$ women (i.e., $\overline{J}_s \in \{d+1, \ldots, n\}$) and increase that woman's counter $\overline{W}_{\overline{J}_s}$. Otherwise (i.e., when $J_s \in \{d+1, \ldots, n\}$), we count the proposal as in the original process. In any case, we have $c_{\overline{J}_s} \geq c_{J_s}$.

Note that we do not alter the proposal mechanism in this coupled process, but just count the proposals in a different way. Therefore, we have

$$\sum_{j \in \mathcal{W}} c_j(W_{j,T} - W_{j,t}) \leq \sum_{j \in \mathcal{W}} c_j(\overline{W}_{j,T} - \overline{W}_{j,t}), \tag{33}$$

Also note that the (re-)sampling distribution $p_s(\cdot)$ was constructed in a way that $\overline{J}_s$ is chosen uniformly at random among $\{d+1, \ldots, n\}$, unconditioned on J_s, independently of $\mathcal{H}_s$. More formally, we have for any $j \in \{d+1, \ldots, n\} \setminus \mathcal{H}_s$,

$$\mathbb{P}(\overline{J}_s = j|\mathcal{H}_s) = \mathbb{P}(J_s = j|\mathcal{H}_s) + \mathbb{P}(\overline{J}_s = j|\mathcal{H}_s, J_s \in \{1, \ldots, d\}) \cdot \mathbb{P}(J_s \in \{1, \ldots, d\}|\mathcal{H}_s)$$

$$= \frac{1}{|\mathcal{W} \setminus \mathcal{H}_s|} + \left(\frac{1}{n-d} - \frac{1}{|\mathcal{W} \setminus \mathcal{H}_s|}\right) = \frac{1}{n-d}.$$

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Similarly it can be verified that $\mathbb{P}(\overline{J}_s = j | \mathcal{H}_s) = \frac{1}{n-d}$ also for any $j \in \{d+1, \dots, n\} \cap \mathcal{H}_s$. The fact that $|\mathcal{H}_s| < d$ guarantees that $p_s(\cdot)$ is a well-defined probability mass function. Therefore,

$$\sum_{j \in \mathcal{W}} c_j(\overline{W}_{j,T} - \overline{W}_{j,t}) \stackrel{\mathrm{d}}{=} \sum_{j=d+1}^n c_j X_j,$$

where $X_j \sim \mathrm{Binomial}\left(T - t, \frac{1}{n-d}\right)$ for $j \in \{d+1, \dots, n\}$. Although X_j's are not independent, they are negatively associated as in the balls-into-bins process (see Section A.3). For any $\lambda \in \mathbb{R}$, $\exp(\lambda c_j X_j)$'s are also NA due to Lemma 3–(iii), and therefore,

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(\lambda \sum_{j=d+1}^n c_j X_j\right)\right] &\leq \prod_{j=d+1}^n \mathbb{E}\left[e^{\lambda c_j X_j}\right] \\
&= \prod_{j=d+1}^n \left(1 - \frac{1}{n-d} + \frac{1}{n-d}e^{\lambda c_j}\right)^{T-t} \\
&\leq \prod_{j=d+1}^n \exp\left(-\frac{1}{n-d} + \frac{1}{n-d}e^{\lambda c_j}\right)^{T-t} \\
&= \exp\left\{(T-t)\left(-1 + \frac{1}{n-d}\sum_{j=d+1}^n e^{\lambda c_j}\right)\right\}.
\end{aligned}
$$

Since $c_j \in [0, 1]$ and $e^x \leq 1 + x + x^2$ for any $x \in (-\infty, 1]$, we have for any $\lambda \in [0, 1]$,

$$-1 + \frac{1}{n-d}\sum_{j=d+1}^n e^{\lambda c_j} \leq -1 + \frac{1}{n-d}\sum_{j=d+1}^n (1 + \lambda c_j + \lambda^2 c_j^2) \leq \frac{\lambda + \lambda^2}{n-d}\sum_{j=d+1}^n c_j.$$

By Markov's inequality, for any $\lambda \in [0, 1]$,

$$
\begin{aligned}
\mathbb{P}\left(\sum_{j=d+1}^n c_j X_j \geq (1+\epsilon)\frac{T-t}{n-d}\sum_{j=d+1}^n c_j\right) &\leq \frac{\mathbb{E}\left[\exp\left(\lambda \sum_{j=d+1}^n c_j X_j\right)\right]}{\exp\left(\lambda(1+\epsilon)\frac{T-t}{n-d}\sum_{j=d+1}^n c_j\right)} \\
&\leq \exp\left\{(T-t)\cdot\frac{\lambda + \lambda^2}{n-d}\sum_{j=d+1}^n c_j - \lambda(1+\epsilon)\frac{T-t}{n-d}\sum_{j=d+1}^n c_j\right\} \\
&\leq \exp\left\{(\lambda^2 - \lambda\epsilon)\cdot\frac{T-t}{n-d}\sum_{j=d+1}^n c_j\right\}.
\end{aligned}
$$

By taking $\lambda \triangleq \frac{\epsilon}{2}$, we obtain

$$\mathbb{P}\left(\sum_{j=d+1}^n c_j X_j \geq (1+\epsilon)\frac{T-t}{n-d}\sum_{j=d+1}^n c_j\right) \leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{T-t}{n-d}\sum_{j=d+1}^n c_j\right).$$

Also note that

$$\frac{S}{n} = \frac{1}{n}\sum_{j=1}^n c_j \leq \frac{1}{n-d}\sum_{j=d+1}^n c_j.$$

Therefore, together with (33),

$$\mathbb{P}\left(Y_{t,T} \geq (1+\epsilon)\frac{(T-t)S}{n-d}\,\Big|\, W_{1,t},\ldots,W_{n,t}\right) \leq \mathbb{P}\left(Y_{t,T} \geq (1+\epsilon)\frac{T-t}{n-d}\sum_{j=d+1}^{n} c_j\,\Big|\, W_{1,t},\ldots,W_{n,t}\right)$$

$$\leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{T-t}{n-d}\sum_{j=d+1}^{n} c_j\right)$$

$$\leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{(T-t)S}{n}\right).$$

Proof of (32): Similarly to above, we can construct a coupled process $\left(\underline{W}_{j,s}\right)_{s\geq t}$ under which $\underline{J}_s$ is resampled among $\{1,\ldots,n-d\}$ whenever a proposal goes to one of d women who have largest c_j values (i.e., when $J_s \in \{n-d+1,\ldots,n\}$) while $\mathbb{P}\left(\underline{J}_s = j\,|\,\mathcal{H}_s\right) = \frac{1}{n-d}$ for any $j \in \{1,\cdots,n-d\}$ and any $\mathcal{H}_s$. With this process, we have

$$\sum_{j\in\mathcal{W}} c_j(W_{j,T} - W_{j,t}) \geq \sum_{j\in\mathcal{W}} c_j(\underline{W}_{j,T} - \underline{W}_{j,t}) \overset{\mathrm{d}}{=} \sum_{j=1}^{n-d} c_j X_j,$$

where $X_j \sim \mathrm{Binomial}\left(T-t, \frac{1}{n-d}\right)$ for $j \in \{1,\ldots,n-d\}$ and X_j's are NA.

For any $\lambda \in [-1,0]$,

$$\mathbb{E}\left[\exp\left(\lambda\sum_{j=1}^{n-d} c_j X_j\right)\right] \leq \prod_{j=1}^{n-d} \mathbb{E}\left[e^{\lambda c_j X_j}\right]$$

$$= \prod_{j=1}^{n-d}\left(1 - \frac{1}{n-d} + \frac{1}{n-d}e^{\lambda c_j}\right)^{T-t}$$

$$\leq \prod_{j=1}^{n-d}\exp\left(-\frac{1}{n-d} + \frac{1}{n-d}e^{\lambda c_j}\right)^{T-t}$$

$$= \exp\left\{(T-t)\left(-1 + \frac{1}{n-d}\sum_{j=1}^{n-d}e^{\lambda c_j}\right)\right\}.$$

Since $c_j \in [0,1]$ and $e^x \leq 1 + x + x^2$ for any $x \in (-\infty,1]$, we have for any $\lambda \in [-1,0]$,

$$-1 + \frac{1}{n-d}\sum_{j=1}^{n-d}e^{\lambda c_j} \leq -1 + \frac{1}{n-d}\sum_{j=1}^{n-d}(1 + \lambda c_j + \lambda^2 c_j^2) \leq \frac{\lambda + \lambda^2}{n-d}\sum_{j=1}^{n-d}c_j.$$

Using Markov's inequality, we have

$$\mathbb{P}\left(\sum_{j=1}^{n-d} c_j X_j \leq (1-\epsilon)\frac{T-t}{n-d}\sum_{j=1}^{n-d}c_j\right) = \mathbb{P}\left(\exp\left(\lambda\sum_{j=1}^{n-d} c_j X_j\right) \geq \exp\left(\lambda(1-\epsilon)\frac{T-t}{n-d}\sum_{j=1}^{n-d}c_j\right)\right)$$

$$\leq \frac{\mathbb{E}\left[\exp\left(\lambda\sum_{j=1}^{n-d} c_j X_j\right)\right]}{\exp\left(\lambda(1-\epsilon)\frac{T-t}{n-d}\sum_{j=1}^{n-d}c_j\right)}$$

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$$\leq \exp\left\{ (T-t) \cdot \frac{\lambda + \lambda^2}{n-d} \sum_{j=1}^{n-d} c_j - \lambda(1-\epsilon)\frac{T-t}{n-d}\sum_{j=1}^{n-d} c_j \right\}$$

$$\leq \exp\left\{ (\lambda^2 + \lambda\epsilon) \cdot \frac{T-t}{n-d} \sum_{j=1}^{n-d} c_j \right\} .$$

With $\lambda \triangleq -\frac{\epsilon}{2}$, we obtain

$$\mathbb{P}\left(\sum_{j=1}^{n-d} c_j X_j \leq (1-\epsilon)\frac{T-t}{n-d}\sum_{j=1}^{n-d} c_j \right) \leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{T-t}{n-d}\sum_{j=1}^{n-d} c_j \right) .$$

Consequently, since $S - d = \sum_{j=1}^{n} c_j - d \leq \sum_{j=1}^{n-d} c_j$,

$$\mathbb{P}\left(Y_{t,T} \leq (1-\epsilon)\frac{(T-t)(S-d)}{n-d} \,\bigg|\, W_{1,t}, \ldots, W_{n,t} \right) \leq \mathbb{P}\left(Y_{t,T} \leq (1-\epsilon)\frac{T-t}{n-d}\sum_{j=1}^{n-d} c_j \,\bigg|\, W_{1,t}, \ldots, W_{n,t} \right)$$

$$\leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{T-t}{n-d}\sum_{j=1}^{n-d} c_j \right)$$

$$\leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{(T-t)(S-d)}{n-d} \right) .$$

$\square$

Lemma 15. *Consider the setting of Theorem 1 and $\bar{R}[t]$ defined in (30). There exists $n_0 < \infty$ such that for all $n > n_0$, we have*

$$\mathbb{P}\left(\bar{R}\left[n(\sqrt{d} + d^{\frac{1}{4}}) \right] \leq \sqrt{d} - 2.3d^{\frac{1}{4}} \right) \leq \exp\left(-\frac{n}{8} \right) . \tag{34}$$

$$\mathbb{P}\left(\bar{R}\left[n(\sqrt{d} - 5d^{\frac{1}{4}}) \right] \geq \sqrt{d} + 7.5d^{\frac{1}{4}} \right) \leq \exp\left(-\frac{n}{d^4} \right) . \tag{35}$$

Proof. **Proof of** (34): Fix $t = n\left(\sqrt{d} + d^{\frac{1}{4}} \right)$ and let $S \triangleq \sum_{j\in\mathcal{W}} \frac{1}{W_{j,t}+1}$. Due to the convexity of $f(x) \triangleq \frac{1}{x+1}$, we have for large enough d (i.e. large enough n since $d = \omega(1)$),

$$\frac{S}{n} = \frac{1}{n}\sum_{j\in\mathcal{W}} f(W_{j,t}) \geq f\left(\frac{1}{n}\sum_{j\in\mathcal{W}} W_{j,t} \right) = f\left(\frac{t}{n} \right) = \frac{1}{t/n+1} \geq \frac{1}{\sqrt{d} + 1.05d^{\frac{1}{4}}} . \tag{36}$$

Given the asymptotic condition, for large enough n,

$$\frac{|k|}{n} = O(e^{-\sqrt{d}}) \leq 0.1d^{-\frac{1}{4}}, \qquad\qquad \frac{t}{nd} \leq \frac{\sqrt{d} + d^{\frac{1}{4}}}{d} = d^{-\frac{1}{2}} + d^{-\frac{3}{4}} \leq 0.1d^{-\frac{1}{4}},$$

$$\frac{S-d}{n/\sqrt{d}} \geq \frac{1}{1 + 1.05d^{-\frac{1}{4}}} - \frac{d^{\frac{3}{2}}}{n} \geq 1 - 1.1d^{-\frac{1}{4}} .$$

Therefore,

$$\frac{((n+k)d - t)(S - d)}{n - d} \geq \frac{((n+k)d - t)(S - d)}{n}$$

$$\geq d \left(1 - \frac{|k|}{n} - \frac{t}{nd} \right) \times \frac{n}{\sqrt{d}} \cdot \frac{S - d}{n/\sqrt{d}}$$

$$\geq n\sqrt{d} \times \left(1 - 0.1d^{-\frac{1}{4}} - 0.1d^{-\frac{1}{4}} \right) \times \left(1 - 1.1d^{-\frac{1}{4}} \right)$$

$$\geq n\sqrt{d} \left(1 - 1.3d^{-\frac{1}{4}} \right). \tag{37}$$

Utilizing Lemma 14, with[16] $c_j \triangleq \frac{1}{W_{j,t}+1}$, $T \triangleq (n+k)d$ and $\epsilon \triangleq d^{-\frac{1}{4}}$, we obtain

$$\mathbb{P}\left(\bar{R}[t] \leq \sqrt{d} - 2.3d^{\frac{1}{4}} \,\Big|\, \vec{W}_t \right) = \mathbb{P}\left(n\bar{R}[t] \leq n\sqrt{d} \left(1 - 2.3d^{-\frac{1}{4}} \right) \,\Big|\, \vec{W}_t \right)$$

$$\leq \mathbb{P}\left(n\bar{R}[t] \leq (1 - \epsilon) \times n\sqrt{d} \left(1 - 1.3d^{-\frac{1}{4}} \right) \,\Big|\, \vec{W}_t \right)$$

$$\leq \mathbb{P}\left(n\bar{R}[t] \leq (1 - \epsilon) \times \frac{((n+k)d - t)(S - d)}{n - d} \,\Big|\, \vec{W}_t \right)$$

$$\leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{((n+k)d - t)(S - d)}{n - d} \right)$$

$$\leq \exp\left(-\frac{1}{4}d^{-\frac{1}{2}} \times n\sqrt{d} \left(1 - 1.3d^{-\frac{1}{4}} \right) \right)$$

$$\leq \exp(-n/8),$$

where the last inequality follows from the fact that $1 - 1.3d^{-\frac{1}{4}} \geq \frac{1}{2}$ for large enough d. Since the above result holds for any realization of $\vec{W}_t$, the claim follows.

Proof of (35): Fix $t = n(\sqrt{d} - 5d^{\frac{1}{4}})$. Define $\mathcal{A} \triangleq \left\{ \vec{W}_t \,\Big|\, \frac{1}{n} \sum_{j \in \mathcal{W}} \frac{1}{W_{j,t}+1} \leq \frac{n}{t} + \frac{d^2}{n} + d^{-\frac{3}{4}} \right\} \subset \mathbb{N}^{|\mathcal{W}|}$. Applying Lemma 8 with $\Delta \triangleq d^{-\frac{3}{4}}$, we obtain

$$\mathbb{P}\left(\vec{W}_t \notin \mathcal{A} \right) = \mathbb{P}\left(\frac{S}{n} > \frac{n}{t} + \frac{d^2}{n} + d^{-\frac{3}{4}} \right) \leq 2 \exp\left(-\frac{n\Delta^2}{8d} \right) = 2 \exp\left(-\frac{1}{8}nd^{-\frac{5}{2}} \right).$$

Regarding the last term, observe that for large enough d, we have

$$2 \exp\left(-\frac{1}{8}nd^{-\frac{5}{2}} \right) \leq \frac{1}{2} \exp\left(-nd^{-4} \right).$$

For any $\vec{W}_t \in \mathcal{A}$, we have for large enough n,

$$\frac{S}{n} \leq \frac{n}{t} + \frac{d^2}{n} + d^{-\frac{3}{4}} = \frac{1}{\sqrt{d} - 5d^{\frac{1}{4}}} + \frac{d^2}{n} + d^{-\frac{3}{4}} = \frac{1}{\sqrt{d}} \left(\frac{1}{1 - 5d^{-\frac{1}{4}}} + \frac{d^{\frac{5}{2}}}{n} + d^{-\frac{1}{4}} \right) \leq \frac{1}{\sqrt{d}} \left(1 + 6.1d^{-\frac{1}{4}} \right),$$

Furthermore, given the asymptotic conditions,

$$\frac{((n+k)d - t)S}{n - d} \leq \frac{(n+k)dS}{n - d}$$

---

[16]In Lemma 14, we assume that c_j's are some deterministic constants whereas we set $c_j \triangleq \frac{1}{W_{j,t}+1}$ here. This is fine because the results of Lemma 14 are stated in terms of conditional probability given $\vec{W}_t$.

$$\leq \frac{n+|k|}{n} \cdot \frac{n}{n-d} \cdot dS$$

$$\leq \frac{n+|k|}{n} \cdot \frac{n}{n-d} \cdot n\sqrt{d}\left(1+6.1d^{-\frac{1}{4}}\right)$$

$$= \left(1+\frac{|k|}{n}\right) \cdot \frac{1}{1-d/n} \cdot n\sqrt{d}\left(1+6.1d^{-\frac{1}{4}}\right)$$

$$\leq \left(1+0.1d^{-\frac{1}{4}}\right) \cdot \left(1+0.1d^{-\frac{1}{4}}\right) \cdot n\sqrt{d}\left(1+6.1d^{-\frac{1}{4}}\right)$$

$$\leq n\sqrt{d}\left(1+6.4d^{-\frac{1}{4}}\right),$$

where we used the fact that $\frac{|k|}{n} = O(e^{-\sqrt{d}}) \leq 0.1d^{-\frac{1}{4}}$ and $\frac{d}{n} \leq 0.1d^{-\frac{1}{4}}$ for large enough n. We further utilize Lemma 14: By taking $c_j \triangleq \frac{1}{W_{j,t}+1}$, $T \triangleq (n+k)d$ and $\epsilon \triangleq d^{-\frac{1}{4}}$, we obtain

$$\mathbb{P}\left(\bar{R}[t] \geq \sqrt{d}+7.5d^{\frac{1}{4}}\,\Big|\,\vec{W}_t\right) = \mathbb{P}\left(n\bar{R}[t] \geq n\sqrt{d}\left(1+7.5d^{-\frac{1}{4}}\right)\,\Big|\,\vec{W}_t\right)$$

$$\leq \mathbb{P}\left(n\bar{R}[t] \geq (1+\epsilon) \times n\sqrt{d}\left(1+6.4d^{-\frac{1}{4}}\right)\,\Big|\,\vec{W}_t\right)$$

$$\leq \mathbb{P}\left(n\bar{R}[t] \geq (1+\epsilon) \times \frac{((n+k)d-t)S}{n-d}\,\Big|\,\vec{W}_t\right)$$

$$\leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{((n+k)d-t)S}{n}\right).$$

for any $\vec{W}_t \in \mathcal{A}$. Also note that, from (37), for large enough n,

$$\frac{((n+k)d-t)S}{n} \geq \left(\left(1-\frac{|k|}{n}\right)d - \frac{t}{n}\right)S \geq \left(\left(1-0.1d^{-\frac{1}{4}}\right)d - \sqrt{d}+5d^{\frac{1}{4}}\right)S,$$

where we used the fact that $\frac{|k|}{n} = O(e^{-\sqrt{d}}) \leq 0.1d^{-\frac{1}{4}}$. Because $\left(1-0.1d^{-\frac{1}{4}}\right)d-\sqrt{d}+5d^{\frac{1}{4}} \geq d-1.3d^{\frac{3}{4}}$ for large enough n, and that $S \geq \frac{n}{t/n+1}$ as derived in (36), we have

$$\frac{((n+k)d-t)S}{n} \geq \left(d-1.3d^{\frac{3}{4}}\right)\frac{n}{t/n+1} \geq \left(d-1.3d^{\frac{3}{4}}\right)\frac{n}{\sqrt{d}} \geq n\sqrt{d}\left(1-1.3d^{-\frac{1}{4}}\right),$$

and therefore,

$$\exp\left(-\frac{1}{4}\epsilon^2 \times \frac{((n+k)d-t)S}{n}\right) \leq \exp\left(-\frac{1}{4}d^{-\frac{1}{2}} \times n\sqrt{d}\left(1-1.3d^{-\frac{1}{4}}\right)\right) \leq \exp\left(-n/8\right).$$

Combining all results, we obtain the desired result: for large enough n,

$$\mathbb{P}\left(\bar{R}[t] \geq \sqrt{d}+7.5d^{\frac{1}{4}}\right) \leq \mathbb{P}\left(\bar{R}[t] \geq \sqrt{d}+7.5d^{\frac{1}{4}}\,\Big|\,\vec{W}_t \in \mathcal{A}\right) \cdot \mathbb{P}\left(\vec{W}_t \in \mathcal{A}\right) + \mathbb{P}\left(\vec{W}_t \notin \mathcal{A}\right)$$

$$\leq \exp\left(-n/8\right) + \frac{1}{2}\exp\left(-nd^{-4}\right)$$

$$\leq \exp\left(-nd^{-4}\right),$$

where the last inequality follows from that $\frac{n}{8} \geq \frac{n}{d^4} + \log 2$ for large enough n and d. $\qquad\square$

B.3.2 Concentration of women's average rank R_{WOMEN}

The following lemma states that conditioned on $(\vec{W}_\tau, \vec{W}_{(n+k)d})$, $R_{\text{WOMEN}}(\text{MOSM})$ is concentrated around $\bar{R}[\tau]$.

Lemma 16. *For any given n, k and d and $(\vec{W}_\tau, \vec{W}_{(n+k)d})$ which arises with positive probability, we have $\mathbb{E}[R_{\text{WOMEN}}(\text{MOSM})|\vec{W}_\tau, \vec{W}_{(n+k)d}] = 1 + \bar{R}[\tau]$. Furthermore, for any $\epsilon > 0$ we have*

$$\mathbb{P}\left(R_{\text{WOMEN}}(\text{MOSM}) \geq 1 + (1+\epsilon)\bar{R}[\tau] \,\big|\, \vec{W}_\tau, \vec{W}_{(n+k)d} \right) \leq \exp\left(-\frac{2\epsilon^2 n^2 \bar{R}[\tau]^2}{\sum_{j\in\mathcal{W}} W_{j,(n+k)d}^2} \right), \qquad (38)$$

$$\mathbb{P}\left(R_{\text{WOMEN}}(\text{MOSM}) \leq 1 + (1-\epsilon)\bar{R}[\tau] \,\big|\, \vec{W}_\tau, \vec{W}_{(n+k)d} \right) \leq \exp\left(-\frac{2\epsilon^2 n^2 \bar{R}[\tau]^2}{\sum_{j\in\mathcal{W}} W_{j,(n+k)d}^2} \right). \qquad (39)$$

Proof. Within this proof, we assume that τ, $\vec{W}_\tau = (W_{j,\tau})_{j\in\mathcal{W}}$, and $\vec{W}_{(n+k)d} = \left(W_{j,(n+k)d}\right)_{j\in\mathcal{W}}$ are revealed (and hence so is $\bar{R}[\tau]$). In what follows, $\mathbb{P}(\cdot)$ and $\mathbb{E}[\cdot]$ denote the associated conditional probability and the conditional expectation, respectively.

For brevity, let $w_j \triangleq W_{j,\tau}$, $w_j' \triangleq W_{j,(n+k)d} - W_{j,\tau}$, and $R_j \triangleq \text{Rank}_j(\text{MOSM})|\vec{W}_\tau, \vec{W}_{(n+k)d}$. Note that a woman j receives w_j proposals until time τ and receives w_j' proposals after time τ (the total number of proposals $w_j + w_j' = W_{j,(n+k)d}$ equals to her degree). Under MOSM, each woman j is matched to her most preferred one among the first w_j proposals, and the rank of her matched partner under MOSM, R_j, can be determined by the number of men among the remaining (at time τ) w_j' men on her list that she prefers to her matched partner.

More specifically, fix j and let Z_t^j be the indicator that the woman j prefers her t^{th} proposal to all of her first w_j proposals for $t \in \{w_j + 1, \ldots, w_j + w_j'\}$. Then, the rank R_j can be represented as

$$R_j = 1 + \sum_{t=w_j+1}^{w_j+w_j'} \mathbb{I}\left(\text{woman } j \text{ prefers her } t^{\text{th}} \text{ proposal to all of her first } w_j \text{ proposals} \right)$$

$$= 1 + \sum_{t=w_j+1}^{w_j+w_j'} Z_t^j .$$

Note that $(Z_t^j)_{t=w_j+1}^{w_j+w_j'}$ has the same distribution as $(\mathbb{I}\{U_t^j > V_j\})_{t=w_j+1}^{w_j+w_j'}$, where $(U_t^j)_{t=w_j+1}^{w_j+w_j'}$ are i.i.d. Uniform$[0,1]$ random variables, V_j is the largest order statistic of w_j i.i.d. Uniform$[0,1]$ random variables, and V_j is independent of U_t^j's. Therefore,

$$\mathbb{E}[R_j] = 1 + w_j' \cdot \mathbb{E}[Z_{w_j+1}^j] = 1 + w_j' \cdot \mathbb{P}(U_{w_j+1}^j > V_j) = 1 + \frac{w_j'}{w_j + 1},$$

and

$$\mathbb{E}[R_{\text{WOMEN}}(\text{MOSM})|\vec{W}_\tau, \vec{W}_{(n+k)d}] = \frac{1}{n}\sum_{j\in\mathcal{W}} \mathbb{E}[R_j] = 1 + \bar{R}[\tau],$$

which proves the first claim in Lemma 16.

Note that $(R_j)_{j\in\mathcal{W}}$ are i.i.d. and that $R_j \in [0, w_j']$. Applying Hoeffding's inequality, we have

$$\mathbb{P}\left(R_{\text{WOMEN}}(\text{MOSM}) \geq 1 + (1+\epsilon)\bar{R}[\tau] \,\big|\, \vec{W}_\tau, \vec{W}_{(n+k)d} \right) = \mathbb{P}\left(\frac{1}{n} R_j \geq 1 + (1+\epsilon)\mathbb{E}[\bar{R}[\tau]] \right)$$

$$\leq \ \exp\left(-\frac{2\epsilon^2 n^2 \bar{R}[\tau]^2}{\sum_{j\in\mathcal{W}} W_{j,(n+k)d}^2}\right) .$$

Similarly, we can show that

$$\mathbb{P}\left(R_{\text{WOMEN}}(\text{MOSM}) \leq 1 + (1-\epsilon)\bar{R}[\tau] \,|\, \vec{W}_\tau, \vec{W}_{(n+k)d}\right) \leq \ \exp\left(-\frac{2\epsilon^2 n^2 \bar{R}[\tau]^2}{\sum_{j\in\mathcal{W}} W_{j,(n+k)d}^2}\right) .$$

This concludes the proof.

$\square$

Proof of Proposition 5. We obtain a high probability lower bound on R_{WOMEN} by combining the results of Proposition 1, and Lemmas 15 and 16. By Proposition 1 and Lemma 15 and by the fact that $\bar{R}[t]$ is decreasing on each sample path,

$$\begin{aligned}
\mathbb{P}\left(\bar{R}[\tau] \leq \sqrt{d} - 2.3d^{\frac{1}{4}}\right) &\leq \mathbb{P}\left(\bar{R}[\tau] \leq \sqrt{d} - 2.3d^{\frac{1}{4}}, \tau < n(\sqrt{d}+d^{\frac{1}{4}})\right) + \mathbb{P}\left(\tau \geq n(\sqrt{d}+d^{\frac{1}{4}})\right) \\
&\leq \mathbb{P}\left(\bar{R}[n(\sqrt{d}+d^{\frac{1}{4}})] \leq \sqrt{d} - 2.3d^{\frac{1}{4}}, \tau < n(\sqrt{d}+d^{\frac{1}{4}})\right) + \mathbb{P}\left(\tau \geq n(\sqrt{d}+d^{\frac{1}{4}})\right) \\
&\leq \exp\left(-\frac{n}{8}\right) + O(\exp(-\sqrt{n})) = O(\exp(-\sqrt{n})) . \tag{40}
\end{aligned}$$

We also need a high probability upper bound on $\sum_{j\in\mathcal{W}} W_{j,(n+k)d}^2$. Since $W_{j,(n+k)d} \sim \text{Binomial}((n+k)d, \frac{1}{n})$, we have for large enough n,

$$\mathbb{E}\left[W_{j,(n+k)d}^2\right] = \ \mathbb{E}^2\left[W_{j,(n+k)d}\right] + \text{Var}\left[W_{j,(n+k)d}\right] = \ (n+k)^2 d^2 \frac{1}{n^2}\left(1 + (1-\frac{1}{n})^2\right) \leq \ 2d^2 .$$

Denote $\mu \triangleq \mathbb{E}[W_{1,(n+k)d}] = \frac{(n+k)d}{n}$. Looking up the table of the central moments of Binomial distribution, we have

$$\mathbb{E}[(W_{1,(n+k)d} - \mu)^4] = \ (n+k)d\frac{1}{n}\left(1 - \frac{1}{n}\right)\left(1 + (3(n+k)d - 6)\frac{1}{n}\left(1 - \frac{1}{n}\right)\right) .$$

Using the fact that $k = o(n)$, $d = o(n)$ and $d = \omega(1)$, we have for large enough n,

$$\mathbb{E}[(W_{1,(n+k)d} - \mu)^4] \leq \ 2d\left(1 + 3(n+k)d\frac{1}{n}\right) \leq \ 2d \cdot 4d = \ 8d^2 .$$

Therefore, for large enough n,

$$\begin{aligned}
\text{Var}[W_{1,(n+k)d}^2] &\leq \mathbb{E}[W_{1,(n+k)d}^4] \\
&= \mathbb{E}[(\mu + (W_{1,(n+k)d} - \mu))^4] \\
&\leq 8\mu^4 + 8\mathbb{E}[(W_{1,(n+k)d} - \mu)^4] \\
&= 8\frac{(n+k)^4 d^4}{n^4} + 64d^2 \\
&\leq 10d^4 .
\end{aligned}$$

In the proof of Lemma 4, we have shown that $W_{1,(n+k)d}, \dots, W_{n,(n+k)d}$ are NA. By Lemma 3–(iii),

$W^2_{1,(n+k)d}, \ldots, W^2_{n,(n+k)d}$ are NA, hence we have for large enough n,

$$\mathrm{Var}\left[\sum_{j \in \mathcal{W}} W^2_{j,(n+k)d}\right] \leq n\mathrm{Var}\left[W^2_{1,(n+k)d}\right] \leq 10nd^4.$$

Applying Chebyshev's inequality, we have for large enough n,

$$\mathbb{P}\left(\sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} \geq 4nd^2\right) \leq \mathbb{P}\left(\sum_{j \in \mathcal{W}} (W^2_{j,(n+k)d} - \mathbb{E}[W^2_{j,(n+k)d}]) \geq 2nd^2\right)$$

$$= \mathbb{P}\left(\sum_{j \in \mathcal{W}} (W^2_{j,(n+k)d} - \mathbb{E}[W^2_{j,(n+k)d}]) \geq \frac{2\sqrt{n}}{\sqrt{10}}\sqrt{10nd^4}\right)$$

$$\leq \mathbb{P}\left(\sum_{j \in \mathcal{W}} (W^2_{j,(n+k)d} - \mathbb{E}[W^2_{j,(n+k)d}]) \geq \frac{2\sqrt{n}}{\sqrt{10}}\sqrt{\mathrm{Var}\left[\sum_{j \in \mathcal{W}} W^2_{j,(n+k)d}\right]}\right)$$

$$\leq \frac{5}{2n} \leq \frac{3}{n}. \tag{41}$$

Given that $\bar{R}[\tau] > \sqrt{d} - 2.3d^{\frac{1}{4}}$, by plugging $\epsilon \triangleq 0.5d^{-\frac{1}{4}}$ in (39) of Lemma 16, we obtain for large enough n,

$$1 + (1 - \epsilon)\bar{R}[\tau] \geq 1 + (1 - 0.5d^{-\frac{1}{4}}) \cdot \sqrt{d}(1 - 2.3d^{-\frac{1}{4}}) \geq \sqrt{d}(1 - 3d^{-\frac{1}{4}}) = \sqrt{d} - 3d^{\frac{1}{4}}. \tag{42}$$

Therefore,

$$\mathbb{P}\left(R_{\mathrm{WOMEN}} \leq \sqrt{d} - 3d^{\frac{1}{4}}\right) \leq \mathbb{P}\left(R_{\mathrm{WOMEN}} \leq \sqrt{d} - 3d^{\frac{1}{4}} \Big| \bar{R}[\tau] > \sqrt{d} - 2.3d^{\frac{1}{4}}, \sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} < 4nd^2\right)$$

$$+ \mathbb{P}\left(\bar{R}[\tau] \leq \sqrt{d} - 2.3d^{\frac{1}{4}}\right) + \mathbb{P}\left(\sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} \geq 4nd^2\right)$$

$$\overset{(a)}{\leq} \mathbb{P}\left(R_{\mathrm{WOMEN}} \leq 1 + (1 - \epsilon)\bar{R}[\tau] \Big| \bar{R}[\tau] > \sqrt{d} - 2.3d^{\frac{1}{4}}, \sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} < 4nd^2\right)$$

$$+ O(\exp(-\sqrt{n})) + \frac{3}{n}$$

$$\overset{(b)}{\leq} \mathbb{E}\left[\exp\left(-\frac{\frac{1}{2}d^{-\frac{1}{2}}n^2\bar{R}[\tau]^2}{4nd^2}\right) \Big| \bar{R}[\tau] > \sqrt{d} - 2.3d^{\frac{1}{4}}\right] + \frac{4}{n}$$

$$\leq \exp\left(-\frac{1}{8}d^{-\frac{5}{2}}n \cdot d(1 - 2.3d^{-\frac{1}{4}})^2\right) + \frac{4}{n}$$

$$\leq \exp\left(-\frac{nd^{-\frac{3}{2}}}{16}\right) + \frac{4}{n}$$

$$\leq \frac{5}{n}.$$

Here inequality (a) follows from (42), (40), and (41); inequality (b) follows from Lemma 16. $\quad\square$

Proof of Proposition 6. We obtain a high probability lower bound on R_{WOMEN} by combining the results of Proposition 4, and Lemma 15 and 16. By Proposition 4 and Lemma 15,

$$
\begin{aligned}
\mathbb{P}\left(\bar{R}[\tau] \geq \sqrt{d} + 7.5d^{\frac{1}{4}}\right) &\leq \mathbb{P}\left(\bar{R}[\tau] \geq \sqrt{d} + 7.5d^{\frac{1}{4}}, \tau > n(\sqrt{d} - 5d^{\frac{1}{4}})\right) + \mathbb{P}\left(\tau \leq n(\sqrt{d} - 5d^{\frac{1}{4}})\right) \\
&\leq \mathbb{P}\left(\bar{R}[n(\sqrt{d} - 5d^{\frac{1}{4}})] \geq \sqrt{d} + 7.5d^{\frac{1}{4}}, \tau > n(\sqrt{d} - 5d^{\frac{1}{4}})\right) + \mathbb{P}\left(\tau \leq n(\sqrt{d} - 5d^{\frac{1}{4}})\right) \\
&\leq \exp\left(-\frac{n}{d^4}\right) + O(\exp(-d^{\frac{1}{4}})) \\
&\leq O(\exp(-d^{\frac{1}{4}})).
\end{aligned}
$$

Given that $\bar{R}[\tau] < \sqrt{d} + 7.5d^{\frac{1}{4}}$, by plugging $\epsilon \triangleq 0.1d^{-\frac{1}{4}}$ in (38) of Lemma 16, we obtain

$$
1 + (1+\epsilon)\bar{R}[\tau] \leq 1 + (1 + 0.1d^{-\frac{1}{4}}) \cdot \sqrt{d}(1 + 7.5d^{-\frac{1}{4}}) \leq \sqrt{d}(1 + 8d^{-\frac{1}{4}}) = \sqrt{d} + 8d^{\frac{1}{4}},
$$

for large enough n. Recall that we have shown in the proof of Proposition 5 that

$$
\mathbb{P}\left(\sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} \geq 4nd^2\right) \leq \frac{3}{n}.
$$

Therefore, similar to the proof of Proposition 5, we have

$$
\begin{aligned}
&\mathbb{P}\left(R_{\text{WOMEN}} \geq \sqrt{d} + 8d^{\frac{1}{4}}\right) \\
&\leq \mathbb{P}\left(R_{\text{WOMEN}} \geq \sqrt{d} + 8d^{\frac{1}{4}} \,\middle|\, \sqrt{d} - 2.3d^{\frac{1}{4}} < \bar{R}[\tau] < \sqrt{d} + 7.5d^{\frac{1}{4}}, \sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} < 4nd^2\right) \\
&\quad + \mathbb{P}\left(\bar{R}[\tau] \leq \sqrt{d} - 2.3d^{\frac{1}{4}}\right) + \mathbb{P}\left(\bar{R}[\tau] \geq \sqrt{d} + 7.5d^{\frac{1}{4}}\right) + \mathbb{P}\left(\sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} \geq 4nd^2\right) \\
&\leq \mathbb{P}\left(R_{\text{WOMEN}} \geq 1 + (1+\epsilon)\bar{R}[\tau] \,\middle|\, \sqrt{d} - 2.3d^{\frac{1}{4}} < \bar{R}[\tau] < \sqrt{d} + 7.5d^{\frac{1}{4}}, \sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} < 4nd^2\right) \\
&\quad + O(\exp(-d^{\frac{1}{4}})) \\
&\leq \mathbb{E}\left[\exp\left(-\frac{2\epsilon^2 n^2 \bar{R}[\tau]^2}{\sum_{j \in \mathcal{W}} W^2_{j,(n+k)d}}\right) \,\middle|\, \sqrt{d} - 2.3d^{\frac{1}{4}} < \bar{R}[\tau] < \sqrt{d} + 7.5d^{\frac{1}{4}}, \sum_{j \in \mathcal{W}} W^2_{j,(n+k)d} < 4nd^2\right] \\
&\quad + O(\exp(-d^{\frac{1}{4}})) \\
&\leq \exp\left(-\frac{1}{200}d^{-\frac{5}{2}}n \cdot d(1 - 2.3d^{-\frac{1}{4}})^2\right) + O(\exp(-d^{\frac{1}{4}})) \\
&\leq \exp\left(-\frac{nd^{-\frac{3}{2}}}{300}\right) + O(\exp(-d^{\frac{1}{4}})) = O(\exp(-d^{\frac{1}{4}})).
\end{aligned}
$$

$\quad\square$

B.4 Proof of Theorem 3

Theorem 3 immediately follows from Propositions 1, 2, 3, 4, 5, and 6.

C Proof for Large Sized d: the Case of $d = \omega(\log^2 n)$, $d = o(n)$

In this section, we consider the case such that $d = \omega(\log^2 n)$ and $d = o(n)$. We will prove a quantitative version of Theorem 2.

Theorem 4 (Quantitative version of Theorem 2). *Consider a sequence of random matching markets indexed by n, with $n + k$ men and n women ($k = k(n)$ is negative), and the men's degrees are $d = d(n)$. If $|k| = o(n)$, $d = \omega(\log^2 n)$ and $d = o(n)$, we have the following results.*

1. Men's average rank of wives. *With probability $1 - \exp(-\sqrt{\log n})$, we have*

$$R_{\text{MEN}}(\text{MOSM}) \leq \left(1 + 2\frac{|k|}{n} + 2\frac{1}{\sqrt{\log n}}\right)\log n\,.$$

2. Women's average rank of husbands. *With probability $1 - O(\exp(-\sqrt{\log n}))$, we have*

$$R_{\text{WOMEN}}(\text{MOSM}) \geq \left(1 - 1.1\left(\frac{|k|}{n} + \frac{3}{\sqrt{\log n}} + \frac{d}{n/\log n}\right)\right)\frac{d}{\log n}\,.$$

Proof of Theorem 4. **Proof of Theorem 4 part 1.** Recall that τ is the *the total number of proposals* that are made until the end of MPDA, i.e., the time at which the men-optimal stable matching (MOSM) is found. We introduce an extended process (which is *different* from the one defined in Appendix A.5) as a natural continuation of the MPDA procedure that continues to evolve even after the MOSM is found (i.e., the extended process continues for $t > \tau$). To define the extended process, we start by defining an extended market, which has the same n women and $n + k$ men, but each man has a complete preference list, i.e. each man ranks all n women. We call the first d women of a man's preference list his "real" preferences and the last $n - d$ women his "fake" preferences. The distribution of preferences in the extended market is again as described in Section 2. We then define the *extended process* as tracking the progress of Algorithm 1 on the extended market: the $n + k$ men enter first in Algorithm 1 with only their real preferences, as before. After time τ, we let the men see their fake preferences and continue Algorithm 1 until the MOSM with full preferences is found. We denote by τ' the total number of proposals to find the MOSM with full preferences. It is easy to see that τ is stochastically dominated by τ'.

Note that τ' is the total number of proposals needed to find the MOSM in a completely-connected market, which has been studied in previous works including Ashlagi et al. (2015), Pittel (2019b). It is well-known that τ' is stochastically dominated by the number of draws in a coupon collector's problem, in which one coupon is chosen out of n coupons uniformly at random at a time and it runs until n distinct coupons are collected. Let X be the number of draws in the coupon collector's problem. A widely used tail bound of X is the following: for $\beta > 1$, $\mathbb{P}(X \geq \beta n \log n) \leq n^{-\beta+1}$. By taking $\beta = 1 + \frac{1}{\sqrt{\log n}}$, we have

$$\mathbb{P}\left(X \geq n \log n + n\sqrt{\log n}\right) \leq n^{-\frac{1}{\sqrt{\log n}}} = e^{-\sqrt{\log n}} = o(1)\,.$$

Hence with probability $1 - e^{-\sqrt{\log n}}$, we have $\tau \leq n(\log n + \sqrt{\log n})$. Because X stochastically

dominates τ, we have, with probability $1 - e^{-\sqrt{\log n}}$,

$$R_{\mathrm{MEN}}(\mathrm{MOSM}) \leq \frac{n}{n+k}(\log n + \sqrt{\log n}) + 1 \,.$$

Because $k = o(n)$ and $k < 0$, for large enough n we have $\frac{n}{n+k} \leq 1 + \frac{2|k|}{n}$, $\frac{|k|}{n} < \frac{1}{3}$, $1 \leq \frac{1}{3}\sqrt{\log n}$, hence

$$\begin{aligned}
\frac{n}{n+k}&(\log n + \sqrt{\log n}) + 1 \\
&\leq \left(1 + 2\frac{|k|}{n}\right)\log n + (1 + \frac{2}{3})\sqrt{\log n} + \frac{1}{3}\sqrt{\log n} \\
&= \left(1 + 2\frac{|k|}{n} + 2\frac{1}{\sqrt{\log n}}\right)\log n \,.
\end{aligned}$$

This concludes the proof.

Proof of Theorem 4 part 2.

The proof is similar to that of Proposition 5. Recall that the proof of Proposition 5 relies on Proposition 1, Lemma 15, and Lemma 16. In the following, we first establish the counterparts of these results in dense markets.

Counterpart of Proposition 1 in dense markets. We have shown in the proof of Theorem 4(1) that with probability $1 - \exp(-\sqrt{\log n})$, we have

$$\tau \leq n\left(\log n + \sqrt{\log n}\right) \,. \tag{43}$$

Counterpart of Lemma 15 in dense markets. Fix $t = n\left(\log n + \sqrt{\log n}\right)$. Given the asymptotic condition, we have for large enough n,

$$\frac{t}{nd} = \frac{\log n + \sqrt{\log n}}{d} \leq 0.1(\log n)^{-1} \,.$$

By examining the proof of Lemma 14, we can see that we have proved the following result (see the statement of Lemma 14 for the definition of the notations), which is stronger than than (32):

$$\mathbb{P}\left(Y_{t,T} \leq (1-\epsilon)\frac{T-t}{n-d}\sum_{j=1}^{n-d}c_j \,\middle|\, \vec{W}_t\right) \leq \exp\left(-\frac{1}{4}\epsilon^2 \times \frac{T-t}{n-d}\sum_{j=1}^{n-d}c_j\right) \tag{44}$$

Let $c_j \triangleq \frac{1}{W_{j,t}+1}$ where $W_{1,t} \geq W_{2,t} \geq \cdots \geq W_{n,T}$, and $T \triangleq (n+k)d$. Due to the convexity of $f(x) \triangleq \frac{1}{x+1}$, we have for large enough n,

$$\begin{aligned}
\frac{1}{n-d}\sum_{j=1}^{n-d}c_j &= \frac{1}{n-d}\sum_{j=1}^{n-d}f(W_{j,t}) \geq f\left(\frac{1}{n-d}\sum_{j=1}^{n-d}W_{j,t}\right) \geq f\left(\frac{t}{n-d}\right) \\
&= \frac{1}{t/(n-d)+1} \geq \frac{1}{\log n\left(1 + 1.05\frac{1}{\sqrt{\log n}} + 1.05\frac{d}{n}\right)} \,.
\end{aligned} \tag{45}$$

Therefore, for large enough n,

$$\frac{T-t}{n-d}\sum_{j=1}^{n-d}c_j \geq n\frac{(n+k)d-t}{n}\frac{1}{\log n\left(1+1.05\frac{1}{\sqrt{\log n}}+1.05\frac{d}{n}\right)}$$

$$\geq nd\left(1-\frac{|k|}{n}-\frac{t}{nd}\right)\frac{1}{\log n\left(1+1.05\frac{1}{\sqrt{\log n}}+1.05\frac{d}{n}\right)}$$

$$\geq nd\left(1-\frac{|k|}{n}-\frac{0.1}{\log n}\right)\frac{1}{\log n}\left(1-1.05\frac{1}{\sqrt{\log n}}-1.05\frac{d}{n}\right)$$

$$\geq \frac{nd}{\log n}\left(1-1.1\frac{|k|}{n}-1.1\frac{1}{\sqrt{\log n}}-1.1\frac{d}{n}\right). \tag{46}$$

Utilizing Lemma 14 (which does not use assumptions on d) with $\epsilon \triangleq \frac{1}{\sqrt{\log n}}$, we obtain

$$\mathbb{P}\left(\bar{R}[t]\leq\frac{d}{\log n}\left(1-\left(1.1\frac{|k|}{n}+\frac{2.1}{\sqrt{\log n}}+1.1\frac{d}{n}\right)\right)\middle|\vec{W}_t\right)$$

$$=\mathbb{P}\left(n\bar{R}[t]\leq\frac{nd}{\log n}\left(1-\left(1.1\frac{|k|}{n}+\frac{2.1}{\sqrt{\log n}}+1.1\frac{d}{n}\right)\right)\middle|\vec{W}_t\right)$$

$$\leq\mathbb{P}\left(n\bar{R}[t]\leq(1-\epsilon)\times\frac{nd}{\log n}\left(1-1.1\left(\frac{|k|}{n}+\frac{1}{\sqrt{\log n}}+\frac{d}{n}\right)\right)\middle|\vec{W}_t\right)$$

$$\overset{(a)}{\leq}\mathbb{P}\left(n\bar{R}[t]\leq(1-\epsilon)\times\frac{(n+k)d-t}{n-d}\sum_{j=1}^{n-d}c_j\middle|\vec{W}_t\right)$$

$$\overset{(b)}{\leq}\exp\left(-\frac{1}{4}\epsilon^2\times\frac{(n+k)d-t}{n-d}\sum_{j=1}^{n-d}c_j\right)$$

$$\overset{(c)}{\leq}\exp\left(-\frac{1}{4}\frac{1}{\log n}\times\frac{nd}{\log n}\left(1-1.1\left(\frac{|k|}{n}+\frac{1}{\sqrt{\log n}}+\frac{d}{n}\right)\right)\right)$$

$$\leq\exp(-n/8).$$

Here inequalities (a) and (c) follow from (46), inequality (b) follows from Lemma 14, and the last inequality follows from the fact that $d=\omega(\log^2 n)$. Since the above result holds for any realization of $\vec{W}_t$, we have

$$\mathbb{P}\left(\bar{R}\left[n(\log n+\sqrt{\log n})\right]\leq\frac{d}{\log n}\left(1-1.1\left(\frac{|k|}{n}+\frac{2}{\sqrt{\log n}}+\frac{d}{n}\right)\right)\right)\leq\exp\left(-\frac{n}{8}\right). \tag{47}$$

Counterpart of Lemma 16 in dense markets. Note that the proof of Lemma 16 does not make any assumption on d, hence (39) still holds.

Proof of Theorem 4 part 2. Using (43) and (47), and the fact that $\bar{R}[t]$ is decreasing on each sample path,

$$\mathbb{P}\left(\bar{R}[\tau]\leq\frac{d}{\log n}\left(1-1.1\left(\frac{|k|}{n}+\frac{2}{\sqrt{\log n}}+\frac{d}{n}\right)\right)\right)$$

$$\leq\mathbb{P}\left(\bar{R}[\tau]\leq\frac{d}{\log n}\left(1-1.1\left(\frac{|k|}{n}+\frac{2}{\sqrt{\log n}}+\frac{d}{n}\right)\right),\tau<n(\log n+\sqrt{\log n})\right)$$

$$+\mathbb{P}\left(\tau\geq n(\log n+\sqrt{\log n})\right)$$

$$\leq \mathbb{P}\left(\bar{R}\left[n(\log n + \sqrt{\log n})\right] \leq \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{2}{\sqrt{\log n}} + \frac{d}{n}\right)\right)\right)$$

$$+ \mathbb{P}\left(\tau \geq n(\log n + \sqrt{\log n})\right)$$

$$\overset{(a)}{\leq} \exp\left(-\frac{n}{8}\right) + O(\exp(-\sqrt{\log n})) = O(\exp(-\sqrt{\log n})). \tag{48}$$

Here inequality (a) follows from (47). Recall inequality (41): for large enough n

$$\mathbb{P}\left(\sum_{j \in \mathcal{W}} W_{j,(n+k)d}^2 \geq 4nd^2\right) \leq \frac{3}{n}.$$

In the derivation of the above inequality, we only used the fact that $d = \omega(1), d = o(n)$ and $k = o(n)$, which also holds in dense markets.

Given that $\bar{R}[\tau] > \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{2}{\sqrt{\log n}} + \frac{d}{n}\right)\right)$, by plugging $\epsilon \triangleq 0.5\frac{1}{\sqrt{\log n}}$ in (39) of Lemma 16, we obtain for large enough n,

$$1 + (1-\epsilon)\bar{R}[\tau] \geq 1 + \left(1 - 0.5\frac{1}{\sqrt{\log n}}\right)\frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{2}{\sqrt{\log n}} + \frac{d}{n}\right)\right)$$

$$\geq \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{3}{\sqrt{\log n}} + \frac{d}{n}\right)\right). \tag{49}$$

Therefore,

$$\mathbb{P}\left(R_{\text{WOMEN}} \leq \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{3}{\sqrt{\log n}} + \frac{d}{n}\right)\right)\right)$$

$$\leq \mathbb{P}\left(R_{\text{WOMEN}} \leq \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{3}{\sqrt{\log n}} + \frac{d}{n}\right)\right)\right|$$

$$\bar{R}[\tau] > \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{2}{\sqrt{\log n}} + \frac{d}{n}\right)\right), \sum_{j \in \mathcal{W}} W_{j,(n+k)d}^2 < 4nd^2\right)$$

$$+ \mathbb{P}\left(\bar{R}[\tau] \leq \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{2}{\sqrt{\log n}} + \frac{d}{n}\right)\right)\right) + \mathbb{P}\left(\sum_{j \in \mathcal{W}} W_{j,(n+k)d}^2 \geq 4nd^2\right)$$

$$\overset{(a)}{\leq} \mathbb{P}\left(R_{\text{WOMEN}} \leq 1 + (1-\epsilon)\bar{R}[\tau]\middle|\bar{R}[\tau] > \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{2}{\sqrt{\log n}} + \frac{d}{n}\right)\right),\right.$$

$$\left.\sum_{j \in \mathcal{W}} W_{j,(n+k)d}^2 < 4nd^2\right) + O(\exp(-\sqrt{\log n})) + \frac{3}{n}$$

$$\overset{(b)}{\leq} \mathbb{E}\left[\exp\left(-\frac{\frac{1}{2\log n}n^2\bar{R}[\tau]^2}{4nd^2}\right)\middle|\bar{R}[\tau] > \frac{d}{\log n}\left(1 - 1.1\left(\frac{|k|}{n} + \frac{2}{\sqrt{\log n}} + \frac{d}{n}\right)\right)\right]$$

$$+ O(\exp(-\sqrt{\log n}))$$

$$\leq \exp\left(-\frac{1}{8}\frac{n}{d^2\log n}\cdot\frac{d^2}{2\log^2 n}\right) + O(\exp(-\sqrt{\log n}))$$

$$\leq \exp\left(-\frac{n(\log n)^{-3}}{16}\right) + O(\exp(-\sqrt{\log n}))$$

$$\leq O(\exp(-\sqrt{\log n})).$$

Here inequality (a) follows from (49), (48), and (41); inequality (b) follows from Lemma 16. This concludes the proof. $\qquad\square$