

Dynamic Assignment Control of a Closed Queueing Network under Complete Resource Pooling*

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Abstract

We study the design of dynamic assignment controls in networks with a fixed number of circulating resources (supply units). Each time a demand arises, the controller has (limited) flexibility in choosing the node from which to assign a supply unit. If no supply units are available at any compatible node, the demand is lost. If the demand is served, this causes the supply unit to relocate to the “destination” of the demand. We study how to minimize the proportion of lost requests in steady state (or over a finite horizon) via a large deviations analysis.

We propose a family of simple state-dependent policies called Scaled MaxWeight (SMW) policies that dynamically manage the distribution of supply in the network. We prove that under a complete resource pooling condition (analogous to the condition in Hall’s marriage theorem), any SMW policy leads to exponential decay of demand-loss probability as the number of supply units scales to infinity. Further, there is an SMW policy that achieves the **optimal** loss exponent among all assignment policies, and we analytically specify this policy in terms of the demand arrival rates for all origin-destination pairs. The optimal SMW policy maintains high supply levels adjacent to structurally under-supplied nodes. We discuss two applications: (i) Shared transportation platforms (like ride-hailing and bikesharing): We incorporate travel delays in our model and show that SMW policies for assignment control continue to have exponentially small loss. Simulations of ride-hailing based on the NYC taxi dataset demonstrate excellent performance. (ii) Service provider selection in scrip systems (like for babysitting or for kidney exchange): With only cosmetic modifications to the setup, our results translate fully to a model of scrip systems and lead to strong performance guarantees for a “Scaled Minimum Scrip” service provider selection rule.

Keywords: network; circulating (reusable) resources; queueing; control; shared transportation systems; scrip systems; assignment; maximum weight policy; large deviations; Lyapunov function.

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1 Introduction

Several real-world systems such as shared transportation platforms and scrip systems involve resource (supply) units circulating in a network. The hallmark of such systems is that serving a demand unit causes a (reusable) supply unit to be relocated. Closed queueing networks provide a powerful abstraction for these applications (see, e.g., Waserhole and Jost 2016, Banerjee et al. 2016, Braverman et al. 2016, Johnson et al. 2014, Kash et al. 2015). The platform operator makes tactical control decisions with the aim of maximizing longer-term system performance, which necessitates that the operator *manage the distribution of the supply* to ensure continued availability of supply throughout the network. In this paper, we focus on *dynamic assignment control* of a closed queueing network given *limited flexibility*, i.e., when a demand unit arrives at a node, from which compatible (e.g., nearby) node should a supply unit be assigned to serve it?

A central challenge in such systems is that of distributional mismatch between supply and demand: to fulfill a demand which arrives at a node, there has to be an available supply unit at a compatible node when the demand arrives. There are two sources of distributional supply-demand asymmetry: *structural imbalance* (some nodes may have a tendency to have a systematic net inflow, or outflow, of supply units) and *stochasticity*. Previous works have studied assignment (or control) decisions made in a *state-independent* manner which handles structural imbalance by solving the fluid limit problem which arises as the number of supply units K is taken to ∞ . However, this approach fails to react to stochasticity leading to optimality gap (fraction of demand lost) which shrinks to zero only (slowly) as $1/K$ (Banerjee et al. 2016) as K grows if demand arrival rates are *exactly* known, and non-vanishing optimality gap as $K \rightarrow \infty$ if demand arrival rates are not perfectly known (see Proposition 4 in Section 4.2). In this paper we propose simple and practical *state-dependent* assignment control policies which automatically handle both structural imbalance and stochasticity. Our policies come with a strong performance guarantee and do not require demand arrival rates to be known (if these rates are known, even better performance can be obtained).

We focus on demand arrival rates satisfying an approximate balance condition (very similar to Hall’s condition in matching and Complete Resource Pooling in queueing), which ensures that in the absence of stochasticity (i.e., in the fluid limit), all demand can be satisfied. The control problem remains non-trivial: all state-independent policies provide unsatisfactory performance as summarized above (Proposition 4), and a naive state-dependent policy similarly suffers $\Omega(1)$ optimality gap as $K \rightarrow \infty$ (Example 4). We provide a very simple “maximum weight” (MaxWeight) control policy which does not use demand arrival rate information and achieves optimality gap (loss) which decays *exponentially* in K . This result motivates the large deviations question: *Which policy maximizes the loss exponent?* We propose a natural family of Scaled MaxWeight (SMW) policies generalizing MaxWeight, and show that all SMW policies achieve exponentially small loss. We then prove the surprising result that there is always an SMW policy which is *exponent-optimal* among all assignment control policies, and characterize how the parameters of the optimal SMW policy are determined by the demand arrival rates.

Our Model. We adopt a stylized model which isolates the challenge of managing the distribution of (reusable) supply in the network given limited flexibility. (Later, we suitably augment this baseline model to incorporate salient features of specific applications.)

In our model, the system consists of a network with two sets of nodes, namely, the supply nodes and the demand nodes. A fixed number of supply units circulate among the supply nodes. Demand units arrive stochastically at demand nodes with supply node destinations, at some time-invariant rates. For each demand node, a subset of the supply nodes are *compatible* with it, and the platform dynamically decides from which compatible supply node to assign a supply unit to serve the incoming demand unit. Thus, compatibilities capture the *limited flexibility* available to the platform. After a supply unit is assigned to a demand unit, it becomes available again at the destination of the demand unit. (Supply units relocate only while serving demand.) Supply units do not enter or leave the system. The platform’s goal is to meet as much demand as possible in steady state. (Our results will extend to transient performance as well.)

Our model assumes that the supply units relocate instantaneously in the process of serving a demand unit. This assumption facilitates a sharp theoretical analysis of general network structures, and moreover ensures transparency about the role of supply units: all K supply units are free when a demand unit arrives, and thus K quantifies the total available “buffer” of free supply units. The controller’s challenge is that of managing the distribution of the K supply units to ensure the continued availability of supply throughout the network.

To obtain tight characterizations, we consider the asymptotic regime where the number of supply units in the system K goes to infinity, and perform a large deviations analysis.

Complete Resource Pooling condition. A main assumption in our model is an approximate balance condition on the demand arrival rates. This condition is very similar to the complete resource pooling (CRP) condition in the queueing literature, therefore we will refer to it as CRP hereafter. CRP is a standard assumption in the heavy traffic analysis of queueing systems (see, e.g., Harrison and López 1999, Dai and Lin 2008, Shi et al. 2015). It can be interpreted as requiring enough overlap in the processing ability of servers (demand nodes in our model) so that they form a “pooled server”. The CRP condition under our model is closely related to the condition in Hall’s marriage theorem in bipartite matching theory. If any CRP inequality is strictly violated, this forces a positive fraction of demand to be lost even as $K \rightarrow \infty$.

Analogy with a classic closed queueing network scheduling problem. Using the terminology of classic queueing theory, the K supply units are “jobs”, each demand location is a “server”, each supply location is a “buffer”, inter-arrival times of demand units with origin i are “service times” at server i . The distribution of demand destinations given an origin node captures “routing probabilities”. “Servers” are flexible (i.e., they can serve multiple queues), and assignment is equivalent to “scheduling”. See Section 8 for a detailed discussion of the analogy. We emphasize the reversal of the usual mapping: in our setup supply units are “jobs” and demand units act as service tokens. As a consequence, intuition based on traditional queueing systems does not easily extend to our setup.

1.1 Main Contributions

We show that a simple and practical MaxWeight assignment policy effectively manages the distribution of supply in the network, leading to a fraction of demand lost that decays exponentially fast in K . Each time a demand arrives, MaxWeight simply assigns a supply unit from the compatible node which currently has the largest number of supply units. In particular, MaxWeight requires no knowledge of demand arrival rates.

This finding motivates a thorough *large deviations analysis* which yields surprisingly elegant results. As a function of system primitives, we derive a large deviations rate-optimal assignment policy that minimizes lost demand. Our optimal policy is a close cousin of MaxWeight and its parameters depend in a natural way on demand arrival rates. Our contribution is threefold:

1. **A family of simple policies.** We propose a family of state-dependent assignment policies called Scaled MaxWeight (SMW) policies, and prove that all of them guarantee exponential decay of demand-loss probability under the CRP condition. An SMW policy is parameterized by a vector of scaling factors, one for each (supply) node; each demand is served by assigning a supply from the compatible node with the largest scaled number of supply units. SMW policies are simple, explicit and promising for practical applications (Section 6.2 and Appendix J demonstrate stellar performance in a realistic simulation environment).
2. **The value of (intelligent) state-dependent control.** We show (Proposition 4) that no state-independent assignment policy can achieve loss which decays exponentially in K , and that if demand arrival rates are not perfectly known, then the loss of a state-independent policy (generically) does not vanish as $K \rightarrow \infty$. Also, a naive state-dependent control policy suffers $\Omega(1)$ loss as $K \rightarrow \infty$ (Example 4). Our SMW policies provide vastly superior performance: even the naive unscaled (“vanilla”) MaxWeight assignment policy requiring no knowledge of demand arrival rates achieves loss which decays exponentially in K .
3. **Exponent-optimal policy and qualitative insights.** For general network structures, we obtain an explicit specification for the optimal scaling factors for SMW based on compatibilities and demand arrival rates. Further, we obtain the surprising finding that the optimal SMW policy is, in fact, *exponent-optimal* among all state-dependent policies (Theorem 1). A key ingredient of this result is that SMW policies satisfy the *critical subset* property: for each SMW policy, there is a corresponding (fluid) equilibrium state, and for this state there are “critical” subsets of demand nodes that are most vulnerable to the depletion of supply in compatible supply nodes. Each SMW policy simultaneously “protects” all critical subsets maximally by maintaining high supply levels near structurally under-supplied nodes.

We consider the natural “large market” scaling where the demand arrival rate is proportional to K , and show that each supply unit is frequently in use.

Technical contributions. To the best of our knowledge, we are the first to perform a large deviations analysis under CRP, leading to the challenging problem of deriving an exponent optimal control. One key difficulty in the mathematical analysis is the necessity to deal with a multi-dimensional system even in the limit. Usually CRP renders the control problem “easy”

because it leads to the “collapse” of the system state to a lower dimensional space in the heavy traffic limit, as in many existing works that establish the asymptotic optimality of a certain policy in minimizing the workload/holding costs of a queueing system. In contrast, in our setting, the limit system remains m -dimensional, where m is the number of supply nodes. A second key challenge we face is that the ideal state for the system is a priori unknown, making it unclear how to define a Lyapunov function. We overcome these difficulties via a novel approach. We construct a *policy-specific* Lyapunov function to facilitate a sharp large deviations analysis of a given SMW policy leveraging the machinery of Venkataramanan and Lin (2013). The analysis applies to general network structures, and reveals that the SMW policy maximally protects all the “critical subsets” of demand nodes. We deduce the existence of an exponent optimal SMW policy, and characterize its scaling factors in terms of demand arrival rates. Happily, the fluid equilibrium for this optimal policy is revealed as the ideal state.

Though our setting considers a closed network, we think that it could inspire similar analyses in open networks, e.g., when there is a shared finite buffer (e.g., a common waiting room) for multiple queues. Our technical machinery may also be broadly useful in deriving large-deviation optimal controls in settings where the ideal state is a priori unclear.

1.2 Applications

Our main model and analysis can serve as a building block towards studying various applications. We discuss two broad applications later in the paper.

Shared transportation systems. Shared transportation platforms such as those for ride-hailing and bikesharing make assignment control decisions under limited flexibility to manage the distribution of supply. In these applications, the nodes in our model correspond to geographical locations,¹ while supply units and demand units correspond to vehicles and customers, respectively. The assignment control in ride-hailing takes the form of dispatch, i.e., the platform can decide where (near the demand’s origin) to dispatch a car from. Bikesharing platforms can execute assignment control by suggesting to the customer where (near the customer’s origin or destination) to pick up (or drop off) a bike.²

We discuss the application to shared transportation systems in Section 6. Transportation involves positive travel times. We incorporate travel times into our theory and show that SMW policies retain their good performance, and also demonstrate excellent performance in realistic simulations of ridehailing:

- (i) We extend our theory by letting demand have independent exponential travel times with mean that can depend on the origin-destination pair, and assume zero pickup times. We consider the large market scaling and assume that the total service requirement (the average number of demands in service at any time assuming no lost demand) is a fraction

¹The set of supply nodes and demand nodes are replicas of each other in these applications.

²For example, the Bike Angels program of CitiBike implicitly makes these suggestions to members by awarding “points for taking bikes from crowded stations and bringing them to empty ones or stations expected to soon become empty”. Notice the resemblance to a MaxWeight approach. A live map of point awards is shown to customers.

of supply which is strictly below 1, consistent with the reality in shared transportation. We prove that for any SMW policy, the loss is again exponentially small in K .

- (ii) We demonstrate excellent performance of SMW policies in simulations of ride-hailing based on the NYC taxi dataset. We propose data-driven approaches for “learning” SMW scaling factors via simulations, and observe close alignment of the resulting SMW scaling factors with those suggested by our theoretical analysis.

We also describe how state-independent “empty” relocation of vehicles can be seamlessly incorporated in our setup.

Scrip systems. A scrip system is a nonmonetary trade economy where agents use scrips (tokens, coupons, artificial currency) to exchange services. These systems are typically implemented when monetary transfer is undesirable or impractical. For example, Agarwal et al. (2019) suggest that in kidney exchange, to align the incentives of hospitals, the exchange should deploy a scrip system that awards points to hospitals that submit donor-patient pairs to the central exchange, and deducts points from hospitals that conduct transplantations. Another well-known example is Capitol Hill Babysitting Co-op (Sweeney and Sweeney 1977, see also Johnson et al. 2014), where married couples pay for babysitting services by another couples with scrips. A key challenge in these markets is the design of the service provider selection rule: among the possible providers for a requested service/trade, who should be selected for service? The platform operator tries to minimize discarded requests (which happen when the service requester runs out of scrips) by choosing this rule appropriately. We will show in Section 7 that with only cosmetic modifications to the setup, our results translate fully to a model of scrip systems; in particular we derive exponent-optimal control policies for these systems.

1.3 Literature Review

MaxWeight scheduling. MaxWeight is a simple scheduling policy in constrained queueing networks which (roughly speaking) chooses the feasible control decision that serves the queues with largest total weight (e.g. queue length, head-of-line waiting time, etc.), at each time. MaxWeight scheduling has been shown to exhibit good performance in various settings (see, e.g., Tassiulas and Ephremides 1992, Dai and Lin 2005, Stolyar 2004, Dai and Lin 2008, Eryilmaz and Srikant 2012, Maguluri and Srikant 2016), including by Shi et al. (2015) who study an open one-hop network version of our setting. In contrast, we find that MaxWeight achieves a suboptimal exponent in our closed network setting.

Large deviations in queueing systems. There is a large literature on characterizing the probability of building up long queues in *open* queueing networks, including controlled (see, e.g., Stolyar and Ramanan 2001, Stolyar 2003) and uncontrolled (see, e.g., Majewski and Ramanan 2008, Blanchet 2013) networks. The work closest to ours is that of Venkataramanan and Lin (2013), who established the relationship between Lyapunov functions and buffer overflow probability for open queueing networks. The key difficulty in extending the Lyapunov approach to closed queueing networks is the lack of a natural reference state where the Lyapunov function

equals to 0 (in an open queueing network the reference state is simply $\mathbf{0}$). It turns out that as we optimize the MaxWeight parameters we are also solving for the best reference state.

Applications: shared transportation systems, scrip systems. Ozkan and Ward (2016) studied revenue-maximizing state-independent assignment control by solving a minimum cost flow problem in the fluid limit. Braverman et al. (2016) modeled the system by a closed queueing network and derived the optimal static routing policy that sends empty vehicles to under-supplied locations. Banerjee et al. (2016) adopted the Gordon-Newell closed queueing network model and considered static pricing/repositioning/matching policies that maximizes throughput, welfare or revenue. In contrast to our work, which studies state-dependent control, these works consider static control that completely relies on system parameters. In terms of convergence rate to the fluid-based solution, Ozkan and Ward (2016) did not study the convergence rate of their policy, Braverman et al. (2016) observed from simulation an $O(1/\sqrt{K})$ convergence rate as the number of supply units in the closed system K goes to infinity,³ while Banerjee et al. (2016) showed finite system bounds with an $O(1/K)$ convergence rate as $K \rightarrow \infty$ in the absence of service times and an $O(1/\sqrt{K})$ convergence rate with service times. All these works propose static policies, and we show that no static policy can achieve exponentially small loss. In contrast, under the CRP condition, we obtain exponentially small loss in K , and further obtain the optimal exponent.

Our approach of studying control while initially ignoring travel delays is mirrored in several papers in this literature, starting with Waserhole and Jost (2016). The main model in Banerjee et al. (2016) ignores travel delays, and the paper subsequently shows that all its findings are robust to that assumption. Similarly, subsequent to the present paper, Balseiro et al. (2019) study the control of (large) networks of circulating resources by ignoring travel delays and then show robustness of their results to delays.

There have been a few papers that model and analyze scrip systems, e.g., Friedman et al. (2006), Kash et al. (2012), Johnson et al. (2014), Kash et al. (2015) etc. The closest paper to ours is Johnson et al. (2014), which considers the case where the compatibility graph is fully connected and the demand arrival rates are identical for each demand type. They propose a service selection rule which is the same as the vanilla version of our proposed policy and show that it is optimal in their symmetric setting. We significantly generalize their model by considering asymmetric demand arrivals and general skill compatibility graphs. For other examples of scrip systems, see, e.g., Sweeney and Sweeney (1977), Agarwal et al. (2019), etc.

Online stochastic bipartite matching. There is a related stream of research on online stochastic bipartite matching, see, e.g., Caldentey et al. (2009), Adan and Weiss (2012), Bušić and Meyn (2015), Mairesse and Moyal (2016). Different types of supplies and demands arrive over time, and the system manager matches supplies with demands of compatible types using a specific matching policy, and then discharges the matched pairs from the system. Our work is

³In the setting of Braverman et al. (2016), the loss probability can remain positive even as K grows, in contrast with our setting where the loss probability can always be sent to 0 because of our CRP condition under which the flows in the network can potentially be balanced. The comparison of convergence rates is most meaningful if we restrict attention to instances in their setting where the loss probability goes to zero as K grows.

different in that we study a *closed* system where supply units never enter or leave the system. Moreover, this literature focuses on the stability and other properties under a given policy instead of looking for the optimal control (except Bušić and Meyn 2015).

Other related work. Jordan and Graves (1995), Désir et al. (2016), Shi et al. (2015) and others study how process flexibility can facilitate improved performance, analogous to our use of dispatch control to improve demand fulfillment. Along similar lines, network revenue management is a classical dynamic resource allocation problem, see, e.g., Gallego and Van Ryzin (1994), Talluri and Van Ryzin (2006), and recent works, e.g., Jasin and Kumar (2012), Bumpensanti and Wang (2018). Different types of demands arrive over time, and a centralized decision is made at each arrival. Again, each of these settings is “open” in that each service token or supply unit can be used only once, in contrast to our closed setting.

1.4 Organization of the paper

The remainder of our paper is organized as follows. In Section 2 we introduce the basic notation and formally describe our baseline model together with the performance metric. In Section 3 we introduce the family of Scaled MaxWeight policies. In Section 4 we present our main theoretical result, i.e., that there is an exponent optimal SMW policy for any set of primitives satisfying our main assumption. In Section 5 we prove the exponent optimality of SMW policies. In Section 6 we discuss the application to shared transportation systems. In Section 7 we discuss the application to scrip systems. We conclude in Section 8.

Notation. We use \mathbf{e}_i to denote the i -th unit vector, and $\mathbf{1}$ the all-1 vector. The dimensions of the vectors will be clear from the context. For a finite index set A , define $\mathbf{1}_A \triangleq \sum_{i \in A} \mathbf{e}_i$. For a set Ω in Euclidean space \mathbb{R}^n , denote its relative interior by $\text{relint}(\Omega)$. For event C , we define the indicator random variable $\mathbb{I}\{C\}$ to equal 1 when C is true, else 0. All vectors are column vectors if not specified otherwise.

2 The Model and Preliminaries

2.1 Basic Setting

We study the dynamic assignment problem in networks with circulating resources. We consider an infinite-horizon continuous-time model, with a fixed number K of identical *supply units* that circulate in the network. Formally, we consider a sequence of systems indexed by $K \in \mathbb{Z}_+$.

The (Assignment) Compatibility Graph. The assignment compatibility structure is described by a bipartite *compatibility graph* $G = (V_S \cup V_D, E)$, where the K supply units are distributed over the *supply nodes* V_S , and demand units arrive at the *demand nodes*⁴ V_D . We add a prime symbol to the indices of nodes in V_D to distinguish between the two. Let $m \triangleq |V_S|$ and $n \triangleq |V_D| \in \mathbb{Z}_+$ be the number of supply and demand nodes, respectively. Each edge

⁴The physical meaning of the nodes depends on the application. For example, in ride-hailing the supply nodes and demand nodes are replicas of each other and both stand for physical locations. However, our result does not require the symmetry between these two sets of nodes.

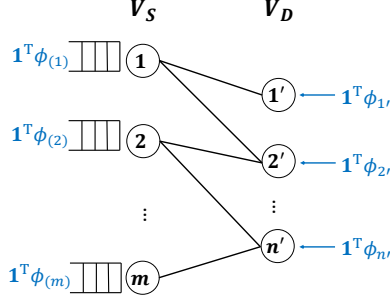


Figure 1: The bipartite (assignment) compatibility graph: On the left are supply nodes $i \in V_S$, and on the right are demand nodes $j' \in V_D$. The edges entering a demand node j' encode compatible (e.g., nearby) supply nodes that can serve node j' . The (normalized) rate of arrival of demand with origin j' is $\mathbf{1}^T \phi_{j'}$. Assuming no demand is lost, the (normalized) rate of arrival of supply units to i is $\mathbf{1}^T \phi_{(i)}$ (this is the normalized arrival rate of demand with destination i).

$(i, j') \in E$ represents a compatible pair of supply and demand nodes, i.e., a supply unit currently stationed at $i \in V_S$ can serve demand arriving at $j' \in V_D$. See Figure 1 for an illustration. We denote the neighborhood of a supply node $i \in V_S$ (resp. demand node $j' \in V_D$) in G as $\partial(i) \subseteq V_D$ (resp. $\partial(j') \subseteq V_S$); thus, for a supply node i , its compatible demand nodes are given by $\partial(i) = \{j' \in V_D \mid (i, j') \in E\}$, and similarly for each demand node. Moreover, for any set of supply nodes $A \subseteq V_S$, we also use $\partial(A)$ to denote its demand neighborhood (and vice versa).

Demand Types and Arrival Process. We denote the type of a demand as $(j', k) \in V_D \times V_S$, where j' is its *origin* node and k is its *destination* node. Demand units of each type (j', k) arrive sequentially following independent Poisson processes with rates $\hat{\phi}_{j'k}^K$. We use $\hat{\phi}^K$ to denote the $n \times m$ matrix of demand arrival rates.

We will consider the asymptotic regime where both the number of supply units K and demand arrival rates $\hat{\phi}^K \triangleq K \hat{\phi}$ (for some $\hat{\phi}$ which does not depend on K) go to infinity together. We call this scaling the *large market regime*. We will later show that the large market scaling ensures that each supply unit waits an $O(1)$ amount of time in expectation between two consecutive assignments under the family of policies we prescribe (see Section 4).

The *demand type distribution* is $\phi \triangleq \frac{\hat{\phi}}{\mathbf{1}^T \hat{\phi} \mathbf{1}}$, which is the normalized version of $\hat{\phi}$. We will find it convenient to carry out our technical development and analysis in terms of $\phi \in \mathbb{R}^{n \times m}$ instead of $\hat{\phi}$ wherever the total arrival rate $\mathbf{1}^T \hat{\phi} \mathbf{1}$ does not play a role. We denote the k -th column of ϕ (i.e., the normalized arrival rates at different origins of demands with destination k) as $\phi_{(k)}$, and the transpose of the j' -th row of ϕ (i.e., the normalized arrival rates of demands with origin j' and different destination nodes) as $\phi_{j'}$. Thus, the (normalized) rate of a demand units arriving at node j' is $\mathbf{1}^T \phi_{j'}$, and, assuming all demands are matched, the (normalized) rate of supply units arriving at node k is $\mathbf{1}^T \phi_{(k)}$. (We exclude demand nodes with zero demand arrival rate $\{j' : \mathbf{1}^T \phi_{j'} = 0\}$ from V_D .)

We use the term *network* to refer to a given set of primitives: an assignment compatibility graph G and demand type distribution matrix ϕ . We make two mild assumptions on the network.

Assumption 1 (Connectedness). *A network (G, ϕ) is connected if for every ordered pair of*

distinct supply nodes $(k_0, i) \in V_S \times V_S$, $k_0 \neq i$, there is a finite sequence of demand types $(j'_1, k_1), \dots, (j'_\ell, k_\ell = i)$ such that $\phi_{j'_r k_r} > 0$ for all $r = 1, \dots, \ell$, and $k_{r-1} \in \partial(j'_r)$ for all $r = 1, \dots, \ell$.

Assumption 1 requires that for every pair of supply nodes, there is a sequence of demand types with positive arrival rates and corresponding compatible supply nodes that would take a supply unit from one node eventually to the other node.

We now observe that if the compatibility graph affords ample flexibility, specifically, if the destination for every demand type belongs to the compatible neighborhood of the origin, then the control problem is trivial.

Proposition 1 (Ample flexibility renders the control problem trivial). *Consider any network (G, ϕ) which satisfies Assumption 1 and such that for all $j' \in V_D$ and $k \in V_S$ such $\phi_{j'k} > 0$ it holds that $k \in \partial(j')$. Then for any $K \geq n \triangleq |V_D|$, there is a control policy which loses an identically zero fraction of demand in the long run. Formally, there is a policy U such that $\mathbb{P}_p^{K,U} = 0$, for $\mathbb{P}_p^{K,U}$ defined in (2) below.*

The reason is simple: we can “reserve” a supply unit for each demand origin node $j' \in V_D$, and each reserved supply unit will never leave the corresponding neighborhood $\partial(j')$, ensuring that no demand is ever lost. The proof of Proposition 1 is in Appendix F.

Proposition 1 motivates the following assumption to ensure that the flexibility available is sufficiently limited that the assignment control problem at hand is non-trivial.

Assumption 2 (Limited flexibility). *A network (G, ϕ) has limited flexibility if there exists an origin-destination pair $j' \in V_D$ and $k \in V_S$ such that $k \notin \partial(j')$ and $\phi_{j'k} > 0$, i.e., the destination k for these demand units is not a supply node compatible with their origin j' .*

Simplifying assumptions regarding relocation of supply. We make the simplifying assumptions that the relocation of a supply unit upon serving a demand is instantaneous, and that a supply unit does not move unless assigned. These assumptions parallel that in an emerging line of works studying control of systems with circulating resources, e.g. Banerjee et al. (2016), Balseiro et al. (2019). The assumptions keep the state space manageable while retaining the complex supply externalities between nodes (namely, serving a demand redistributes the supply by causing a supply unit to relocate to a specific destination), which is the key challenge that we focus on. We relax the instantaneous relocation assumption in Section 6.1 and in Section 6.2 (simulations) and show that our insights are robust to this assumption. In Section 8 we observe that “empty” relocation (as may occur in ride-hailing) which is state independent can be seamlessly integrated into our framework.

System State. For the K -th system, its state at any time is given by \mathbf{X}^K , an m -dimensional vector that tracks the number of supply units at each supply node. The state space of the K -th system is thus given by $\Omega_K \triangleq \{\mathbf{x} \in \{0, 1, 2, \dots\}^m \mid \mathbf{1}^T \mathbf{x} = K\}$. Note that the normalized state $\frac{1}{K} \mathbf{X}^K$ lies in the m -probability simplex $\Omega = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} \geq 0, \mathbf{1}^T \mathbf{x} = 1\}$. We use $\mathbf{X}^K(0)$ to denote the initial state.

2.2 Optimal Assignment Control

Given the above setting, the problem we want to study is how to design assignment policies which minimize the probability of losing demand. For fixed K , this problem can be formulated as an average cost Markov decision process on a finite (albeit, very large) state space, and is thus known to admit a stationary optimal policy (i.e., where the assignment rule at any time only depends on the current system state \mathbf{X}^K ; see Proposition 5.1.3 in Bertsekas 1995).

Assignment policies. Upon the arrival of an incoming demand of type (j', k) , the platform must immediately assign a supply unit from a compatible node of j' ; subsequently, after serving the demand, the supply unit becomes available at the destination node k . If no supply unit is available at any compatible node of j' , then we experience a *demand loss*, wherein the demand unit leaves the system without being served. Let \mathcal{U}^K be the set of stationary policies for the K -th system. An assignment policy $U \in \mathcal{U}$ consists of, for each $j' \in V_D$, $k \in V_S$, a sequence of mappings $(U^K \in \mathcal{U}^K)_{K=1}^\infty$, which map the current queue-length vector \mathbf{X}^K and demand type (j', k) to $U^K[\mathbf{X}^K](j', k) \in \partial(j') \cup \{\emptyset\}$. Here $U^K[\mathbf{X}^K](j', k) = i$ means given the current state \mathbf{X}^K , we assign a supply unit from $i \in \partial(j')$ to fulfill demand with origin j' and destination k , and $U^K[\mathbf{X}^K](j', k) = \emptyset$ means that the platform does not assign supplies to type (j', k) demands and hence any such demand is lost. When $\mathbf{X}_i^K = 0$ for all $i \in \partial(j')$, this forces $U^K[\mathbf{X}^K](j', k) = \emptyset$ since there is no supply at nodes compatible to j' . For simplicity of notation, we refer to the policies by U instead of U^K .

System Evolution. Let t_r be the r -th demand arrival epoch after time 0. Denote the state of the system just before t_r by $\mathbf{X}^K(t_r^-)$ (the initial state is $\mathbf{X}^K(0)$); note that this incorporates the state change due to serving the $(r-1)$ -th demand arrival for $r > 1$. Now suppose the platform uses an assignment policy U , and the r -th demand arrival has origin node $o[r]$ with destination $d[r]$ (sampled from demand type distribution ϕ). Let $S[r] \triangleq U^K[\mathbf{X}^K(t_r^-)](o[r], d[r])$ be the chosen supply node (potentially \emptyset). Then, formally, the system state updates as per

$$\mathbf{X}^K(t_r) \triangleq \begin{cases} \mathbf{X}^K(t_r^-) - \mathbf{e}_{S[r]} + \mathbf{e}_{d[r]} & \text{if } S[r] \in V_S, \\ \mathbf{X}^K(t_r^-) & \text{if } S[r] = \emptyset. \end{cases}$$

Performance Measure. The platform's goal is to find an assignment policy that loses as few demands as possible in steady state. A natural performance measure is the *long-run average demand-loss probability*. Formally, for $U \in \mathcal{U}$ we define

$$\mathbb{P}_o^{K,U} \triangleq \min_{\mathbf{X}^{K,U}(0) \in \Omega_K} \mathbb{E} \left(\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{r=1}^T \mathbb{I} \left\{ U^K[\mathbf{X}^{K,U}(t_r^-)](o[r], d[r]) = \emptyset \right\} \right), \quad (1)$$

$$\mathbb{P}_p^{K,U} \triangleq \max_{\mathbf{X}^{K,U}(0) \in \Omega_K} \mathbb{E} \left(\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{r=1}^T \mathbb{I} \left\{ U^K[\mathbf{X}^{K,U}(t_r^-)](o[r], d[r]) = \emptyset \right\} \right). \quad (2)$$

Here (1) is an *optimistic* (subscript “o” for optimistic) performance measure (which underestimates demand-loss probability), whereas (2) is a *pessimistic* (subscript “p” for pessimistic)

performance measure (which overestimates demand-loss probability). Since $U \in \mathcal{U}$ is a stationary policy, the limits in (1) and (2) exist. Note that $\mathbb{P}_o^{K,U} \leq \mathbb{P}_p^{K,U}$. We will establish the exponent optimality of our policy by showing that its pessimistic measure decays as fast with K as any policy's optimistic measure can possibly decay.

The exact values of (1) and (2) for fixed K are challenging to study. To this end, the main performance measures of interest in this work are the decay rates of $\mathbb{P}_o^{K,U}$ and $\mathbb{P}_p^{K,U}$ as $K \rightarrow \infty$:

$$\gamma_o(U) \triangleq -\liminf_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_o^{K,U}, \quad (3)$$

$$\gamma_p(U) \triangleq -\limsup_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_p^{K,U}. \quad (4)$$

For brevity, we henceforth refer to these as the *demand-loss exponents*. Note that $\gamma_o(U) \geq \gamma_p(U)$. The definition (3) uses \liminf so that we can state a strong converse result by upper bounding $\sup_{U \in \mathcal{U}} \gamma_o(U)$, since no policy can achieve a larger demand-loss exponent. Similarly, the definition (4) uses \limsup so that we can state a strong achievability result (for our proposed policies the limit will exist; when the limit exists we write $\gamma(U) \triangleq \gamma_o(U) = \gamma_p(U)$).

2.3 The Complete Resource Pooling (CRP) Condition

We now make a few additional definitions to allow us to state our main assumption.

We say that a subset of demand nodes $J \subsetneq V_D$ has *limited flexibility* if there is some demand node $j' \in J$ and supply node $k \notin \partial(J)$ such that $\phi_{j'k} > 0$. (Informally, there is a demand type which requires supply units to leave the neighborhood of J .) We denote the set of limited-flexibility subsets by \mathcal{J} . Assumption 2 guarantees that there is at least one non-trivial singleton J and hence that $\mathcal{J} \neq \emptyset$.

Observe that J has limited flexibility if and only if

$$\mu_J \triangleq \sum_{j' \in J} \sum_{k \notin \partial(J)} \phi_{j'k} > 0. \quad (5)$$

We call μ_J the *net demand* of J , since it captures the probability that a demand arrival has origin in J and destination outside $\partial(J)$ (and hence requires a supply unit to leave $\partial(J)$). Similarly, we define the (optimistic) *net supply* to J as

$$\lambda_J \triangleq \sum_{j' \notin J} \sum_{k \in \partial(J)} \phi_{j'k}. \quad (6)$$

Informally, λ_J the probability that a demand arrival is such that it can (depending on the assignment decision) cause a supply unit to enter $\partial(J)$.

The following is the main assumption of this paper.

Assumption 3 (Complete Resource Pooling). *We assume that for all subsets of demand nodes J with limited flexibility (i.e., $J \subsetneq V_D$ with positive net demand $\mu_J > 0$) we have that $\lambda_J > \mu_J$, where the net supply λ_J was defined in (6), and the net demand μ_J was defined in (5).*

The intuition behind this assumption is simple: it assumes the system is “balanceable” in

that for each subset $J \subsetneq V_D$ of demand nodes, supply arrives sufficiently fast at neighboring nodes to meet the demand arriving to J , on average. Assumption 3 is equivalent to a strict version of the condition in Hall’s marriage theorem. It is also closely related to the Complete Resource Pooling (CRP) condition in queueing: we show (formalized in Proposition 7 that in Appendix I) if the “open queueing network counterpart” of network $(G, \hat{\phi})$ satisfies the CRP condition defined in Dai and Lin (2008), then the network $(G, \hat{\phi})$ satisfies Assumption 3. The control problem under CRP is non-trivial: In Section 4.2 we will show that all state-independent policies and a naive state-dependent policy perform inadequately.

We remark that the condition $\lambda_J > \mu_J$ is equivalent to $\sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)} > \sum_{j' \in J} \mathbf{1}^T \phi_{j'}$ (informally, that the total supply to J exceeds total demand of J), but the representation $\lambda_J > \mu_J$ will turn out to be more closely related to our analysis and our main theorem. We will find that the limited-flexibility subsets J with ratio λ_J/μ_J close to 1, i.e., only a small excess of supply over demand, will be pivotal in determining the performance of our policies and optimal policy design. We illustrate the quantities involved (\mathcal{J} , λ_J and μ_J) and their impact on policy performance and design via an example at the end of the next section (Example 1).

We show that Assumption 3 is necessary in order to obtain exponentially small loss in Proposition 2.

Proposition 2. *For any G and ϕ ’s such that Assumption 3 is violated, it holds that for any policy U , the demand loss probability does not decay exponentially,⁵ i.e., $\gamma_o(U) = \gamma_p(U) = 0$ where $\gamma_o(U)$ and $\gamma_p(U)$ are defined in (3) and (4).*

In other words, if Assumption 3 is violated, this means the system has significant distributional imbalance of demand and demand loss is unavoidable. The intuition is similar to that of Hall’s marriage theorem (Marshall Hall 1986): if there is a limited-flexibility subset J with net supply (weakly) less than the net demand, then it is impossible for any policy to ensure that all but an exponentially small fraction of demand originating in J is served. The proof of Proposition 2 is in Appendix F.

2.4 Sample Path Large Deviation Principle

Our main theoretical result is the culmination of a sharp *large deviations* analysis, characterizing the best possible demand loss exponent. We provide a brief introduction to classical large deviations theory in this subsection.

For each fixed $K \in \mathbb{Z}_+$ and $T \in (0, \infty)$, define a *scaled sample path* of accumulated demand arrivals $\bar{\mathbf{A}}^K(\cdot) \in (L^\infty[0, T])^{n \times m}$ as follows.⁶ Let $\{A_{j'k}^K(\cdot)\}_{j' \in V_D, k \in V_S}$ be independent Poisson processes where $A_{j'k}^K(\cdot)$ has rate $K\hat{\phi}_{j'k}$. Let

$$\bar{\mathbf{A}}_{j'k}^K(t) \triangleq \frac{1}{K} \mathbf{A}_{j'k}^K(t) \quad \forall t \in [0, T]. \quad (7)$$

⁵If the inequality in Assumption 3 is strictly *reversed* for some $J \subsetneq V_D$, i.e., $\lambda_J < \mu_J$ then we have a demand loss probability which is at least $\epsilon > 0$ for all K , where $\epsilon = \sum_{j' \in J} \mathbf{1}^T \phi_{j'} - \sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)}$.

⁶Here $L^\infty[0, T]$ denotes the space of bounded functions on $[0, T]$ equipped with the supremum norm.

Let μ_K be the law of $\bar{\mathbf{A}}^K(\cdot)$ in $(L^\infty[0, T])^{n \times m}$. For all $\mathbf{f} \in \mathbb{R}_+^{n \times m}$, let

$$\Lambda^*(\mathbf{f}) \triangleq \begin{cases} \sum_{j' \in V_D} \sum_{k \in V_S} \left(f_{j'k} \log \frac{f_{j'k}}{\hat{\phi}_{j'k}} - f_{j'k} + \hat{\phi}_{j'k} \right) & \text{if } \mathbf{f} > \mathbf{0}, \\ \infty & \text{otherwise.} \end{cases} \quad (8)$$

For any set Γ , let $\bar{\Gamma}$ denote its closure, and Γ° denote its interior. Below is the sample path large deviation principle (also known as Mogulskii's Theorem, see Dembo and Zeitouni 1998):

Fact 1. *For measures $\{\mu_K\}$ defined above, and any arbitrary measurable set $\Gamma \subseteq (L^\infty[0, T])^{n \times m}$, we have*

$$-\inf_{\bar{\mathbf{A}} \in \Gamma^\circ} I_T(\bar{\mathbf{A}}) \leq \liminf_{K \rightarrow \infty} \frac{1}{K} \log \mu_K(\Gamma) \leq \limsup_{K \rightarrow \infty} \frac{1}{K} \log \mu_K(\Gamma) \leq -\inf_{\bar{\mathbf{A}} \in \bar{\Gamma}} I_T(\bar{\mathbf{A}}), \quad (9)$$

where the rate function⁷ is:

$$I_T(\bar{\mathbf{A}}) \triangleq \begin{cases} \int_0^T \Lambda^*\left(\dot{\bar{\mathbf{A}}}(t)\right) dt & \text{if } \bar{\mathbf{A}}(\cdot) \in \text{AC}[0, T], \bar{\mathbf{A}}(0) = \mathbf{0}, \\ \infty & \text{otherwise.} \end{cases} \quad (10)$$

Here $\text{AC}[0, T]$ is the space of absolutely continuous functions on $[0, T]$, and $\dot{\bar{\mathbf{A}}}(t)$ is the derivative of $\bar{\mathbf{A}}$ at time t when the derivative exists.

Informally, this fact says the following. (Suppose the leftmost term and rightmost term in (9) are equal.) The probability exponent (with respect to K) for the event Γ is equal to the exponent for the most likely fluid sample path (a limit of scaled sample paths, see Section 5.1) of demand $\bar{\mathbf{A}}$ such that the event occurs. The exponent for $\bar{\mathbf{A}}$ is the *time integral of the exponent for its time derivative*, and the latter is given by the function (8) where the summand is the large deviations exponent of a (sequence of) Poisson random variable(s) with mean $\hat{\phi}_{j'k}$.

In the present work, the relevant Γ will be the demand-loss event. The reason the sample paths of accumulated demand arrivals fully determine whether this event occurs is because given any deterministic policy (as the policies we propose will be), the arrival process $\mathbf{A}(\cdot)$ and the initial configuration $\mathbf{X}(0)$ uniquely determine the evolution of the system state $\mathbf{X}(\cdot)$, and hence determine demand loss. The key will be to understand the most likely sample paths of the arrival process which lead to demand loss. Our converse (impossibility) bound on the exponent will be established by constructing a fluid sample path of demand arrivals that *always* leads to demand loss regardless of the policy.

3 Scaled MaxWeight Policies

The traditional MaxWeight policy is a celebrated approach to scheduling which has been effectively deployed in many applications such as cloud computing, communication networks, traffic management, etc., (see, e.g., Tassiulas and Ephremides 1992, Maguluri et al. 2012). MaxWeight (hereafter referred to as vanilla MaxWeight) allocates the service capacity to the queue(s) with

⁷Since absolutely continuous functions are differentiable almost everywhere, the rate function is well-defined.

largest “weight” (where weight can be any relevant parameter such as queue length, head-of-the-line waiting time, etc.). In our setting, supply units form queues and demand is like service tokens, and vanilla MaxWeight would correspond to assigning from the compatible supply node with most supply units (with appropriate tie-breaking rules).

Besides its simplicity, one reason for the popularity of MaxWeight is that it is known to be asymptotically optimal in many problem settings (e.g., see Stolyar 2003, 2004, Shi et al. 2015, Maguluri and Srikant 2016). In our setting too, we will find that vanilla MaxWeight is asymptotically optimal. In fact, we will show that it achieves an exponentially small loss. However, we will find that, in general, vanilla MaxWeight does not achieve the largest possible loss exponent. (We will provide a concrete example at the end of this section.) Suboptimality of the exponent prompts us to consider alternate control policies.

We generalize vanilla MaxWeight by attaching a positive scaling parameter α_i to each queue $i \in V_S$, and assign from the compatible queue with largest *scaled* queue length \mathbf{X}_i/α_i . Without loss of generality, we normalize $\boldsymbol{\alpha}$ s.t. $\mathbf{1}^T \boldsymbol{\alpha} = 1$, or equivalently, $\boldsymbol{\alpha} \in \text{relint}(\Omega)$. We call this family of policies *Scaled MaxWeight (SMW) policies*, and use $\text{SMW}(\boldsymbol{\alpha})$ to denote SMW with parameter $\boldsymbol{\alpha}$.

The formal definition of SMW is as follows.

Definition 1 (Scaled MaxWeight $\text{SMW}(\boldsymbol{\alpha})$). *Fix $\boldsymbol{\alpha} \in \text{relint}(\Omega)$, i.e., $\boldsymbol{\alpha} \in \mathbb{R}^m$ such that $\alpha_i > 0 \forall i \in V_S$ and $\sum_{i \in V_S} \alpha_i = 1$. Given system state $\mathbf{X}(t_r^-)$ just before the r -th demand arrival and for demand arriving at demand node j' , $\text{SMW}(\boldsymbol{\alpha})$ assigns from*

$$\text{argmax}_{i \in \partial(j')} \frac{\mathbf{X}_i(t_r^-)}{\alpha_i}$$

if $\max_{i \in \partial(j')} \frac{\mathbf{X}_i(t_r^-)}{\alpha_i} > 0$; otherwise the demand is lost. (If there are ties when determining the argmax, it assigns from the location with highest index.⁸)

As may be expected, SMW policies tend to equalize the scaled queue lengths if CRP holds. The following fact is formalized later in Proposition 6 in Section 5.

Remark 1 (Resting state under $\text{SMW}(\boldsymbol{\alpha})$). *If Assumptions 1, 2 and 3 hold then for any $\boldsymbol{\alpha} \in \text{relint}(\Omega)$, the $\text{SMW}(\boldsymbol{\alpha})$ policy has a “resting state” $\boldsymbol{\alpha}$: Specifically, consider using $\text{SMW}(\boldsymbol{\alpha})$ on a sequence of systems indexed by the number of supply units K . Then there exists $T_0 = T_0(\boldsymbol{\alpha}) > 0$ which does not depend on K , such that for any $T > T_0$,*

$$\limsup_{K \rightarrow \infty} \left(\max_{\mathbf{X}^K(0) \in \Omega_K} \left\| \frac{1}{K} \mathbf{X}^{K, \boldsymbol{\alpha}}(T) - \boldsymbol{\alpha} \right\|_2 \right) = 0 \quad \text{almost surely,}$$

where $\mathbf{X}^{K, \boldsymbol{\alpha}}(T)$ is the state of the K -th system at time T .

We conclude this section with an example which illustrates our model and SMW policies, and provides a brief preview of our main result.

⁸Our analysis and results are unchanged if any other deterministic tie-breaking rule is employed instead.

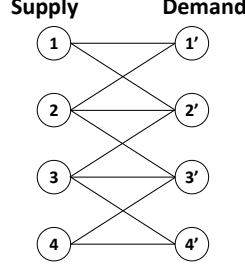


Figure 2: An example compatibility graph.

Example 1. Consider a network with “line-of-four-nodes” compatibility graph given as

$$G = (V_S \cup V_D, E) = (\{1, 2, 3, 4\} \cup \{1', 2', 3', 4'\}, \{11', 12', 21', 22', 23', 32', 33', 34', 43', 44'\});$$

see Figure 2. Let the demand type distribution ϕ , supported on types $\{1'3, 2'4, 3'1, 4'2\}$, be

$$\phi_{1'3} = \phi_{2'4} = 0.25, \quad \phi_{3'1} = 0.1, \quad \phi_{4'2} = 0.4.$$

It is easy to verify that the network (G, ϕ) satisfies Assumptions 1 and 2. It also satisfies the CRP condition (Assumption 3): Table 3 lists the limited-flexibility subsets \mathcal{J} , i.e., the demand node subsets J whose net demand $\mu_J > 0$, and their neighborhoods, net supply λ_J and net demand. For example, $\lambda_{\{1'\}} = \phi_{3'1} + \phi_{4'2} = 0.5$ and $\mu_{\{1'\}} = \phi_{1'3} = 0.25$. We see that the net supply exceeds net demand $\lambda_J > \mu_J$ for each limited-flexibility subset, as required. We also observe that the log ratio $\xi_J \triangleq \log\left(\frac{\lambda_J}{\mu_J}\right)$ is smallest for $J = \{4'\}$.

Our main result (in the next section) will tell us that because this network satisfies our assumptions, for any $\alpha \in \text{relint}(\Omega)$, the $\text{SMW}(\alpha)$ policy achieves a loss which decays exponentially in K . The result will moreover say that the loss exponent achieved by $\text{SMW}(\alpha)$ is explicitly given by $\gamma(\alpha) = \min_{J \in \mathcal{J}} \mathbf{1}_{\partial(J)}^T \alpha \cdot \xi_J > 0$, and establish that there is an SMW policy which is globally exponent optimal. In particular, in this example:

- (Optimal SMW policy) The SMW policy with

$$\alpha = \bar{\alpha} = \begin{bmatrix} \frac{b}{2} & \frac{b}{2} & \frac{1-b}{2} & \frac{1-b}{2} \end{bmatrix}^T \quad \text{for } b = \frac{\log 1.25}{\log 2 + \log 1.25} \approx 0.244 \quad (11)$$

has (normalized) resting state $\bar{\alpha}$ and achieves loss exponent $\gamma(\bar{\alpha}) = \frac{\log 1.25 \cdot \log 2}{\log 2 + \log 1.25} \approx 0.169$. $\text{SMW}(\bar{\alpha})$ maximizes $\gamma(\alpha)$ and is, in fact, exponent optimal among all possible policies.

- (Vanilla MaxWeight achieves a suboptimal exponent) The vanilla MaxWeight policy has (normalized) resting state $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T$ and achieves a loss exponent $0.5 \log 1.25 \approx 0.112$.

Note that the resting state $\bar{\alpha}$ of the exponent optimal policy “protects” the subset $\{4'\}$ which has the smallest λ_J/μ_J by putting $\alpha_3 + \alpha_4 = 1 - b \approx 75.6\%$ fraction of supply in its neighborhood.⁹

⁹In this example, it turns out that the achieved exponent $\gamma(\alpha) = \max((\alpha_1 + \alpha_2)\xi_{\{1'\}}, (\alpha_3 + \alpha_4)\xi_{\{4'\}})$ hinges entirely on the tradeoff between protecting $\{1'\}$ and $\{4'\}$. Specifically, SMW with any $\alpha \in \text{relint}(\Omega)$ satisfying $\alpha_3 + \alpha_4 = 1 - b \approx 75.6\%$ is exponent optimal, and α defined in (11) represents one such choice.

Table 1: Limited-flexibility subsets $J \in \mathcal{J}$ in Example 1, their neighborhood $\partial(J)$, net demand μ_J and net supply λ_J .

J	$\partial(J)$	μ_J	λ_J	$\xi_J \triangleq \log\left(\frac{\lambda_J}{\mu_J}\right)$
$\{1'\}$	$\{1, 2\}$	0.25	0.5	0.69
$\{1', 2'\}$	$\{1, 2, 3\}$	0.25	0.5	0.69
$\{3', 4'\}$	$\{2, 3, 4\}$	0.1	0.5	1.61
$\{4'\}$	$\{3, 4\}$	0.4	0.5	0.22

4 Main Result

In this section we present our main result, which says that for any network such that CRP holds: (i) All Scaled Maxweight (SMW) policies yield exponential decay of demand loss in the number of supply units K , with an exponent which we explicitly specify. (ii) For scaling parameter vector α which maximizes the exponent among SMW policies, the $\text{SMW}(\alpha)$ policy is exponent optimal among all possible policies. In sharp contrast, we show in Section 4.2 that that no state-independent assignment policy can achieve loss which decays exponentially in K , and moreover that if demand arrival rates are not perfectly known, then the loss of a state-independent policy (generically) does not vanish as $K \rightarrow \infty$. Also, a naive state-dependent control policy suffers $\Omega(1)$ loss as $K \rightarrow \infty$.

Recall from Section 2.3 the set of subsets of demand nodes with limited flexibility

$$\mathcal{J} = \left\{ J \subsetneq V_D : \sum_{j' \in J} \sum_{k \notin \partial(J)} \phi_{j'k} > 0 \right\}. \quad (12)$$

The following is our main result.

Theorem 1 (Main Result). *For any network (G, ϕ) satisfying Assumptions 1, 2 and 3, we have:*

1. **Exponentially small loss under any SMW policy:** *For any $\alpha \in \text{relint}(\Omega)$, $\text{SMW}(\alpha)$ achieves exponential decay of the demand loss probability with exponent^{10,11}*

$$\gamma(\alpha) = \min_{J \in \mathcal{J}} B_J \log\left(\frac{\lambda_J}{\mu_J}\right) > 0, \quad (13)$$

$$\text{where } B_J \triangleq \mathbf{1}_{\partial(J)}^T \alpha, \quad \lambda_J \triangleq \sum_{j' \notin J} \sum_{k \in \partial(J)} \phi_{j'k}, \quad \text{and} \quad \mu_J \triangleq \sum_{j' \in J} \sum_{k \notin \partial(J)} \phi_{j'k}.$$

2. **There is an exponent optimal SMW policy:** *Under any policy U , it must be that*

$$\gamma_p(U) \leq \gamma_o(U) \leq \bar{\gamma}, \quad \text{where } \bar{\gamma} = \sup_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha). \quad (14)$$

Thus, there is an SMW policy that achieves an exponent arbitrarily close to the optimal one.

¹⁰We show that for SMW policies, the \liminf in (3) and \limsup in (4) are equal, i.e., $\gamma_o(\alpha) = \gamma_p(\alpha)$. (We use α to represent the policy $\text{SMW}(\alpha)$ in the argument of the γ s.)

¹¹Note that the argument of the logarithm has a strictly larger numerator than denominator for every $J \subsetneq V_D$ since Assumption 3 holds, implying that $\gamma(\alpha)$ is the minimum of finitely many positive numbers, and hence is positive.

The first part of the theorem states that for any SMW policy with α in the relative interior of Ω , the policy achieves an explicitly specified positive demand loss exponent $\gamma(\alpha)$, i.e., the demand loss probability decays as $e^{-(\gamma(\alpha)-o(1))K}$ as $K \rightarrow \infty$. The second part of the theorem provides a universal upper bound $\bar{\gamma}$ on the exponent that *any* policy can achieve, i.e., for any assignment policy U , the demand loss probability is at least $e^{-(\bar{\gamma}+o(1))K}$. Crucially, $\bar{\gamma}$ is identical to the supremum over α of $\gamma(\alpha)$. In other words, *there is an (almost) exponent optimal SMW policy*, and moreover, the scaling parameters for this policy can be obtained as the solution to the explicit problem: $\text{maximize}_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha)$.

We note that Theorem 1 is qualitatively different from the numerous results showing near optimality of (vanilla) maximum weight matching in various open queueing network settings (e.g., Stolyar 2004, Dai and Lin 2008, show that vanilla MaxWeight asymptotically minimizes workload in heavy-traffic in certain open queueing networks under the CRP condition). Despite our objective (minimize demand loss) being symmetric in all the m queues, our result says that there is an optimal *scaled* maximum weight policy, that is *not* symmetric in the m queues; rather, it uses asymmetric scaling factors that optimally account for the network primitives.

Intuition for $\gamma(\alpha)$. Consider the expression for $\gamma(\alpha)$ in (13). It is a minimum over subsets $J \in \mathcal{J}$ of demand nodes of a certain “robustness” of the subset to demand loss. For subset J , the robustness of SMW(α)’s ability to serve demand arising in J is the product of two terms $B_J \times \log\left(\frac{\lambda_J}{\mu_J}\right)$ (see Figure 3 for an illustration of the quantities involved):

- “Protection” due to α : At the resting point α (see Remark 1) of SMW(α), the supply at neighboring nodes is $B_J = \mathbf{1}_{\partial(J)}^T \alpha$, and the larger that is, the more unlikely it is that the subset will be deprived of supply.
- “Inherent robustness” arising from excess of supply over demand: The logarithmic term $\xi_J \triangleq \log(\lambda_J/\mu_J)$ captures the *inherent robustness* of that subset is to being drained of supply. Recall that λ_J is the (optimistic) net supply coming in to $\partial(J)$, and that μ_J is the net demand taking supply out of $\partial(J)$. The larger the ratio λ_J/μ_J , the more oversupplied and hence robust J is.

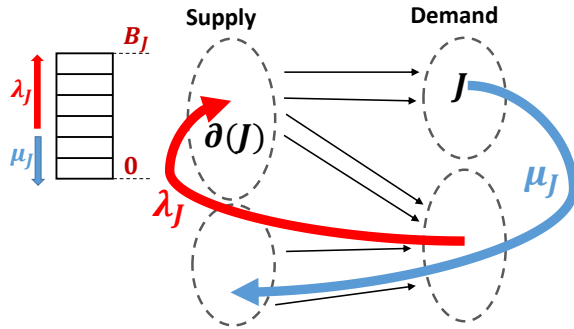


Figure 3: An illustration of the terms B_J , λ_J , and μ_J in Theorem 1.

Remarkably, the expression for robustness of subset J under SMW(α) is as large (i.e., as good) as the demand loss exponent for subset J alone would be, with starting state α , under a “protect- J ” policy which *exclusively protects J at the expense of all other nodes*. (Similar to stan-

dard buffer overflow probability calculations, the likelihood of the supply at $\partial(J)$ being depleted by KB_J units under a protect- J policy is $\Theta((\lambda_J/\mu_J)^{-KB_J}) = \Theta(\exp(-KB_J \log(\lambda_J/\mu_J)))$. We then set B_J to the starting scaled supply at $\partial(J)$, i.e., $B_J = \mathbf{1}_{\partial(J)}^T \boldsymbol{\alpha}$, to establish the claim.) Thus, Theorem 1 part 1 says that given the resting state $\boldsymbol{\alpha}$, $\text{SMW}(\boldsymbol{\alpha})$ achieves an exponent such that *it suffers no loss from the need to protecting multiple subsets J simultaneously*. Given this remarkable property, it is intuitive that the globally optimal exponent can be achieved via an SMW policy by choosing $\boldsymbol{\alpha}$ suitably (part 2 of the theorem).

Structural insights. The choice of scaling factors (resting state) $\boldsymbol{\alpha}$ for SMW which maximizes the exponent $\gamma(\boldsymbol{\alpha})$ as a function of network primitives (G, ϕ) is discussed in Section 4.1.

Proof approach. We establish Theorem 1 via a *novel Lyapunov analysis for a closed queueing network*. A key technical challenge we face in our closed queueing network setting is that it is a priori unclear what the ideal state for the system is. This is in contrast to open queueing network settings in which the ideal state is typically the one in which all queues are empty, and the Lyapunov functions considered typically achieve their minimum at this state. We overcome the challenge of unknown ideal state via an innovative approach as follows: We define a *policy-specific* Lyapunov function that achieves its minimum at the resting point of the SMW policy we are analyzing, and use this Lyapunov function to characterize its exponent $\gamma(\boldsymbol{\alpha})$. Moreover, given the optimal choice of $\boldsymbol{\alpha}$, our tailored Lyapunov function corresponding to this choice of $\boldsymbol{\alpha}$ helps us establish our converse result. In particular, the ideal state is finally revealed as a byproduct of our analysis to be equal to the optimal choice of $\boldsymbol{\alpha}$. Our technical machinery may be broadly useful in deriving large-deviation optimal controls in settings where the appropriate target state is a priori unclear. Our analysis is described in Section 5.

Transient performance. Our analysis extends readily to finite horizon performance: Considering transient behavior over a finite horizon (which is not too short), under a starting scaled state $\frac{\mathbf{x}^K(0)}{K} = \boldsymbol{\alpha} \in \text{relint}(\Omega)$, we find that the optimal demand loss exponent is $\gamma(\boldsymbol{\alpha})$ given by (13) and $\text{SMW}(\boldsymbol{\alpha})$ achieves it. The formal statement is provided in Appendix D.4.

Utilization Rate of Supply Units. Recall that we consider the large market regime where the number of supply units K and the demand arrival rates $\hat{\phi}^K \triangleq K\hat{\phi}$ scale up at the same rate. The next proposition shows that in this regime under any SMW policy, supply units are “frequently” in use, in the sense that is formalized below.

Definition 2 (Resource utilization rate). *Given a policy $U \in \mathcal{U}$, the resource utilization rate $\xi^{K,U}$ is the average number of demands served per supply unit per unit time in steady state in the K -th system.*

Proposition 3. *Consider any network (G, ϕ) satisfying Assumptions 1, 2 and 3 and any $\boldsymbol{\alpha} \in \text{relint}(\Omega)$. Consider the $\text{SMW}(\boldsymbol{\alpha})$ policy and denote its resource utilization rate by $\xi^{K,\boldsymbol{\alpha}}$.*

1. (Utilization rate) *There exists $c > 0$ such that for any $K > 0$ we have $\xi^{K,\boldsymbol{\alpha}} > c$.*

2. (Waiting time) Suppose the head-of-line unit from the queue at the supply location is chosen in a first-in-first-out (FIFO) manner when implementing $SMW(\alpha)$, then there exists $w < \infty$ such that for every $K > 0$, for every current state $\mathbf{X}(t)$, and every supply unit (distinguished by its location in V_S and its queue position), the expected waiting time before the supply unit is assigned is at most¹² w .

Proposition 3 tells us that for any SMW policy, the resource utilization rate is bounded below by a positive constant which does not depend on K . See Appendix E for the proof.

4.1 Optimal choice of scaling factors

In this subsection, we discuss the optimal choice of the scaling factors (resting state) α based on Theorem 1. We illustrate the structure of the optimal α via two examples (formal corollaries generalizing each example to arbitrary compatibility graphs are provided in Appendix E).

We start by defining a *vulnerable subset* as one with small inherent robustness.

Definition 3 (Vulnerable subset). *Given a compatibility graph G and a sequence of demand type distributions $(\phi^n)_{n \in \mathbb{Z}_+}$, we say that a limited-flexibility subset of demand nodes $J \subset \mathcal{J}$ is vulnerable if its inherent robustness vanishes as n grows:*

$$\xi_J \triangleq \log \left(\frac{\lambda_J^n}{\mu_J^n} \right) \xrightarrow{n \rightarrow \infty} 0. \quad (15)$$

Our first example considers the case of exactly one vulnerable subset.

Example 2 (If one subset of nodes is vulnerable, the optimal α protects it). *Consider the “line-of-four-nodes” compatibility graph introduced in Example 1 and Figure 2, and the sequence of demand type distribution matrices*

$$\phi^n = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1' \\ 2' \\ 3' \\ 4' \end{matrix} & \begin{bmatrix} 0 & 0 & 1/4 & 1/4 - \eta_n \\ 0 & 0 & 0 & \eta_n \\ \delta_n & 0 & 0 & \\ 1/4 - \delta_n & 1/4 & 0 & 0 \end{bmatrix} \end{matrix} \quad \text{for } n \in \mathbb{Z}_+. \quad (16)$$

We set $\delta_n = 1/n$ and $\eta_n = 1/8$ in this example (and consider $n > 4$). Note that (G, ϕ^n) satisfies Assumptions 1, 2 and 3 for all $n > 4$.

The subsets of demand locations with limited flexibility are the same for all ϕ^n in the sequence $\mathcal{J} = \{\{1'\}, \{1', 2'\}, \{3', 4'\}, \{4'\}\}$. Consider these subsets one by one. We have $\lambda_{\{4'\}} = \frac{1}{2}$ and $\mu_{\{4'\}} = \frac{1}{2} - \frac{1}{n}$, which tells us that $\{4'\}$ is a “vulnerable” subset since

$$\xi_{\{4'\}} \triangleq \log \left(\frac{\lambda_{\{4'\}}}{\mu_{\{4'\}}} \right) = \frac{2}{n} + O\left(\frac{1}{n^2}\right) \xrightarrow{n \rightarrow \infty} 0^+.$$

Meanwhile, the other subsets are not vulnerable in the sense that $\xi_J \triangleq \log(\lambda_J/\mu_J)$ remains bounded away from zero: $\xi_{\{1'\}} = \log\left(\frac{1/2}{3/8}\right) \xrightarrow{n \rightarrow \infty} \log(4/3) > 0$, and $\xi_{\{1', 2'\}} = \xi_{\{3', 4'\}} =$

¹²The same result also holds when the supply unit is chosen uniformly at random from the queue.

$\log\left(\frac{1/2}{1/4}\right) = \log 2 > 0$. We deduce from Theorem 3 (as formalized in Corollary 2 in Appendix E), that for any $\epsilon > 0$, there exists $n_0 < \infty$ such that, for all $n > n_0$, for network (G, ϕ^n) we have

- (i) (Optimal exponent) The best achievable exponent $\bar{\gamma}$ is close to $\xi_{\{4'\}}$. Formally, $\bar{\gamma} \in [(1 - \epsilon)\xi_{\{4'\}}, \xi_{\{4'\}}]$ and, as always, SMW policies suffice to achieve it $\bar{\gamma} = \sup_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha)$.
- (ii) (Near optimal α protects vulnerable subset $\{4'\}$.) If SMW with scaling factors $\alpha \in \text{relint}(\Omega)$ achieves a demand-loss exponent $\gamma(\alpha) \geq (1 - \epsilon)\xi_{\{4'\}}$, then it must be that $\alpha_3 + \alpha_4 \geq 1 - \epsilon$. (Note that $\partial(4') = \{3, 4\}$.)
- (iii) (Example of near optimal α .) The SMW policy with $\alpha = \begin{bmatrix} \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{1-\epsilon}{2} & \frac{1-\epsilon}{2} \end{bmatrix}^T$ achieves $\gamma(\alpha) = (1 - \epsilon)\xi_{\{4'\}}$.

Example 2 illustrates Corollary 2 in Appendix E, which demonstrates that if there is just one vulnerable subset of demand nodes J_1 , then the exponent optimal SMW policy has a resting state which puts almost all the supply in the neighborhood of J_1 . The intuition is that the total supply located in $\partial(J_1)$ follows a random walk which has only slightly positive drift even if the assignment rule protects it (recall that the definition of the net supply λ_{J_1} assumes that the policy protects J_1), and hence it is optimal to keep the total supply in $\partial(J_1)$ at a high resting point, to minimize the likelihood of depletion.

Our next example illustrates the case of two non-overlapping vulnerable subsets.

Example 3 (If there are two non-overlapping vulnerable subsets, the optimal α protects them in inverse proportion to their inherent robustness). Once again consider the same compatibility graph as in Example 2. We further take the sequence ϕ^n given by (16) again with $\delta_n = 1/n$ but change the definition of η_n to $\eta_n = \eta/n$ for some fixed $\eta > 0$ (we consider $n > 4/\min(1, \eta)$). Note that $\lim_{n \rightarrow \infty} \phi^n = \phi^*$ where ϕ^* is given by (16) with δ_n and η_n both replaced by 0.

The limited-flexibility subsets of demand locations are the same for all ϕ^n in the sequence $\mathcal{J} = \{\{1'\}, \{1', 2'\}, \{3', 4'\}, \{4'\}\}$. The two singleton subsets are vulnerable:

$$\xi_{\{4'\}} \triangleq \log\left(\frac{\lambda_{\{4'\}}}{\mu_{\{4'\}}}\right) = \log\left(\frac{1/2}{1/2 - 1/n}\right) = \frac{2}{n} + O\left(\frac{1}{n^2}\right) \xrightarrow{n \rightarrow \infty} 0^+, \quad \xi_{\{1'\}} = \frac{2\eta}{n} + O\left(\frac{1}{n^2}\right) \xrightarrow{n \rightarrow \infty} 0^+,$$

and $\frac{\xi_{\{1'\}}}{\xi_{\{4'\}}} = \eta + O\left(\frac{1}{n}\right)$. The other subsets are not vulnerable since $\xi_{\{1', 2'\}} = \xi_{\{3', 4'\}} = \log\left(\frac{1/2}{1/4}\right) = \log 2 > 0$. We deduce from Theorem 3 (formalized in Corollary 3 in Appendix E), that for any $\epsilon > 0$, there exists $n_0 < \infty$ such that, for all $n > n_0$, for network (G, ϕ^n) we have

- (i) (Optimal exponent) The best achievable exponent $\bar{\gamma}$ is close to $H \triangleq \frac{\xi_{\{4'\}}\xi_{\{1'\}}}{\xi_{\{4'\}} + \xi_{\{1'\}}} = \frac{1}{n} \cdot \frac{\eta}{1+\eta} + O\left(\frac{1}{n^2}\right)$. Formally, $\bar{\gamma} \in [(1 - \epsilon)H, H]$, and, as always, SMW policies suffice to achieve it $\bar{\gamma} = \sup_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha)$.
- (ii) (Near optimal α protects vulnerable subsets in inverse proportion to their inherent robustness.) If SMW with scaling factors $\alpha \in \text{relint}(\Omega)$ achieves a demand-loss exponent $\gamma(\alpha) \geq (1 - \epsilon)H$, then it must be that

$$\alpha_1 + \alpha_2 \stackrel{\epsilon}{=} \frac{\xi_{\{4'\}}}{\xi_{\{4'\}} + \xi_{\{1'\}}} = \frac{1}{1+\eta} + O\left(\frac{1}{n}\right) \quad \text{and} \quad \alpha_3 + \alpha_4 \stackrel{\epsilon}{=} \frac{\xi_{\{1'\}}}{\xi_{\{4'\}} + \xi_{\{1'\}}} = \frac{\eta}{1+\eta} + O\left(\frac{1}{n}\right),$$

where $a \stackrel{\epsilon}{=} b$ represents $|a - b| \leq \epsilon$. (Recall that $\partial(1') = \{1, 2\}$ and $\partial(4') = \{3, 4\}$.)

(iii) (Example of near optimal α .) The SMW policy with

$$\alpha = \left[\frac{\eta'}{2(1+\eta')} \quad \frac{\eta'}{2(1+\eta')} \quad \frac{1}{2(1+\eta')} \quad \frac{1}{2(1+\eta')} \right]^T \quad \text{for} \quad \eta' = \frac{\xi_{\{1'\}}}{\xi_{\{4'\}}} = \eta + O\left(\frac{1}{n}\right) \quad (17)$$

achieves $\gamma(\alpha) \geq (1 - \epsilon)H$.

Example 3 illustrates Corollary 3 in Appendix E which tells us that if there are two non-overlapping vulnerable subsets of demand nodes J_1 and J_2 , then the exponent optimal SMW policy has a resting state which divides the supply between the two neighborhoods in inverse proportion to the inherent robustness of the vulnerable subsets

$$\frac{\mathbf{1}_{\partial(J_2)}^T \alpha}{\mathbf{1}_{\partial(J_1)}^T \alpha} \approx \frac{\xi_{J_1}}{\xi_{J_2}} \approx \eta.$$

In this simple example, $\partial(J_1) \cup \partial(J_2) = V_S$. More generally, if $\partial(J_1) \cup \partial(J_2) \subsetneq V_S$, then the optimal α places very little supply at nodes outside the union of neighborhoods $\partial(J_1) \cup \partial(J_2)$; see Corollary 3.

While the examples above (and the corollaries they illustrate) focusing on the cases of one or two vulnerable subsets are interesting in themselves; we highlight that the optimal policy characterized in Theorem 1 goes much beyond to solve the general m -dimensional problem considering *all* subsets of V_S simultaneously. SMW with the optimal α balances between the demands of protecting different subsets and is (provably) globally exponent optimal.

Knowledge requirements. We remark that choosing the exponent optimal α requires exact knowledge of ϕ . However, if a noisy estimate of the demand type distribution is employed to choose α (by maximizing the exponent for the estimated distribution), the resulting SMW policy will nevertheless perform well: (i) it will achieve exponentially small loss (as long as the true ϕ satisfies our assumptions), (ii) if the estimate of ϕ is close to the true distribution, then the exponent achieved by the chosen α will be close to the estimated exponent based on the estimated distribution, since $\gamma(\alpha)$ given by (13) varies continuously in ϕ for each $\alpha \in \text{relint}(\Omega)$.

4.2 State-independent policies and naive state-dependent policies are inferior

State-independent policies. Previous works studying control of circulating resources in networks, e.g., Ozkan and Ward (2016) and Banerjee et al. (2016), have proposed state-independent control policies. We show that in our setting, such policies are not competitive with the SMW policies we have proposed.

We first formally define state-independent policies.

Definition 4 (State independent policy). *We call an assignment policy U state independent if, for each¹³ $K \geq 1$, it maps each $j' \in V_D$, $k \in V_S$, $r \in \mathbb{Z}_+$ to a distribution $u_{j'k}(t_r^-)$ over $\partial(j') \cup \{\emptyset\}$; for the r -th demand arrival with origin j' and destination k , the platform dispatches*

¹³We suppress the dependence on K in our notation.

from i drawn independently from distribution $u_{j'k}(t_r^-)$, ignoring the current state $\mathbf{X}(t_r^-)$ and the history. If $i = \emptyset$ or there is no supply at the dispatch node, the demand is lost.

The next proposition formalizes that for any state independent policy: (i) Exponentially small loss is impossible (even if demand arrival rates are exactly known), (ii) Given a compatibility graph G and a state independent policy, for “almost all” demand type distributions ϕ the loss incurred under the policy does not vanish as $K \rightarrow \infty$; informally, asymptotic optimality fails if ϕ is not exactly known. The proof is in Appendix F.

Proposition 4 (All state independent policies have inferior performance). *Fix a compatibility graph G and any state-independent dispatch policy U . We have:*

1. (Exponentially small loss is impossible.) For any demand type distribution ϕ , $\mathbb{P}_o^{K,U} = \Omega\left(\frac{1}{K^2}\right)$. In particular, $\gamma_o(U) = 0$, where $\gamma_o(\cdot)$ is the optimistic exponent defined in (3).
2. (For almost all ϕ , asymptotic optimality fails.) Let $\text{Supp}(\phi) \triangleq \{(j', k) \in V_D \times V_S : \phi_{j'k} > 0\}$. Fix any subset of demand types $S \subseteq V_D \times V_S$ such that each demand node $j' \in V_D$ has at least one demand type in S . Let $D(S) \triangleq \{\phi : \text{Supp}(\phi) = S\}$ be the set of demand type distributions with support S . Then, there is a subset of $D(S)$ which is open and dense in $D(S)$ such that for all ϕ in this subset it holds that $\liminf_{K \rightarrow \infty} \mathbb{P}_o^{K,U} > 0$.

Proposition 4 makes it clear that as K grows, any state independent policy suffers from inferior performance. There are two possibilities regarding what is known about the demand type distribution ϕ :

1. **ϕ exactly known.** In this case, part 1 of Proposition 4 tells us that any state independent policy has loss $\Omega(\frac{1}{K^2})$ whereas any SMW policy produces exponentially small loss (Theorem 1 part 1) and moreover $\text{SMW}(\alpha)$ is exponent optimal for α chosen to maximize $\gamma(\alpha)$ in (13).
2. **ϕ is not exactly known.** In this case, any state independent policy typically fails to achieve asymptotic optimality (part 2 of Proposition 4) whereas vanilla MaxWeight (or any fixed SMW policy) achieves exponentially small loss.

A naive state-dependent policy. Would a naive state dependent policy do well in our setting? For a natural state dependent policy, we show via a simple example that the loss is $\Omega(1)$ as $K \rightarrow \infty$, even though the example network satisfies all our assumptions.

Define the *naive* policy as follows: each time a demand arrives, consider the supply nodes compatible with the origin in a uniformly random order (independently of the past), and assign a supply unit from the first compatible supply node which has at least one supply unit.

Example 4 (Naive state-dependent policy loses $\Omega(1)$). *Consider again the “line-of-four-nodes” compatibility graph introduced in Example 1 and Figure 2, and the demand type distribution*

matrix

$$\phi = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1' \\ 2' \\ 3' \\ 4' \end{matrix} & \begin{bmatrix} 0 & 0 & 0.21 & 0.21 \\ 0.08 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \end{bmatrix} \end{matrix}. \quad (18)$$

It is easy to verify that this network satisfies Assumptions 1, 2 and 3. Even so, the naive policy incurs $\Omega(1)$ loss in this network (in fact, this is true for any demand type distribution in a ball of positive radius centered at the right-hand side of (18)). The proof is in Appendix F.

Variants of the naive policy which sample a compatible supply using a non-uniform distribution can similarly be shown to fail in simple examples.

5 Analysis of Scaled MaxWeight Policies: Proof of Theorem 1

In this section, we prove that any SMW policy is exponent optimal given its resting state, and derive explicitly the demand-loss exponent achieved, and the most likely sample paths leading to demand loss. In Section 5.1, we follow the standard approach for large deviations analyses and characterize the system behavior in the fluid scale through fluid sample paths and fluid limits. In Section 5.2 we take a novel approach to define a family of Lyapunov functions parameterized by the desired state, since we do not know the ideal state for the system. In Section 5.3 we follow Venkataramanan and Lin (2013) and show that if the Lyapunov function (centered at the starting state) is scale-invariant and sub-additive, a policy that performs steepest descent on this Lyapunov function is exponent optimal. In Section 5.4 we prove that each SMW policy performs steepest descent on the Lyapunov function centered at its resting state and is hence exponent optimal given its resting state. We also explicitly characterize the optimal exponent, the most likely sample paths leading to demand loss, and the critical subsets (i.e., the subsets that are most likely to be depleted of supply). Finally, we deduce Theorem 1.

5.1 Fluid Sample Paths and Fluid Limits

For any stationary assignment policy $U \in \mathcal{U}$ defined in Section 2, we define the scaled demand and queue-length sample paths by (the former was defined in (7))

$$\bar{\mathbf{A}}_{j'k}^K(t) \triangleq \frac{1}{K} \mathbf{A}_{j'k}^K(t), \quad \bar{\mathbf{X}}_i^{K,U}(t) \triangleq \frac{1}{K} \mathbf{X}_i^{K,U}(t), \quad (19)$$

Note that for a fixed policy (with specified tie-breaking rules), each given demand sample path and initial state uniquely determines the state sample path. We denote this correspondence by $\Psi^{K,U} : (\bar{\mathbf{A}}^K(\cdot), \bar{\mathbf{X}}^{K,U}(0)) \mapsto \bar{\mathbf{X}}^{K,U}(\cdot)$.

To obtain a large deviation result, we need to study the demand process and the queue-length process in the fluid scaling, as captured in (19). We take the standard approach of *fluid sample paths* (FSP) (see Stolyar 2003, Venkataramanan and Lin 2013).

Definition 5 (Fluid sample paths). *We call a pair $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot))_T \triangleq (\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot))_{t \in [0, T]}$ a fluid sample path on $[0, T]$ (under stationary policy U) if there exists a sequence*

$$((\bar{\mathbf{A}}^K(\cdot))_{t \in [0, T]}, \bar{\mathbf{X}}^{K, U}(0), (\Psi^{K, U}(\bar{\mathbf{A}}^K(\cdot), \bar{\mathbf{X}}^{K, U}(0)))_{t \in [0, T]})$$

where $\bar{\mathbf{A}}^K(\cdot)$ are scaled demand sample paths and $\bar{\mathbf{X}}^{K, U}(0) \in \Omega$, such that it has a subsequence which converges to $((\bar{\mathbf{A}}(\cdot))_{t \in [0, T]}, \bar{\mathbf{X}}^U(0), (\bar{\mathbf{X}}^U(\cdot))_{t \in [0, T]})$ uniformly on $[0, T]$.

In short, FSPs include both typical and atypical sample paths. Recall Fact 1, which gives the likelihood for an unlikely event to occur based on the most likely fluid sample path that causes the event. Accordingly, the large deviations analysis in Section 5.4 will identify the most likely FSP that leads to demand loss. We comment on the existence of FSPs in Appendix A.2.

Fluid limits are fluid sample paths that characterize *typical* system behavior, as they are the formal limits in the Functional Law of Large Numbers (Dai 1995).

Definition 6 (Fluid limits). *We call a pair $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot))_T$ a fluid limit on $[0, T]$ (under stationary policy U) if (i) the pair $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot))_T$ is a fluid sample path; (ii) we have $\bar{\mathbf{A}}_{j'k}(t) = \hat{\phi}_{j'k}t$, for all $j' \in V_D, k \in V_S$ and all $t \in [0, T]$.*

5.2 A Family of Lyapunov Functions

Lyapunov functions are a useful tool for analyzing complex stochastic systems. In open queueing networks the ideal state is one in which all queues are empty, and correspondingly the Lyapunov function is chosen to achieve its minimum value in the ideal state, e.g., the sum of squared queue lengths. Lyapunov function is a popular choice (Tassiulas and Ephremides 1992, Eryilmaz and Srikant 2012, etc.), while others have also used piecewise linear Lyapunov functions (Bertsimas et al. 2001, Venkataramanan and Lin 2013, etc.). Since our setting is a closed queueing network and ideal state is unknown, we instead construct a novel approach. We define a family of piecewise linear Lyapunov functions, parameterized by the desired state α , such that the function achieves its minimum at α .

Definition 7. *For each $\alpha \in \text{relint}(\Omega)$, define Lyapunov function $L_\alpha(\mathbf{x}) : \Omega \rightarrow [0, 1]$ as $L_\alpha(\mathbf{x}) \triangleq 1 - \min_i \frac{x_i}{\alpha_i}$.*

The intuition behind our definition is as follows. The Lyapunov function value is jointly determined by the desired state α of the system (under some policy) and our objective of avoiding demand loss, and can be interpreted as the energy of the system at each state. The desired state should have minimum energy, and the most undesirable states should have maximum energy. In our case the boundary $\partial\Omega$ of Ω is most undesirable since demand loss only happens there, and correspondingly, $L_\alpha(\mathbf{x}) = 1$ for $\mathbf{x} \in \partial\Omega$, whereas $L_\alpha(\alpha) = 0$ as we want. In general, for $\mathbf{x} \in \Omega$, $L_\alpha(\mathbf{x})$ is one minus the smallest scaled queue length, given scaling factors α . See Figure 4 for an illustration.

These functions moreover have the following properties which play a key role in our analysis:

Lemma 1 (Key properties of $L_\alpha(\cdot)$). *For $L_\alpha(\cdot)$ with $\alpha \in \text{relint}(\Omega)$, we have:*

1. *Scale-invariance (about α).* $L_\alpha(\alpha + c\Delta\mathbf{x}) = cL_\alpha(\alpha + \Delta\mathbf{x})$ for any $c > 0$ and $\Delta\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{1}^T \Delta\mathbf{x} = 0$ and $\alpha + \Delta\mathbf{x} \in \Omega$, $\alpha + c\Delta\mathbf{x} \in \Omega$.
2. *Sub-additivity (about α).* $L_\alpha(\alpha + \Delta\mathbf{x} + \Delta\mathbf{x}') \leq L_\alpha(\alpha + \Delta\mathbf{x}) + L_\alpha(\alpha + \Delta\mathbf{x}')$ for any $\Delta\mathbf{x}, \Delta\mathbf{x}' \in \mathbb{R}^m$ such that $\mathbf{1}^T \Delta\mathbf{x} = \mathbf{1}^T \Delta\mathbf{x}' = 0$ and $\alpha + \Delta\mathbf{x}, \alpha + \Delta\mathbf{x}', \alpha + \Delta\mathbf{x} + \Delta\mathbf{x}' \in \Omega$.

The proof of Lemma 1 is in Appendix A.

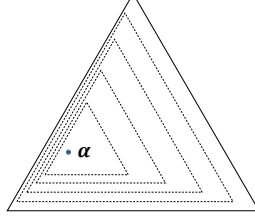


Figure 4: Sub-level sets of L_α when $|V_S| = |V_D| = 3$. State space Ω is the probability simplex in \mathbb{R}^3 , and its boundary coincides with $\{\mathbf{x} : L_\alpha(\mathbf{x}) = 1, \mathbf{1}^T \mathbf{x} = 1\}$. The minimum value is achieved at α ; $L_\alpha(\alpha) = 0$.

A time $t \in (0, T)$ is said to be a *regular point* of an FSP $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot))_T$ if $\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot), L_\alpha(\bar{\mathbf{X}}^U(\cdot))$ are all differentiable at time t .

Because of the Large Deviations Principle (Fact 1), it will suffice in our analysis to consider only the FSPs that have absolutely continuous demand sample paths. Now, if $\bar{\mathbf{A}}(\cdot)$ is absolutely continuous, then so are $\bar{\mathbf{X}}^U(\cdot)$ and $L_\alpha(\bar{\mathbf{X}}^U(\cdot))$, and as a result almost all t are regular: For any policy $U \in \mathcal{U}$ and FSP $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot))_T$, it holds that for any t, t' , $\|\bar{\mathbf{X}}^U(t) - \bar{\mathbf{X}}^U(t')\|_1 \leq 2\|\bar{\mathbf{A}}(t) - \bar{\mathbf{A}}(t')\|_1$ because supply units relocate only when demand arrives, and $|L_\alpha(\bar{\mathbf{X}}^U(t)) - L_\alpha(\bar{\mathbf{X}}^U(t'))| \leq \frac{1}{\min_{i \in V_S} \alpha_i} \|\bar{\mathbf{X}}(t) - \bar{\mathbf{X}}(t')\|_\infty \leq \frac{1}{\min_{i \in V_S} \alpha_i} \|\bar{\mathbf{X}}(t) - \bar{\mathbf{X}}(t')\|_1$ (see Appendix A for the short proof). As a result, if $\bar{\mathbf{A}}(\cdot)$ is absolutely continuous then so are $\bar{\mathbf{X}}(\cdot)$ and $L_\alpha(\bar{\mathbf{X}}^U(\cdot))$. Therefore for these FSPs, $\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot), L_\alpha(\bar{\mathbf{X}}^U(\cdot))$ have derivatives almost everywhere with respect to Lebesgue measure on $[0, T]$ for any $T < \infty$.

5.3 Sufficient Conditions for Exponent Optimality

In this section, given a starting state, we provide a converse bound on the exponent for any stationary policy $U \in \mathcal{U}$, and derive sufficient conditions for a policy to achieve this bound.

We use the intuition from differential games (see, e.g., Atar et al. 2003) to informally illustrate the interplay between the control and the most likely sample path leading to demand loss. Consider a zero-sum game between the adversary (nature) who chooses the fluid-scale demand arrival process $\bar{\mathbf{A}}(\cdot)$, and the controller who decides the assignment rule U , where the adversary minimizes the large-deviation “cost” of a demand sample path that leads to demand loss. Specifically, the adversary’s cost for a demand sample path $\bar{\mathbf{A}}(\cdot)$ is the rate function defined in (10), i.e., the exponent. The converse bound we will obtain next will correspond to the adversary playing first and choosing the minimum cost *time-invariant* demand sample path that ensures demand loss. The following pleasant surprises will emerge subsequently: (i) we will find an equi-

librium in pure strategies to the aforementioned zero-sum game, (ii) the converse will turn out to be tight, i.e., the adversary's equilibrium demand sample path will be time invariant, (iii) the controller's equilibrium assignment strategy will be an SMW policy with specific α (this simple policy will satisfy the sufficient conditions for achievability we will state immediately after our converse, in Proposition 5).

We provide a policy-independent upper bound on the exponent that only depends on the starting state. First, for any $\mathbf{f} \in \mathbb{R}_+^{n \times m}$, define

$$\mathcal{X}_{\mathbf{f}} \triangleq \left\{ \Delta \mathbf{x} \left| \begin{array}{ll} \Delta x_i = \sum_{j' \in V_D} f_{j'i} - \sum_{j' \in \partial(i)} d_{ij'} \left(\sum_{k \in V_S} f_{j'k} \right), & \forall i \in V_S \\ \sum_{i \in \partial(j')} d_{ij'} = 1, \quad d_{ij'} \geq 0, & \forall i \in V_S, j' \in V_D \end{array} \right. \right\}, \quad (20)$$

which is the attainable change of (normalized) state in unit time, given that the average demand arrival rates during this period are \mathbf{f} and assuming no demand is lost. (Here $(d_{ij'})_{i \in \partial(j')}$ is the chosen *assignment distribution* over supply nodes neighboring j' for assigning supply units to serve demand originating at j' .) Then given starting state α , the attainable states at time T belong to $\alpha + T\mathcal{X}_{\mathbf{f}} \triangleq \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \alpha + T\mathbf{x}, \mathbf{x} \in \mathcal{X}_{\mathbf{f}}\}$, if no demand is lost during $[0, T]$ and the average demand arrival rate is \mathbf{f} . We obtain an upper bound on the demand-loss exponent by considering the most likely \mathbf{f} and T such that $\alpha + T\mathcal{X}_{\mathbf{f}}$ lies entirely outside the state space Ω . Because the true state must lie in Ω , there must be demand loss during $[0, T]$, no matter the assignment rule \mathbf{d} used by the controller.

Lemma 2 (Converse bound on the exponent). *For any stationary policy $U \in \mathcal{U}$, it holds that*

$$-\liminf_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_o^{K,U} \leq \sup_{\alpha \in \text{relint}(\Omega)} \gamma_{\text{CB}}(\alpha), \quad (21)$$

where, for $\Lambda^*(\cdot)$ given by (8), $\gamma_{\text{CB}}(\alpha) \triangleq \inf_{\mathbf{f} \in \mathbb{R}_+^{n \times m} : v_{\alpha}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{v_{\alpha}(\mathbf{f})}$, and $v_{\alpha}(\mathbf{f}) \triangleq \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} L_{\alpha}(\alpha + \Delta \mathbf{x})$.

We now provide an informal explanation for the form of this key lemma. The α in (21) captures the most frequently visited (normalized) state (the “resting” state) in steady state under U , and $\gamma_{\text{CB}}(\alpha)$ is an upper bound on the exponent given the most frequent state α . Let us informally describe the expression for $\gamma_{\text{CB}}(\alpha)$. Suppose the system starts in state α . Then $v_{\alpha}(\mathbf{f})$ is the minimum rate of increase of $L_{\alpha}(\cdot)$ under demand arrival rates \mathbf{f} , no matter the assignment distributions \mathbf{d} . So, starting at α and under time-invariant demand arrival rates \mathbf{f} , the state hits Ω and demand is lost in time at most $1/v_{\alpha}(\mathbf{f})$, implying a demand-loss exponent of at most $\frac{\Lambda^*(\mathbf{f})}{v_{\alpha}(\mathbf{f})}$. The upper bound $\gamma_{\text{CB}}(\alpha)$ follows from minimizing over \mathbf{f} since nature can choose any \mathbf{f} . Finally, the bound in (21) takes the supremum over α since the policy can choose its resting state. The proof of Lemma 2 is in Appendix B.

Recall that for a function $g(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ for some positive integer d , we use $\dot{g}(t)$ to denote the derivative of g at time t when the derivative exists.

The following proposition provides sufficient conditions for a policy to achieve the converse bound exponent $\gamma_{\text{CB}}(\alpha)$. The conditions are requirements on the time derivative of $L_{\alpha}(\bar{\mathbf{X}}^U(t))$.

Proposition 5 (Sufficient conditions). *Fix $\alpha \in \text{relint}(\Omega)$. Let $U \in \mathcal{U}$ be a stationary, non-idling policy. Suppose that for each regular point t , the following hold:*

1. (Steepest descent). *For any demand fluid sample path $\bar{\mathbf{A}}(\cdot)$, we have*

$$\dot{L}_\alpha(\bar{\mathbf{X}}^U(t)) = \inf_{U' \in \mathcal{U}_{\text{ni}}} \left\{ \dot{L}_\alpha(\bar{\mathbf{X}}^{U'}(t)) \mid \bar{\mathbf{X}}^{U'}(t) = \bar{\mathbf{X}}^U(t) \right\},$$

for corresponding queue-length sample paths satisfying $\bar{\mathbf{X}}^U(t) \neq \alpha$ and $L_\alpha(\bar{\mathbf{X}}^U(t)) < 1$, where \mathcal{U}_{ni} is the set of non-idling policies;

2. (Negative drift). *There exists $\eta > 0$ and $\epsilon > 0$ such that for all FSPs $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^U(\cdot))$ satisfying $\dot{\bar{\mathbf{A}}}(t) \in B(\phi, \epsilon)$ and $\bar{\mathbf{X}}(t) \neq \alpha$, we have $\dot{L}_\alpha(\bar{\mathbf{X}}^U(t)) \leq -\eta$. Here $B(\phi, \epsilon)$ is a ball with radius ϵ centered on the typical demand type distribution ϕ .*

Then we have $\gamma_o(U) = \gamma_p(U) = \gamma_{\text{CB}}(\alpha)$, i.e., $\gamma(U) = \gamma_{\text{CB}}(\alpha)$.

Informally, the negative drift property requires the policy to have negative Lyapunov drift for near typical demand arrival rates, as long as the current state is not α . This property forces the state to return to α .

The full proof of Proposition 5 is quite technical and is included in Appendix C, but the key idea is straightforward. Given starting state α , the (i) steepest descent property of U and (ii) the scale-invariance and sub-additivity of $L_\alpha(\cdot)$, together ensure that the speed at which $L_\alpha(\cdot)$ increases under U cannot exceed the minimum speed $v_\alpha(\mathbf{f})$ in the converse construction (Lemma 2) for $\mathbf{f} \triangleq \dot{\bar{\mathbf{A}}}(t)$. Mathematically,

$$\begin{aligned} & \dot{L}_\alpha(\bar{\mathbf{X}}^U(t)) \Big|_{\dot{\bar{\mathbf{A}}}(t)=\mathbf{f}} \\ &= \inf_{U' \in \mathcal{U}_{\text{ni}}} \left\{ \dot{L}_\alpha(\bar{\mathbf{X}}^{U'}(t)) \Big|_{\dot{\bar{\mathbf{A}}}(t)=\mathbf{f}} \right\} && \text{(steepest descent)} \\ &= \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} \lim_{\Delta t \rightarrow 0} \frac{L_\alpha(\bar{\mathbf{X}}^U(t) + \Delta \mathbf{x} \Delta t) - L_\alpha(\bar{\mathbf{X}}^U(t))}{\Delta t} && \text{(definition of } \mathcal{X}_{\mathbf{f}} \text{)} \\ &\leq \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} \lim_{\Delta t \rightarrow 0} \frac{L_\alpha(\alpha + \Delta \mathbf{x} \Delta t)}{\Delta t} && \text{(sub-additivity of } L_\alpha, \text{ Lemma 1)} \quad (22) \\ &= \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} L_\alpha(\alpha + \Delta \mathbf{x}) = v_\alpha(\mathbf{f}). && \text{(scale-invariance of } L_\alpha, \text{ Lemma 1)} \end{aligned}$$

As a result, the demand loss exponent under U is no worse than $\gamma_{\text{CB}}(\alpha)$.

Faced with a policy satisfying the above sufficient conditions, the adversary wants to force equality in (22) by forcing the queue-length sample path $\bar{\mathbf{X}}^U$ to go *radially* outward starting at α . This is why our converse in Lemma 2 based on a time invariant demand arrival process will turn out to be tight. We will formalize this intuition in Section 5.4 and explicitly characterize the most likely demand FSP forcing demand loss.

5.4 Optimality of SMW Policies, Explicit Exponent, and Critical Subsets

In this section, we verify that SMW policies satisfy the sufficient conditions in Proposition 5. In doing so, we reveal the critical subset structure of the most-likely sample paths for demand loss and derive the explicit exponent for $\text{SMW}(\alpha)$. Proofs for this section are in Appendix D.

The following lemma shows that the Lyapunov drift only depends on the nodes with shortest

scaled queue lengths, and that $\text{SMW}(\alpha)$ minimizes its use of supplies from these queues.

Lemma 3 ($\text{SMW}(\alpha)$ causes steepest descent). *Let $(\bar{\mathbf{A}}, \bar{\mathbf{X}}^U)$ be any FSP under any non-idling policy U on $[0, T]$, and consider any $\alpha \in \text{relint}(\Omega)$. For a regular $t \in [0, T]$, define:*

$$S_1(\bar{\mathbf{X}}^U(t)) \triangleq \left\{ k \in V_S : k \in \underset{\alpha_k}{\text{argmin}} \frac{\bar{\mathbf{X}}_k^U(t)}{\alpha_k} \right\},$$

$$S_2(\bar{\mathbf{X}}^U(t), \dot{\bar{\mathbf{X}}}^U(t)) \triangleq \left\{ k \in S_1(\bar{\mathbf{X}}^U(t)) : k \in \underset{\alpha_k}{\text{argmin}} \frac{\dot{\bar{\mathbf{X}}}_k^U(t)}{\alpha_k} \right\}.$$

All the derivatives are well defined since t is regular. We have

$$\dot{L}_\alpha(\bar{\mathbf{X}}^U(t)) = -\frac{\dot{\bar{\mathbf{X}}}_k^U(t)}{\alpha_k} \quad \text{for any } k \in S_2(\bar{\mathbf{X}}^U(t), \dot{\bar{\mathbf{X}}}^U(t)) \quad (23)$$

$$\geq -\frac{1}{\mathbf{1}_{S_2}^T \alpha} \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j' \in V_D : \partial(j') \subseteq S_2, k \in V_S} \dot{A}_{j'k}(t) \right) \quad (24)$$

for $\bar{\mathbf{X}}^U(t) \neq \alpha$ and $L_\alpha(\bar{\mathbf{X}}^U(t)) < 1$. Inequality (24) holds with equality under $\text{SMW}(\alpha)$, i.e., $\text{SMW}(\alpha)$ satisfies the steepest descent property in Proposition 5.

In Lemma 4, we prove that $\text{SMW}(\alpha)$ satisfies the negative drift property. In particular, the drift η is related to the Hall's gap (i.e., the slack in the CRP condition) of the network; see Appendix D for details.

Lemma 4 ($\text{SMW}(\alpha)$ satisfies negative drift). *For any $\alpha \in \text{relint}(\Omega)$, under Assumptions 1, 2 and 3, the policy $\text{SMW}(\alpha)$ satisfies the negative drift condition in Proposition 5.*

Before proceeding with our analysis, we point out that Lemma 4 implies that α is the unique resting state of $\text{SMW}(\alpha)$ policy.

Proposition 6 (Resting state of $\text{SMW}(\alpha)$). *Suppose Assumptions 1, 2 and 3 hold. For any $\alpha \in \text{relint}(\Omega)$, there exists $T_0 > 0$ such that any fluid limit $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ on $[0, T]$ (where $T > T_0$) under $\text{SMW}(\alpha)$ satisfies $\bar{\mathbf{X}}(t) = \alpha$ for all $t \in [T_0, T]$.*

Combining Proposition 5 with Lemmas 3 and 4, we immediately deduce that $\text{SMW}(\alpha)$ achieves the best possible exponent given resting state α .

Corollary 1. *For any $\alpha \in \text{relint}(\Omega)$, we have $\gamma(\alpha) = \gamma_{\text{CB}}(\alpha)$.*

We argued in Section 5.3 that the most likely queue-length sample path leading to demand loss with initial state α should be radial: when the controller chooses an exponent-optimal policy, the adversary picks a constant arrival rate \mathbf{f} such that the sample path of queue lengths is radial starting at α , and the Lyapunov function increases at a constant rate. From Lemma 3 we see that the rate at which the Lyapunov function increases depends on the (scaled) inflow and outflow rate of supply in each subset. Since the most likely queue-length sample path is radial, this sample path should drain the supply of one subset (the critical subset), and that subset will determine the demand loss exponent. We next lemma obtains an explicit expression for $\gamma_{\text{CB}}(\alpha)$ and the most likely demand FSP forcing demand loss.

Lemma 5. Recall the definitions of \mathcal{J} in (12) and B_J, λ_J and μ_J in (13). For any $\alpha \in \text{relint}(\Omega)$, we have $\gamma_{\text{CB}}(\alpha) = \min_{J \in \mathcal{J}} B_J \log(\lambda_J/\mu_J)$. Moreover, the infimum in the definition of $\gamma_{\text{CB}}(\alpha)$ in Lemma 2 is achieved by the following \mathbf{f}^* : for any $J^* \in \text{argmin}_{J \in \mathcal{J}} B_J \log(\lambda_J/\mu_J)$,

$$f_{j'k}^* \triangleq \begin{cases} \hat{\phi}_{j'k} \lambda_{J^*} / \mu_{J^*} & \text{for } j' \in J^*, k \notin \partial(J^*), \\ \hat{\phi}_{j'k} \mu_{J^*} / \lambda_{J^*} & \text{for } j' \notin J^*, k \in \partial(J^*), \\ \hat{\phi}_{j'k} & \text{otherwise.} \end{cases} \quad (25)$$

Remark 2 (Critical Subset Property). Lemma 5 provides the most likely demand sample path that leads to demand loss under any dispatch policy that is exponent optimal, starting at state¹⁴ α . We observe the critical subset property:

- (Adversary's strategy) For each starting state $\alpha \in \text{relint}(\Omega)$, there is (are) corresponding critical subset(s) $J^* \in \text{argmin}_{J \in \mathcal{J}} B_J \log(\lambda_J/\mu_J)$, such that the most likely demand sample path forcing demand loss drains a critical subset.
- (Controller's strategy) If the current state \mathbf{x} is on the most likely sample path forcing demand loss in critical subset J^* starting at α , an exponent optimal policy (for given α) will maximally protect J^* at \mathbf{x} , i.e., the policy will use supply in $\partial(J^*)$ exclusively to serve demand originating in J^* . Lemma 3 tells us that $\text{SMW}(\alpha)$ is such a policy.

We can now prove the main theorem.

Proof of Theorem 1. Lemma 2 along with the explicit expression for $\gamma_{\text{CB}}(\alpha)$ provided by Lemma 5 yields the converse result (part 2 of the theorem).

Achievability (part 1 of the theorem) follows from Corollary 1 along with the explicit expression for $\gamma_{\text{CB}}(\alpha)$ provided by Lemma 5. □

6 Application to shared transportation systems

In this section we discuss the application of our findings to shared transportation systems including ride-hailing and bike sharing systems, focusing on assignment control. In these systems, for each customer (demand unit), the platform must assign a vehicle (supply unit) which is sufficiently close to their origin location, and this limited flexibility leads to the compatibility graph G in our model. (In bikesharing, customers are willing to walk only a certain amount for pickup; within these constraints, they do respond to suggestions to prefer a given pickup location as in the Bike Angels program of CitiBike; see Section 1.2.) The number of bikes in a bikesharing system is typically held constant as in our model, and in ride-hailing drivers typically do a substantial number of trips in a session,¹⁵ and so it is common for theoretical investigations

¹⁴Remark 2 applies to demand lost over a (long) finite horizon given starting state α . $\text{SMW}(\alpha)$ further forces the state to return to α (negative drift), so our observations carry over to the steady state as well under that policy.

¹⁵For example, the average number of trips per session is over 12 in New York City <https://toddschneider.com/dashboards/nyc-taxi-ridehailing-uber-lyft-data/>.

of tactical control levers to make the approximation that cars do not enter or leave the system, e.g., Braverman et al. (2016), Balseiro et al. (2019). Shared transportation platforms typically aim to meet as much demand as possible.¹⁶

Notably, in shared transportation systems, a supply unit must spend positive time serving a demand before becoming available again at the destination. In Section 6.1, we incorporate travel times into our theory and show that SMW policies retain their superior performance and ensure loss which decays exponentially in K . In Section 6.2, we provide a summary of simulation experiments for ridehailing based on New York City yellow cab data. The simulation results validate our theoretical results and demonstrate excellent performance of our policies (a full description is provided in Appendix J). Finally in Section 6.3 we briefly discuss additional aspects of ride-hailing and bike sharing systems.

6.1 Incorporating Travel Delays

In this subsection, we relax the assumption that supply units move instantaneously between nodes by adding travel delays. Even in the presence of travel delays, we will show that any SMW policy with scaling parameters $\alpha \in \text{relint}(\Omega)$ achieves exponential decay of the demand loss probability in the large market regime (the practically relevant regime).

We first describe the model with travel delays. The following model inherits all the components of the model defined in Section 2 where K is the number of supply units, except that it has an enlarged state space to keep track of in-transit supply units, and additional parameters to characterize travel times.

Model with travel delays. Following a standard way to model travel delays which preserves tractability (see, e.g., George 2012, Braverman et al. 2016, Banerjee et al. 2016), we assume that the travel delays of serving demand units are independent random variables drawn from exponential distributions with means which depend on the source and destination of the demand. Let the mean travel time from node $j' \in V_D$ to node $k \in V_S$ be denoted by $\tau_{j'k} \in \mathbb{R}_+$. We assume the τ s do not depend on K . We make the simplifying assumption that pickup remains instantaneous, because travel times between neighboring locations are short relative to travel times to all other locations. The primitives of the extended model are $(G, \hat{\phi}, \tau)$ and the demand type distribution is again $\phi = \frac{\hat{\phi}}{\mathbf{1}^\top \hat{\phi} \mathbf{1}}$.

The augmented state space. The state of the K -th system is now $(\mathbf{X}^K(t), \mathbf{Y}^K(t))$, where $X_i^K(t)$ is the number of *available* supply units at (supply) node i at time t , and $Y_{j'k}^K(t)$ is the number of supply units *in transit* from node j' to node k at time t . Note that the travel delays follow exponential distributions, which have the memoryless property, and therefore $(\mathbf{X}^K(t), \mathbf{Y}^K(t))$ fully characterizes the system state.

Large market regime. As before, we consider the large market regime where the number

¹⁶Though the formal objective in Section 2 was to maximize the fraction of demand served, note that all our results are unchanged if the platform is payoff-maximizing where the payoff of serving a demand depends on the demand's origin and destination. This is because we perform a large deviations analysis, and the payoff values have no impact on the large-deviation asymptotics.

of supply units K and the demand arrival rates $\hat{\phi}^K \triangleq K\hat{\phi}$ scale up proportionally. Since the mean travel times $(\tau_{j'k})_{j' \in V_D, k \in V_S}$ do not depend on K , if a $\Theta(1)$ fraction of demand is served on average, a $\Theta(K)$ number of supply units is in transit at any time, on average, meaning that an $\Theta(1)$ fraction of supply units is in service, consistent with the reality in shared transportation.

In order to order to serve (almost) all the demand, we need sufficiently many supply units. By Little's law, if all demand units are served, the expected number of in-transit supply units is $K \sum_{j' \in V_D} \sum_{k \in V_S} \hat{\phi}_{j'k} \tau_{j'k}$. This number must be smaller than K to satisfy all demand even if stochasticity is ignored. In order to obtain an exponentially small loss despite stochasticity, we will need a slightly stronger assumption:

Assumption 4. *The model primitives $(G, \hat{\phi}, \tau)$ satisfy $\sum_{j' \in V_D} \sum_{k \in V_S} \hat{\phi}_{j'k} \tau_{j'k} < 1$.*

Let $\beta \triangleq 1 - \sum_{j' \in V_D} \sum_{k \in V_S} \hat{\phi}_{j'k} \tau_{j'k}$. Here β is the proportion of free supply units if all demands are served, and $1 - \beta$ is the ideal utilization rate (the utilization rate if all demands are served). Here utilization rate is the average proportion of time during which a supply unit is engaged in serving demand. Assumption 4 requires that $\beta \in (0, 1)$, which is consistent with the reality in shared transportation, e.g., the ride-hailing industry in New York City has an average driver utilization rate of 58% (Parrott and Reich 2018, NYC TLC and DoT 2019), i.e., on average 42% of drivers are free at any given time (moreover, most of these free drivers are not travelling to pick up a passenger¹⁷). In most bikesharing systems, the fraction of bikes in transit at any time is typically quite small (under 10%).¹⁸

The following is our main result for the setting with travel delay. For any assignment policy U , define the pessimistic performance measure $\gamma_p(U)$ by (4).

Theorem 2 (Result with Travel Delays). *Consider any network with travel delays $(G, \hat{\phi}, \tau)$. If the network satisfies Assumptions 1, 2, 3 and 4, then for any $\alpha \in \text{relint}(\Omega)$, $\text{SMW}(\alpha)$ achieves exponential decay of the demand loss probability with strictly positive demand loss exponent, i.e., $\gamma_p(\text{SMW}(\alpha)) > 0$.*

Theorem 2 shows that a key finding obtained from the analysis in previous sections (where there is no travel delay), i.e., that SMW policies achieve exponentially decaying demand loss probability as the number of supply units increases, is preserved when delay is incorporated. The scaling regime is the natural large market regime, along with the natural assumption that the system has a fleet size (of supply units) that is strictly larger than what is necessary to satisfy all demand (Assumption 4). Thus, SMW policies are able to deploy excess supply to effectively manage the stochasticity caused by travel time and demand uncertainty in the system.

¹⁷ NYC TLC and DoT (2019) reports that the average trip duration is 20 minutes, and for each trip that occurs a driver spends nearly 14 minutes “cruising” (free), and less than half of that time, about 5.5 minutes, is the driver traveling to pick up a passenger. Thus a driver spends roughly 8 minutes waiting for their next trip.

¹⁸The report <https://nacto.org/bike-share-statistics-2017/> tells us that U.S. dock-based systems produced an average of 1.7 rides/bike/day, while dockless bike share systems nationally had an average of about 0.3 rides/bike/day. Average trip duration was 12 minutes for pass holders (subscribers) and 28 mins for casual users. In other words, for most systems, each bike was used less than 1 hour per day, which implies that less than 10% of bikes are in use at any given time during day hours (in fact the utilization is below 10% even during rush hours).

Meanwhile, the negative results in Section 4.2 on state-independent policies and naive state-dependent policies are also preserved with travel delay, i.e., any state-independent policy can only achieve polynomially decaying demand loss and moreover (typically) fails asymptotic optimality if exact demand arrival rates are not known, and similarly a naive state-dependent policy can incur $\Omega(1)$ demand loss.

Remark 3 (State-independent/naive state-dependent policies remain inferior and utilization rate remains high). *Augment the system in Propositions 2, 3 and 4 (and Example 4) to incorporate travel delays τ as above. Then Propositions 2, 3 and 4 (and the claim in Example 4) continue to hold, and the proofs are unchanged.*

Thus, SMW policies remain substantially superior to alternative policies under travel delays.

We prove Theorem 2 in Appendix H. Similar to the previous analysis, the proof of Theorem 2 is based on a novel Lyapunov analysis. The analysis is more involved than the one in Section 5 because of the enlarged state space. For each $\alpha \in \text{relint}(\Omega)$, we construct a Lyapunov function that augments the prior Lyapunov function (see Definition 7) with additional terms that capture how much the number of in-transit supply units deviate from their typical values. We show that in the fluid limit, the Lyapunov function exhibits a strictly negative drift if the current state is not at its unique minimum. Using similar methodology as in Section 5, we show that the demand loss exponent can be lower bounded by a variational problem (more complicated than the one in Section 5) that has strictly positive value, leading to Theorem 2.

6.2 Simulation experiments

We use NYC yellow cab data (to estimate demand) and Google Maps (to estimate travel times) to simulate SMW-based dispatch policies in an environment that resembles the real-world ride-hailing system in Manhattan, New York City. In the interest of space, we provide only a brief summary of these experiments here and refer the interested reader to Appendix J for a full description.

Our theoretical model in Section 2 made several simplifying assumptions:

1. Service is *instantaneous* (i.e., vehicles travel to their destination with no delay).
2. Pickup is *instantaneous* (i.e., vehicles travel to matched customers with no delay).
3. The objective is to minimize lost demand *in steady state* (though our characterization extends to transient performance as shown in Appendix D.4).

We relax these assumptions one by one in our numerical experiments. We study three settings: (i) steady state performance with *Service times* (Section J.2); (ii) steady state performance with *Service+Pickup times* (Section J.3); and (iii) *Transient* performance with *Service+Pickup times* (Section J.4). For the second and third settings, we modify SMW policies heuristically to incorporate pickup times. In each case, we let the number of cars in the system be only slightly ($\sim 3\%$) above the “fluid requirement” (see Appendix J.5 for a formal definition of the fluid requirement) to meet demand, and find that we are able to meet almost all demand nevertheless

(the number of free cars in real systems is typically much larger and hence the real problem is easier along this dimension, see the paragraph following Assumption 4 in Section 6.1).

A highlight of SMW policies is that they are a simple family of policies with a manageable number of parameters (one per location). We propose a simulation-based optimization approach to choose the scaling parameters α in a practical setting.

Summary of findings (Appendix J). Consistently across all three settings, we find that the vanilla MaxWeight policy, which requires no knowledge of the demand arrival rates, outperforms static (fluid-based) control proposed in prior work by up to an order of magnitude, and loses very little demand even with small K (just ~ 10 free cars per location, whereas the static policy has a lot more free cars to work with since it loses so much more demand). Furthermore, in each of the settings, the SMW policy obtained using simulation-based optimization further significantly outperforms vanilla MaxWeight. Overall, we deduce that non-zero service times, non-exponential pickup times, and finite K do not diminish the effectiveness of the SMW family policies at managing the spatial distribution of supply. In addition, we observe that the simulation-based optimal scaling factors α in the Service time setting are similar to the theory-based optimal α , indicating robustness of our structural results (Section 4.1) to travel time.

6.3 Additional discussion

Role of supply as a buffer. As mentioned, less than 10% of bikes in a typical bike sharing system are in use at any time. The vast majority of bikes serve as a “buffer” against distributional mismatch between supply and demand, and *not* merely to fulfil the “service requirement”. This aligns well with our focus in this paper on the role of supply as a buffer. In ride-hailing systems a larger fraction $\sim 60\%$ of cars are typically carrying passengers at any time, but this still leaves a substantial fraction $\sim 40\%$ free, and these free cars again serve as a buffer.

Empty relocation. It is quite costly for bike share system operators to relocate bikes, and they generally prefer to avoid (or minimize) this. In ride-hailing, empty relocation incurs gas costs (it also costs driver effort and causes road congestion), and may be beneficial to drivers in some settings and not in others.¹⁹

Incorporating empty relocation in our theory. Drivers may independently choose to relocate without a passenger, or the platform may make relevant suggestions to drivers (or incentivize drivers to relocate). For example, if CRP is violated in the absence of empty relocation, the ride-hailing platform may employ empty relocation to ensure that CRP holds.

We point out that state-independent relocation of free supply units can be seamlessly incorporated into our framework following the approach in Banerjee et al. (2016, Section 5.1): For every trip ending at node $k \in V_S$, the car is redirected to node $i \in V_S$ with probability r_{ki} for all $i \in V$, independently. Call $(r_{ki})_{k \in V_S, i \in V_S}$ the empty-relocation rule and i

¹⁹For instance, this online article by Uber data scientists <https://www.uber.com/newsroom/semi-automated-science-using-an-ai-simulation-framework> finds that “...when dispatch distances are relatively longer, drivers maximize their earnings by using less gas by remaining stationary between trips” instead of gravitating to high demand areas, and that this behavior causes only a few additional trips to be lost.

the “effective destination”. This generalization of our model is straightforward to incorporate. Throughout the paper, the demand type distribution ϕ is replaced with the “effective demand type distribution” ϕ^{eff} whose definition is immediate from the empty-relocation rule: $\phi_{j'i}^{\text{eff}} \triangleq \sum_{k \in V_S} \phi_{j'k} r_{ki}$, and our entire formulation, analysis and results in Sections 2-5 remain unchanged. Section 6.1 incorporating travel delays also extends unchanged with the modified definition $\beta \triangleq 1 - \sum_{j' \in V_D} \sum_{k \in V_S} \sum_{i \in V_S} \hat{\phi}_{j'k} r_{ki} (\tau_{j'k} + \tau_{ki})$ and the assumption that this $\beta > 0$ in place of Assumption 4.

Future directions related to bike sharing. Our model in Section 2 captures pickup flexibility in dockless bike-sharing systems (e.g., Mobike in China, the world’s largest shared bicycle operator by number of bicycles). Beyond our model, bike sharing may afford the platform the additional control lever of suggesting to customers where to drop off their bike, in which case we expect that SMW policies retain their guarantees with the recommended dropoff location being the location near the destination with the fewest (scaled) number of bikes. In docked bike-sharing systems (e.g., CitiBike in New York City), there is an additional wrinkle, namely, stations have a limited number of docks, and a bike cannot be dropped off at a location if no dock is available. We are optimistic that our analysis can be extended to such a setting, leading to generalized SMW policies which seek to ensure that both bikes and free docks remain available throughout the network.

7 Application to Scrip Systems

Scrip systems allow agents to exchange services like babysitting, and have been proposed as a way to improve the functioning of kidney exchanges (here hospitals play the role of agents). In a scrip system, a fixed amount of artificial currency (scrips) circulates among a set of agents, and when agent i services a request by agent k , then agent k “pays” agent i in scrip. Given a service request, the platform has limited flexibility in assigning the provider since, typically, only a subset of agents are able to provide the requested service. A loss occurs when an agent runs out of scrips and is hence unable to request service. We show that with only cosmetic modifications, our model and results translate fully to a model of a scrip system with heterogeneous services, thus providing novel prescriptive insights into dynamic assignment control of such systems. We show that for any scrip system such that CRP (formally reintroduced for this application later) holds, we can construct a family of simple service provider selection rules, which we name *Scaled Minimum Scrips* (SMS) policies, and prove a very strong performance guarantee analogous to Theorem 1 for these policies. In particular, SMS policies achieve exponentially small loss under complete resource pooling, and moreover, there is an SMS policy (which we characterize) which is exponent optimal among all policies.

We note that many features of our model align with real-world scrip systems. Transactions in scrip systems are typically quick, which justifies our instantaneous relocation assumption. Scrips only relocate as a result of transactions (no “empty” relocation). The number of scrips is typically

held nearly constant over significant periods of time. Finally, the CRP assumption appears reasonable for many scrip systems: In the proposed scrip system between hospitals for kidney exchanges (Agarwal et al. 2019), approximate similarity of patient pools across hospitals and partial flexibility in matching donor-patient pairs with each other should ensure CRP. One would also expect CRP to hold for scrip systems in contexts like babysitting, as long as participants make themselves available as providers sufficiently often.

7.1 Model of Scrip Systems

We now provide a detailed description of our model of a scrip system.

Service exchange. The set of primitives is the same as in the previous model, i.e., it consists of a compatibility graph $G(V_S \cup V_D, E)$ and Poisson arrivals with a demand arrival rate matrix $\hat{\phi}$ and consequent demand type distribution (normalized demand arrival rate) matrix $\phi = \hat{\phi}/(\mathbf{1}^T \hat{\phi} \mathbf{1})$ (let $m = |V_S|, n = |V_D|$). Here V_S is the finite set of agents, and V_D is the finite set of heterogeneous *types* of service. Each agent has a skill set, i.e., the service types he²⁰ can provide. The skill set structure is modeled by the *skill compatibility graph* G (see Figure 5 for an illustration). The neighborhood of $i \in V_S$ in G is his skill set, which is denoted by $\partial(i) \subseteq V_D$. The neighborhood of $j' \in V_D$ in G consists of the providers of type j' service, which is denoted by $\partial(j') \subseteq V_S$.

The main difference between the current model and the previous model is in the types of requests (i.e., demand). In the previous model, each demand originates from a demand node and has a supply node destination. The situation is reversed here: each service request originates from an agent (i.e., “supply node”) and requires a certain service type (i.e., “demand node”). Therefore, the arrival rate matrix ϕ is of dimension $m \times n$, and $\phi_{ij'}$ is the probability of a request to be of type (i, j') requests, i.e., it comes from agent i and requests type j' service. We assume that agent i does not request service types in $\partial(i)$ (i.e., service types belonging to i ’s own skill set); formally, $\phi_{ij'} = 0$ for all $i \in V_S, j' \in \partial(i)$. (This assumption does not impose any restriction, since, if $i \in \partial(j')$ but i wants to request service type j' , one can formally define an additional service type k' such that $\partial(k') = \partial(j') \setminus \{i\}$ and classify the request as type (i, k') .) We also assume that each agent has a positive arrival rate of requesting *some* service type.

Scrips. There are a fixed number (denoted by K) of scrips in the K -th system, which are distributed among the agents. Denote the number of scrips each agent has at time t as $\mathbf{X}^K(t) = [X_1^K(t), \dots, X_m^K(t)]$, hence $\mathbf{X}^K(t) \in \Omega_K$ where Ω_K is defined in Section 2.

We informally point out that there is a natural constraint on the total number of scrips a system operator can introduce: Whereas it is tempting to think that the efficiency of a scrip system can be increased simply by increasing the total number of scrips in circulation, this is the case only up to the point where the system experiences a “monetary crash”, where money is sufficiently devalued that no agent is willing to perform a service; see, e.g., Kash et al. (2012).

Service provider selection rule. The central planner’s control lever is the *provider se-*

²⁰For expository simplicity, we refer to an agent as “he” and the central planner as “she”.

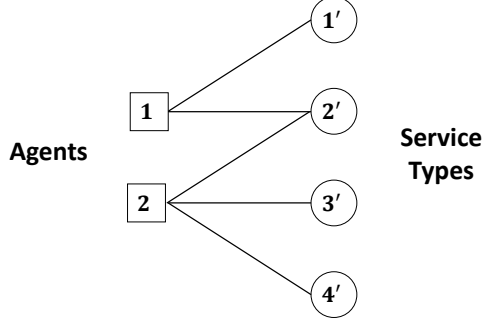


Figure 5: An example of skill compatibility graph in a service exchange with two service types and four agents.

lection rule: when a request of type (i, j') arrives, the planner chooses the provider of type j' service. Subsequently, after providing the service, agent i pays a scrip to the service provider. As is typical in scrip systems, if an agent i has no scrip, then his request is lost. As in the previous model, it suffices to consider stationary policies U , which is formally defined as a sequence of mappings, indexed by the total number of scrips K , that map the current distribution of scrips \mathbf{X}^K and request type (i, j') to $\partial(j') \cup \{\emptyset\}$.

Let t_r be the r -th service request arrival epoch after time 0. Denote the state of the system just before t_r by $\mathbf{X}^K(t_r^-)$ (the initial state is $\mathbf{X}^K(0)$). Now suppose the platform uses an assignment policy U , and the r -th request comes from agent $o[r]$ and the requested service type is $d[r]$. Let $S[r] \triangleq U^K[\mathbf{X}^K(t_r^-)](o[r], d[r])$ be the chosen service provider (potentially \emptyset). Formally,

$$\mathbf{X}^K(t_r) \triangleq \begin{cases} \mathbf{X}^K(t_r^-) - \mathbf{e}_{o[r]} + \mathbf{e}_{S[r]} & \text{if } S[r] \in V_S, \\ \mathbf{X}^K(t_r^-) & \text{if } S[r] = \emptyset. \end{cases}$$

Performance measure. We consider a central planner who tries to maximize the fraction of requests served. We define the *optimistic* and *pessimistic* performance measures in exactly the same way as in (1) and (2). Similarly, for policy U , we define *demand-loss exponents* $\gamma_o(U)$ and $\gamma_p(U)$ in the same way as in (3) and (4).

Complete Resource Pooling condition (for scrip systems). We require the following CRP condition on the network primitives G and ϕ for our main result in this section.

Assumption 5. We assume that for all subsets $I \subsetneq V_S$ where $I \neq \emptyset$, it holds that $\lambda_I > \mu_I$ for $\lambda_I \triangleq \sum_{i \notin I} \sum_{j' \in \partial(I)} \phi_{ij'}$ and $\mu_I \triangleq \sum_{i \in I} \sum_{j' \notin \partial(I)} \phi_{ij'}$.

Intuitively, Assumption 5 assumes that for each subset $I \subsetneq V_S$ of agents, requests (from outside I) which belong to the union of their skill sets arrive fast enough that they can earn enough scrips to finance their own service requests.²¹

²¹Let us clarify the relationship between Assumption 5 and the assumptions we made in the main model in Section 2: Assumption 5 is slightly stronger than Assumption 3 in that it requires strict inequality for *all* strict subsets of V_S and not just for subsets with $\mu_I > 0$. Though we do not need this stronger assumption for our analysis, we make it to simplify the exposition in this section by eliminating the need for other assumptions. In particular, Assumption 5 automatically implies connectivity (the analog of Assumption 1). Also, the analog of Assumption 2 (limited flexibility) holds automatically in the present setup since each individual agent forms a

Discussion of the model. The skill compatibility graph can capture intricate compatibility structures. For example, in scrip systems for kidney exchange, for each service (i.e., exchange) request, the ability of each other agent (hospital) to service the request may be thought of as stochastic or else arbitrary. Happily, arbitrary compatibilities can be captured in our framework by including a node in V_D for each element in 2^{V_S} , i.e., the power set of V_S .

7.2 Scaled Minimum Scrips (SMS) selection rules and main result

Leveraging the similarity between the current model and the previous model introduced in Section 2, we are easily able to define the following Scaled Minimum Scrip selection rule which is similar to SMW in spirit and achieves exponentially decaying demand loss. The formal definition of SMS is as follows.

Definition 8 (Scaled Minimum Scrip selection rule $\text{SMS}(\alpha)$). *Fix $\alpha \in \text{relint}(\Omega)$, i.e., $\alpha \in \mathbb{R}^m$ such that $\alpha_i > 0 \forall i \in V_S$ and $\sum_{i \in V_S} \alpha_i = 1$. Given system state $\mathbf{X}(t_r^-)$ just before the r -th demand arrival and for demand with type (i, j') , $\text{SMS}(\alpha)$ chooses service provider*

$$\text{argmin}_{k \in \partial(j')} \frac{X_k(t_r^-)}{\alpha_k}$$

if $X_i(t_r^-) > 0$; otherwise the request is lost. (If there are ties when determining the argmin, it assigns from the location with highest index.)

The following performance guarantee similar to Theorem 1 holds for Scaled Minimum Scrip(α) under the CRP condition (Assumption 5).

Theorem 3 (Result for Scrip Systems). *For any scrip system (G, ϕ) satisfying Assumption 5, we have:*

1. **Exponentially small loss under any SMS policy:** *For any $\alpha \in \text{relint}(\Omega)$, $\text{SMS}(\alpha)$ achieves exponential decay of the demand loss with exponent,*

$$\gamma(\alpha) = \min_{I \subseteq V_S, I \neq \emptyset} B_I \log \left(\frac{\lambda_I}{\mu_I} \right) > 0, \quad (26)$$

$$\text{where } B_I \triangleq \mathbf{1}_I^T \alpha, \quad \lambda_I \triangleq \sum_{i \notin I} \sum_{j' \in \partial(I)} \phi_{ij'}, \quad \text{and} \quad \mu_I \triangleq \sum_{i \in I} \sum_{j' \notin \partial(I)} \phi_{ij'}. \quad (27)$$

2. **There is an exponent optimal SMS policy:** *Under any policy U , it must be that*

$$\gamma_p(U) \leq \gamma_o(U) \leq \bar{\gamma}, \quad \text{where } \bar{\gamma} = \sup_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha). \quad (28)$$

Thus, there is an SMS rule that achieves an exponent arbitrarily close to the optimal one.

The proof of Theorem 3 is very similar to that of Theorem 1; see Appendix G.

Remark 4 (Comparison with the model in Johnson et al. (2014)). *Johnson et al. (2014) consider the case where there is only one type of service which all agents can provide (i.e., G is*

“limited flexibility” subset, i.e., for all $i \in V_S$ we have $\mu_{\{i\}} > 0$, which holds since $\forall i \in V_S \exists j' \in V_D$ such that $\phi_{ij'} > 0$, and moreover $\phi_{ij'} > 0 \Rightarrow j' \notin \partial(i) \Rightarrow \mu_{\{i\}} > 0$.

a star graph), and $\phi_{ij'}$ is equal for all agents i . On one hand, we significantly generalize their model by considering heterogeneous services, asymmetric service request arrivals, and general skill compatibility graphs. They show that the minimum scrip selection rule, a special case of our SMS rule, is optimal for their symmetric setting, whereas we show that the family of SMS selection rules achieve exponentially small demand loss and that there exists an SMS rule that is globally exponent-optimal. On the other hand, our analysis of scrip systems is meant to illustrate the versatility of SMW type policies, hence we only focused on the central planner setting and leave a study of the incentives of agents for future work.

8 Discussion

In this paper we study state-dependent assignment control of a shared transportation system modeled as a closed queueing network. We introduce a family of state-dependent assignment policies called Scaled MaxWeight (SMW) and prove that they have superior performance in terms of maximizing throughput, comparing with state-independent policies including previously proposed policies. In particular, we construct an SMW policy that (almost) achieves the optimal large deviation rate of decay of demand loss. Our analysis also uncovers the structure of the problem: given system state, demand loss is most likely to happen within state-dependent critical subsets of locations. The optimal SMW policy protects all critical subsets simultaneously.

SMW policies are simple and explicit, and hence have the potential to influence practice. We discuss two applications: Towards shared transportation applications, we show the SMW policies continue to have exponentially small loss if there are positive travel times, and obtain promising simulation results in a realistic ridehailing environment. We also provide a model of a scrip system, and show that our entire formulation and results translate to that model with only cosmetic changes, leading us to propose Scaled Minimum Scrip (SMS) policies for service provider assignment in such systems.

Connection with closed queueing networks. There is a subtle difference between our model and “classical” closed queueing network, but our results extend to the “classical” setting, as we clarify below. We define our model’s “classical” (controlled) closed queueing network (CQN) counterpart as follows: There are a sequence of systems indexed by $K \in \mathbb{Z}_+$. Consider the K -th system. There are m buffers indexed by $i \in V_S$ and n servers indexed by $j' \in V_D$. When server j' becomes free, the system controller must immediately decide which buffer from $\partial(j')$ to serve; also, when a job joins buffer i and at least one server in $\partial(i)$ is empty, the system controller needs to decide which server will serve the job (“scheduling”). The service time by server j' is an independent $\text{Exponential}(K\hat{\mu}_{j'})$ random variable where $\hat{\mu}_{j'} \triangleq \sum_{k \in V_S} \hat{\phi}_{j'k}$. When the service ends, with probability $\hat{\phi}_{j'k}/\hat{\mu}_{j'}$ the job is routed to buffer k , and the server becomes free again. There are K jobs in the system, and jobs do not enter or leave the system.

A quick reminder about the analogy between our model and the classical CQN above: the K supply units are “jobs”, each demand location is a “server”, each supply location is a “buffer”, inter-arrival times of customers with origin i are “service times” at “server” i . The distribution

of customers’ destinations given an origin node captures “routing probabilities”. “Servers” are flexible (i.e., they can serve multiple “buffers”), and assignment is equivalent to “scheduling”.

We now explain the subtle difference between our model (Section 2) and its classical counterpart. In our model, the server (demand location) is modeled by a Poisson process of “service tokens” (demand requests). Scheduling (assignment) decisions are made only when a service token is generated: if the queue to which the token is assigned is non-empty, then one job (supply unit) from that queue “consumes” the service token and relocates, otherwise, the service token is “wasted” (demand is lost). The service token formulation is not new, see Spencer et al. (2014) and the references therein. When the queue to which the token is assigned is non-empty, the generation of a service token can be interpreted as the completion of a previous job, upon which the server is ready to fetch the next job. The time between two consecutive tokens generated at the same server corresponds to the service time (of the job fetched when the earlier token of the two is generated). The waste of service token can be interpreted as the server starting to serve a “dummy job”. Service of dummy jobs corresponds to server idleness in the “classical” model. There are two main differences between our model and the “classical” model: (i) In the “classical” model, a job joins the destination queue at the end of its service time, while in our model the supply unit joins the destination queue at the beginning of its “service time”. (ii) In the “classical” model, SMW policies are work-conserving, i.e. a server will not idle if at least one of its neighboring queues is non-empty. In our model, however, if the server is serving a “dummy job” and a job arrives at the server’s neighborhood, the server has to wait until the “dummy job” is finished to start service of the newly arriving job.

Despite these differences, our main result (Theorem 1) extends unchanged to the classical closed queueing network as follows. Define the SMW policies in the same way as in this paper and let them be non-preemptive.²² Let the objective be the fraction of time some server is idle in steady state.²³ Suppose $(\hat{\phi}, G)$ satisfies Assumptions 1, 2, and 3. Then we have (achievability): as $K \rightarrow \infty$, under the SMW(α) policy (where $\alpha \in \text{relint}(\Omega)$), the objective decays to zero exponentially fast with exponent $\gamma(\alpha)$ specified in Theorem 1 of this paper. We also have (converse): no scheduling policy can lead to an exponentially decaying objective as $K \rightarrow \infty$ with exponent strictly larger than $\bar{\gamma} \triangleq \sup_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha)$.

Our work may inspire similar analyses in open networks, e.g., obtaining exponent optimal controls when there is a shared finite buffer (e.g., a common waiting room) for multiple queues.

²²As required in the classical setting, the decision is made at the beginning of the “service time” (interarrival time). Supply units that enter “service” are removed from the buffer from which they are assigned, and join the “destination” buffer k only upon completion of service. We emphasize that the “service time” here has nothing to do with the relocation time; the latter is again assumed to be zero. (We can also incorporate positive relocation times as in Section 6.1, by incorporating, for each destination k with $\hat{\phi}_{j'k} > 0$, the need to be served by one of infinitely many “relocation servers” with $\text{Exponential}(1/\tau_{j'k})$ service times, immediately after the assignment service is complete. The job joins the buffer at destination k only after the relocation service is complete. Theorem 2 remains intact under this CQN model which includes relocation servers.)

²³Demand drops in the original model correspond to service tokens which arrive in the classical CQN when the corresponding server is idle. Using the PASTA property (Wolff 1982), the fraction of such service tokens at a given server is identical to the fraction of time that server is idle, and the exponent (with respect to K) for the latter fraction, minimized across the n servers, is equal to the exponent for the fraction of time *some* server is idle in steady state.

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Appendix

This technical appendix is organized as follows.

- We prove our main result, Theorem 1, in Appendices A-D. In particular:
 - Appendix A discusses fluid sample paths in detail and establishes key properties of our Lyapunov functions, including the proof of Lemma 1.
 - Appendix B includes the proof of Lemma 2, a converse bound on the demand loss exponent.
 - Appendix C includes the proof of Proposition 5, containing sufficient conditions for a policy to achieve the optimal exponent.
 - Appendix D shows that the SMW policy satisfies the sufficient conditions for exponent optimality, and derives explicitly the optimal exponent and most-likely sample paths, including the proofs of Lemma 3, Lemma 4, and Lemma 5. It also formally establishes exponent optimality of SMW policies for transient performance.
- Appendix E includes the proof of Proposition 3 showing frequent utilization of supply units under SMW, and provides the structural corollaries (of Theorem 1) illustrated in Section 4.1.
- Appendix F shows the necessity of the assumptions and state-dependent control, including the proofs of Propositions 1, 2 and 4, and the claim in Example 4.
- Appendix G proves Theorem 3, the extension of our main result to scrip systems.
- Appendix H proves Theorem 2, the extension of our main result to the shared transportation setting with travel delays.
- Appendix I proves that the Assumption 3 in our paper is implied by the CRP condition defined in Dai and Lin (2008).
- Appendix J provides the full description of our simulation experiments.

A Lyapunov Functions and Fluid Sample Paths

A.1 Properties of the Lyapunov Functions $L_\alpha(\mathbf{x})$

A.1.1 Scale-invariance and sub-additivity (about α): proof of Lemma 1

Proof of Lemma 1. (i) For $c > 0$, $\alpha \in \text{relint}(\Omega)$, we have

$$L_\alpha(\alpha + c\Delta\mathbf{x}) = 1 - \min_i \frac{\alpha_i + c\Delta x_i}{\alpha_i} = -\min_i \frac{c\Delta x_i}{\alpha_i} = -c \min_i \frac{\Delta x_i}{\alpha_i} = cL_\alpha(\alpha + \Delta\mathbf{x}).$$

(ii) For $\alpha \in \text{relint}(\Omega)$, we have

$$\begin{aligned} L_\alpha(\alpha + \Delta\mathbf{x} + \Delta\mathbf{x}') &= 1 - \min_i \frac{\alpha_i + \Delta x_i + \Delta x'_i}{\alpha_i} = -\min_i \frac{\Delta x_i + \Delta x'_i}{\alpha_i} \\ &\leq -\min_i \frac{\Delta x_i}{\alpha_i} - \min_i \frac{\Delta x'_i}{\alpha_i} = L_\alpha(\alpha + \Delta\mathbf{x}) + L_\alpha(\alpha + \Delta\mathbf{x}'). \end{aligned}$$

□

A.1.2 Regularity properties

The following lemma is a collection of regularity properties of $L_{\alpha}(\mathbf{x})$ that are useful in the following proofs.

Lemma 6. *For $\alpha \in \text{relint}(\Omega)$ and $L_{\alpha}(\mathbf{x})$ specified in Definition 7, we have*

1. $L_{\alpha}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Omega$, and $L_{\alpha}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \alpha$.
2. $L_{\alpha}(\mathbf{x})$ is globally Lipschitz on Ω , i.e. for any $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$, we have

$$|L_{\alpha}(\mathbf{x}_1) - L_{\alpha}(\mathbf{x}_2)| \leq \frac{1}{\min_i \alpha_i} \|\mathbf{x}_1 - \mathbf{x}_2\|_{\infty}.$$

Proof of Lemma 6. Property 1 is easy to verify hence we omit the proof.

For property 2, note that

$$|L_{\alpha}(\mathbf{x}_1) - L_{\alpha}(\mathbf{x}_2)| = \left| \min_i \frac{x_{1,i}}{\alpha_i} - \min_i \frac{x_{2,i}}{\alpha_i} \right| \leq \min_i \frac{|x_{1,i} - x_{2,i}|}{\alpha_i} \leq \frac{1}{\min_i \alpha_i} \|\mathbf{x}_1 - \mathbf{x}_2\|_{\infty}.$$

□

A.2 Discussion of FSPs

In this section, we discuss the existence of *fluid sample paths (FSPs)* and techniques related to FSP in large deviations analysis. FSP is a technique used to establish large deviation bounds of the *queue lengths* using the sample path large deviation principle of *demand arrival processes* (Fact 1), see, e.g., Stolyar (2008), Venkataramanan and Lin (2013).

We briefly comment on the existence of FSP. Consider a sequence of demand sample paths $\{\bar{\mathbf{A}}^K(\cdot)\}_{K=1}^{\infty}$ where in the K -th system the interarrival times of type (j', k) demand are deterministic with value $\frac{1}{K\hat{\phi}_{j'k}}$. It is trivial to show that $\{\bar{\mathbf{A}}^K(\cdot)\}_{K=1}^{\infty}$ converges uniformly on compact intervals (u.o.c.) to the fluid limit $\bar{\mathbf{A}}(t) = t\hat{\phi}$. For any policy $U \in \mathcal{U}$, because at most one relocation happens at each demand arrival, each (normalized) queue length process $\bar{\mathbf{X}}^K(\cdot) = \Psi^{K,U}(\bar{\mathbf{A}}^K(\cdot), \bar{\mathbf{X}}^K(0))$ is Lipschitz continuous with Lipschitz constant $\mathbf{1}^T \hat{\phi} \mathbf{1}$, hence equicontinuous; see, for example, Royden and Fitzpatrick (1988). Thus, there must exist a subsequence of $\{\bar{\mathbf{X}}^K(\cdot)\}_{K=1}^{\infty}$ that converges u.o.c. to a continuous function $\bar{\mathbf{X}}(\cdot)$. Therefore $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(\cdot))$ is an FSP. This establishes the existence of FSP.

In the large-deviations literature, a technique named the “contraction principle” is often used to translate large deviations principles (LDP) for the arrival process to LDP for the state process, see Dembo and Zeitouni (1998). The translation step is important in most of the large deviations analysis in the literature, including the one in this paper. However, to apply the contraction principle one needs to prove that the mapping from demand sample path $\bar{\mathbf{A}}(\cdot)$ to queue length sample path $\bar{\mathbf{X}}(\cdot)$ is continuous with respect to suitable topologies for the corresponding functional spaces. The continuity is usually technically challenging to establish (see Subramanian (2010) for an application of the contraction principle to MaxWeight policies under a different setting). The FSP technique partly circumvents this issue.

B Converse Bound on the Exponent: Proof of Lemma 2

In this section, we prove Lemma 2, the converse bound on the exponent for any policy $U \in \mathcal{U}$. The proof consists of three steps:

- Step 1: For each stationary policy $U \in \mathcal{U}$ we define a state $\tilde{\alpha} \in \text{relint}(\Omega)$ such that the state visits the neighborhood of $\tilde{\alpha}$ frequently enough. In the following steps we will bound the demand loss exponent of U by $\gamma_{\text{CB}}(\tilde{\alpha})$.
- Step 2: Given that the system's initial state is close to $\tilde{\alpha}$, we construct a set of demand sample paths that are guaranteed to lead to a demand loss regardless of the policy used. To this end, we compute $v_{\tilde{\alpha}}(\mathbf{f})$, which is the minimum rate of increase of $L_{\tilde{\alpha}}(\cdot)$ under demand arrival rates \mathbf{f} no matter the assignment distributions. This step is used to lower bound the “one-shot” probability of demand-loss.
- Step 3: We use renewal-reward theorem to translate the one-shot demand loss probability to steady-state demand loss probability. The final bound in (21) takes the supremum over α since the policy can choose its resting state.

The technique used in step 2 follows from Proposition 9 in Venkataramanan and Lin (2013). The approach in steps 1 and 3 is novel (to the best of our knowledge) and tackles the key challenge of our closed network model, i.e., the policy has the flexibility to choose a resting state, as opposed to open network settings where the resting state is always $\mathbf{0}$.

Proof of Lemma 2. Step 1: Find the “frequently visited” state α . Fix a stationary policy $U \in \mathcal{U}$. For each K , the K -th system under policy U is a finite-state Markov chain, whose state space has cardinality smaller than K^m . Since we are considering the optimistic exponent, let the K -th system start within a communication class that minimizes steady state demand loss among all communication classes. Denote the stationary distribution (henceforth it refers to the stationary distribution of the communication class where the initial state belongs to) of (normalized) states as $\pi^K(\bar{\mathbf{X}}^K)$. Then there must exist a (normalized) state $\tilde{\mathbf{X}}^K$ such that $\pi^K(\tilde{\mathbf{X}}^K) \geq K^{-m}$. Take a subsequence $\{K_r\}$ of $\{K\}$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{K_r} \log \mathbb{P}_o^{K_r, U} = \liminf_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_o^{K, U}.$$

By compactness of Ω , there must exist a further subsequence of $\{K_r\}$, which we denote by $\{K_{r'}\}$, and $\alpha \in \Omega$ such that $\lim_{r' \rightarrow \infty} \tilde{\mathbf{X}}^{K_{r'}} = \alpha$.

For any $0 < \epsilon_1 < \frac{1}{2} \left(\min_{j: \alpha_j > 0} \alpha_j \right)$, define $\tilde{\alpha} \in \text{relint}(\Omega)$ such that

$$\begin{aligned} 0 &< \tilde{\alpha}_j < \epsilon_1/2 \quad \text{for } j \text{ such that } \alpha_j = 0, \\ |\tilde{\alpha}_j - \alpha_j| &< \epsilon_1/2 \quad \text{for } j \text{ such that } \alpha_j > 0. \end{aligned}$$

Since α is the limit point of $\tilde{\mathbf{X}}^{K_{r'}}$, there exists $r'_0(\epsilon) > 0$ such that $\forall r' \geq r'_0(\epsilon)$,

$$0 \leq \tilde{X}_j^{K_{r'}} < \tilde{\alpha}_j \quad \text{for } j \text{ such that } \alpha_j = 0, \tag{29}$$

$$|\tilde{X}_j^{K_{r'}} - \alpha_j| < \epsilon_1/2 \quad \text{for } j \text{ such that } \alpha_j > 0. \tag{30}$$

Inequalities (29) and (30) imply that for $r' \geq r'_0(\epsilon)$

$$|\tilde{X}_j^{K_{r'}} - \tilde{\alpha}_j| \leq \tilde{\alpha}_j < \epsilon_1, \quad \text{for } j \text{ such that } \alpha_j = 0$$

$$|\tilde{X}_j^{K_{r'}} - \tilde{\alpha}_j| \leq |\tilde{X}_j^{K_{r'}} - \alpha_j| + |\tilde{\alpha}_j - \alpha_j| < \epsilon_1, \quad \text{for } j \text{ such that } \alpha_j > 0.$$

Hence $\|\tilde{\mathbf{X}}^{K_{r'}} - \tilde{\boldsymbol{\alpha}}\|_\infty < \epsilon_1$ for $r' \geq r'_0(\epsilon)$.

We quantify the fact that $\tilde{\boldsymbol{\alpha}}$ is a “frequently visited” state in the following claim.

Claim: Fix $K = K_{r'}$ that comes from the subsequence defined above. In the K -th system, define

$$\tau^K \triangleq \inf \left\{ t > 0 : \bar{\mathbf{X}}^K(t) = \tilde{\mathbf{X}}^K | \bar{\mathbf{X}}^K(0) = \tilde{\mathbf{X}}^K \right\}, \quad (31)$$

then we have

$$\mathbb{E}[\tau^K] \leq \frac{K^m}{\mathbf{1}^\top \hat{\boldsymbol{\phi}} \mathbf{1}}.$$

Proof of claim: Consider the discrete-time embedded chain of $\{\bar{\mathbf{X}}^K(\cdot)\}$. Since the initial state $\tilde{\mathbf{X}}^K$ is positive recurrent within its communication class, the expected number of jumps between two consecutive visits to $\tilde{\mathbf{X}}^K$ is inversely proportional to its steady state measure $\pi^K(\tilde{\mathbf{X}}^K)$. By definition of $\tilde{\mathbf{X}}^K$, the expected number of jumps must be no larger than K^m . Since the time between two jumps are i.i.d. exponential variables with mean $(\mathbf{1}^\top \hat{\boldsymbol{\phi}} \mathbf{1})^{-1}$, this concludes the proof.

Step 2: Lower bound on the “one-shot” demand-loss probability. Fix $K_{r'}$ and a demand sample path $\bar{\mathbf{A}}^{K_{r'}}(\cdot)$. For $t > 0$, define $f_{j'k}(t) \triangleq \frac{1}{t} \bar{\mathbf{A}}^{K_{r'}}(t)$, i.e. the average arrival rate of type (j', k) demand during $[0, t]$. For stationary policy U , denote the average fraction of demand arriving at j' that is served by supply at i during this period as $d_{ij'}^U(t)$ (we omit the superscript U in the following for notational simplicity). For $t \geq 0$, if $\bar{\mathbf{X}}^{K_{r'}}(0) = \tilde{\mathbf{X}}^{K_{r'}}$ and no demand is lost prior to t , we have for any $i \in V_S$

$$\bar{X}_i^{K_{r'}}(t) - \tilde{X}_i^{K_{r'}} = t \left(\sum_{j' \in V_D} f_{j'i}(t) - \sum_{j' \in \partial(i)} d_{ij'}(t) \left(\sum_{k \in V_S} f_{j'k}(t) \right) \right).$$

Since $\tilde{\alpha}_j > 0$ for any $j \in V_S$, the Lyapunov function $L_{\tilde{\boldsymbol{\alpha}}}(\cdot)$ is well-defined. Evaluate the Lyapunov function at $(\tilde{\boldsymbol{\alpha}} + \bar{\mathbf{X}}^{K_{r'}}(t) - \tilde{\mathbf{X}}^{K_{r'}})$, we have:

$$\begin{aligned} & L_{\tilde{\boldsymbol{\alpha}}}(\tilde{\boldsymbol{\alpha}} + \bar{\mathbf{X}}^{K_{r'}}(t) - \tilde{\mathbf{X}}^{K_{r'}}) \\ &= L_{\tilde{\boldsymbol{\alpha}}} \left(\tilde{\boldsymbol{\alpha}} + t \left(\sum_{j' \in V_D} f_{j'i}(t) - \sum_{j' \in \partial(i)} d_{ij'}(t) \left(\sum_{k \in V_S} f_{j'k}(t) \right) \right) \right)_{i \in V_S} \\ &\stackrel{(a)}{=} t L_{\tilde{\boldsymbol{\alpha}}} \left(\tilde{\boldsymbol{\alpha}} + \left(\sum_{j' \in V_D} f_{j'i}(t) - \sum_{j' \in \partial(i)} d_{ij'}(t) \left(\sum_{k \in V_S} f_{j'k}(t) \right) \right) \right)_{i \in V_S} \\ &\geq t \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} L_{\tilde{\boldsymbol{\alpha}}}(\tilde{\boldsymbol{\alpha}} + \Delta \mathbf{x}). \end{aligned} \quad (32)$$

Equality (a) holds because the Lyapunov function is scale-invariant with respect to $\tilde{\boldsymbol{\alpha}}$. Here $\Delta \mathbf{x}$ is the change of (normalized) state in unit time given average demand arrival rate during this period \mathbf{f} , and $\mathcal{X}_{\mathbf{f}}$ is defined in (20).

Define $v_{\tilde{\boldsymbol{\alpha}}}(\mathbf{f}) \triangleq \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} L_{\tilde{\boldsymbol{\alpha}}}(\tilde{\boldsymbol{\alpha}} + \Delta \mathbf{x})$, which is the minimum rate the Lyapunov function

increases under any policy, given demand arrival rate \mathbf{f} . Now we construct a set of demand sample paths that must lead to demand loss before the system returns to the starting state. First note that $\{\mathbf{f} : v_{\tilde{\alpha}}(\mathbf{f}) > 0\}$ is non-empty. To see this, let $f'_{j',k}$ equal to 1 for some j' and $k \notin \partial(j')$, and 0 otherwise (such a pair (j', k) exists by Assumption 2). This \mathbf{f}' results in a strictly positive²⁴ $v_{\tilde{\alpha}}(\mathbf{f}')$. Therefore for any $\epsilon_2 > 0$ there exists demand arrival rate $\tilde{\mathbf{f}}$ such that

$$v_{\tilde{\alpha}}(\tilde{\mathbf{f}}) > 0 \quad \text{and} \quad \frac{\Lambda^*(\tilde{\mathbf{f}})}{v_{\tilde{\alpha}}(\tilde{\mathbf{f}})} \leq \inf_{\mathbf{f}: v_{\tilde{\alpha}}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{v_{\tilde{\alpha}}(\mathbf{f})} + \epsilon_2.$$

It is not hard to show that $v_{\tilde{\alpha}}(\mathbf{f})$ is continuous in \mathbf{f} , hence there exists $\epsilon_3 > 0$ such that for any $\hat{\mathbf{f}} : \|\hat{\mathbf{f}} - \tilde{\mathbf{f}}\|_{\infty} < \epsilon_3$, we have

$$v_{\tilde{\alpha}}(\hat{\mathbf{f}}) > (1 - \epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}}) > 0.$$

Denote $T \triangleq \frac{1 + \frac{\epsilon_1}{\min_{j: \alpha_j > 0} \alpha_j}}{(1 - \epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}})}$, define

$$\mathcal{B}_{\tilde{\alpha}} \triangleq \left\{ \bar{\mathbf{A}}(\cdot) \in C[0, T] \left| \sup_{t \in [0, T]} \|\bar{\mathbf{A}}(t) - t\tilde{\mathbf{f}}\|_{\infty} \leq \epsilon_3 \right. \right\}.$$

For any demand arrival sample path $\bar{\mathbf{A}}(\cdot) \in \mathcal{B}_{\tilde{\alpha}}$, we will show that for $t \in [0, T]$ the followings are true: (i) normalized state $\bar{\mathbf{X}}^{K_{r'}}(t)$ does not hit $\tilde{\mathbf{X}}^{K_{r'}}$ before any demand is lost; (ii) at least one demand is lost.

To prove (i), define function $\tilde{L}_{\tilde{\alpha}}(\bar{\mathbf{X}}) \triangleq L_{\tilde{\alpha}}(\tilde{\alpha} + \bar{\mathbf{X}} - \tilde{\mathbf{X}}^{K_{r'}})$. By definition, we have $L_{\tilde{\alpha}}(\mathbf{x}) > 0$ for any $\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^m : \mathbf{1}^T \mathbf{x} = 1\} \setminus \{\tilde{\alpha}\}$, hence we have that $\tilde{L}_{\tilde{\alpha}}(\bar{\mathbf{X}}) > 0$ for any $\bar{\mathbf{X}} \in \Omega \setminus \{\tilde{\mathbf{X}}^{K_{r'}}\}$. By inequality (33), if no demand is lost during $[0, T]$ we have:

$$\tilde{L}_{\tilde{\alpha}}(\bar{\mathbf{X}}^{K_{r'}}(t)) \geq tv \left(\frac{1}{t} \bar{\mathbf{A}}(t) \right) \geq t \min_{\bar{\mathbf{A}}(\cdot) \in \mathcal{B}} v \left(\frac{1}{t} \bar{\mathbf{A}}(t) \right) > t(1 - \epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}}) > 0.$$

We prove (ii) by contradiction. Suppose no demand is lost given (fluid scale) demand arrival sample path $\bar{\mathbf{A}}(\cdot) \in \mathcal{B}$, then

$$\tilde{L}_{\tilde{\alpha}}(\bar{\mathbf{X}}^{K_{r'}}(T)) \geq T \min_{\bar{\mathbf{A}}(\cdot) \in \mathcal{B}} v \left(\frac{1}{T} \bar{\mathbf{A}}(T) \right) > \frac{1 + \frac{\epsilon_1}{\min_{j: \alpha_j > 0} \alpha_j}}{(1 - \epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}})} (1 - \epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}}) = 1 + \frac{\epsilon_1}{\min_{j: \alpha_j > 0} \alpha_j}.$$

Expand the expression of $\tilde{L}_{\tilde{\alpha}}(\bar{\mathbf{X}}^{K_{r'}}(T))$ on the LHS, we have

$$1 - \min_j \frac{\bar{X}_j^{K_{r'}}(T) + (\tilde{\alpha}_j - \tilde{x}_j^{K_{r'}})}{\tilde{\alpha}_j} > 1 + \frac{\epsilon_1}{\min_{j: \alpha_j > 0} \alpha_j}.$$

Therefore

$$\begin{aligned} \min \left\{ \min_{j: \alpha_j = 0} \frac{\bar{X}_j^{K_{r'}}(T)}{\tilde{\alpha}_j}, \min_{j: \alpha_j > 0} \frac{\bar{X}_j^{K_{r'}}(T) - \epsilon_1/2}{\tilde{\alpha}_j} \right\} &\leq \min_j \frac{\bar{X}_j^{K_{r'}}(T) + (\tilde{\alpha}_j - \tilde{x}_j^{K_{r'}})}{\tilde{\alpha}_j} \\ &< -\frac{\epsilon_1}{\min_{j: \alpha_j > 0} \alpha_j}. \end{aligned} \quad (34)$$

Note that the first inequality in (34) holds because of (29) and (30). Inequality (34) implies that $\min_j \bar{X}_j^{K_{r'}}(T) < 0$, which is impossible as queue lengths must be non-negative.

²⁴To see this, notice that $L_{\tilde{\alpha}}(\mathbf{x}) > 0$ for any $\mathbf{x} \in \Omega \setminus \{\tilde{\alpha}\}$, hence it suffices to show that $\mathbf{0} \notin \mathcal{X}_{\tilde{\mathbf{f}}'}$. Because for any $\Delta \mathbf{x} \in \mathcal{X}_{\tilde{\mathbf{f}}'}$, we have $\Delta x_k = f'_{j',k} > 0$, hence $\mathbf{0} \notin \mathcal{X}_{\tilde{\mathbf{f}}'}$. This concludes the proof.

Step 3: Asymptotic steady-state lower bound on demand loss probability. We use renewal-reward theorem (see, e.g., Ross 1996) to lower bound the demand-loss probability. Consider the regenerative process that restarts each time $\bar{\mathbf{X}}^{K_{r'}}(t) = \tilde{\mathbf{X}}^{K_{r'}}$. Without loss of generality, let $\bar{\mathbf{X}}^{K_{r'}}(0) = \tilde{\mathbf{X}}^{K_{r'}}$. Recall the definition of τ^K in (31). Using the claim in step 1 and the result in step 2, we have:

$$\begin{aligned}\mathbb{P}_o^{K_{r'}, U} &= \frac{\mathbb{E}[\#\{\text{demand lost during } [0, \tau]\}]}{\mathbb{E}[\tau]} \\ &\geq K_{r'}^{-m}(\mathbf{1}^T \hat{\phi} \mathbf{1}) \mathbb{E}[\#\{\text{demand lost during } [0, \tau]\}] \\ &\geq K_{r'}^{-m}(\mathbf{1}^T \hat{\phi} \mathbf{1}) \mathbb{P}(\#\{\text{demand lost during } [0, \tau]\} \geq 1) \\ &\geq K_{r'}^{-m}(\mathbf{1}^T \hat{\phi} \mathbf{1}) \mathbb{P}(\bar{\mathbf{A}}^{K_{r'}}(\cdot) \in \mathcal{B}_{\tilde{\alpha}}).\end{aligned}$$

Take asymptotic limit on both sides, we have:

$$\begin{aligned}\liminf_{r' \rightarrow \infty} \frac{1}{K_{r'}} \log \mathbb{P}_o^{K_{r'}, U} &\geq \liminf_{r' \rightarrow \infty} \frac{1}{K_{r'}} \log \mathbb{P}(\bar{\mathbf{A}}^{K_{r'}}(\cdot) \in \mathcal{B}_{\tilde{\alpha}}) \\ &\stackrel{(a)}{\geq} - \inf_{\bar{\mathbf{A}}(\cdot) \in \mathcal{B}_{\tilde{\alpha}}^0 \cap \text{AC}[0, T]} \int_0^T \Lambda^*(\dot{\bar{\mathbf{A}}}(t)) dt \\ &\stackrel{(b)}{\geq} -T \Lambda^*(\tilde{\mathbf{f}}) \\ &= - \frac{1 + \frac{\epsilon_1}{\min_{j: \alpha_j > 0} \alpha_j}}{(1 - \epsilon_2) v_{\tilde{\alpha}}(\tilde{\mathbf{f}})} \Lambda^*(\tilde{\mathbf{f}}) \\ &\geq - \frac{1 + \frac{\epsilon_1}{\min_{j: \alpha_j > 0} \alpha_j}}{1 - \epsilon_2} \left(\inf_{\mathbf{f}: v_{\tilde{\alpha}}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{v_{\tilde{\alpha}}(\mathbf{f})} + \epsilon_2 \right).\end{aligned}$$

Here (a) holds because of Mogulskii's Theorem (Fact 1), (b) holds because demand sample path $\bar{\mathbf{A}}(t) = t\tilde{\mathbf{f}} \in \text{AC}[0, T]$ is a member of $\mathcal{B}_{\tilde{\alpha}}$. For any $\delta > 0$, by choosing small enough $\epsilon_1(\delta), \epsilon_2(\delta) > 0$, we have

$$- \liminf_{r' \rightarrow \infty} \frac{1}{K_{r'}} \log \mathbb{P}_o^{K_{r'}, U} \leq (1 + \delta)(\gamma_{\text{CB}}(\tilde{\alpha}(\delta)) + \delta).$$

Here the choice of $\tilde{\alpha}$ depends on δ . To get rid of the multiplicative term $(1 + \delta)$, it suffices to show that $\sup_{\alpha \in \text{relint}(\Omega)} \gamma_{\text{CB}}(\alpha) < \infty$. This can be proved by the following construction: let $\bar{\mathbf{A}}(t) = t\mathbf{f}'$ for $t \in [0, 1]$ where $f_{j'k} = 1$ for some $j' \in V_D$ and $k \notin \partial(j')$. Because $\gamma_{\text{CB}}(\alpha)$ is defined by an infimum $\gamma_{\text{CB}}(\alpha) \triangleq \inf_{\mathbf{f} \in \mathbb{R}_+^{nm}: v_{\alpha}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{v_{\alpha}(\mathbf{f})}$, we have $\gamma_{\text{CB}}(\alpha) \leq \frac{\Lambda^*(\mathbf{f}')}{v_{\alpha}(\mathbf{f})}$. By definition, $v_{\alpha}(\mathbf{f}') = 1 - \max_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}'}} \min_i \frac{\tilde{\alpha}_i + \Delta x_i}{\tilde{\alpha}_i} = - \max_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}'}} \min_i \frac{\Delta x_i}{\tilde{\alpha}_i}$. Note that

$$\begin{aligned}\mathcal{X}_{\mathbf{f}'} &= \{\Delta \mathbf{x} \in \mathbb{R}^{|V_S|} : \sum_{i \in \partial(j')} \Delta x_i = -1, \Delta x_i \leq 0 \text{ for } i \in \partial(j'), \Delta x_k = 1 \\ &\quad \Delta x_i = 0 \text{ for } i \notin \partial(j') \cup \{k\}\}.\end{aligned}$$

Therefore

$$\max_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}'}} \min_{i \in V_S} \frac{\Delta x_i}{\tilde{\alpha}_i} = \max_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}'}} \min_{i \in \partial(j')} \frac{\Delta x_i}{\tilde{\alpha}_i} \leq \max_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}'}} \min_{i \in \partial(j')} \Delta x_i \leq -\frac{1}{|\partial(j')|} \leq -\frac{1}{m}.$$

Hence $v_{\tilde{\alpha}}(\mathbf{f}') \geq \frac{1}{m}$. Hence $\gamma_{\text{CB}}(\alpha) \leq \frac{\Lambda^*(\mathbf{f}')}{v_{\tilde{\alpha}}(\mathbf{f}')} \leq m \Lambda^*(\mathbf{f}') < \infty$. Therefore by choosing a small enough δ , we have

$$- \liminf_{r' \rightarrow \infty} \frac{1}{K_{r'}} \log \mathbb{P}_o^{K_{r'}, U} \leq \gamma_{\text{CB}}(\tilde{\alpha}(\epsilon)) + \epsilon.$$

By the definition of subsequence $\{K_{r'}\}$, we have

$$-\liminf_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_o^{K,U} \leq \gamma_{\text{CB}}(\tilde{\alpha}(\epsilon)) + \epsilon.$$

As a result, for any $\epsilon > 0$ there exists $\alpha \in \Omega$ such that $-\liminf_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_o^{K,U} \leq \sup_{\alpha \in \text{relint}(\Omega)} \gamma_{\text{CB}}(\alpha) + \epsilon$, therefore $-\liminf_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_o^{K,U} \leq \sup_{\alpha \in \text{relint}(\Omega)} \gamma_{\text{CB}}(\alpha)$. \square

C Sufficient Conditions for Exponent Optimality: Proof of Proposition 5

The proof of Proposition 5 consists of two parts. We first derive an achievability bound for policies that, for a given $\alpha \in \text{relint}(\Omega)$, satisfy the negative drift property in Proposition 5; we then show it matches the converse bound in Lemma 2 for that specific α (i.e., $\gamma_{\text{CB}}(\alpha)$) if the steepest descent property in Proposition 5 is also satisfied.

C.1 An achievability bound

The following lemma is an adaptation of Theorem 5 and Proposition 7 in Venkataramanan and Lin (2013) to our setting. It gives the achievability bound for the exponent of the steady state demand-loss probability, for any policy such that the negative drift condition in Proposition 5 is met for $L_\alpha(\cdot)$ where $\alpha \in \text{relint}(\Omega)$. The main technical difficulty comes from the fact that it characterizes the *steady state* of the system. The analysis uses Freidlin-Wentzell theory and follows from Stolyar (2003), Venkataramanan and Lin (2013). While the main proof idea follows that in Venkataramanan and Lin (2013), we refine the results there by dropping the assumption that all FSPs are Lipschitz continuous with a universal Lipschitz constant. This allows us to deal with Poisson-driven demand arrival processes which does not satisfy this assumption.

Lemma 7 (Achievability bound). *For the system being considered, if policy U satisfies the negative drift condition in Proposition 5 for $L_\alpha(\cdot)$ where $\alpha \in \text{relint}(\Omega)$, we have (the subscript “AB” stands for achievability bound)*

$$-\limsup_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_p^{K,U} \geq \gamma_{\text{AB}}(\alpha). \quad (35)$$

Here for fixed²⁵ $T > 0$,

$$\gamma_{\text{AB}}(\alpha) \triangleq \inf_{v > 0, \mathbf{f}, \bar{\mathbf{A}}, \bar{\mathbf{X}}} \frac{\Lambda^*(\mathbf{f})}{v},$$

where $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ is a FSP on $[0, T]$ under U such that for some regular $t \in [0, T]$

$$\dot{\bar{\mathbf{A}}}(t) = \mathbf{f}, \quad L_\alpha(\bar{\mathbf{X}}(t)) < 1, \quad \dot{L}_\alpha(\bar{\mathbf{X}}(t)) = v.$$

Proof of Lemma 7. *Step 1.* Define stopping times and consider the sampling chain. In this step, we mostly follow the approach in Venkataramanan and Lin (2013) (Freidlin-Wentzell theory) and decompose the expression for the likelihood of the Lyapunov function taking on a large value. There are minor differences between our proof and proof of Theorem 4 in Venkatara-

²⁵The definition of quantity $\gamma_{\text{AB}}(\alpha)$ is based on the local behavior of $\bar{\mathbf{A}}$ and $\bar{\mathbf{X}}$ for times close to t . In particular, the value of T plays no role.

manan and Lin (2013) because of our closed queueing network setting, so we will write down each step for completeness.

Let $\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}(\infty)$ be a random vector distributed as the stationary distribution of recurrent class associated with initial (normalized) state $\mathbf{z} \in \Omega$. For notation simplicity, we suppress the dependence on \mathbf{z} and U and keep them fixed. We want to upper bound:

$$\limsup_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P} \left(L_{\alpha}(\bar{\mathbf{X}}^K(\infty)) \geq 1 \right).$$

Choose positive constants δ, ϵ such that $0 < \delta < \epsilon < 1$. Consider the following stopping times defined on a sample path $\bar{\mathbf{X}}^K(\cdot)$:

$$\begin{aligned} \beta_1^K &\triangleq \inf\{t \geq 0 : L_{\alpha}(\bar{\mathbf{X}}^K(t)) \leq \delta\}, \\ \eta_i^K &\triangleq \inf\{t \geq \beta_i^K : L_{\alpha}(\bar{\mathbf{X}}^K(t)) \geq \epsilon\}, \quad i = 1, 2, \dots \\ \beta_i^K &\triangleq \inf\{t \geq \eta_{i-1}^{K,U} : L_{\alpha}(\bar{\mathbf{X}}^K(t)) \leq \delta\}, \quad i = 2, 3, \dots \end{aligned}$$

Let the discrete-time Markov chain $\hat{\mathbf{X}}^K[i]$ be obtained by sampling $\bar{\mathbf{X}}^K(t)$ at the stopping times η_i^K . Since $\bar{\mathbf{X}}^K(\cdot)$ is stationary, there must also exist a stationary distribution for Markov chain $\hat{\mathbf{X}}^K[\cdot]$. Let Θ^K denote the state space of the sampled chain $\hat{\mathbf{X}}^K[\cdot]$, $\hat{\pi}^K$ is the sampled chain's stationary distribution.

The above construction was based on the following idea: first divide time into cycles, where the i -th cycle is the interval of time between consecutive η_i 's, i.e., a cycle is completed each time the value of $L_{\alpha}(\bar{\mathbf{X}}^K)$ goes down below δ and then rises above ϵ . Then the fraction of time the Lyapunov function spent above 1 is equal to the ratio

$$\mathbb{E}[\text{time for which } L_{\alpha}(\bar{\mathbf{X}}^K) \geq 1 \text{ during a cycle}] / (\mathbb{E}[\text{length of cycle}])$$

in steady state. We sample the initial state as $\bar{\mathbf{X}}^K(0) = \mathbf{x} \sim \hat{\pi}^K$, hence the first cycle itself characterizes the steady state ratio. Therefore, the stationary likelihood of event $\{L_{\alpha}(\bar{\mathbf{X}}^K) \geq 1\}$ can be expressed as (see Lemma 10.1 in Stolyar 2003):

$$\mathbb{P} \left(L_{\alpha}(\bar{\mathbf{X}}^K) \geq 1 \right) = \frac{\int_{\Theta^K} \hat{\pi}^K(d\mathbf{x}) \cdot \mathbb{E} \left(\int_0^{\eta_1^K} \mathbb{I} \left\{ L_{\alpha}(\bar{\mathbf{X}}^K(t)) \geq 1 \right\} dt \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right)}{\int_{\Theta^K} \hat{\pi}^K(d\mathbf{x}) \cdot \mathbb{E}(\eta_1^K \mid \bar{\mathbf{X}}^K(0) = \mathbf{x})}. \quad (36)$$

Step 2. Bounding the RHS of (36). To upper bound $\mathbb{P} \left(L_{\alpha}(\bar{\mathbf{X}}^K) \geq 1 \right)$, we lower bound the denominator in the RHS of (36) and upper bound the numerator.

- *Step 2a. Bounding the Denominator.* To lower bound the denominator, we focus on the discrete-time embedded chain of $\{\bar{\mathbf{X}}^K(\cdot)\}$. Note each exactly one demand arrives at each jump of the chain, therefore $\|\bar{\mathbf{X}}^K(\cdot)\|_{\infty}$ change by at most $\frac{1}{K}$ at each jump. Using property 2 of $L_{\alpha}(\cdot)$ in Lemma 6, we further have that $L_{\alpha}(\bar{\mathbf{X}}^K(\cdot))$ change by at most $\frac{1}{K \cdot \min_i \alpha_i}$ at each jump. Since the Lyapunov function $L_{\alpha}(\bar{\mathbf{X}}^K(\cdot))$ has to increase from δ to ϵ during $[0, \eta_1^K]$, there exists $K_1 = K_1(\epsilon, \delta) > 0$ such that for any $K > K_1$, at least $\frac{K \cdot \min_i \alpha_i}{2}(\epsilon - \delta)$ jumps occur during $[0, \eta_1^K]$. Because the times between two consecutive jumps follow i.i.d. exponential distribution with rate $K \mathbf{1}^T \hat{\phi} \mathbf{1}$, therefore for any $K > K_1$,

$$\mathbb{E}(\eta_1^K \mid \bar{\mathbf{X}}^K(0) = \mathbf{x}) \geq \frac{K \cdot \min_i \alpha_i}{2}(\epsilon - \delta) \frac{1}{K \mathbf{1}^T \hat{\phi} \mathbf{1}} = \frac{\min_i \alpha_i}{2 \cdot \mathbf{1}^T \hat{\phi} \mathbf{1}}(\epsilon - \delta). \quad (37)$$

- *Step 2b. Bounding the Numerator.* This part is more complex, and we first decompose the numerator into several terms. Let $\rho \in (\epsilon, 1)$. Because each (normalized) queue length change by at most $\frac{1}{K}$ at each jump almost surely, and that $L_\alpha(\cdot)$ is Lipschitz continuous, there exists $K_2 = K_2(\epsilon, \rho) > 0$, such that for all $K \geq K_2$, we have $L(\bar{\mathbf{X}}^K(\eta_i^K)) \leq \rho$.

We define another stopping time:

$$\eta^{K,\uparrow} \triangleq \inf\{t \geq 0 : L_\alpha(\bar{\mathbf{X}}^K(t)) \geq 1\}.$$

Then for any $\mathbf{x} \in \Theta^K$, we must have:

$$\mathbb{E} \left(\int_0^{\eta_1^K} \mathbb{I}\{L_\alpha(\bar{\mathbf{X}}^K(t)) \geq 1\} dt \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \leq \mathbb{E} \left(\mathbb{I}\{\eta^{K,\uparrow} \leq \beta_1^K\} (\beta_1^K - \eta^{K,\uparrow}) \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right).$$

The above inequality holds because:

- if $\beta_1^K \leq \eta^{K,\uparrow}$, then both sides are zero (because the Lyapunov function will hit ϵ before 1);
- if $\beta_1^K > \eta^{K,\uparrow}$, then $L_\alpha(\bar{\mathbf{X}}^K(t)) \geq 1$ can occur only for a subset of $t \in [\eta^{K,\uparrow}, \beta_1^K]$, and this time interval has length $\beta_1^K - \eta^{K,\uparrow}$.

Hence

$$\begin{aligned} & \mathbb{E} \left(\int_0^{\eta_1^K} \mathbb{I}\{L_\alpha(\bar{\mathbf{X}}^K(t)) \geq 1\} dt \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \\ & \leq \mathbb{E} \left(\beta_1^K - \eta^{K,\uparrow} \mid \eta^{K,\uparrow} \leq \beta_1^K, \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \mathbb{P} \left(\eta^{K,\uparrow} \leq \beta_1^K \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right). \end{aligned}$$

Define

$$\beta^K(\mathbf{x}) \triangleq \inf \left\{ t \geq 0 : L_\alpha(\bar{\mathbf{X}}^K(t)) \leq \delta \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right\}.$$

Using the properties of Markov chains and conditional expectation, we have:

$$\begin{aligned} & \mathbb{E} \left(\beta_1^K - \eta^{K,\uparrow} \mid \eta^{K,\uparrow} \leq \beta_1^K, \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \\ & = \mathbb{E} \left(\mathbb{E} \left(\beta^K(\bar{\mathbf{X}}^K(\eta^{K,\uparrow})) \mid \eta^{K,\uparrow} \leq \beta_1^K, \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \right) \\ & \leq \sup_{\mathbf{x} \in \Omega} \mathbb{E} \left(\beta_1^K \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right). \end{aligned}$$

Let T be a positive number which will be chosen later. Recall that $L_\alpha(\mathbf{x}) \leq \rho$ for all $\mathbf{x} \in \Theta^K$ almost surely when $K \geq K_2$. Hence, for any such $\mathbf{x} \in \Theta^K$, we have,

$$\begin{aligned} & \mathbb{E} \left(\int_0^{\eta_1^K} \mathbb{I}\{L_\alpha(\bar{\mathbf{X}}^K(t)) \geq 1\} dt \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \\ & \leq \mathbb{E} \left(\beta_1^K - \eta^{K,\uparrow} \mid \eta^{K,\uparrow} \leq \beta_1^K, \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \mathbb{P} \left(\eta^{K,\uparrow} \leq \beta_1^K \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \\ & \leq \left(\sup_{\mathbf{x} \in \Omega} \mathbb{E} \left(\beta_1^K \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \right) \left[\mathbb{P} \left(\eta^{K,\uparrow} \leq T \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \right. \\ & \quad \left. + \mathbb{P} \left(\beta_1^K \geq T \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \right] \quad (\text{using } \eta^{K,\uparrow} \leq \beta_1^K \Rightarrow \eta^{K,\uparrow} \leq T \text{ or } T \leq \beta_1^K) \end{aligned}$$

$$\begin{aligned}
& \leq \underbrace{\left(\sup_{\mathbf{x} \in \Omega} \mathbb{E} \left(\beta_1^K \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \right)}_{(a)} \left[\underbrace{\sup_{\mathbf{x}: L_{\alpha}(\mathbf{x}) \leq \rho} \mathbb{P} \left(\eta^{K, \uparrow} \leq T \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right)}_{(b)} \right. \\
& \quad \left. + \underbrace{\sup_{\mathbf{x}: L_{\alpha}(\mathbf{x}) \leq \rho} \mathbb{P} \left(\beta_1^K \geq T \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right)}_{(c)} \right]. \tag{38}
\end{aligned}$$

- *Step 2b(i). Bounding term (a).* Term (a) is the upper bound of the expected time for the Lyapunov function to hit a lower level δ starting from a higher level ϵ . Because the policy U satisfies the negative drift condition, it follows from standard argument (see Part B(1) of the proof of Theorem 4 in Venkataramanan and Lin (2013), which applies the classical results in Dai (1995)) that there exists $K_3 = K_3(\delta, \epsilon)$ and constant $C > 0$ such that for $K \geq K_3$, we have $(a) \leq C$.
- *Step 2b(ii). Asymptotics for (b).* Let $K \rightarrow \infty$ and apply Proposition 2 in Venkataramanan and Lin (2013) to $\bar{\mathbf{X}}^K(\cdot)$. We have:

$$\begin{aligned}
& \limsup_{K \rightarrow \infty} \frac{1}{K} \log \left(\sup_{\mathbf{x}: L_{\alpha}(\mathbf{x}) \leq \rho} \mathbb{P} \left(\eta^{K, \uparrow} \leq T \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \right) \\
& \leq - \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}} \int_0^T \Lambda^* \left(\dot{\bar{\mathbf{A}}}(t) \right) dt, \text{ where } (\bar{\mathbf{A}}, \bar{\mathbf{X}}) \text{ is an FSP} \\
& \text{such that } L_{\alpha}(\bar{\mathbf{X}}(0)) \leq \rho, L_{\alpha}(\bar{\mathbf{X}}(t)) \geq 1 \text{ for some } t \in [0, T].
\end{aligned}$$

- *Step 2b(iii). Asymptotics for (c).* Intuitively, term (c) is the tail probability of the duration of a cycle that terminates when the Lyapunov function hit δ . It remains to be shown that this term is negligible comparing to (b) as $T \rightarrow \infty$. Let $K \rightarrow \infty$ and apply Proposition 2 in Venkataramanan and Lin (2013) to $\bar{\mathbf{X}}^K(\cdot)$. We obtain:

$$\begin{aligned}
& \limsup_{K \rightarrow \infty} \frac{1}{K} \log \left(\sup_{\mathbf{x}: L_{\alpha}(\mathbf{x}) \leq \rho} \mathbb{P} \left(\beta_1^K \geq T \mid \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \right) \\
& \leq - \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}} \int_0^T \Lambda^* \left(\dot{\bar{\mathbf{A}}}(t) \right) dt, \text{ where } (\bar{\mathbf{A}}, \bar{\mathbf{X}}) \text{ is an FSP} \\
& \text{such that } L_{\alpha}(\bar{\mathbf{X}}(0)) \leq \rho, L_{\alpha}(\bar{\mathbf{X}}(t)) \geq \delta \text{ for all } t \in [0, T].
\end{aligned}$$

We focus on the variational problem on the RHS. Note that any FSP that is feasible to the variational problem must satisfy:

$$\delta \leq L_{\alpha}(\bar{\mathbf{X}}(0)) + \int_{t=1}^T \dot{L}(\bar{\mathbf{X}}(t)) dt \leq \rho + \int_{t=1}^T \dot{L}(\bar{\mathbf{X}}(t)) dt.$$

For any fixed FSP, define $\mathcal{T}_0 \triangleq \{t \in [0, T] : \dot{L}(\bar{\mathbf{X}}(t)) > -\eta\}$, where η is the negative drift parameter in the statement of Proposition 5. Denote the measure of \mathcal{T}_0 by t_0 . Therefore it must hold that:

$$\rho + \int_{t=1}^T \dot{L}(\bar{\mathbf{X}}(t)) dt = \rho + \int_{t \notin \mathcal{T}_0} \dot{L}(\bar{\mathbf{X}}(t)) dt + \int_{t \in \mathcal{T}_0} \dot{L}(\bar{\mathbf{X}}(t)) dt$$

$$\leq \rho - \eta(T - t_0) + \int_{t \in \mathcal{T}_0} \dot{L}(\bar{\mathbf{X}}(t)) dt.$$

Hence

$$\int_{t \in \mathcal{T}_0} \dot{L}(\bar{\mathbf{X}}(t)) dt \geq \eta(T - t_0) + \delta - \rho \geq \eta(T - t_0) - 1.$$

There are two cases:

Case 1: When $t_0 > \frac{T}{2}$. Define

$$\begin{aligned} J_{\min} &\triangleq \min \Lambda^*(\dot{\bar{\mathbf{A}}}(t)) \\ \text{subject to } &\dot{L}(\bar{\mathbf{A}}(t)) \geq -\eta, \quad t \in [0, T], \quad (\bar{\mathbf{A}}(t), \bar{\mathbf{X}}(t)) \text{ is an FSP.} \end{aligned} \quad (39)$$

Note that $J_{\min} \geq \min_{\mathbf{f} \notin B(\phi, \epsilon')} \Lambda^*(\mathbf{f}) > 0$ and ϵ' is the ϵ specified in condition (2) of Proposition 5. Therefore a lower bound of the exponent of these sample paths is

$$\int_0^T \Lambda^*(\dot{\bar{\mathbf{A}}}(t)) dt \geq \frac{T}{2} J_{\min}.$$

Case 2: When $t_0 \leq \frac{T}{2}$. We have

$$\int_{t \in \mathcal{T}_0} \dot{L}(\bar{\mathbf{X}}(t)) dt \geq \eta(T - t_0) - 1 \geq \frac{\eta T}{2} - 1.$$

We choose $T > \frac{4}{\eta}$, therefore $\frac{\eta T}{2} - 1 \geq \frac{\eta T}{4}$. A lower bound of the exponent of these sample paths is the value of the following variational problem:

$$\begin{aligned} J(T) &\triangleq - \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}} \int_0^T \Lambda^*(\dot{\bar{\mathbf{A}}}(t)) dt, \quad \text{where } (\bar{\mathbf{A}}, \bar{\mathbf{X}}) \text{ is an FSP} \\ &\quad \text{such that } \int_0^T \max\{\dot{L}_{\alpha}(\bar{\mathbf{X}}(t)), 0\} dt \geq \frac{\eta T}{4}. \end{aligned}$$

We claim that $J(T) \rightarrow \infty$ as $T \rightarrow \infty$ and prove the claim in step 3.

Combine the two cases, we have:

$$\limsup_{K \rightarrow \infty} \frac{1}{K} \log \left(\sup_{\mathbf{x}: L_{\alpha}(\mathbf{x}) \leq \rho} \mathbb{P}(\beta_1^K \geq T \mid \bar{\mathbf{X}}^K(0) = \mathbf{x}) \right) \leq - \min \left\{ \frac{T}{2} J_{\min}, J(T) \right\}.$$

It is not hard to see that as $T \rightarrow \infty$, the exponent of term (c) tends to $-\infty$ hence is negligible.

Now combine all the terms. For fixed ϵ, δ, ρ , note that the denominator of (36) and (a) in (38) are bounded by a constant term, so they have no contribution to the exponent of (36). Since as $T \rightarrow \infty$, (c) in (38) have an exponent that is at most $-\liminf_{T \rightarrow \infty} J(T)$, we have

$$\begin{aligned} &\limsup_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_{\mathbf{p}}^{K, U} \\ &\leq - \liminf_{T \rightarrow \infty} J(T), \quad \limsup_{K \rightarrow \infty} \frac{1}{K} \log \left(\max_{\bar{\mathbf{X}}^K(0) \in \Omega} \mathbb{P}(L_{\alpha}(\bar{\mathbf{X}}^K(\infty)) \geq 1) \right) \end{aligned} \quad (40)$$

$$\begin{aligned} &\leq - \inf_{T > 0} \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}} \int_0^T \Lambda^*(\dot{\bar{\mathbf{A}}}(t)) dt \\ &\quad \text{where } (\bar{\mathbf{A}}, \bar{\mathbf{X}}) \text{ is an FSP such that } L_{\alpha}(\bar{\mathbf{X}}(0)) = \rho, \quad L_{\alpha}(\bar{\mathbf{X}}(T)) \geq 1. \end{aligned} \quad (41)$$

Finally, let $\delta, \epsilon, \rho \rightarrow 0$, we have

$$\begin{aligned} & \limsup_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_p^{K,U} \\ & \leq - \inf_{T > 0} \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}} \int_0^T \Lambda^* \left(\dot{\bar{\mathbf{A}}}(t) \right) dt \\ & \quad \text{where } (\bar{\mathbf{A}}, \bar{\mathbf{X}}) \text{ is an FSP such that } L_{\alpha}(\bar{\mathbf{X}}(0)) = 0, L_{\alpha}(\bar{\mathbf{X}}(T)) \geq 1. \end{aligned}$$

We briefly summarize Step 2 and provide some intuition. The goal is to upper bound the stationary likelihood that the Lyapunov function equals 1. To study the stationary behavior, we first divide time into cycles, where a cycle is completed each time the Lyapunov function goes down below δ then rises above ϵ , where $\delta < \epsilon \ll 1$. Then using a variant of renewal-reward theorem (equation (36)), we only need to lower bound the expected cycle duration, and upper bound the expected time the Lyapunov function stays at 1 during a cycle. The Lipschitz property of the Lyapunov function ensures that the cycle duration is bounded away from 0 hence has no contribution to the *exponent* of the desired likelihood (Lemma 6). Meanwhile, the negative drift condition ensures the expected time until the Lyapunov function returns to δ after hitting 1. This leaves the exponent of the desired likelihood to be solely dependent on the probability that the Lyapunov function ever hit 1 during a cycle. Finally we apply the sample path large deviation principle (Fact 1) to bound this quantity.

Step 3. Reduce (40) to an one-dimensional variational problem. This rest of the proof is exactly the same as the proof of Theorem 5 and Proposition 7 in Venkataramanan and Lin (2013); we provide the intuition and omit the details.

The proof up until this point dealt with the *steady state* of the system. Recall the link between the exponent and value of a differential game described in Section 5.3. We now lower bound the exponent of the steady state demand loss probability by a variational problem (differential game), namely, (41). Since we are trying to lower bound the adversary's cost, we consider an "ideal adversary" who can increase $L_{\alpha}(\mathbf{x})$ at the minimum cost at *each* level set. Mathematically,

$$\text{The quantity in (41)} \leq - \inf_{T > 0} \theta_T, \quad (42)$$

where

$$\begin{aligned} \theta_T & \triangleq \inf_{L_{\alpha}(\cdot)} \int_0^T l_{\alpha,T} \left(L(t), \dot{L}(t) \right) dt \\ & \text{s.t. } L(\cdot) \text{ is absolutely continuous and } L(0) = 0, \quad L(T) \geq 1. \end{aligned}$$

$$\begin{aligned} l_{\alpha,T}(y, v) & \triangleq \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}} \Lambda^*(\mathbf{f}) \\ & \text{s.t. } (\bar{\mathbf{A}}, \bar{\mathbf{X}}) \text{ is an FSP on } [0, T] \text{ such that for some regular } t \in [0, T] \\ & \quad \dot{\bar{\mathbf{A}}}(t) = \mathbf{f}, \quad L_{\alpha}(\bar{\mathbf{X}}(t)) = y, \quad \dot{L}_{\alpha}(\bar{\mathbf{X}}(t)) = v. \end{aligned}$$

Using the scale-invariance property of $L_{\alpha}(\mathbf{x})$ (Lemma 1), we can show that $l_{\alpha,T}(y, v)$ is independent of y (Proposition 7 in Venkataramanan and Lin 2013). As a result, the above variational problem reduces to an one-dimensional problem where the "ideal adversary" chooses a single rate (i.e., v in the statement of Lemma 7) at which $L_{\alpha}(\mathbf{x})$ increases. This problem is exactly the one in the statement of Lemma 7.

(We prove the claim in step 2 that $\liminf_{T \rightarrow \infty} J(T) = \infty$ here. Using exactly the same

argument as in step 3, we can show that $J(T) \geq \frac{\eta T}{4} \gamma_{\text{AB}}(\boldsymbol{\alpha})$ where the RHS is defined in (35). This concludes the proof.) \square

C.2 Converse Bound Matches Achievability Bound

In Lemma 2 we obtain a converse bound which holds for any state-dependent policy. However, for a given policy U can we obtain a tighter policy-specific converse bound? In the following Lemma, we show that for policies that satisfy the negative drift property in Proposition 5 for Lyapunov function $L_{\boldsymbol{\alpha}}(\cdot)$ where $\boldsymbol{\alpha} \in \text{relint}(\Omega)$, there is a tighter converse bound given by $\gamma_{\text{CB}}(\boldsymbol{\alpha})$.

Lemma 8. *For policies $U \in \mathcal{U}$ that satisfy the negative drift condition in the statement of Proposition 5 for $\boldsymbol{\alpha} \in \text{relint}(\Omega)$, we have*

$$-\liminf_{K \rightarrow \infty} \frac{1}{K} \log P_o^{K,U} \leq \gamma_{\text{CB}}(\boldsymbol{\alpha}).$$

Proof. The following proof is very similar to the proof of Lemma 2. We will emphasize the parts that are different and skip the repetitive arguments. In the proof of Lemma 2, we divide the process into cycles and apply the renewal-reward theorem. We follow the same approach here except that we define the cycles differently.

Step 1: Show that $\boldsymbol{\alpha}$ is the “resting point” of U . Fix $\epsilon_1 > 0$ and define

$$\tau^K \triangleq \inf \left\{ t \geq 0 : L_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}^K(t)) \leq \epsilon_1 \right\}.$$

Using the argument in Step 2b(i) of the proof of Lemma 7, we can show that there exists $K_0 = K_0(\epsilon_1) > 0$ and constant $C > 0$ such that for $K \geq K_0$,

$$\sup_{\mathbf{x} \in \Omega} \mathbb{E} \left(\tau^K | \bar{\mathbf{X}}^K(0) = \mathbf{x} \right) \leq C.$$

In other words, starting from any state, the expected time for the system state to reach the $O(\epsilon_1)$ -neighborhood of $\boldsymbol{\alpha}$ is bounded from above by a constant.

Step 2: Lower bound the demand-loss probability. Proceed exactly as Step 2 and Step 3 in the proof of Lemma 2, we explicitly construct a demand sample path that guarantees a demand loss within $\Theta(1)$ units of time given the starting state satisfies $L_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}^K(T + \tau^K)) < \epsilon_1$. Then we obtain the desired result. \square

Now we combine Lemma 7 and Lemma 8 to prove Proposition 5 by showing that $\gamma_{\text{AB}}(\boldsymbol{\alpha}) = \gamma_{\text{CB}}(\boldsymbol{\alpha})$. Lemma 1 and the steepest descent property in Proposition 5 are crucial in showing $\gamma_{\text{AB}}(\boldsymbol{\alpha}) \geq \gamma_{\text{CB}}(\boldsymbol{\alpha})$ (the other direction is obvious).

Proof of Proposition 5. Let $U \in \mathcal{U}$ satisfy the conditions in Proposition 5. Then for regular t we have

$$\begin{aligned} \dot{L}_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}(t)) &\leq \inf_{U' \in \mathcal{U}} \left\{ \dot{L}_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}^{U'}(t)) \mid \dot{\mathbf{A}}'(t) = \mathbf{f} \right\} && \text{(steepest descent)} \\ &= \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} \lim_{\Delta t \rightarrow 0} \frac{L_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}^{U'}(t) + \Delta \mathbf{x} \Delta t) - L_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}^{U'}(t))}{\Delta t} \\ &\leq \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} \lim_{\Delta t \rightarrow 0} \frac{L_{\boldsymbol{\alpha}}(\boldsymbol{\alpha} + \Delta \mathbf{x} \Delta t)}{\Delta t} && \text{(sub-additivity, Lemma 1)} \end{aligned}$$

$$= \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} L_{\alpha}(\alpha + \Delta \mathbf{x}) = v_{\alpha}(\mathbf{f}). \quad (\text{scale-invariance, Lemma 1})$$

Let $v = \dot{L}_{\alpha}(\bar{\mathbf{X}}(t))$, from $v \leq v_{\alpha}(\mathbf{f})$ we have $\{v > 0\} \subset \{v_{\alpha}(\mathbf{f}) > 0\}$, hence using Lemma 7 we have

$$\gamma_{AB}(\alpha) = \inf_{v > 0, \mathbf{f}, \bar{\mathbf{A}}, \bar{\mathbf{X}}} \frac{\Lambda^*(\mathbf{f})}{v} \geq \inf_{\mathbf{f}: v_{\alpha}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{v_{\alpha}(\mathbf{f})} = \gamma_{CB}(\alpha).$$

But since by Lemma 8 we know $\gamma_{CB}(\alpha)$ is a converse bound for policy U , hence $\gamma_{AB}(\alpha) \leq \gamma_{CB}(\alpha)$. Therefore $\gamma_{AB}(\alpha) = \gamma_{CB}(\alpha)$. \square

D SMW Policies and Explicit Exponent

Appendix D shows that the SMW policy satisfies the sufficient conditions for exponent optimality, and derives explicitly the optimal exponent and most-likely sample paths, including the proofs of Lemma 3, Lemma 4, and Lemma 5. The last subsection formally establishes exponent optimality of SMW policies for transient performance.

D.1 Lyapunov Drift of FSPs under SMW: Proof of Lemma 3

In this subsection we prove Lemma 3 which establishes that $\text{SMW}(\alpha)$ policies perform steepest descent on $L_{\alpha}(\cdot)$.

Proof of Lemma 3. For notation simplicity, we will write $S_1(\bar{\mathbf{X}}(t))$ as S_1 , $S_2(\bar{\mathbf{X}}(t), \dot{\bar{\mathbf{X}}}(t))$ as S_2 , and $\min_{k \in S_1} \frac{\dot{X}_k(t)}{\alpha_k}$ as c in the following. Let $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ be an FSP under policy $U \in \mathcal{U}$.

- *Proof of (23).* Note that t is a *regular* time, hence $L_{\alpha}(\bar{\mathbf{X}}(\cdot))$ and $\bar{\mathbf{X}}(\cdot)$ are differentiable at t . It follows from the definition of derivatives that $\dot{L}_{\alpha}(\bar{\mathbf{X}}(t))$ is determined by the queues in S_2 alone, hence we have $\dot{L}_{\alpha}(\bar{\mathbf{X}}(t)) = -\min_{k \in S_1} \frac{\dot{X}_k(t)}{\alpha_k} = -c$.
- *Proof of (24).* For the K -th system, define auxiliary processes:

$$\bar{E}_{ij'k}^{K,U}(t) \triangleq \# \{ \text{Type } (j', k) \text{ demand units that arrive during } [0, t] \\ \text{and are served by supply units at } i \text{ under policy } U \in \mathcal{U} \} \quad i, k \in V_S, j' \in V_D.$$

Using standard argument (see, e.g., Dai and Lin 2005), we can extend the definition of FSP (Definition 5) to $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(\cdot), \bar{\mathbf{E}}(\cdot))$, where a subsequence of $\bar{\mathbf{E}}^{K,U}(\cdot)$ converges u.o.c. to $\bar{\mathbf{E}}(\cdot)$. We focus on the regular times t where $\dot{\bar{\mathbf{E}}}(t)$ exists, which includes almost all regular times because $\dot{\bar{\mathbf{E}}}(t)$ is differentiable almost everywhere.

Consider any non-idling policy $U' \in \mathcal{U}$, and $\bar{\mathbf{X}}^{U'}(t)$ such that $\bar{\mathbf{X}}^{U'}(t) \neq \alpha$, $L_{\alpha}(\bar{\mathbf{X}}^{U'}(t)) < 1$. The flow of supply units entering S_2 is $\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t)$ because U' is non-idling. The flow of supply units leaving S_2 is at least $\sum_{j' \in V_D: \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t)$ because U' is non-idling and that the supply units in $V_S \setminus S_2$ cannot be used to serve demand originating from $\{j' \in V_D : \partial(j') \subset S_2\}$. Therefore,

$$\sum_{k \in S_2} \dot{X}_k^{U'}(t) \leq \sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j' \in V_D: \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t). \quad (43)$$

Now we consider $\text{SMW}(\alpha)$ policies and $\bar{\mathbf{X}}^{\text{SMW}(\alpha)}(t)$ such that $\bar{\mathbf{X}}^{\text{SMW}(\alpha)}(t) \neq \alpha$. For the process $\bar{\mathbf{E}}(t)$ (resp. $\bar{\mathbf{X}}(t)$), we use notation $\Delta \bar{\mathbf{E}}(t)$ (resp. $\Delta \bar{\mathbf{X}}(t)$) to denote $\bar{\mathbf{E}}(t + \Delta t) - \bar{\mathbf{E}}(t)$

(resp. $\bar{\mathbf{X}}(t + \Delta t) - \bar{\mathbf{X}}(t)$). It holds that

$$\sum_{k \in S_2} \Delta \bar{X}_k^{K,U}(t) = \sum_{j' \in V_D, k \in S_2} \sum_{i \in \partial(j')} \Delta \bar{E}_{ij'k}^{K,U}(t) - \sum_{i \in S_2, k \in V_S} \sum_{j' \in \partial(i)} \Delta \bar{E}_{ij'k}^{K,U}(t).$$

For regular t , it follows from the definition of derivative that

$$\sum_{k \in S_2} \dot{\bar{X}}_k^U(t) = \sum_{j' \in V_D, k \in S_2} \sum_{i \in \partial(j')} \dot{\bar{E}}_{ij'k}^U(t) - \sum_{i \in S_2, k \in V_S} \sum_{j' \in \partial(i)} \dot{\bar{E}}_{ij'k}^U(t).$$

For SMW(α) policy, using exactly the same argument as in Lemma 4 of Dai and Lin (2005), we have

$$\dot{\bar{E}}_{ij'k}^{\text{SMW}(\alpha)}(t) = 0 \quad \text{if } \frac{\bar{X}_i^{\text{SMW}(\alpha)}(t)}{\alpha_i} < \max_{\ell \in \partial(j')} \frac{\bar{X}_\ell^{\text{SMW}(\alpha)}(t)}{\alpha_\ell}. \quad (44)$$

By definition of S_2 , there exists $\epsilon > 0$ such that any (scaled) queue length in S_2 is strictly smaller than all (scaled) queue lengths in $V_S \setminus S_2$ in $(t, t + \epsilon)$, which also implies that the queue lengths in $V_S \setminus S_2$ remain strictly positive during $(t, t + \epsilon)$. Apply (44), we know that the system will use the supplies within $V_S \setminus S_2$ to serve all demands arriving at $\partial(V_S \setminus S_2)$ during $(t, t + \epsilon)$. Hence we have

$$\begin{aligned} \sum_{k \in V_S \setminus S_2} \dot{\bar{X}}_k^{\text{SMW}(\alpha)}(t) &= \sum_{j' \in V_D, k \in V_S \setminus S_2} \sum_{i \in \partial(j')} \dot{\bar{E}}_{ij'k}^{\text{SMW}(\alpha)}(t) - \sum_{j' \in \partial(V_S \setminus S_2), k \in V_S} \dot{\bar{A}}_{j'k}(t) \\ &\leq \sum_{j' \in V_D, k \in V_S \setminus S_2} \dot{\bar{A}}_{j'k}(t) - \sum_{j' \in \partial(V_S \setminus S_2), k \in V_S} \dot{\bar{A}}_{j'k}(t). \end{aligned}$$

Since it is a closed system, we have:

$$\begin{aligned} \sum_{k \in S_2} \dot{\bar{X}}_k^{\text{SMW}(\alpha)}(t) &= - \sum_{k \in V_S \setminus S_2} \dot{\bar{X}}_k^{\text{SMW}(\alpha)}(t) \\ &\geq \sum_{j' \in \partial(V_S \setminus S_2), k \in V_S} \dot{\bar{A}}_{j'k}(t) - \sum_{j' \in V_D, k \in V_S \setminus S_2} \dot{\bar{A}}_{j'k}(t). \end{aligned} \quad (45)$$

Note that

$$\begin{aligned} \text{RHS of (45)} &= \sum_{j' \in \partial(V_S \setminus S_2), k \in V_S} \dot{\bar{A}}_{j'k}(t) - \sum_{j' \in V_D, k \in V_S \setminus S_2} \dot{\bar{A}}_{j'k}(t) \\ &= \left(\sum_{j' \in V_D, k \in V_S} \dot{\bar{A}}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{\bar{A}}_{j'k}(t) \right) \\ &\quad - \left(\sum_{j' \in V_D, k \in V_S} \dot{\bar{A}}_{j'k}(t) - \sum_{j' \in V_D, k \in S_2} \dot{\bar{A}}_{j'k}(t) \right) \\ &= \sum_{j' \in V_D, k \in S_2} \dot{\bar{A}}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{\bar{A}}_{j'k}(t) \\ &= \text{RHS of (43)}. \end{aligned}$$

Finally, observe that for any $k \in S_2$,

$$\dot{L}_\alpha(\bar{\mathbf{X}}(t)) = -\frac{\dot{\bar{X}}_k^{U'}(t)}{\alpha_k} = -\frac{1}{\mathbf{1}_{S_2}^\top \alpha} \sum_{k \in S_2} \alpha_k \frac{\dot{\bar{X}}_k^{U'}(t)}{\alpha_k} = -\frac{1}{\mathbf{1}_{S_2}^\top \alpha} \sum_{k \in S_2} \dot{\bar{X}}_k^{U'}(t). \quad (46)$$

Plug (45) and (43) into (46), we know that inequality (23) holds, and it becomes equality for

SMW(α) policy.

□

D.2 Lyapunov Drift of Fluid Limits under SMW: Proof of Lemma 4

Proof of Lemma 4. Negative drift. Let $(\bar{\mathbf{A}}, \bar{\mathbf{X}})$ be a fluid limit of the system under SMW(α), and t be its regular point. Simply plug in Lemma 3, and replace $\dot{\bar{\mathbf{A}}}_{j'k}(t)$ with $\hat{\phi}_{j'k}$, we have (S_2 is defined in Lemma 3, $S_2 \neq \emptyset$)

$$\begin{aligned} \dot{L}_\alpha(\bar{\mathbf{X}}(t)) &= -\frac{1}{\mathbf{1}_{S_2}^\top \alpha} \left(\sum_{j' \in V_D, k \in S_2} \dot{\bar{\mathbf{A}}}_{j'k}(t) - \sum_{j' \in V_D: \partial(j') \subseteq S_2, k \in V_S} \dot{\bar{\mathbf{A}}}_{j'k}(t) \right) \\ &\leq -\min_{S_2 \subsetneq V_S, S_2 \neq \emptyset} \frac{1}{\mathbf{1}_{S_2}^\top \alpha} \left(\sum_{j' \in V_D, k \in S_2} \hat{\phi}_{j'k} - \sum_{j' \in V_D: \partial(j') \subseteq S_2, k \in V_S} \hat{\phi}_{j'k} \right) \\ &\leq -\min_{S_2 \subsetneq V_S, S_2 \neq \emptyset} \left(\sum_{j' \in V_D, k \in S_2} \hat{\phi}_{j'k} - \sum_{j' \in V_D: \partial(j') \subseteq S_2, k \in V_S} \hat{\phi}_{j'k} \right) \\ &\stackrel{(a)}{\leq} -\min\{\xi, \hat{\phi}_{\min}\}. \end{aligned}$$

Here (a) holds for the following reason. First note that when $\bar{\mathbf{X}}(t) \neq \alpha$, we have $S_2 \neq V_S$. Let $J \triangleq \{j' \in V_D : \partial(j') \subset S_2; \exists k \in V_S \setminus S_2 \text{ s.t. } \phi_{j'k} > 0\}$. If $J = \emptyset$, we have

$$\begin{aligned} &\sum_{j' \in V_D, k \in S_2} \hat{\phi}_{j'k} - \sum_{j' \in V_D: \partial(j') \subseteq S_2, k \in V_S} \hat{\phi}_{j'k} \\ &= \sum_{j' \in V_D: \partial(j') \cap (V_S \setminus S_2) \neq \emptyset, k \in S_2} \hat{\phi}_{j'k} - \sum_{j' \in V_D: \partial(j') \subseteq S_2, k \in V_S \setminus S_2} \hat{\phi}_{j'k} \\ &\geq \sum_{j' \in V_D: \partial(j') \cap (V_S \setminus S_2) \neq \emptyset, k \in S_2} \hat{\phi}_{j'k} \geq \hat{\phi}_{\min}, \end{aligned}$$

where $\hat{\phi}_{\min} \triangleq \min_{j' \in V_S, k \in V_S, \hat{\phi}_{j'k} > 0} \hat{\phi}_{j'k}$ is the minimum positive arrival rate for any demand type (j', k) (the last inequality holds because of Assumption 1). If $J \neq \emptyset$, we must have $J \in \mathcal{J}$, hence

$$\sum_{j' \in V_D, k \in S_2} \hat{\phi}_{j'k} - \sum_{j' \in V_D: \partial(j') \subseteq S_2, k \in V_S} \hat{\phi}_{j'k} \geq \sum_{j' \in V_D, k \in \partial(J)} \hat{\phi}_{j'k} - \sum_{j' \in J, k \in V_S} \hat{\phi}_{j'k} \geq \xi,$$

where $\xi \triangleq \min_{J \in \mathcal{J}} \left(\sum_{i \in \partial(J)} \mathbf{1}^\top \hat{\phi}_{(i)} - \sum_{j' \in J} \mathbf{1}^\top \hat{\phi}_{j'} \right) > 0$ is the Hall's gap of the system.

Robustness of drift. Define

$$G(\mathbf{f}) \triangleq \min_{S \subsetneq V_S, S \neq \emptyset} \left(\sum_{j' \in V_D, k \in S} f_{j'k} - \sum_{j': \partial(j') \subseteq S, k \in V_S} f_{j'k} \right).$$

Note that $G(\mathbf{f})$ is continuous in \mathbf{f} . Since $G(\hat{\phi}) \leq -\min\{\xi, \hat{\phi}_{\min}\} < 0$, by continuity there exists ϵ such that for any $\dot{\bar{\mathbf{A}}}(t) \in B(\hat{\phi}, \epsilon)$,

$$\dot{L}_\alpha(\bar{\mathbf{X}}(t)) = G(\dot{\bar{\mathbf{A}}}(t)) \leq -\frac{1}{2} \min\{\xi, \hat{\phi}_{\min}\}.$$

□

D.3 Explicit Exponent and Most Likely Sample Path: Proof of Lemma 5

Proof of Lemma 5. Explicit exponent. Let $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(\cdot))$ be a fluid sample path under $\text{SMW}(\boldsymbol{\alpha})$. For a regular point t of this FSP, denote $\mathbf{f} \triangleq \dot{\bar{\mathbf{A}}}(t)$.

For notation simplicity, for $S \subset V_S$ denote

$$\text{gap}_S(\mathbf{f}) \triangleq \sum_{j': \partial(j') \subseteq S, k \in V_S} f_{j'k} - \sum_{j' \in V_D, k \in S} f_{j'k}.$$

In words, $\text{gap}_S(\mathbf{f})$ is the minimum net rate at which supply in S is drained given current demand arrival rate \mathbf{f} , assuming no demand is dropped. Using the result of Lemma 3, we have:

$$\dot{L}_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}(t)) = \frac{\text{gap}_{S_2}(\mathbf{f})}{\mathbf{1}_{S_2}^T \boldsymbol{\alpha}}, \quad (47)$$

where $S_2 \triangleq S_2(\bar{\mathbf{X}}(t), \dot{\bar{\mathbf{X}}}(t))$ and the latter is defined in Lemma 3. Given $\dot{\bar{\mathbf{A}}}(t) = \mathbf{f}$, we define

$$\bar{v}(\mathbf{f}) \triangleq \sup_{\bar{\mathbf{X}}(t) \in \Omega \setminus \{\boldsymbol{\alpha}\}} \dot{L}_{\boldsymbol{\alpha}}(\bar{\mathbf{X}}(t)) = \max_{S \neq \emptyset, S \subseteq V_S} \frac{\text{gap}_S(\mathbf{f})}{\mathbf{1}_S^T \boldsymbol{\alpha}}.$$

Recall the definition of $\gamma_{\text{AB}}(\boldsymbol{\alpha})$ in Lemma 7, we have

$$\begin{aligned} \gamma_{\text{AB}}(\boldsymbol{\alpha}) &= \inf_{\mathbf{f} \geq \mathbf{0}: \bar{v}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{\bar{v}(\mathbf{f})} \\ &= \inf_{\mathbf{f} \geq \mathbf{0}: \max_{S \subseteq V_S} \text{gap}_S(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{\max_{S \subseteq V_S} \frac{\text{gap}_S(\mathbf{f})}{\mathbf{1}_S^T \boldsymbol{\alpha}}} \\ &= \inf_{\mathbf{f} \geq \mathbf{0}: \max_{S \subseteq V_S} \text{gap}_S(\mathbf{f}) > 0} \left\{ \min_{S \subseteq V_S: \text{gap}_S(\mathbf{f}) > 0} \left(\mathbf{1}_S^T \boldsymbol{\alpha} \right) \frac{\Lambda^*(\mathbf{f})}{\text{gap}_S(\mathbf{f})} \right\} \end{aligned} \quad (48)$$

$$\stackrel{(a)}{=} \min_{S \subseteq V_S} \left\{ \inf_{\mathbf{f} \geq \mathbf{0}: \text{gap}_S(\mathbf{f}) > 0} \left(\mathbf{1}_S^T \boldsymbol{\alpha} \right) \frac{\Lambda^*(\mathbf{f})}{\text{gap}_S(\mathbf{f})} \right\}. \quad (49)$$

For completeness, define the minimum over the empty set as $+\infty$. Here (a) holds because: For a minimizer $\mathbf{f}^* \geq \mathbf{0}$ of the outer problem of (48) and a minimizer $S^* \subseteq V_S$ of the inner problem of (48), $S^* \subseteq V_S$ is feasible for the inner problem of (49) while $\mathbf{f}^* \geq \mathbf{0}$ is feasible for the outer problem of (49), hence (48) \geq (49). Similarly we can show (48) \leq (49).

We claim that

$$(49) = \min_{J \in \mathcal{J}} \left\{ \inf_{\mathbf{f} \geq \mathbf{0}: \text{gap}_{\partial(J)}(\mathbf{f}) > 0} \left(\mathbf{1}_{\partial(J)}^T \boldsymbol{\alpha} \right) \frac{\Lambda^*(\mathbf{f})}{\text{gap}_{\partial(J)}(\mathbf{f})} \right\}. \quad (50)$$

Recall that the definition of \mathcal{J} :

$$\mathcal{J} = \left\{ J \subsetneq V_D : \sum_{j' \in J} \sum_{k \notin \partial(J)} \phi_{j'k} > 0 \right\}.$$

To see (50), first note that for $S \subseteq V_S$ where $\{j' \in V_D : \partial(j') \subset S\}$ is empty, $\text{gap}_S(\mathbf{f})$ is non-positive regardless of $\mathbf{f} \geq \mathbf{0}$, hence such S can never be the minimizer. For other S , let $J \triangleq \{j' \in V_D : \partial(j') \subset S\}$, then $\partial(J) \subset S$. Note that

$$\begin{aligned} \text{gap}_{\partial(J)}(\mathbf{f}) &= \sum_{j' \in J, k \in V_S} f_{j'k} - \sum_{j' \in V_D, k \in \partial(J)} f_{j'k} \\ &= \sum_{j': \partial(j') \subset S, k \in V_S} f_{j'k} - \sum_{j' \in V_D, k \in S} f_{j'k} + \sum_{j' \in V_D, k \in S \setminus \partial(J)} f_{j'k} \end{aligned}$$

$$\begin{aligned}
&= \text{gap}_S(\mathbf{f}) + \sum_{j' \in V_D, k \in S \setminus \partial(J)} f_{j'k} \\
&\geq \text{gap}_S(\mathbf{f}).
\end{aligned}$$

As a result, for \mathbf{f} such that $\text{gap}_S(\mathbf{f}) > 0$, we have

$$\left(\mathbf{1}_S^\top \boldsymbol{\alpha} \right) \frac{\Lambda^*(\mathbf{f})}{\text{gap}_S(\mathbf{f})} \geq \left(\mathbf{1}_{\partial(J)}^\top \boldsymbol{\alpha} \right) \frac{\Lambda^*(\mathbf{f})}{\text{gap}_{\partial(J)}(\mathbf{f})}.$$

Hence only those $S \subseteq V_S$ where $S = \partial(J)$ for $J \subseteq V_D$ can be the minimizer. If $J \notin \mathcal{J}$, then $\text{gap}_{\partial(J)}(\mathbf{f}) \leq 0$ regardless of $\mathbf{f} \geq \mathbf{0}$, so these sets are also ruled out. Therefore (50) holds.

Suppose the outer minimum of (50) is achieved by $J^* \in \mathcal{J}$. Denote the optimal value of the inner infimum of (50) as $(\mathbf{1}_{\partial(J^*)}^\top \boldsymbol{\alpha})g(\hat{\phi}, J) > 0$, then we have:

$$\inf_{\mathbf{f} \geq \mathbf{0}; \text{gap}_{\partial(J)}(\mathbf{f}) > 0} \Lambda^*(\mathbf{f}) - g(\hat{\phi}, J) \left(\sum_{j' \in J, k \in V_S} f_{j'k} - \sum_{j' \in V_D, k \in \partial(J)} f_{j'k} \right) = 0. \quad (51)$$

We can get rid of the constraint on \mathbf{f} because for \mathbf{f} where $\text{gap}_{\partial(J)}(\mathbf{f}) \leq 0$, the argument of minimization in (51) is negative; and for \mathbf{f} that has negative components, its rate function is ∞ by definition. Using Legendre transform, we have:

$$\begin{aligned}
&\inf_{\mathbf{f}} \Lambda^*(\mathbf{f}) - g(\hat{\phi}, J) \left(\sum_{j' \in J, k \in V_S} f_{j'k} - \sum_{j' \in V_D, k \in \partial(J)} f_{j'k} \right) \\
&= \inf_{\mathbf{f}} \Lambda^*(\mathbf{f}) - \mathbf{f}^\top \left(g(\hat{\phi}, J) \sum_{j' \in J, k \in V_S} \mathbf{e}_{j'k} - g(\hat{\phi}, J) \sum_{j' \in V_D, k \in \partial(J)} \mathbf{e}_{j'k} \right) \\
&= -\Lambda \left(g(\hat{\phi}, J) \sum_{j' \in J, k \in V_S} \mathbf{e}_{j'k} - g(\hat{\phi}, J) \sum_{j' \in V_D, k \in \partial(J)} \mathbf{e}_{j'k} \right) \\
&\stackrel{(b)}{=} - \sum_{j' \in V_D, k \in V_S} \hat{\phi}_{j'k} \left(e^{g(\hat{\phi}, J) \mathbb{I}\{j' \in J\} - g(\hat{\phi}, J) \mathbb{I}\{k \in \partial(J)\}} - 1 \right).
\end{aligned}$$

In (b) we use the fact that the dual function of $\Lambda^*(\mathbf{f})$ is $\Lambda(\mathbf{x}) = \sum_{j' \in V_D, k \in V_S} \hat{\phi}_{j'k} (e^{x_{j'k}} - 1)$ where $\mathbf{x} \in \mathbb{R}^{n \times m}$. Hence Eq. (51) reduces to the nonlinear equation

$$\left(\sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k} \right) e^{-g(\hat{\phi}, J)} + \left(\sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k} \right) e^{g(\hat{\phi}, J)} = \sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k} + \sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k}.$$

Let $y \triangleq e^{g(\hat{\phi}, J)}$, this becomes a quadratic equation:

$$\left(\sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k} \right) y^2 - \left(\sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k} + \sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k} \right) y + \left(\sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k} \right) = 0.$$

Hence

$$y = \frac{\sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k}}{\sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k}} \text{ or } 1.$$

Since $g(\hat{\phi}, J) > 0$, we have

$$g(\hat{\phi}, J) = \log \left(\frac{\sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k}}{\sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k}} \right) = \log \left(\frac{\sum_{j' \notin J, k \in \partial(J)} \phi_{j'k}}{\sum_{j' \in J, k \notin \partial(J)} \phi_{j'k}} \right).$$

Plugging into (50), we have:

$$\gamma_{AB}(\alpha) = \min_{J \in \mathcal{J}} \left(\mathbf{1}_{\partial(J)}^T \alpha \right) \log \left(\frac{\sum_{j' \notin J, k \in \partial(J)} \phi_{j'k}}{\sum_{j' \in J, k \notin \partial(J)} \phi_{j'k}} \right).$$

Remark: For $J \in \mathcal{J}$, if there exists $j' \in V_D$ such that $j' \notin J$ but $\partial(j') \subseteq \partial(J)$, then such subsets J are “spurious” in the sense that they cannot achieve the minimum in the expression of $\gamma_{AB}(\alpha)$ (the term corresponding to $J \cup \{j'\}$ is no larger than the term corresponding to J). Therefore only the “maximal” J ’s matter to the value of exponent.

Most likely demand sample path leading to demand loss. Denote

$$\mathbf{c} \triangleq g(\hat{\phi}, J) \left(\sum_{j' \in J, k \in V_S} \mathbf{e}_{j'k} - \sum_{j' \in V_D, k \in \partial(J)} \mathbf{e}_{j'k} \right),$$

denote \mathbf{f}_J as the minimizer of the inner minimization problem on the RHS of (50). We have

$$\begin{aligned} \mathbf{f}_J &= \operatorname{argmin}_{\mathbf{f} \geq \mathbf{0}} \sum_{j' \in V_D} \sum_{k \in V_S} \left(\Lambda_{j'k}^*(f_{j'k}) - c_{j'k} f_{j'k} \right) \\ &= \operatorname{argmin}_{\mathbf{f} \geq \mathbf{0}} \sum_{j' \in V_D} \sum_{k \in V_S} \left(f_{j'k} \log \frac{f_{j'k}}{\hat{\phi}_{j'k}} + \hat{\phi}_{j'k} - f_{j'k} - c_{j'k} f_{j'k} \right). \end{aligned}$$

First order condition implies: $(\mathbf{f}_J)_{j'k} = \hat{\phi}_{j'k} \frac{e^{c_{j'k} + 1}}{\sum_{j', k} \hat{\phi}_{j'k} e^{c_{j'k} + 1}} = \hat{\phi}_{j'k} \frac{e^{c_{j'k}}}{\sum_{j', k} \hat{\phi}_{j'k} e^{c_{j'k}}}$. Recall the definition of λ_J, μ_J in (13), we have

$$\begin{aligned} \sum_{j', k} \hat{\phi}_{j'k} e^{c_{j'k}} &= \sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k} \frac{\lambda_J}{\mu_J} + \sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k} \frac{\mu_J}{\lambda_J} + \left(1 - \sum_{j' \in J, k \notin \partial(J)} \hat{\phi}_{j'k} - \sum_{j' \notin J, k \in \partial(J)} \hat{\phi}_{j'k} \right) \\ &= \mu_J \frac{\lambda_J}{\mu_J} + \lambda_J \frac{\mu_J}{\lambda_J} + (1 - \lambda_J - \mu_J) \\ &= 1. \end{aligned}$$

Hence

$$(\mathbf{f}_J)_{j'k} = \begin{cases} \hat{\phi}_{j'k} (\lambda_J / \mu_J), & \text{for } j' \in J, k \notin \partial(J) \\ \hat{\phi}_{j'k} (\mu_J / \lambda_J), & \text{for } j' \notin J, k \in \partial(J) \\ \hat{\phi}_{j'k}, & \text{otherwise} \end{cases}.$$

Let $J^* = \operatorname{argmin}_{J \in \mathcal{J}} B_J \log(\lambda_J / \mu_J)$, then demand sample path with constant derivative \mathbf{f}_{J^*} is the most likely sample path leading to demand drop. \square

D.4 Transient behavior

Consider transient behavior over $[0, T]$ of our model with starting state $\mathbf{X}^K(0) \in \Omega_K$. We modify our objective appropriately: For any policy U which may be time dependent, we define

$$\mathbb{P}^{K,U}(\mathbf{X}^K(0), T) \triangleq \mathbb{E} \left(\frac{1}{A_\Sigma(T)} \sum_{r: t_r \in [0, T]} \mathbb{I} \left\{ U_{t_r}^K[\mathbf{X}^{K,U}(t_r^-)](o[r], d[r]) = \emptyset \right\} \right), \quad (52)$$

where $A_\Sigma \triangleq \sum_{j' \in V_D, k \in V_S} A_{j'k}(T)$ is the total number of demand arrivals during $[0, T]$, t_r is the r -th demand arrival epoch. We then define

$$\gamma_o(U) \triangleq - \liminf_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}^{K,U}(\mathbf{X}^K(0), T), \quad (53)$$

$$\gamma_p(U) \triangleq - \limsup_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}^{K,U}(\mathbf{X}^K(0), T). \quad (54)$$

If $\gamma_o(U) = \gamma_p(U)$, we denote this value by $\gamma(U)$ and call it the exponent achieved by policy U .

Theorem 4. Fix any $\alpha \in \text{relint}(\Omega)$ and any $T \geq T_0$ for $T_0 = \frac{1}{v_\alpha(\mathbf{f}^*)}$, where $v_\alpha(\cdot)$ was defined in Lemma 2 and \mathbf{f}^* is given by Lemma 5. Consider a sequence of initial states $\mathbf{X}^K(0) \in \Omega_K$ such that $\frac{\mathbf{X}^K(0)}{K} \xrightarrow{K \rightarrow \infty} \alpha$ and transient behavior over $[0, T]$. Then, the SMW(α) policy achieves exponent $\gamma(\alpha)$ as given by (13). No other policy can do better: for any policy U , we have $\gamma_p(U) \leq \gamma_o(U) \leq \gamma(\alpha)$.

Sketch of proof of Theorem 4. The converse bound $\gamma_o(U) \leq \gamma(\alpha)$ follows from the proof of Lemma 2. The adversary (nature) can ensure at least this much demand loss by using the demand arrival rates \mathbf{f}^* given in Lemma 5.

Achievability is straightforward to show. The sufficient conditions for exponent optimality in Proposition 5 (steepest descent and negative drift) apply to transient behavior starting at scaled state α and for any finite horizon $T \geq 1/v_\alpha(\mathbf{f}^*)$: The proof of the proposition goes through verbatim since it is fundamentally an argument about what happens over a finite horizon. It then remains to check that SMW(α) satisfies these conditions, but we know this is true from Lemmas 3 and 4. \square

E Proof of Proposition 3 and appendix to Section 4.1

In this appendix, the first subsection provides the proof of Proposition 3 showing frequent utilization of supply units under SMW. The second subsection provides the structural corollaries (of Theorem 1) illustrated in Section 4.1.

E.1 Utilization rate of supply units: Proof of Proposition 3

Proof of Proposition 3. 1. Because supply units relocate only when assigned to an incoming demand, we have

$$\begin{aligned} \xi^{K,\alpha} &= \frac{\mathbb{E}(\text{number of demand fulfilled in unit time in steady state})}{(\text{number of supply units})} \\ &= \frac{K \cdot \mathbf{1}^T \hat{\phi} \mathbf{1} - \mathbb{E}(\text{number of lost demand in unit time in steady state})}{K} \end{aligned}$$

$$\geq \mathbf{1}^T \hat{\phi} \mathbf{1} - \mathbb{P}_p^{K, \alpha},$$

where $\mathbb{P}_p^{K, \alpha}$ is the pessimistic demand loss probability defined in (2). Apply Theorem 1, we have for any $\alpha \in \text{relint}(\Omega)$, $\lim_{K \rightarrow \infty} \xi^{K, \alpha} = \mathbf{1}^T \hat{\phi} \mathbf{1} > 0$. Note that the above argument only uses the fact that the probability of losing demand is diminishing as $K \rightarrow \infty$, hence it holds with travel delays as well (apply Theorem 2).

2. *Sketch of proof.* The key observation is that under FIFO, if a supply unit is not assigned, neither do all the supply units that join the same queue later. Fix a supply unit which is the end-of-line unit in the i -th queue at time 0. Let ϵ, ζ be positive constants to be specified later. Let $T' \triangleq \frac{4}{\epsilon\eta} + \max\{T_0, \frac{2}{\lambda_{\min}}\} > 0$ where η is the Lyapunov drift under SMW(α) defined in Lemma 4, T_0 is defined in Theorem 4, and $\lambda_{\min} \triangleq \min_{i \in V_S} \sum_{j' \in V_D} \hat{\phi}_{j'i}$. We have $\eta > 0$, $\lambda_{\min} > 0$, where the former is ensured by Lemma 4, and the latter holds because of Assumption 1. We consider the time intervals $[0, T'), [T', 2T'), \dots$.

In the following, we upper bound the probability that the fixed unit is not assigned during $[kT', (k+1)T')$ given it is not assigned during $[0, kT')$ (here $k \geq 0$). Let $kT' + \tau^K$ be the first time $L_\alpha(\bar{\mathbf{X}}^K(t))$ hit level $\frac{\zeta}{K}$ or below during $[kT', (k+1)T')$. Define the following three events:

$$\begin{aligned} \mathcal{E}_1^K &\triangleq \left\{ \tau^K \leq \frac{4}{\epsilon\eta} \middle| \bar{\mathbf{X}}^K(kT') \right\}, \\ \mathcal{E}_2^K &\triangleq \left\{ L_\alpha(\bar{\mathbf{X}}^K(t)) < 1 \text{ for all } t \in [kT' + \tau^K, (k+1)T' + \tau^K] \right\}, \\ \mathcal{E}_3^K &\triangleq \left\{ \sum_{j' \in V_D} (\bar{A}_{j'i}((k+1)T') - \bar{A}_{j'i}(kT' + \tau)) \leq \frac{3}{2} \right\}, \end{aligned}$$

Note that if event $\mathcal{E}_1^K \cap \mathcal{E}_2^K \cap \mathcal{E}_3^K$ happens, then the fixed supply unit must be assigned during $[kT', (k+1)T')$: otherwise, the length of the i -th queue will exceed $\frac{3}{2}K$, which is impossible. Now we use union bound to lower bound $\mathcal{E}_1^K \cap \mathcal{E}_2^K \cap \mathcal{E}_3^K$.

Using the argument in the proof of Theorem 4 in Venkataramanan and Lin (2013), there exists $\epsilon > 0, \zeta > 0$ independent of $\bar{\mathbf{X}}^K(kT')$ such that for large enough K , $\mathbb{E}[\tau^K] \leq \frac{1}{\epsilon\eta}$. Let the undetermined constants ϵ, ζ to be such ϵ, ζ . Using Markov's inequality we have $\mathbb{P}(\mathcal{E}_1^K) \geq 1 - \frac{1}{4} = \frac{3}{4}$. Using Theorem 4 we have for large enough K , the probability of \mathcal{E}_2^K converges to 1, hence $\mathbb{P}(\mathcal{E}_2^K) \geq \frac{3}{4}$ for large enough K . Using Chernoff bound of Poisson arrivals, we have for large enough K , $\mathbb{P}(\mathcal{E}_3^K) \geq \frac{3}{4}$. As a result,

$$\mathbb{P}(\mathcal{E}_1^K \cap \mathcal{E}_2^K \cap \mathcal{E}_3^K) \geq \frac{1}{4}.$$

Let $\omega_i^K(\mathbf{x})$ be the waiting time of the fixed supply unit given the (normalized) initial state converges to \mathbf{x} as $K \rightarrow \infty$. Then for large enough K , we have

$$\mathbb{E}[\omega_i^K(\mathbf{x})] \leq \sum_{k=0}^{\infty} \left(1 - \frac{1}{4}\right)^k \frac{1}{4}(k+1)T' = 4T' < \infty.$$

This concludes the proof. Note that the above argument only uses the fact that the probability of losing demand is diminishing as $K \rightarrow \infty$, and that the Lyapunov drift in

fluid limit is negative, hence it should hold with travel delays as well (apply Theorem 2). \square

E.2 Appendix to Section 4.1: optimal choice of scaling factors α

The following corollary of Theorem 1 considers the case where there is exactly one vulnerable subset of demand nodes (Definition 3).

Corollary 2 (If one subset of nodes is vulnerable, the optimal α protects it). *Fix a compatibility graph G . Consider a sequence of demand type distributions $(\phi^n)_{n=1}^\infty$ satisfying the following properties:*

- (Limiting distribution) *There is a demand type distribution ϕ^* such that $\lim_{n \rightarrow \infty} \phi^n = \phi^*$ and such that (G, ϕ^*) satisfies Assumptions 1 and 2.*
- (Vulnerable subset) *There is a subset $J_1 \in \mathcal{J}^*$ such that $\lambda_{J_1}^* = \mu_{J_1}^*$, whereas for all other subsets $J \in \mathcal{J}^* \setminus J_1$, we have $\lambda_J^* > \mu_J^*$, cf. Assumption 3 (here λ_J^* , μ_J^* and \mathcal{J}^* are the quantities under distribution ϕ^*). The distributions ϕ^n satisfy Assumption 3; in particular, $\lambda_{J_1}^n / \mu_{J_1}^n \rightarrow 1^+$.*

Fix any $\epsilon \in (0, 1/2)$. There exists $n_0 = n_0(\epsilon) < \infty$ such that, for all $n > n_0$, the following holds on network (G, ϕ^n) :

- (i) (Optimal exponent) *The best achievable exponent $\bar{\gamma}$ satisfies*

$$\bar{\gamma} \in [(1 - \epsilon)\xi_{J_1}, \xi_{J_1}] \quad \text{for } \xi_{J_1} \triangleq \log(\lambda_{J_1}^n / \mu_{J_1}^n).$$

As always, SMW policies suffice to achieve it, i.e., $\bar{\gamma} = \sup_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha)$.

- (ii) (Near optimal α protects supply near J_1 .) *If SMW with scaling factors $\alpha \in \text{relint}(\Omega)$ achieves a demand-loss exponent $\gamma(\alpha) \geq (1 - \epsilon)\xi_{J_1}$, then it must be that*

$$\mathbf{1}_{\partial(J_1)}^T \alpha \geq 1 - \epsilon.$$

- (iii) (Example of near optimal α .) *The SMW(α) policy with*

$$\alpha_i \triangleq \begin{cases} \frac{1-\epsilon}{|\partial(J_1)|} & \text{for all } i \in \partial(J_1), \\ \frac{\epsilon}{m-|\partial(J_1)|} & \text{for all } i \in V_S \setminus \partial(J_1). \end{cases} \quad (55)$$

achieves $\gamma(\alpha) = (1 - \epsilon)\xi_{J_1}$.

Informally speaking, Corollary 2 says that if there is just one vulnerable subset of demand nodes J_1 , then the exponent optimal SMW policy has a resting state which puts almost all the supply in the neighborhood of J_1 . The intuition is that the supply at $\partial(J_1)$ follows a random walk which has only slightly positive drift even if the assignment rule protects it (recall that the definition of the net supply λ_{J_1} is optimistic), and hence it is optimal to keep the total supply at these nodes at a high resting point, to minimize the likelihood of depletion.

It is easy to verify that Example 2 satisfies the conditions in Corollary 2: Note that in the example $\lim_{n \rightarrow \infty} \phi^n = \phi^*$ where ϕ^* is given by (16) with δ_n replaced by 0 and η_n replaced by $1/8$. Clearly, the limit demand type distribution ϕ^* satisfies Assumptions 1 and 2, and

ϕ^n satisfies Assumption 3 for all $n > 4$. Furthermore, the limited-flexibility subset $\{4'\}$ is vulnerable, whereas all the other limited-flexibility subsets (namely, $\{1'\}$, $\{1', 2'\}$ and $\{3', 4'\}$) are not vulnerable.

We now prove the corollary.

Proof of Corollary 2. We are given that (G, ϕ^n) satisfies Assumption 3 for all $n \in \mathbb{Z}_+$. We start by showing that for all large enough n , we have that (G, ϕ^n) also satisfies Assumptions 1 and 2: We are given that (G, ϕ^*) satisfies Assumptions 1 and 2. For any demand type distribution ϕ , let the *support* of ϕ be the set of demand types which occur with positive probability

$$\text{support}(\phi) \triangleq \{(j', i) \in V_D \times V_S : \phi_{j'i} > 0\}.$$

Since $\lim_{n \rightarrow \infty} \phi^n = \phi^*$, it is clear that there exists n_0 such that for all $n > n_0$, the support of ϕ^n is a superset of the support of ϕ^* , i.e., $\text{support}(\phi^n) \supseteq \text{support}(\phi^*)$. It is then clear from the form of Assumptions 1 and 2 that (G, ϕ^n) satisfies them, given that (G, ϕ^*) satisfies them (the assumptions are requirements on the *support* of the demand type distribution, and if a given distribution satisfies them, then it is easy to see that any distribution supported on a superset of demand types also satisfies them).

For all $n > n_0$, since (G, ϕ^n) satisfies all three assumptions, Theorem 1 is applicable. From Theorem 1 part 1, we know $\gamma(\alpha) \leq \mathbf{1}_{\partial(J_1)}^T \alpha \log(\lambda_{J_1}^n / \mu_{J_1}^n) = \mathbf{1}_{\partial(J_1)}^T \xi_{J_1}$. We deduce both part (ii) of the corollary, as well as $\bar{\gamma} \leq \xi_{J_1}$ towards part (i) (to reach the latter conclusion we further use $\mathbf{1}_{\partial(J_1)}^T \alpha \leq 1$ and Theorem 3 part 2).

We now prove part (iii), namely, that for α defined in (55), $\text{SMW}(\alpha)$ achieves an exponent

$$\gamma(\alpha) = (1 - \epsilon) \log(\lambda_{J_1}^n / \mu_{J_1}^n). \quad (56)$$

(It will follow immediately that $\bar{\gamma} \geq (1 - \epsilon) \log(\lambda_{J_1}^n / \mu_{J_1}^n)$, completing the proof of part (i) as well.) We will again use Theorem 1 part 1 to establish (56). It is clear from the definition (55) that $\mathbf{1}_{\partial(J_1)}^T \alpha = 1 - \epsilon$ and hence $\mathbf{1}_{\partial(J_1)}^T \alpha \log(\lambda_{J_1}^n / \mu_{J_1}^n) = (1 - \epsilon) \log(\lambda_{J_1}^n / \mu_{J_1}^n)$. Hence, to show that (56) holds, it suffices to show that we have

$$\mathbf{1}_{\partial(J)}^T \alpha \cdot \log(\lambda_J^n / \mu_J^n) \geq (1 - \epsilon) \log(\lambda_{J_1}^n / \mu_{J_1}^n) \quad (57)$$

for all $J \in \mathcal{J}^n \setminus \{J_1\}$. We will show that this holds for all large enough n .

Consider any $J \neq J_1$ such that $J \in \mathcal{J}^n$ for infinitely many n (if $J \in \mathcal{J}^n$ for finitely many n , we can eliminate it from consideration simply by taking n large enough). We will show that (57) holds for J for all n large enough. Note that for the chosen α we have $\mathbf{1}_{\partial(J)}^T \alpha \geq \epsilon/m > 0$ (since $|\partial(J)| \geq 1$, using Assumption 3), and so it suffices to show that

$$\liminf_{n \rightarrow \infty} \log(\lambda_J^n / \mu_J^n) > 0, \quad (58)$$

since the right-hand side of (57) tends to 0 as $n \rightarrow \infty$. (Here we define any positive number divided by 0 as ∞ .) If $J \in \mathcal{J}^*$, it is easy to see that (58) holds: we know that $\lambda_J^n \rightarrow \lambda_J^*$ and $\mu_J^n \rightarrow \mu_J^* > 0$, and so $\log(\lambda_J^n / \mu_J^n) \rightarrow \log(\lambda_J^* / \mu_J^*) > 0$. To complete the proof consider the complementary case $J \notin \mathcal{J}^*$, i.e., $\mu_J^* = 0$. We will establish (58) by showing that $\lambda_J^* > 0$. Since $J \in \mathcal{J}^n$ for some $n > n_0$, by definition of \mathcal{J}^n we know that $\partial(J)$ is a strict subset of V_S (else there cannot be a demand type with origin in J and destination in $V_S \setminus \partial(J)$). Consider any $i_1 \in V_S \setminus \partial(J)$ and any $i_2 \in \partial(J)$. Since we know that ϕ^* satisfies Assumption 1, there is a path

to move supply from i_1 to i_2 , and so there must exist a demand type (j', k) with $j' \in V_D \setminus J$ and $k \in \partial(J)$ with $\phi_{j'k}^* > 0$, which immediately implies $\lambda_J^* > 0$. We deduce from $\lambda_J^n \rightarrow \lambda_J^* > 0$ and $\mu_J^n \rightarrow \mu_J^* = 0$ that $\log(\lambda_J^n/\mu_J^n) \rightarrow \infty$, and hence that (58) holds.

Since there are only finitely many subsets J to consider, we deduce from (58) that there exists n_0 such that, for all $n > n_0$, (57) holds for all $J \in \mathcal{J}^n \setminus \{J_1\}$. \square

The second corollary considers the case of two non-overlapping vulnerable subsets of nodes.

Corollary 3 (If there are two non-overlapping vulnerable subsets, the optimal α protects them in inverse proportion to their inherent robustness). *Fix a compatibility graph G . Consider a sequence of demand type distributions $(\phi^n)_{n=1}^\infty$ satisfying the following properties:*

- (Limiting distribution) *There is a demand type distribution ϕ^* such that $\lim_{n \rightarrow \infty} \phi^n = \phi^*$ and such that (G, ϕ^*) satisfies Assumptions 1 and 2.*
- (Vulnerable subsets) *There are two non-overlapping subsets $J_1, J_2 \in \mathcal{J}^*$, $J_1 \cap J_2 = \emptyset$, $\partial(J_1) \cap \partial(J_2) = \emptyset$ such that $\lambda_{J_1}^* = \mu_{J_1}^*$ and $\lambda_{J_2}^* = \mu_{J_2}^*$, whereas for all other subsets $J \in \mathcal{J}^* \setminus \{J_1, J_2\}$, we have $\lambda_J^* > \mu_J^*$, cf. Assumption 3 (here λ_J^* , μ_J^* and \mathcal{J}^* are the quantities under distribution ϕ^*). The distributions ϕ^n satisfy Assumption 3; in particular, $\lambda_{J_1}^n/\mu_{J_1}^n \rightarrow 1^+$ and $\lambda_{J_2}^n/\mu_{J_2}^n \rightarrow 1^+$.*

Fix any $\epsilon \in (0, 1/2)$. There exists $n_0 = n_0(\epsilon) < \infty$ such that, for all $n > n_0$, the following holds on network (G, ϕ^n) :

- (i) (Optimal exponent) *The best achievable exponent $\bar{\gamma}$ satisfies*

$$\bar{\gamma} \in [(1 - \epsilon)H, H] \quad \text{for } H \triangleq \frac{\xi_{J_1}\xi_{J_2}}{\xi_{J_1} + \xi_{J_2}}, \quad \xi_J \triangleq \log(\lambda_J^n/\mu_J^n).$$

As always, SMW policies suffice to achieve it, i.e., $\bar{\gamma} = \sup_{\alpha \in \text{relint}(\Omega)} \gamma(\alpha)$.

- (ii) (Near optimal α protects supply near J_1 .) *If SMW with scaling factors $\alpha \in \text{relint}(\Omega)$ achieves a demand-loss exponent $\gamma(\alpha) \geq (1 - \epsilon)H$, then it must be that*

$$\mathbf{1}_{\partial(J_1)}^T \alpha \stackrel{\epsilon}{=} \frac{\xi_{J_2}}{\xi_{J_1} + \xi_{J_2}} \quad \text{and} \quad \mathbf{1}_{\partial(J_2)}^T \alpha \stackrel{\epsilon}{=} \frac{\xi_{J_1}}{\xi_{J_1} + \xi_{J_2}},$$

where $a \stackrel{\epsilon}{=} b$ represents $|a - b| \leq \epsilon$.

- (iii) (Example of near optimal α .) *The SMW(α) policy with*

$$\alpha_i \triangleq \begin{cases} \frac{1-\epsilon_1}{|\partial(J_1)|} \cdot \frac{\xi_{J_2}}{\xi_{J_1} + \xi_{J_2}} & \text{for all } i \in \partial(J_1), \\ \frac{1-\epsilon_1}{|\partial(J_2)|} \cdot \frac{\xi_{J_1}}{\xi_{J_1} + \xi_{J_2}} & \text{for all } i \in \partial(J_2), \\ \frac{\epsilon}{m - |\partial(J_1)| - |\partial(J_2)|} & \text{for all } i \in V_S \setminus (\partial(J_1) \cup \partial(J_2)) \end{cases} \quad (59)$$

for $\epsilon_1 \triangleq \epsilon \cdot \mathbb{I}(V_S \setminus (\partial(J_1) \cup \partial(J_2)) \neq \emptyset)$, achieves $\gamma(\alpha) \geq (1 - \epsilon)H$.

Corollary 3 says that if there are two non-overlapping vulnerable subsets of demand nodes J_1 and J_2 , then the exponent optimal SMW policy has a resting state (i) which puts almost all the supply in the union of their neighborhoods $\partial(J_1) \cup \partial(J_2)$, (ii) divides the supply between the

two neighborhoods in inverse proportion to the inherent robustness of the vulnerable subsets

$$\frac{\mathbf{1}_{\partial(J_2)}^T \boldsymbol{\alpha}}{\mathbf{1}_{\partial(J_1)}^T \boldsymbol{\alpha}} \approx \frac{\xi_{J_1}}{\xi_{J_2}}.$$

Example 3 follows from Corollary 3: Clearly, the limit demand type distribution $\boldsymbol{\phi}^*$ in the example satisfies Assumptions 1 and 2, and $\boldsymbol{\phi}^n$ satisfies Assumption 3 for all $n > 4/\min(1, \eta)$. Furthermore, the limited-flexibility subsets $\{1'\}$ and $\{4'\}$ are non-overlapping and vulnerable, whereas all the other limited-flexibility subsets (namely, $\{1', 2'\}$ and $\{3', 4'\}$) are not vulnerable. Note that $V_S \setminus (\partial(J_1) \cup \partial(J_2)) = \emptyset$ and hence $\epsilon_1 = 0$ in the example.

We now prove the corollary.

Proof of Corollary 3. The proof is analogous to that of Corollary 2.

From Theorem 1 part we know that for any $\boldsymbol{\alpha}$, it holds that

$$\gamma(\boldsymbol{\alpha}) \leq \mathbf{1}_{\partial(J_1)}^T \boldsymbol{\alpha} \cdot \xi_{J_1} \quad \text{and} \quad \gamma(\boldsymbol{\alpha}) \leq \mathbf{1}_{\partial(J_2)}^T \boldsymbol{\alpha} \cdot \xi_{J_2}. \quad (60)$$

Since $\partial(J_1) \cap \partial(J_2) = \emptyset$, we know that

$$\mathbf{1}_{\partial(J_1)}^T \boldsymbol{\alpha} + \mathbf{1}_{\partial(J_2)}^T \boldsymbol{\alpha} \leq \mathbf{1}^T \boldsymbol{\alpha} = 1.$$

We then deduce from (60) that

$$\gamma(\boldsymbol{\alpha}) \leq H = \frac{\xi_{J_1} \xi_{J_2}}{\xi_{J_1} + \xi_{J_2}}.$$

holds for all $\boldsymbol{\alpha} \in \text{relint}(\Omega)$, and hence, using Theorem 1 part 2, we obtain $\bar{\gamma} \leq H$. This is the upper bound in part (i) of the corollary.

We now prove part (ii). If $\gamma(\boldsymbol{\alpha}) \geq (1 - \epsilon)H$ then using (60) we have

$$\begin{aligned} \mathbf{1}_{\partial(J_1)}^T \boldsymbol{\alpha} \cdot \xi_{J_1} &\geq (1 - \epsilon) \cdot \frac{\xi_{J_1} \xi_{J_2}}{\xi_{J_1} + \xi_{J_2}} \\ \Rightarrow \quad \mathbf{1}_{\partial(J_1)}^T \boldsymbol{\alpha} &\geq (1 - \epsilon) \cdot \frac{\xi_{J_2}}{\xi_{J_1} + \xi_{J_2}} \geq \frac{\xi_{J_2}}{\xi_{J_1} + \xi_{J_2}} - \epsilon \end{aligned} \quad (61)$$

and similarly

$$\mathbf{1}_{\partial(J_2)}^T \boldsymbol{\alpha} \geq \frac{\xi_{J_1}}{\xi_{J_1} + \xi_{J_2}} - \epsilon. \quad (62)$$

But (62) further implies

$$\mathbf{1}_{\partial(J_1)}^T \boldsymbol{\alpha} \leq 1 - \mathbf{1}_{\partial(J_2)}^T \boldsymbol{\alpha} \leq \frac{\xi_{J_2}}{\xi_{J_1} + \xi_{J_2}} + \epsilon.$$

Combining with (61) we have shown $\mathbf{1}_{\partial(J_1)}^T \boldsymbol{\alpha} \stackrel{\epsilon}{=} \frac{\xi_{J_2}}{\xi_{J_1} + \xi_{J_2}}$, and analogously obtain $\mathbf{1}_{\partial(J_2)}^T \boldsymbol{\alpha} \stackrel{\epsilon}{=} \frac{\xi_{J_1}}{\xi_{J_1} + \xi_{J_2}}$. This completes the proof of part (ii).

It remains to show part (iii) which will further imply the lower bound $\bar{\gamma} \geq H(1 - \epsilon)$ in part (i). Part (iii) states that $\boldsymbol{\alpha}$ defined in (59), we have $\gamma(\boldsymbol{\alpha}) \geq (1 - \epsilon)H$ for large enough n . Using Theorem 1 part 1, it suffices to show that for large enough n , we have

$$\mathbf{1}_{\partial(J)}^T \boldsymbol{\alpha} \xi_J \geq (1 - \epsilon)H \quad (63)$$

for all $J \in \mathcal{J}^n$. For $J = J_1$, it clear that the left-hand side of (63) is $(1 - \epsilon_1)H \geq (1 - \epsilon)H$, and similarly for J_2 . It remains to consider the other subsets. Note that $H \xrightarrow{n \rightarrow \infty} 0$. Now to prove

that for large enough n , (63) holds for all $J \in \mathcal{J}^n \setminus \{J_1, J_2\}$, we can use the proof of (57) (in the proof of Corollary 2) verbatim. \square

F Necessity of the Assumptions and the Inferiority of State-Independent Control

This section shows the necessity of our assumptions, and of state-dependent control, including the proofs of Propositions 1, 2 and 4. It also demonstrates poor performance of the naive state-dependent policy by establishing the claim in Example 4.

F.1 Necessity of Assumption 2: Proof of Proposition 1

Proof of Proposition 1. We define the following policy U which ensures no demand loss in the long run, i.e., $\mathbb{P}_p^{K,U} = 0$. Arbitrarily choose n of the K supply units and dedicate one of the chosen supply units to each of the demand nodes. Suppose the supply unit dedicated to demand node j' is initially at supply node i . Since Assumption 1 is satisfied, there is a way to move the supply unit from i to a supply node compatible with j' in a finite (random) time. Move the supply unit to some node in $\partial(j')$. Similarly, move each of the n dedicated demand units into the neighborhood of the corresponding demand node. All this is completed in an initial transient of finite (random) duration (the expected duration is also finite). Thereafter, for each demand arrival, use the supply unit dedicated to the origin of the demand to serve it. We are guaranteed that the destination $k \in \partial(j')$, i.e., the supply unit remains within the neighborhood of j' after completing service (we are told that demand types with $k \notin \partial(j')$ have zero arrival rate $\phi_{j'k} = 0$). \square

F.2 Necessity of CRP Condition: Proof of Proposition 2

Proof of Proposition 2. There are two cases:

Case 1: *There exists $J \subsetneq V_D$ s.t. $\lambda_J < \mu_J \iff \sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)} < \sum_{j' \in J} \mathbf{1}^T \phi_{j'}$.*

The main proof idea in this case is simply that since the net supply to $\partial(J)$ is less than the net demand originating in J , a positive fraction of demand must be lost.

Consider the following balance equation:

$$\begin{aligned}
& \#\{\text{demands originating in } J \text{ during } [0, T] \text{ which are lost}\} \\
&= \#\{\text{demands originating in } J \text{ during } [0, T]\} \\
&\quad - \#\{\text{demands originating in } J \text{ during } [0, T] \text{ which are fulfilled}\} \\
&\geq \#\{\text{demands originating in } J \text{ during } [0, T]\} - \#\{\text{supplies assigned from } \partial(J) \text{ during } [0, T]\} \\
&\geq \sum_{r: t_r \in [0, T]} \mathbb{I}\{o[r] \in J\} - \sum_{r: t_r \in [0, T]} \mathbb{I}\{d[r] \in \partial(J)\} - \#\{\text{initial supply in } \partial(J)\}.
\end{aligned}$$

The first inequality holds because the demands originating in J can only be fulfilled by supply units from $\partial(J)$. The second inequality holds because the total number of supply units assigned from $\partial(J)$ during $[0, T]$ cannot exceed the initial supply there plus the number of demand

arrivals with destination in $\partial(J)$. Divide both sides by $A_\Sigma(T)$ which is the total number of demand arrivals during $[0, T]$, and let $T \rightarrow \infty$. By the strong law of large numbers, we have:

$$\liminf_{T \rightarrow \infty} \{\text{fraction of lost in } [0, T]\} \geq \sum_{j' \in J} \mathbf{1}^T \phi_{j'} - \sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)} > 0.$$

Hence a positive fraction of demand will be lost in the long run, and the loss exponent is 0.

Case 2: We have $\lambda_{J'} \geq \mu_{J'}$ for all $J' \in \mathcal{J}$ but there exists $J \in \mathcal{J}$ such that $\lambda_J = \mu_J \Leftrightarrow \sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)} = \sum_{j' \in J} \mathbf{1}^T \phi_{j'}$.

The high-level idea in this case is that if all the demand originating in J is served (if possible), then, at best, the total quantity of supply in $\partial(J)$ follows an unbiased random walk on $0, 1, \dots, K$. Such a random walk spends a positive fraction of time at 0, and all demand originating in J when there is zero supply in $\partial(J)$ is lost. The proof is somewhat more intricate than this argument may suggest; in particular because we need to allow for idling policies (those which sometimes lose demand even though supply is available at a neighboring node).

Divide the demand arrivals into cycles with MK^2 arrivals each, where

$$M \triangleq \frac{1}{\mu_J},$$

for $\mu_J = \sum_{j' \in J, k \notin \partial(J)} \phi_{j'k} > 0$ as before. Without loss of generality, consider the first cycle t_1, \dots, t_{MK^2} . Define random walk S_r with the following dynamics:

- $S_0 = \mathbf{1}_{\partial(J)}^T \mathbf{X}(0)$.
- $S_{r+1} = S_r + 1$ if $o[r] \notin J, d[r] \in \partial(J)$.
- $S_{r+1} = S_r - 1$ if $o[r] \in J, d[r] \notin \partial(J)$.
- $S_{r+1} = S_r$ otherwise.

It is not hard to see that if no demand is lost during $r \leq MK^2$ under some policy U , then S_r is a pathwise upper bound on the number of supply units in $\partial(J)$, namely, $\mathbf{1}_{\partial(J)}^T \mathbf{X}(t_r)$, for any $r \leq MK^2$. With this observation, we have:

$$\begin{aligned} & \mathbb{P} \left(\text{some demand is lost during } r \leq MK^2 \right) \\ & \geq \mathbb{P} \left(S_{r'} = 0 \text{ for some } r' < MK^2 \right) \cdot (\mathbf{1}^T \phi_{j'}). \end{aligned} \tag{64}$$

The above is true because when the event on RHS happens, either (1) some demand is lost before t'_r , or (2) no demand is lost before t'_r , then since $0 = S_{r'} \geq \mathbf{1}_{\partial(J)}^T \mathbf{X}(t'_r)$ we have $\mathbf{1}_{\partial(J)}^T \mathbf{X}(t'_r) = 0$ and so any demand with origin in J is lost at t'_{r+1} . Importantly, (64) holds for *any* policy.

For the given J we have $\lambda_J = \mu_J > 0$ and so S_r is a “lazy” simple random walk, which takes a step with probability $2\mu_J$ independently at each r . Define the stopping time τ as

$$\tau \triangleq \inf \left\{ r \in \mathbb{Z}_+ : S_r \in \left\{ \mathbf{1}_{\partial(J)}^T \mathbf{X}(0) - K, \mathbf{1}_{\partial(J)}^T \mathbf{X}(0) + K \right\} \right\}.$$

Using (Example 4.1.6, Durrett 2010) on the lazy simple random walk $S_r - \mathbf{1}_{\partial(J)}^T \mathbf{X}(0)$, we obtain²⁶

$$\mathbb{E}[\tau] = \frac{K^2}{2\mu_J}.$$

Using Markov's inequality, we have

$$\mathbb{P}(\tau \geq MK^2) \leq \frac{\mathbb{E}[\tau]}{MK^2} = \frac{1}{2}$$

By symmetry

$$\mathbb{P}(S_\tau - \mathbf{1}_{\partial(J)}^T \mathbf{X}(0) = -K \text{ and } \tau < mK^2) = \frac{1}{2} \mathbb{P}(\tau < mK^2) \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}. \quad (65)$$

Now, $S_\tau - \mathbf{1}_{\partial(J)}^T \mathbf{X}(0) = -K$ and $\tau < MK^2$, i.e., S_r hits $\mathbf{1}_{\partial(J)}^T \mathbf{X}(0) - K$ during $r < MK^2$, implies that S_r hits 0 during $t < MK^2$, since S_r must hit 0 (weakly) before it hits $\mathbf{1}_{\partial(J)}^T \mathbf{X}(0) - K$. Hence, plugging (65) into (64) we obtain that

$$\mathbb{P}(\text{some demand is lost during } r \leq MK^2) \geq \frac{\mathbf{1}^T \phi_{j'}}{4},$$

and this uniform and strictly positive lower bound holds for *any* policy, during *any* cycle consisting of MK^2 consecutive arrivals.

It follows that

$$\mathbb{P}_p^{K,U} \geq \mathbb{P}_o^{K,U} = \Omega\left(\frac{1}{K^2}\right),$$

and hence $\gamma_p(U) = \gamma_o(U) = 0$ for any U . □

F.3 Necessity of State-Dependent Control: Proof of Proposition 4

Proof of Proposition 4.

- Proof of first part. For notation simplicity, denote $X(t_r)$ by $X[r]$, similar for another notations. Denote the probability mass function of distribution $u_{j'k}[t]$ by $u_{j'k}[t](\cdot)$. We first define an “augmented” policy \tilde{U} for any state-independent policy U . Policy \tilde{U} is also state independent with distribution $\tilde{u}_{j'k}[t]$, where:

$$\begin{aligned} \tilde{u}_{j'k}[t](i) &= u_{j'k}[t](i) + \frac{1}{|\partial(j')|} u_{j'k}[t](\emptyset) \quad \text{for } i \in \partial(j'), \\ \tilde{u}_{j'k}[t](\emptyset) &= 0. \end{aligned}$$

In the following analysis, we couple U and \tilde{U} in such a way that if U dispatches from i to serve the t -th demand, then \tilde{U} will do the same.

Divide the demand arrivals into cycles with K^2 arrivals each. We will lower bound the probability of demand loss in any cycle. Without loss of generality, consider the first cycle $[1, K^2]$. Suppose $\mathbf{X}^{K,U}[0] = \mathbf{X}_0$. By Assumption 2, $\exists j' \in V_D$, $k \notin \partial(j') \subset V_S$ such that $\phi_{j'k} > 0$. Consider the random walk S_t with the following dynamics, which is the “virtual” net change of supply in $\partial(j')$:

²⁶Since $S_r - \mathbf{1}_{\partial(J)}^T \mathbf{X}(0)$ is a lazy version of a simple random walk, which takes a step with probability $2\mu_J$ independently at each time, the expectation of the time τ to hit $\pm K$ is inflated by a factor of $1/(2\mu_J)$ relative to that of a simple random walk (this follows from using the natural coupling between the steps in the two walks, and noting that the lazy walk takes expected time $1/(2\mu_J)$ between consecutive steps).

- $S_0 = 0$.
- $S_{t+1} = S_t + 1$ if $d[t] \in \partial(j')$ and policy \tilde{U} assigns a supply unit from outside of $\partial(j')$ to serve it (regardless of whether there is available supply to assign).
- $S_{t+1} = S_t - 1$ if $d[t] \notin \partial(j')$ and policy \tilde{U} assigns a supply unit from $\partial(j')$ to serve it (regardless of whether there is available supply to assign).
- $S_{t+1} = S_t$ if otherwise.

Using similar argument as in eq. (64), we have

$$\begin{aligned} & \mathbb{P}\left(\text{some demand is lost in epoch } [1, K^2]\right) \\ & \geq \mathbb{P}\left(S_{K^2} + \mathbf{1}_{\partial(j')}^T \mathbf{X}_0 > K \text{ or } < 0\right) \geq \mathbb{P}(|S_{K^2}| > K). \end{aligned} \quad (66)$$

Note that S_{K^2} is the sum of K^2 independent random variables Z_t , where $Z_t = S_t - S_{t-1}$. Here independence holds because we ignore demand losses in the definition of the process. Here Z_t has support $\{-1, 0, 1\}$ and satisfies:

$$\mathbb{P}(Z_t = -1) \geq \delta \triangleq \phi_{j'k} > 0, \quad (67)$$

where $k \notin \partial(j)$. There are two cases:

1. If $\mathbb{E}[S_{K^2}] \leq -\frac{K^2}{2}$, then for $K \geq 8$, we have

$$\begin{aligned} & 1 - \mathbb{P}(S_{K^2} \in [-K, K]) \\ & \geq 1 - \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \geq -K + \frac{K^2}{2}\right) \\ & \geq 1 - 2 \exp\left(-\frac{K^2}{32}\right) \quad (\text{Hoeffding's inequality, } -K + K^2/2 \geq K^2/4) \\ & \geq \frac{1}{2}. \end{aligned}$$

Plugging into (66) establishes that demand is lost with likelihood at least $1/2$.

2. If $\mathbb{E}[S_{K^2}] > -\frac{K^2}{2}$, then using linearity of expectation and simple algebra we obtain that the number of t 's such that $\mathbb{E}[Z_t] \geq -\frac{3}{4}$ is at least $\frac{K^2}{7}$.

Denote the set of these t 's as \mathcal{T} . Hence

$$K^2 \geq \text{Var}(S_{K^2}) = \sum_{t=1}^{K^2} \text{Var}(Z_t) \geq \sum_{t \in \mathcal{T}} \text{Var}(Z_t) \geq \frac{K^2}{7} \cdot \delta \left(1 - \frac{3}{4}\right)^2 = \frac{\delta}{102} K^2, \quad (68)$$

using (67).

Note from (66) that to show a constant lower bound of demand-loss probability on $[1, K^2]$, it suffices to derive a uniform upper bound on $\mathbb{P}(S_{K^2} \in [-K, K])$ that is strictly smaller than 1. To this end, apply Theorem 7.4.1 in Chung (2001) (Berry-Esseen Theorem) to obtain:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \leq x \sqrt{\text{Var}[S_{K^2}]}\right) - \Phi(x) \right| \leq \frac{\sum_{t=1}^{K^2} \mathbb{E}|Z_t - \mathbb{E}Z_t|^3}{(\text{Var}[S_{K^2}])^{3/2}} \leq \frac{5000}{K\delta^{3/2}}, \quad (69)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Denote $B(x, a) \triangleq [x - a, x + a]$. Note that there are two subcases (indexed 2(i) and

2(ii)):

$$[-K, K] \subset B(\mathbb{E}[S_{K^2}], 4K), \quad [-K, K] \cap B(\mathbb{E}[S_{K^2}], 2K) = \emptyset.$$

In subcase 2(i),

$$\mathbb{P}(S_{K^2} \in [-K, K]) \leq \mathbb{P}(S_{K^2} \in B(\mathbb{E}[S_{K^2}], 4K)),$$

whereas in subcase 2(ii),

$$\mathbb{P}(S_{K^2} \in [-K, K]) \leq 1 - \mathbb{P}(S_{K^2} \in B(\mathbb{E}[S_{K^2}], 2K)).$$

Hence

$$\mathbb{P}(S_{K^2} \in [-K, K]) \leq \max \left\{ \mathbb{P}(S_{K^2} \in B(\mathbb{E}[S_{K^2}], 4K)), 1 - \mathbb{P}(S_{K^2} \in B(\mathbb{E}[S_{K^2}], 2K)) \right\}. \quad (70)$$

Use (69) and $\text{Var}(S_{K^2}) \leq K^2$ to obtain

$$\begin{aligned} \mathbb{P}(S_{K^2} \in B(\mathbb{E}[S_{K^2}], 4K)) &\leq \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \leq \sqrt{\text{Var}[S_{K^2}]} \frac{4K}{\sqrt{\text{Var}[S_{K^2}]}}\right) \\ &\leq 5000\delta^{-3/2}K^{-1} + \Phi\left(\frac{4K}{\sqrt{\text{Var}[S_{K^2}]}}\right) \\ &\leq 5000\delta^{-3/2}K^{-1} + \Phi(50\delta^{-1/2}), \\ 1 - \mathbb{P}(S_{K^2} \in B(\mathbb{E}[S_{K^2}], 2K)) &= \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \leq \sqrt{\text{Var}[S_{K^2}]} \frac{-2K}{\sqrt{\text{Var}[S_{K^2}]}}\right) \\ &\quad + \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \geq \sqrt{\text{Var}[S_{K^2}]} \frac{2K}{\sqrt{\text{Var}[S_{K^2}]}}\right) \\ &\leq 10000\delta^{-3/2}K^{-1} + 2\Phi\left(\frac{-2K}{\sqrt{\text{Var}[S_{K^2}]}}\right) \\ &\leq 10000\delta^{-3/2}K^{-1} + 2\Phi(-2). \end{aligned}$$

Hence for $K > \max \left\{ \frac{10000\delta^{-3/2}}{\Phi(50\delta^{-1/2})}, \frac{10000\delta^{-3/2}}{\frac{1}{2} - \Phi(-2)} \right\}$, plugging into (70) and then into (66), we obtain

$$\mathbb{P}(\text{some demand is lost in } [1, K^2]) \geq \min \left\{ \frac{1}{2}\bar{\Phi}(50\delta^{-1/2}), \frac{1}{2} - \Phi(-2) \right\} > 0.$$

Since we obtained a uniform lower bound on the likelihood of dropping demand in both cases, we conclude that the steady state demand-loss probability is $\Omega(1/K^2)$ as $K \rightarrow \infty$.

- Proof of second part. Consider any $k \in V_S$ such that $\exists j' \in V_D$ such that $(j', k) \in S$. Given a demand type distribution $\phi \in D(S)$, suppose U achieves asymptotic optimality $\mathbb{P}_o^{K,U} = o(1)$, i.e., $1 - o(1)$ fraction of demand is served. This implies that a fraction $\sum_{j' \in V_D: (j', k) \in S} \phi_{j'k} - o(1)$ of demand has destination k and is served under U . And that a fraction $\sum_{(j', i) \in S} \phi_{j'i} u_{j'k} - o(1)$ of demand is assigned a supply unit from k and is served under U . (Our proof will focus on the case where $u_{j'k}$ is time invariant and independent of K . The proof for the general case of time varying $u_{j'k}(t)$ which can depend on K is

very similar, though the latter fraction can now vary over time, increasing the notational burden. We omit the details.) But in steady state, the inflow of supply units to node k must be equal to the outflow of supply units, i.e., it must be that

$$\sum_{j' \in V_D: (j', k) \in S} \phi_{j'k} = \sum_{(j', i) \in S} \phi_{j'i} u_{j'k}.$$

This is a knife edge requirement. In particular, the set of $\phi \in D(S)$ which *do not* satisfy this condition is clearly an open and dense subset of $D(S)$. For all such ϕ , the above argument implies that $\liminf_{K \rightarrow \infty} \mathbb{P}_o^{K,U} > 0$, completing the proof. \square

F.4 Proof of Example 4

We will prove by contradiction that the naive policy incurs an $\Omega(1)$ loss. Suppose the loss is vanishing $\mathbb{P}_o^K = o(1)$, i.e., all but a $o(1)$ fraction of demands are served. Consider the subset of supply nodes $\{3, 4\}$ (demand type (4'1) is entirely dependent on this subset). We will show that supply units arrive at these nodes slower than they are assigned from these nodes, which cannot possibly be the case in steady state: The fraction of demands which lead to a supply unit arriving to $\{3, 4\}$ is at most $\sum_{j' \in V_D} \sum_{k \in \{3, 4\}} \phi_{j'k} = \phi_{1'3} + \phi_{1'4} = 0.42$. All demands of type (4'1) which are served are assigned a supply unit from $\{3, 4\}$. Since all but $o(1)$ fraction of demands of type (4'1) are served:

- (i) There is a supply unit present in at least one of $\{3, 4\}$ a $1 - o(1)$ fraction of the time.
- (ii) A fraction of demands $0.4 - o(1)$ are of type (4'1) *and* are assigned a supply unit from $\{3, 4\}$.

Now consider demands of type (3'2): When such a demand arrives, using point (i) above, with probability $1 - o(1)$ there is a supply unit present in at least one of $\{3, 4\}$. The other compatible supply (with the origin 3') is 2. In all cases where there is a supply unit present in at least one of $\{3, 4\}$, the naive policy assigns a supply unit from one of $\{3, 4\}$ with probability at least $1/2$, by definition of the policy. It follows that a fraction $1/2 - o(1)$ of demands of type (3'2) are assigned a supply unit from one of $\{3, 4\}$, and hence a fraction $0.1 \times 1/2 - o(1) = 0.05 - o(1)$ of demands are of type (3'2) *and* are assigned a supply unit from one of $\{3, 4\}$. In total (adding across the demand types (4'1) and (3'2)), a supply unit from one of $\{3, 4\}$ is assigned to serve at least a fraction $0.45 - o(1)$ of all demand. But this (minimum possible) “outflow rate” exceeds the maximum possible “inflow rate” of 0.42 established above, which is impossible in steady state. Thus we have obtained a contradiction. We infer that the naive policy incurs an $\Omega(1)$ loss in this network. We further observe that both the (minimum possible) outflow rate and the maximum possible inflow rate are continuous in ϕ , hence the above argument goes through for any demand type distribution which is sufficiently close to ϕ given by (18).

G Extension to Scrip Systems: Proof of Theorem 3

The proof of Theorem 3 is almost identical to the proof of Theorem 1. To avoid redundancy, we skip the parts of the proof which are mere repetitions of their counterparts in the proof of

Theorem 1.

Proof of Theorem 3. Recall that the converse result in Theorem 1 follows from Lemmas 2 and 5, the achievability result follows from Lemmas 3, 4, 5 and Proposition 5.

Here we can prove a result identical to Lemma 2 except that $v_{\alpha}(\mathbf{f})$ is now defined as

$$v_{\alpha}(\mathbf{f}) \triangleq \min_{\Delta \mathbf{x} \in \mathcal{X}'_{\mathbf{f}}} L_{\alpha}(\alpha + \Delta \mathbf{x}),$$

where

$$\mathcal{X}'_{\mathbf{f}} \triangleq \left\{ \Delta \mathbf{x} \left| \begin{array}{ll} \Delta x_i = \sum_{j' \in \partial(i)} d_{ij'} \left(\sum_{k \in V_S} f_{kj'} \right) - \sum_{j' \in V_D} f_{ij'}, & \forall i \in V_S \\ \sum_{i \in \partial(j')} d_{ij'} = 1, \ d_{ij'} \geq 0, & \forall i \in V_S, j' \in V_D \end{array} \right. \right\}.$$

Here $(d_{ij'})_{i \in \partial(j')}$ is the chosen *service provider distribution* over agents neighboring j' for assigning agents to serve demand of service j' . Lemmas 3, 4, 5 are replaced by Lemmas 9, 10, 11 below, respectively. Proposition 5 continues to hold. This concludes the proof. \square

Lemma 9 (SMS(α) causes steepest descent). *Let $(\bar{\mathbf{A}}, \bar{\mathbf{X}}^U)$ be any FSP under any non-idling policy U on $[0, T]$, and consider any $\alpha \in \text{relint}(\Omega)$. For a regular $t \in [0, T]$, define:*

$$\begin{aligned} S_1(\bar{\mathbf{X}}^U(t)) &\triangleq \left\{ k \in V_S : k \in \text{argmin} \frac{\bar{X}_k^U(t)}{\alpha_k} \right\}, \\ S_2(\bar{\mathbf{X}}^U(t), \dot{\bar{\mathbf{X}}}^U(t)) &\triangleq \left\{ k \in S_1(\bar{\mathbf{X}}^U(t)) : k \in \text{argmin} \frac{\dot{\bar{X}}_k^U(t)}{\alpha_k} \right\}. \end{aligned}$$

All the derivatives are well defined since t is regular. We have

$$\dot{L}_{\alpha}(\bar{\mathbf{X}}^U(t)) = -\frac{\dot{\bar{\mathbf{X}}}_k^U(t)}{\alpha_k} \quad \text{for any } k \in S_2(\bar{\mathbf{X}}^U(t)) \quad (71)$$

$$\geq -\frac{1}{\mathbf{1}_{S_2}^T \alpha} \left(\sum_{i \in V_S, j' \in \partial(S_2)} \dot{A}_{ij'}(t) - \sum_{i \in S_2, j' \in V_D} \dot{A}_{ij'}(t) \right) \quad (72)$$

for $\bar{\mathbf{X}}^U(t) \neq \alpha$ and $L_{\alpha}(\bar{\mathbf{X}}^U(t)) < 1$. Inequality (72) holds with equality under SMS(α), i.e., SMS(α) satisfies the steepest descent property in Proposition 5.

Proof. We will write $S_1(\bar{\mathbf{X}}(t))$ as S_1 , $S_2(\bar{\mathbf{X}}(t), \dot{\bar{\mathbf{X}}}(t))$ as S_2 , and $\min_{k \in S_1} \frac{\dot{\bar{X}}_k(t)}{\alpha_k}$ as c in the following. Let $(\bar{\mathbf{A}}, \bar{\mathbf{X}}^U)$ be an FSP under policy $U \in \mathcal{U}$.

- *Proof of (71).* The proof is exactly the same as the proof of (23).
- *Proof of (72).* For the K -th system, define auxiliary processes:

$$\begin{aligned} \bar{E}_{ij'k}^{K,U}(t) &\triangleq \# \{ \text{Type } (i, j') \text{ demand units that arrive during } [0, t] \\ &\quad \text{and are served by agents at } k \text{ under policy } U \in \mathcal{U} \} \quad i, k \in V_S, j' \in V_D. \end{aligned}$$

Similar to the proof of (24), extend the definition of FSP to $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(\cdot), \bar{\mathbf{E}}(\cdot))$. For regular times t , we have

$$\sum_{i \in S_2} \dot{X}_i^U(t) = \sum_{k \in V_S, i \in S_2} \sum_{j' \in \partial(i)} \dot{E}_{kj'i}^U(t) - \sum_{i \in S_2, j' \in V_D} \sum_{k \in \partial(j')} \dot{E}_{ij'k}^U(t).$$

Consider any non-idling policy $U' \in \mathcal{U}$, it cannot use the agents in S_2 to serve the demand of

service types out of $\partial(S_2)$. Therefore for any policy U' we have

$$\sum_{k \in S_2} \dot{X}_k^{U'}(t) \leq \sum_{i \in V_S, j' \in \partial(S_2)} \dot{A}_{ij'}(t) - \sum_{i \in S_2, j' \in V_D} \dot{A}_{ij'}(t). \quad (73)$$

For $\text{SMS}(\alpha)$ policy, using similar argument as in the proof of (24), we know that all the demands for service type $j' \in \partial(S_2)$ will be served by agents $i \in S_2$ during $(t, t + \epsilon)$ for some $\epsilon > 0$. Hence we have

$$\begin{aligned} \sum_{k \in S_2} \dot{X}_k^{\text{SMS}(\alpha)}(t) &= \sum_{i \in V_S, j' \in \partial(S_2)} \dot{A}_{ij'}(t) - \sum_{i \in S_2, j' \in V_D} \sum_{k \in \partial(j')} \dot{E}_{ij'k}^{\text{SMS}(\alpha)}(t) \\ &\geq \sum_{i \in V_S, j' \in \partial(S_2)} \dot{A}_{ij'}(t) - \sum_{i \in S_2, j' \in V_D} \dot{A}_{ij'}(t). \end{aligned}$$

Finally, observe that for any $k \in S_2$,

$$\dot{L}_\alpha(\bar{\mathbf{X}}(t)) = -\frac{\dot{X}_k^{U'}(t)}{\alpha_k} = -\frac{1}{\mathbf{1}_{S_2}^\top \alpha} \sum_{k \in S_2} \alpha_k \frac{\dot{X}_k^{U'}(t)}{\alpha_k} = -\frac{1}{\mathbf{1}_{S_2}^\top \alpha} \sum_{k \in S_2} \dot{X}_k^{U'}(t). \quad (74)$$

Plug (73) into (74), we know that inequality (71) holds, and it becomes equality for $\text{SMS}(\alpha)$ policy. \square

Lemma 10 ($\text{SMS}(\alpha)$ satisfies negative drift). *For any $\alpha \in \text{relint}(\Omega)$, under Assumption 5, the policy $\text{SMS}(\alpha)$ satisfies the negative drift condition in Proposition 5.*

Proof. It follows from Lemma 9 that for any fluid limit under $\text{SMS}(\alpha)$ ($\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(t)$) and regular t , we have

$$\dot{L}_\alpha(t) \leq -\min_{S_2 \subsetneq V_S, S_2 \neq \emptyset} \left(\sum_{i \in V_S, j' \in \partial(S_2)} \phi_{ij'} - \sum_{i \in S_2, j' \in V_D} \phi_{ij'} \right).$$

Because of Assumption 5, we have $\dot{L}_\alpha(t) < 0$, and the rest of the proof proceeds exactly the same as the proof of Lemma 4. \square

Lemma 11. *Recall the definitions of B_J, λ_J and μ_J in (27). For any $\alpha \in \text{relint}(\Omega)$, we have $\gamma(\alpha) = \min_{I \subsetneq V_S, I \neq \emptyset} B_I \log \left(\frac{\lambda_I}{\mu_I} \right)$.*

Proof. We omit the proof because it is almost identical to the proof of Lemma 5. \square

H SMW with Travel Delays: Proof of Theorem 2

This section provides a proof of Theorem 2, our guarantee of exponentially small loss under SMW in the presence of travel delays (Section 6.1).

H.1 Fluid Sample Paths, Fluid Limits, and Large Deviations Principle

Similar to the development in Section 5.1, we first define the fluid sample paths and fluid limits of the system with delay. Consider the K -th system under $\text{SMW}(\alpha)$ policy. We make the following definitions:

- For $j' \in V_D, k \in V_S$, let $\mathbf{A}_{j'k}^K(\cdot)$ be an independent Poisson process with rate $\hat{\phi}_{j'k}^K = K\hat{\phi}_{j'k}$.
- For $\alpha \in \text{relint}(\Omega)$ and $i \in V_S$, we denote by $X_i^{K,\alpha}(t)$ the number of available supply units at node i at time t .
- For $j' \in V_D, k \in V_S$, we denote by $Y_{j'k}^{K,\alpha}(t)$ the number of supply units transporting type (j', k) demands at time t .
- For $j' \in V_D, k \in V_S$, we denote by $R_{j'k}^{K,\alpha}(t)$ be the cumulative number of supply units that arrive at node k carrying type (j', k) demand during time $[0, t]$.

Define the scaled version of the above sample paths as follows:

$$\begin{aligned} \bar{A}_{j'k}^K(t) &\triangleq \frac{1}{K} A_{j'k}^K(t), & \bar{X}_i^{K,\alpha}(t) &\triangleq \frac{1}{K} \bar{X}_i^{K,\alpha}(t), \\ \bar{Y}_{j'k}^{K,\alpha}(t) &\triangleq \frac{1}{K} Y_{j'k}^{K,\alpha}(t), & \bar{R}_{j'k}^{K,\alpha}(t) &\triangleq \frac{1}{K} \bar{R}_{j'k}^{K,\alpha}(t). \end{aligned} \quad (75)$$

We define fluid sample paths and fluid limits as follows.

Definition 9 (Fluid sample paths). *We call $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^\alpha(\cdot), \bar{\mathbf{Y}}^\alpha(\cdot), \bar{\mathbf{R}}^\alpha(\cdot))_T$ a fluid sample path (under $\text{SMW}(\alpha)$) on $[0, T]$ if there exists a sequence of sample paths $(\bar{\mathbf{A}}^K(\cdot), \bar{\mathbf{X}}^{K,\alpha}(\cdot), \bar{\mathbf{Y}}^{K,\alpha}(\cdot), \bar{\mathbf{R}}^{K,\alpha}(\cdot))_{K=1}^\infty$ (which are defined in (75) and (76)), such that it has a subsequence which converges to $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^\alpha(\cdot), \bar{\mathbf{Y}}^\alpha(\cdot), \bar{\mathbf{R}}^\alpha(\cdot))$ uniformly on $[0, T]$.*

Definition 10 (Fluid limits). *We call $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^\alpha(\cdot), \bar{\mathbf{Y}}^\alpha(\cdot), \bar{\mathbf{R}}^\alpha(\cdot))_T$ a fluid limit (under $\text{SMW}(\alpha)$) on $[0, T]$ if (i) it is a fluid sample path; (ii) we have $\bar{A}_{j'k}(t) = \hat{\phi}_{j'k} t$ and $\bar{R}_{j'k}(t) = \frac{1}{\tau_{j'k}} \int_{s=0}^t \bar{Y}_{j'k}^\alpha(s) ds$, for all $j' \in V_D, k \in V_S$ and all $t \in [0, T]$.*

Large deviations principle for $M/M/\infty$ queue. Because the system with travel delay consists of $M/M/\infty$ queues, the following result (Theorem 12.18, Shwartz and Weiss 1995) is useful.

Let $Y^K(\cdot)$ be the sample path of the content of an $M/M/\infty$ queue with job arrival rate $K\hat{\phi}$ and service rate τ^{-1} ; $A^K(t)$ be the number of job arrivals to the queue during $[0, t]$; $R^K(t)$ be the number of served jobs during $[0, t]$. Let

$$\bar{Y}^K(t) \triangleq \frac{1}{K} Y^K(t), \quad \bar{A}^K(t) \triangleq \frac{1}{K} A^K(t), \quad \bar{R}^K(t) \triangleq \frac{1}{K} R^K(t).$$

Let μ_K be the law of $(\bar{Y}^K(\cdot), \bar{A}^K(\cdot), \bar{R}^K(\cdot))$ in $(L^\infty[0, T])^3$. Let $\Lambda^*(\ell, \cdot)$ be the large deviation rate function of Poisson random variable with mean ℓ :

$$\Lambda^*(\ell, f) \triangleq \begin{cases} f \log \frac{f}{\ell} - f + \ell & \text{if } f > 0, \\ \infty & \text{otherwise.} \end{cases} \quad (77)$$

For any set Γ , let $\bar{\Gamma}$ be its closure, and Γ° be its interior. We have the following sample path large deviations principle.²⁷

Fact 2. *For measures $\{\mu_K\}$ defined above, and any arbitrary measurable set $\Gamma \subseteq (L^\infty[0, T])^3$, we have*

$$-\inf_{(\bar{Y}, \bar{A}, \bar{R}) \in \Gamma^\circ} I_T(\bar{Y}) \leq \liminf_{K \rightarrow \infty} \frac{1}{K} \log \mu_K(\Gamma) \leq \limsup_{K \rightarrow \infty} \frac{1}{K} \log \mu_K(\Gamma) \leq -\inf_{(\bar{Y}, \bar{A}, \bar{R}) \in \bar{\Gamma}} I_T(\bar{Y}), \quad (78)$$

²⁷The original formulation in Shwartz and Weiss (1995) is more compact than the following one, but the following formulation turns out to be more useful in our analysis.

where the rate function is:

$$I_T(\bar{Y}, \bar{A}, \bar{R}) \triangleq \begin{cases} \int_0^T \left(\Lambda^* \left(\hat{\phi}, \dot{A}(t) \right) + \Lambda^* \left(\frac{\bar{Y}(t)}{\tau}, \dot{\bar{R}}(t) \right) \right) dt & \text{if } \bar{Y}(\cdot), \bar{A}(\cdot), \bar{R}(\cdot) \in \text{AC}[0, T], \bar{Y}(0) = 0, \\ \infty & \text{otherwise.} \end{cases} \quad (79)$$

Here $\text{AC}[0, T]$ is the space of absolutely continuous functions on $[0, T]$.

H.2 Lyapunov Functions and Drift

Our analysis relies on a novel family of piecewise linear Lyapunov functions, which we construct below. Let Ω^ℓ be the $(\ell - 1)$ -dimensional simplex.

Definition 11. For each $\alpha \in \text{relint}(\Omega)$, define Lyapunov function $L_\alpha(\mathbf{x}, \mathbf{y}) : \Omega^{m+n \times m} \rightarrow \mathbb{R}$ as

$$L_\alpha(\mathbf{x}, \mathbf{y}) = L_{1,\alpha}(\mathbf{x}) + \frac{2}{\min_{i \in V_S} \alpha_i} L_2(\mathbf{y})$$

where $L_{1,\alpha}(\mathbf{x}) = \beta - \min_{i \in V_S} \frac{x_i}{\alpha_i}$, $L_2(\mathbf{y}) = \sum_{j' \in V_D, k \in V_S} |y_{j'k} - \tau_{jk} \hat{\phi}_{j'k}|$.

The intuition of such choices of Lyapunov functions is as follows. The first part of the Lyapunov function, $L_{1,\alpha}(\mathbf{x})$, is almost identical to the Lyapunov function for the no-delay case (see Definition 7) except for the constant term since only β portion of the cars are available at the system equilibrium. It captures how much the current distribution of available supply units deviates from the distribution at equilibrium. The second part of the Lyapunov function characterizes the deviation of the number of in-transit cars from their typical values. The Lyapunov function attains minimum value 0 at $\Omega_{m+n \times m}$ at $((\beta \alpha_i)_{i \in V_S}, (\tau_{j'k} \hat{\phi}_{j'k})_{j' \in V_D, k \in V_S})$, and is strictly positive elsewhere on $\Omega_{m+n \times m}$.

Same as before, the demand-loss probability can be upper bounded by the probability that the Lyapunov function exceeds a certain value. Note that demand loss only happens when $x_i = 0$ for some $i \in V_S$, which implies $L_{1,\alpha} = \beta$. In the following, we bound the probability of the event where $L_\alpha(\bar{\mathbf{X}}, \bar{\mathbf{Y}}) \geq \beta$.

Because we only need an achievability bound, it suffices to prove a result analogous to Lemma 7. As a first step, we establish in the following lemma that the Lyapunov function has negative drift under $\text{SMW}(\alpha)$ policies in the fluid limit.

A time $t \in (0, T)$ is said to be a *regular point* of an FSP $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}^\alpha(\cdot), \bar{\mathbf{Y}}^\alpha(\cdot), \bar{\mathbf{R}}^\alpha(\cdot))_T$ if $\bar{\mathbf{A}}(\cdot)$, $\bar{\mathbf{X}}^\alpha(\cdot)$, $\bar{\mathbf{Y}}^\alpha(\cdot)$, $\bar{\mathbf{R}}^\alpha(\cdot)$, $L_\alpha(\bar{\mathbf{X}}^\alpha(\cdot), \bar{\mathbf{Y}}^\alpha(\cdot))$ are all differentiable at time t .

Because of the Large Deviations Principle (Facts 1 and 2), it will suffice in our analysis to consider only the FSPs that have absolutely continuous demand sample paths $\bar{\mathbf{A}}(\cdot)$. Now, if $\bar{\mathbf{A}}(\cdot)$ is absolutely continuous, then so are $\bar{\mathbf{X}}^\alpha(\cdot)$ and $L_\alpha(\bar{\mathbf{X}}^\alpha(\cdot))$, and as a result almost all t are regular.

As a first step to bound the drift of L_α we first bound the drift of $L_{1,\alpha}$ in Lemma 12. For notation simplicity, we drop the FSP's superscript α .

Lemma 12. Let $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(\cdot), \bar{\mathbf{Y}}(\cdot), \bar{\mathbf{R}}(\cdot))_T$ be any FSP under $\text{SMW}(\alpha)$ on $[0, T]$, where $\alpha \in \text{relint}(\Omega)$. Define:

$$S_1(\bar{\mathbf{X}}(t)) \triangleq \left\{ k \in V_S : k \in \arg\min \frac{\bar{\mathbf{X}}_k(t)}{\alpha_k} \right\},$$

$$S_2(\bar{\mathbf{X}}(t), \dot{\mathbf{X}}(t)) \triangleq \left\{ k \in S_1(\bar{\mathbf{X}}(t)) : k \in \operatorname{argmin} \frac{\dot{\mathbf{X}}_k(t)}{\alpha_k} \right\}. \quad (80)$$

For a regular $t \in [0, T]$, we have

$$\dot{L}_{1,\alpha}(\bar{\mathbf{X}}(t)) \leq -\frac{1}{\mathbf{1}_{S_2}^T \alpha} \left(\sum_{j' \in V_D, k \in S_2} \dot{R}_{j'k}(t) - \sum_{j' \in V_D: \partial(j) \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right).$$

Proof. From (46) we have

$$\dot{L}_{1,\alpha}(\bar{\mathbf{X}}(t)) = -\frac{1}{\mathbf{1}_{S_2}^T \alpha} \sum_{k \in S_2} \dot{X}_k(t). \quad (81)$$

Because we are considering a closed system, it holds that:

$$\sum_{j' \in V_D, k \in V_S} \dot{Y}_{j'k}(t) + \sum_{k \in V_S} \dot{X}_k(t) = 0. \quad (82)$$

Therefore

$$\sum_{k \in S_2} \dot{X}_k(t) = - \sum_{j' \in V_D, k \in V_S} \dot{Y}_{j'k}(t) - \sum_{k \in V_S \setminus S_2} \dot{X}_k(t). \quad (83)$$

Note that

$$\dot{Y}_{j'k}(t) \leq \dot{A}_{j'k}(t) - \dot{R}_{j'k}(t), \quad (84)$$

where the equality is achieved when no type (j', k) demand is lost at time t . Using the same argument as in the proof of Lemma 3, we know that under $\text{SMW}(\alpha)$ policy all demand in $\partial(V_S \setminus S_2)$ are served by supplies in $V_S \setminus S_2$, and that no demand whose origin is in $\partial(V_S \setminus S_2)$ is lost. We have

$$\sum_{k \in V_S \setminus S_2} \dot{X}_k(t) = \sum_{j' \in V_D, k \in V_S \setminus S_2} \dot{R}_{j'k}(t) - \sum_{j' \in \partial(V_S \setminus S_2), k \in V_S} \dot{A}_{j'k}(t). \quad (85)$$

Plug in (84) and (85) to (83), we have

$$\begin{aligned} \sum_{k \in S_2} \dot{X}_k(t) &\geq \sum_{j' \in V_D, k \in V_S} \left(\dot{R}_{j'k}(t) - \dot{A}_{j'k}(t) \right) - \sum_{j' \in V_D, k \in V_S \setminus S_2} \dot{R}_{j'k}(t) + \sum_{j' \in \partial(V_S \setminus S_2), k \in V_S} \dot{A}_{j'k}(t) \\ &= \sum_{j' \in V_D, k \in S_2} \dot{R}_{j'k}(t) - \sum_{j' \in V_D: \partial(j) \subset S_2, k \in V_S} \dot{A}_{j'k}(t). \end{aligned}$$

Plugging the above to (81) and we conclude the proof. \square

Now we are ready to bound the drift of L_α .

Lemma 13. Let $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(\cdot), \bar{\mathbf{Y}}(\cdot), \bar{\mathbf{R}}(\cdot))_T$ be any FSP under $\text{SMW}(\alpha)$ on $[0, T]$, where $\alpha \in \operatorname{relint}(\Omega)$. Recall the definition of S_2 in (80).

- If for any $i \in S_2$, $\bar{X}_i(t) > 0$ or $\bar{X}_i(t) = 0$, $\dot{\bar{X}}_i(t) > 0$, we have

$$\begin{aligned} &\dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \\ &\triangleq F_1(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t), \dot{\bar{\mathbf{A}}}(t), \dot{\bar{\mathbf{R}}}(t)) \\ &\leq -\frac{1}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| - \frac{1}{\mathbf{1}_{S_2}^T \alpha} \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) \end{aligned}$$

$$+ \frac{3}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right).$$

- If for $i \in S_2$, $\bar{X}_i(t) = 0$ and $\dot{\bar{X}}_i(t) = 0$, we have

$$\begin{aligned} & \dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \\ & \triangleq F_2(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t), \dot{\bar{\mathbf{A}}}(t), \dot{\bar{\mathbf{R}}}(t)) \\ & \leq -\frac{2}{\alpha_{\min}} \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) - \frac{2}{\alpha_{\min}} \sum_{j' \in V_D, k \in S_2} \left[\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right]^- \\ & \quad - \frac{2}{\alpha_{\min}} \sum_{j' \in V_D, k \notin S_2} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| + \frac{4}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right). \end{aligned}$$

Proof. Recall the definition of S_2 in (80). To analyze the Lyapunov drift of L_{α} , we consider two cases depending on, roughly speaking, whether the queues in S_2 are empty at t and shortly after t .

- **Case 1:** for any $i \in S_2$, $\bar{X}_i(t) > 0$ or $\bar{X}_i(t) = 0$, $\dot{\bar{X}}_i(t) > 0$. Let $\alpha_{\min} \triangleq \min_{i \in V_S} \alpha_i$. We have

$$\begin{aligned} & \dot{L}_{\alpha}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \\ & \leq -\frac{1}{\mathbf{1}_{S_2}^T \boldsymbol{\alpha}} \left(\sum_{j' \in V_D, k \in S_2} \dot{R}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) \end{aligned} \quad (86)$$

$$- \frac{2}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\dot{A}_{j'k}(t) - \dot{R}_{j'k}(t) \right) \left(\mathbb{I} \left\{ \bar{Y}_{j'k}(t) \leq \hat{\phi}_{j'k} \tau_{j'k} \right\} - \mathbb{I} \left\{ \bar{Y}_{j'k}(t) > \hat{\phi}_{j'k} \tau_{j'k} \right\} \right) \quad (87)$$

$$\triangleq F_1(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t), \dot{\bar{\mathbf{A}}}(t), \dot{\bar{\mathbf{R}}}(t)).$$

Here the term (86) comes from Lemma 12. Note that

$$\begin{aligned} & \sum_{j' \in V_D, k \in S_2} \dot{R}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \\ & = \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) - \sum_{j' \in V_D, k \in S_2} \left(\dot{A}_{j'k}(t) - \dot{R}_{j'k}(t) \right) \end{aligned}$$

Simple algebra yields that: for $j' \in V_D$, $k \in V_S$.

$$\dot{A}_{j'k}(t) - \dot{R}_{j'k}(t) \leq \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} + \left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right|.$$

Combined, we have

$$\begin{aligned} & \sum_{j' \in V_D, k \in S_2} \dot{R}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \\ & \geq \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) - \sum_{j' \in V_D, k \in S_2} \left(\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right) \\ & \quad - \sum_{j' \in V_D, k \in S_2} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right). \end{aligned}$$

Now we focus on the term (87). For $j' \in V_D$, $k \in V_S$, we have

$$\begin{aligned}
& \left(\dot{\bar{A}}_{j'k}(t) - \dot{\bar{R}}_{j'k}(t) \right) \left(\mathbb{I} \left\{ \bar{Y}_{j'k}(t) \leq \hat{\phi}_{j'k} \tau_{j'k} \right\} - \mathbb{I} \left\{ \bar{Y}_{j'k}(t) > \hat{\phi}_{j'k} \tau_{j'k} \right\} \right) \\
&= \left(\dot{\bar{A}}_{j'k}(t) - \dot{\bar{R}}_{j'k}(t) \right) \left(\mathbb{I} \left\{ \hat{\phi}_{j'k} \geq \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} - \mathbb{I} \left\{ \hat{\phi}_{j'k} < \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} \right) \\
&= \left(\left(\dot{\bar{A}}_{j'k}(t) - \hat{\phi}_{j'k} \right) - \left(\dot{\bar{R}}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right) \right) \left(\mathbb{I} \left\{ \hat{\phi}_{j'k} \geq \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} - \mathbb{I} \left\{ \hat{\phi}_{j'k} < \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} \right) \\
&\quad + \left(\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right) \left(\mathbb{I} \left\{ \hat{\phi}_{j'k} \geq \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} - \mathbb{I} \left\{ \hat{\phi}_{j'k} < \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} \right).
\end{aligned}$$

Note that

$$\left(\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right) \left(\mathbb{I} \left\{ \hat{\phi}_{j'k} \geq \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} - \mathbb{I} \left\{ \hat{\phi}_{j'k} < \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} \right) = \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right|,$$

and that

$$\begin{aligned}
& \left(\left(\dot{\bar{A}}_{j'k}(t) - \hat{\phi}_{j'k} \right) - \left(\dot{\bar{R}}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right) \right) \left(\mathbb{I} \left\{ \hat{\phi}_{j'k} \geq \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} - \mathbb{I} \left\{ \hat{\phi}_{j'k} < \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} \right) \\
&\geq - \left| \dot{\bar{A}}_{j'k}(t) - \hat{\phi}_{j'k} \right| - \left| \dot{\bar{R}}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right|.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& \left(\dot{\bar{A}}_{j'k}(t) - \dot{\bar{R}}_{j'k}(t) \right) \left(\mathbb{I} \left\{ \bar{Y}_{j'k}(t) \leq \hat{\phi}_{j'k} \tau_{j'k} \right\} - \mathbb{I} \left\{ \bar{Y}_{j'k}(t) > \hat{\phi}_{j'k} \tau_{j'k} \right\} \right) \\
&\geq \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| - \left(\left| \dot{\bar{A}}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{\bar{R}}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right).
\end{aligned}$$

Plugging into (86) and (87), we have

$$\begin{aligned}
& F_1(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t), \dot{\bar{\mathbf{A}}}(t), \dot{\bar{\mathbf{R}}}(t)) \\
&\leq -\frac{1}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| - \frac{1}{\mathbf{1}_{S_2}^T \boldsymbol{\alpha}} \left(\sum_{j' \in V_D, k \in S_2} \dot{\bar{A}}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{\bar{A}}_{j'k}(t) \right) \\
&\quad + \frac{3}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{\bar{A}}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{\bar{R}}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right).
\end{aligned}$$

- **Case 2:** for $i \in S_2$, $\bar{X}_i(t) = 0$ and $\dot{\bar{X}}_i(t) = 0$. In this case, $\dot{\bar{L}}_{1,\alpha}(\bar{\mathbf{X}}(t)) = 0$. Similar to the proof of Lemma 3, for $i, k \in V_S$, $j' \in V_D$, let $\bar{E}_{ij'k}(t)$ be the FSP of the number of type (j', k) demand served by supply units at i during $[0, t]$. Define

$$U_{j'k}(t) \triangleq A_{j'k}(t) - \sum_{i \in \partial(j')} \bar{E}_{ij'k}(t)$$

as the number of type (j', k) demand lost during $[0, t]$.

We have

$$\dot{\bar{L}}_2(\bar{\mathbf{Y}}(t))$$

$$\begin{aligned}
&= - \sum_{j' \in V_D, k \in V_S} \left(\dot{A}_{j'k}(t) - \dot{U}_{j'k}(t) - \dot{R}_{j'k}(t) \right) \mathbb{I} \left\{ \bar{Y}_{j'k}(t) \leq \hat{\phi}_{j'k} \tau_{j'k} \right\} \\
&\quad + \sum_{j' \in V_D, k \in V_S} \left(\dot{A}_{j'k}(t) - \dot{U}_{j'k}(t) - \dot{R}_{j'k}(t) \right) \mathbb{I} \left\{ \bar{Y}_{j'k}(t) > \hat{\phi}_{j'k} \tau_{j'k} \right\} \\
&\leq - \sum_{j' \in V_D, k \in V_S} \left(\dot{A}_{j'k}(t) - \dot{R}_{j'k}(t) \right) \left(\mathbb{I} \left\{ \hat{\phi}_{j'k} \geq \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} - \mathbb{I} \left\{ \hat{\phi}_{j'k} < \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} \right) \\
&\quad + \sum_{j' \in V_D, k \in V_S} \dot{U}_{j'k}(t).
\end{aligned}$$

Note that by definition of the set S_2 , no queue in $\partial(V_S \setminus S_2)$ loses demand at time t . We have

$$\sum_{j' \in V_D, k \in V_S} \dot{U}_{j'k}(t) = \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) - \sum_{j' \in V_D, k \in S_2} \dot{R}_{j'k}(t).$$

Combining, and using the same algebra as in Case 1, we have:

$$\begin{aligned}
&\dot{L}_2(\bar{\mathbf{Y}}(t)) \\
&= - \sum_{j' \in V_D, k \in V_S} \left(\dot{A}_{j'k}(t) - \dot{R}_{j'k}(t) \right) \left(\mathbb{I} \left\{ \hat{\phi}_{j'k} \geq \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} - \mathbb{I} \left\{ \hat{\phi}_{j'k} < \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right\} \right) \\
&\quad + \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) - \sum_{j' \in V_D, k \in S_2} \dot{R}_{j'k}(t) \\
&\leq - \sum_{j' \in V_D, k \in V_S} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| + \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right) \\
&\quad - \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) + \sum_{j' \in V_D, k \in S_2} \left(\dot{A}_{j'k}(t) - \dot{R}_{j'k}(t) \right) \\
&\leq - \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) - \sum_{j' \in V_D, k \in S_2} \left[\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right]^- \\
&\quad - \sum_{j' \in V_D, k \notin S_2} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| + 2 \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right).
\end{aligned}$$

Here $[x]^- \triangleq -\min\{x, 0\}$.

Therefore we have

$$\begin{aligned}
&\dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \\
&\triangleq F_2(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t), \dot{\mathbf{A}}(t), \dot{\mathbf{R}}(t)) \\
&\leq - \frac{2}{\alpha_{\min}} \left(\sum_{j' \in V_D, k \in S_2} \dot{A}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \dot{A}_{j'k}(t) \right) - \frac{2}{\alpha_{\min}} \sum_{j' \in V_D, k \in S_2} \left[\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right]^- \\
&\quad - \frac{2}{\alpha_{\min}} \sum_{j' \in V_D, k \notin S_2} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| + \frac{4}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right).
\end{aligned}$$

□

Using the result in Lemma 13, we can show that the system has strictly negative Lyapunov drift in the fluid limit, and that the drift remains negative for perturbed demand arrival rates and travel times given the perturbation is small enough.

Lemma 14. Fix $\alpha \in \text{relint}(\Omega)$. Then there exists $\eta > 0$ and $\epsilon > 0$ such that for all FSPs $(\bar{\mathbf{A}}(\cdot), \bar{\mathbf{X}}(\cdot), \bar{\mathbf{Y}}(\cdot), \bar{\mathbf{R}}(\cdot))_T$ (under the $\text{SMW}(\alpha)$ policy), if $t \in (0, T)$ is regular, $L_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) > \frac{\beta}{2}$, and that $\dot{\bar{\mathbf{A}}}(t) \in B(\hat{\phi}, \epsilon)$, $\max_{j' \in V_D, k \in V_S} |\dot{R}_{j'k} - \bar{Y}_{j'k}(t)/\tau_{j'k}| \leq \epsilon$, we have $\dot{L}_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\eta$.

Proof. Same as in the proof of Lemma 13, we consider two cases. Recall the definition of S_2 in (80).

- If for any $i \in S_2$, $\bar{X}_i(t) > 0$ or $\bar{X}_i(t) = 0$, $\dot{\bar{X}}_i(t) > 0$, we have

$$\begin{aligned} & \dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \\ & \triangleq F_1(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t), \dot{\bar{\mathbf{A}}}(t), \dot{\bar{\mathbf{R}}}(t)) \\ & \leq \underbrace{-\frac{1}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right|}_{\text{(I)}} - \underbrace{\frac{1}{\mathbf{1}_{S_2}^T \alpha} \left(\sum_{j' \in V_D, k \in S_2} \hat{\phi}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \hat{\phi}_{j'k}(t) \right)}_{\text{(II)}}. \end{aligned}$$

Depending on whether $S_2 = V_S$, there are two sub-cases:

- When $S_2 \neq V_S$, it follows from Assumption 3 that (II) > 0 . Since (I) ≥ 0 , we have

$$\dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -(\text{II}) \leq -\min\{\lambda_{\min}, \xi\}.$$

Here $\lambda_{\min} \triangleq \min_{i \in V_S} \mathbf{1}^T \hat{\phi}_{(i)} > 0$ is the minimum supply arrival rate at any node (that has positive arrival rate), and $\xi \triangleq \min_{J \subsetneq V_D, J \neq \emptyset} \left(\sum_{i \in \partial(J)} \mathbf{1}^T \hat{\phi}_{(i)} - \sum_{j' \in J} \mathbf{1}^T \hat{\phi}_{j'} \right) > 0$ is the Hall's gap of the system.

- When $S_2 = V_S$, observe that (II) = 0, hence we only analyze (I). Recall that we focus on the case where $L_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) > \frac{\beta}{2}$. Denote $\kappa \triangleq \sum_{i \in V_S} \bar{X}_i(t)$. We have

$$\sum_{j' \in V_D, k \in V_S} \left(\bar{Y}_{j'k}(t) - \hat{\phi}_{j'k} \tau_{j'k} \right) = \beta - \kappa,$$

hence

$$\sum_{j' \in V_D, k \in V_S} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \geq \frac{|\beta - \kappa|}{\tau_{\max}}. \quad (88)$$

Here $\tau_{\max} \triangleq \max_{j' \in V_D, k \in V_S} \tau_{j'k}$. Plug in to the expression of F_1 , we have

$$\dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\frac{1}{\alpha_{\min}} \frac{|\beta - \kappa|}{\tau_{\max}}.$$

On the other hand, since $S_2 = V_S$, it must be that $\bar{X}_i(t) = \alpha_i \kappa$ for all $i \in V_S$, hence

$$L_{1,\alpha}(\bar{\mathbf{X}}(t)) = \beta - \kappa.$$

Since $L_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) > \frac{\beta}{2}$, we have

$$L_2(\bar{\mathbf{Y}}(t)) > \left(\kappa - \frac{\beta}{2} \right) \frac{\alpha_{\min}}{2}.$$

When $\kappa < \frac{3}{4}\beta$, plugging into (88), we have

$$\dot{L}_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\frac{\beta}{4\tau_{\max}}.$$

When $\kappa \geq \frac{3}{4}\beta$, we have

$$L_2(\bar{\mathbf{Y}}(t)) = \sum_{j' \in V_D, k \in V_S} |\bar{Y}_{j'k}(t) - \tau_{j'k} \hat{\phi}_{j'k}| \geq \frac{\alpha_{\min} \beta}{8}.$$

hence

$$\sum_{j' \in V_D, k \in V_S} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \geq \frac{\alpha_{\min} \beta}{8\tau_{\max}},$$

therefore

$$\dot{L}_{\alpha}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\frac{\alpha_{\min} \beta}{8\tau_{\max}}.$$

Combine all the above analysis, we have

$$\dot{L}_{\alpha}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\min \left\{ \lambda_{\min}, \xi, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\}.$$

- If for $i \in S_2$, $\bar{X}_i(t) = 0$ and $\dot{\bar{X}}_i(t) = 0$, we have

$$\begin{aligned} & \dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \\ & \triangleq F_2(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t), \dot{\bar{\mathbf{A}}}(t), \dot{\bar{\mathbf{R}}}(t)) \\ & \leq -\frac{2}{\alpha_{\min}} \left(\sum_{j' \in V_D, k \in S_2} \hat{\phi}_{j'k}(t) - \sum_{j': \partial(j') \subset S_2, k \in V_S} \hat{\phi}_{j'k}(t) \right) - \frac{2}{\alpha_{\min}} \sum_{j' \in V_D, k \in S_2} \left[\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right]^- \\ & \quad - \frac{2}{\alpha_{\min}} \sum_{j' \in V_D, k \notin S_2} \left| \hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right|. \end{aligned}$$

- When $S_2 \neq V_S$, we have

$$\dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\min\{\lambda_{\min}, \xi\}.$$

- When $S_2 = V_S$. Since all the cars are in-transit, we have

$$\sum_{j' \in V_D, k \in V_S} \hat{\phi}_{j'k} \tau_{j'k} - \sum_{j' \in V_D, k \in V_S} \bar{Y}_{j'k}(t) = -\beta$$

Hence

$$-\sum_{j' \in V_D, k \in V_S} \left[\hat{\phi}_{j'k} \tau_{j'k} - \bar{Y}_{j'k}(t) \right]^- \leq -\beta,$$

and

$$-\sum_{j' \in V_D, k \in V_S} \left[\hat{\phi}_{j'k} - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right]^- \leq -\frac{\beta}{\tau_{\max}},$$

Therefore

$$\dot{L}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\frac{\beta}{\tau_{\max}}.$$

Combine all the cases above, we have for any fluid limit, when $L_{\alpha}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) > \frac{\beta}{2}$, we have

$$\dot{L}_{\alpha}(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq -\min \left\{ \lambda_{\min}, \xi, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\}.$$

Repeat the analysis above for FSP, we have

$$\begin{aligned} \dot{L}_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq & -\min \left\{ \min_{J \in \mathcal{J}} \left(\sum_{j' \in V_D, k \in \partial(J)} \dot{A}_{j'k} - \sum_{j' \in J, k \in V_S} \dot{A}_{j'k} \right), \min_{i \in V_S} \sum_{j' \in V_S} \dot{A}_{j'i}, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\} \\ & + \frac{4}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right). \end{aligned} \quad (89)$$

Using the same argument as at end of proof of Lemma 4, we conclude that the drift is strictly negative for small enough perturbation of demand arrival rates and travel times. \square

H.3 Proof of Theorem 2

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Since we only need an achievability result, it suffices to repeat Steps 1 and 2 in the proof of Lemma 7. Since the technical analysis is almost identical, we make the following claim and omit its proof.

Claim: Consider the system under SMW(α) policy for some $\alpha \in \text{relint}(\Omega)$. Let $\mathbb{P}_p^{K,U}$ be the pessimistic demand-loss probability defined in (1), then we have

$$-\limsup_{K \rightarrow \infty} \frac{1}{K} \log \mathbb{P}_p^{K,U} \geq \frac{\beta}{2} \gamma_{AB}(\alpha). \quad (90)$$

Here for fixed $T > 0$,

$$\gamma_{AB}(\alpha) \triangleq \inf_{v > 0, \mathbf{f}, (\bar{\mathbf{A}}, \bar{\mathbf{X}}, \bar{\mathbf{Y}}, \bar{\mathbf{R}})} \frac{\sum_{j' \in V_D, k \in V_S} \Lambda^*(\hat{\phi}_{j'k}, f_{j'k}) + \sum_{j' \in V_D, k \in V_S} \Lambda^*\left(\frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}}, r_{j'k}\right)}{v},$$

where $(\bar{\mathbf{A}}, \bar{\mathbf{X}}, \bar{\mathbf{Y}}, \bar{\mathbf{R}})$ is a FSP on $[0, T]$ under SMW(α) such that for some regular $t \in (0, T)$

$$\dot{\bar{\mathbf{A}}}(t) = \mathbf{f}, \quad \dot{\bar{\mathbf{R}}}(t) = \mathbf{r}, \quad L_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \in \left(\frac{\beta}{2}, \beta\right), \quad \dot{L}_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) = v.$$

It remains to show that $\gamma_{AB}(\alpha) > 0$. Recall eq. (89):

$$\begin{aligned} \dot{L}_\alpha(\bar{\mathbf{X}}(t), \bar{\mathbf{Y}}(t)) \leq & -\min \left\{ \min_{J \in \mathcal{J}} \left(\sum_{j' \in V_D, k \in \partial(J)} \dot{A}_{j'k} - \sum_{j' \in J, k \in V_S} \dot{A}_{j'k} \right), \min_{i \in V_S} \sum_{j' \in V_S} \dot{A}_{j'i}, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\} \\ & + \frac{4}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\left| \dot{A}_{j'k}(t) - \hat{\phi}_{j'k} \right| + \left| \dot{R}_{j'k}(t) - \frac{\bar{Y}_{j'k}(t)}{\tau_{j'k}} \right| \right). \end{aligned}$$

For $v > 0$, define

$$\begin{aligned} \gamma(v) \triangleq & \min_{\mathbf{f} > 0, \mathbf{r} > 0, \mathbf{y} \in \Omega_n \times m} \sum_{j' \in V_D, k \in V_S} \Lambda^*(\hat{\phi}_{j'k}, f_{j'k}) + \sum_{j' \in V_D, k \in V_S} \Lambda^*\left(\frac{y_{j'k}}{\tau_{j'k}}, r_{j'k}\right) \\ \text{s.t. } & -\min \left\{ \min_{J \in \mathcal{J}} \left(\sum_{j' \in V_D, k \in \partial(J)} f_{j'k} - \sum_{j' \in J, k \in V_S} f_{j'k} \right), \min_{i \in V_S} \sum_{j' \in V_S} f_{j'i}, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\} \\ & + \frac{4}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left(\left| f_{j'k} - \hat{\phi}_{j'k} \right| + \left| r_{j'k} - \frac{y_{j'k}}{\tau_{j'k}} \right| \right) \geq v. \end{aligned}$$

Then it holds that $\gamma_{\text{AB}}(\alpha) \geq \inf_{v>0} \frac{\gamma(v)}{v}$. We define the following three quantities:

$$\begin{aligned}
\gamma_1(v) &\triangleq \min_{\mathbf{f}>\mathbf{0}} \sum_{j' \in V_D, k \in V_S} \Lambda^*(\hat{\phi}_{j'k}, f_{j'k}) \\
&\text{s.t.} \quad - \min \left\{ \min_{J \in \mathcal{J}} \left(\sum_{j' \in V_D, k \in \partial(J)} f_{j'k} - \sum_{j' \in J, k \in V_S} f_{j'k} \right), \min_{i \in V_S} \sum_{j' \in V_S} f_{j'i}, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\} \\
&\quad \geq -\frac{1}{2} \min \left\{ \xi, \lambda_{\min}, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\}. \\
\gamma_2(v) &\triangleq \min_{\mathbf{f}>\mathbf{0}} \sum_{j' \in V_D, k \in V_S} \Lambda^*(\hat{\phi}_{j'k}, f_{j'k}) \\
&\text{s.t.} \quad \frac{4}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} |f_{j'k} - \hat{\phi}_{j'k}| \geq \frac{v}{2} + \frac{1}{4} \min \left\{ \xi, \lambda_{\min}, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\}. \\
\gamma_3(v) &\triangleq \min_{\mathbf{r}>\mathbf{0}, \mathbf{y} \in \Omega_n \times m} \sum_{j' \in V_D, k \in V_S} \Lambda^* \left(\frac{y_{j'k}}{\tau_{j'k}}, r_{j'k} \right) \\
&\text{s.t.} \quad \frac{4}{\alpha_{\min}} \sum_{j' \in V_D, k \in V_S} \left| r_{j'k} - \frac{y_{j'k}}{\tau_{j'k}} \right| \geq \frac{v}{2} + \frac{1}{4} \min \left\{ \xi, \lambda_{\min}, \frac{\alpha_{\min} \beta}{8\tau_{\max}} \right\}.
\end{aligned}$$

Note that if $(\mathbf{f}, \mathbf{r}, \mathbf{y})$ satisfies the constraint in the definition of $\gamma(v)$, then it must satisfy at least one of the constraints in the definition of $\gamma_1(v)$, $\gamma_2(v)$, and $\gamma_3(v)$. Hence

$$\gamma(v) \geq \min\{\gamma_1(v), \gamma_2(v), \gamma_3(v)\}.$$

Therefore

$$\inf_{v>0} \frac{\gamma(v)}{v} \geq \min \left\{ \inf_{v>0} \frac{\gamma_1(v)}{v}, \inf_{v>0} \frac{\gamma_2(v)}{v}, \inf_{v>0} \frac{\gamma_3(v)}{v} \right\}.$$

Now we bound the three quantities on the RHS one by one. Using the same argument as in the no-delay case, we can show that there exists $\delta_1 > 0$ such that $\inf_{v>0} \frac{\gamma_1(v)}{v} > \delta_1$.

For the other two quantities, we first prove the following bound. For $\ell > 0, f > 0$, since $\frac{d^2}{df^2} \Lambda^*(\ell, f) = \frac{1}{f}$, using Taylor expansion we have

$$\Lambda^*(\ell, f) = f \log \frac{f}{\ell} - f + \ell \geq \frac{1}{2f} (f - \ell)^2.$$

If $f \leq 2\phi$ we have

$$\frac{1}{2f} (f - \phi)^2 \geq \frac{1}{4\phi} (f - \phi)^2$$

Otherwise $\frac{f-\phi}{f} \geq \frac{1}{2}$, hence

$$\frac{1}{2f} (f - \phi)^2 \geq \frac{1}{2} (f - \phi).$$

Combined, we have

$$\Lambda^*(\ell, f) \geq \frac{1}{\max\{2, 4\phi\}} \min \left\{ (f - \phi)^2, |f - \phi| \right\}.$$

Looking at the constraint in the definition of $\gamma_2(v)$, it can be deduced that there must exist

$\tilde{j}' \in V_D, \tilde{k} \in V_S$ such that

$$|f_{\tilde{j}'\tilde{k}} - \hat{\phi}_{\tilde{j}'\tilde{k}}| \geq \frac{\alpha_{\min}}{4nm} \left(\frac{v}{2} + \frac{1}{4} \min \left\{ \xi, \lambda_{\min}, \frac{\alpha_{\min}\beta}{8\tau_{\max}} \right\} \right).$$

Denote $g \triangleq \frac{1}{4} \min \left\{ \xi, \lambda_{\min}, \frac{\alpha_{\min}\beta}{8\tau_{\max}} \right\} > 0$. Hence

$$\begin{aligned} \frac{\gamma_2(v)}{v} &\geq \frac{\Lambda_{\tilde{j}'\tilde{k}}^*(\hat{\phi}_{\tilde{j}'\tilde{k}}, f_{\tilde{j}'\tilde{k}})}{v} \\ &\geq \frac{1}{\max\{2, 4\hat{\phi}_{\max}\}} \frac{1}{v} \min \left\{ \frac{\alpha_{\min}^2}{16n^2m^2} \left(\frac{v}{2} + g \right)^2, \frac{\alpha_{\min}}{8nm} v \right\} \\ &\geq \frac{1}{\max\{2, 4\hat{\phi}_{\max}\}} \frac{1}{v} \min \left\{ \frac{\alpha_{\min}^2}{16n^2m^2} (g^2 + gv), \frac{\alpha_{\min}}{8nm} v \right\} \\ &\geq \frac{1}{\max\{2, 4\hat{\phi}_{\max}\}} \min \left\{ \frac{\alpha_{\min}^2 g}{16n^2m^2}, \frac{\alpha_{\min}}{8nm} \right\}. \end{aligned}$$

Note that the last term is independent of v and is strictly positive. Therefore there exists $\delta_2 > 0$ such that $\inf_{v>0} \frac{\gamma_2(v)}{v} > \delta_2$. Similarly, we can show that there exists $\delta_3 > 0$ such that $\inf_{v>0} \frac{\gamma_3(v)}{v} > \delta_3$. This establishes that $\gamma_{AB}(\alpha) > 0$ and concludes the proof. \square

I Classical CRP Condition Implies Assumption 3

In this section, we show that the Assumption 3 in our paper is implied by the CRP condition defined in Dai and Lin (2008). This justifies our naming of Assumption 3 as the CRP condition.

Note that the CRP condition is defined for open networks in the literature. To facilitate the comparison between the CRP condition and Assumption 3, we first define an open network counterpart of our model: Consider an one-hop queueing system with m queues (indexed by $i \in V_S$) and n servers (indexed by $j' \in V_D$). Jobs arrive to the i -th queue at rate

$$\lambda_i \triangleq \sum_{j' \in V_D} \hat{\phi}_{j'i}, \quad (91)$$

and the j' -th server processes jobs at rate

$$\mu_{j'} \triangleq \sum_{k \in V_S} \hat{\phi}_{j'k}. \quad (92)$$

Let $G = (V_S \cup V_D, E)$ be the compatibility graph defined in our paper, and denote the neighborhood of $i \in V_S$ (or $j' \in V_D$) in G by $\partial(i)$ (or $\partial(j')$). To defined the classical CRP condition, we first need to make the following definitions. Define the (primal) static planning problem as the following linear program:

$$\begin{aligned} &\text{minimize}_{\mathbf{x}, \rho} \quad \rho \\ &\text{subject to} \quad \sum_{j' \in \partial(i)} \mu_{j'} x_{ij'} = \lambda_i & \forall i \in V_S, \\ &\quad \sum_{i \in \partial(j')} x_{ij'} \leq \rho & \forall j' \in V_D, \\ &\quad x_{ij'} \geq 0 & \forall i \in V_S, j' \in V_D. \end{aligned}$$

The dual problem of the static planning problem is the following:

$$\begin{aligned}
& \text{maximize}_{\mathbf{y}, \mathbf{z}} && \sum_{i \in V_S} \lambda_i y_i \\
& \text{subject to} && z_{j'} \geq y_i \mu_{j'} && \forall (i, j') \in E, \\
& && \sum_{j' \in V_D} z_{j'} = 1 \\
& && z_{j'} \geq 0 && \forall j' \in V_D.
\end{aligned}$$

Assumption 6 (Heavy-traffic CRP condition (Assumptions 1,2 in Dai and Lin (2008))). A triple $(\boldsymbol{\lambda}, \boldsymbol{\mu}, G)$ is said to be in heavy traffic if the primal static planning problem has a unique solution (\mathbf{x}^*, ρ^*) , where $\sum_{i \in \partial(j')} x_{ij'}^* = 1$ for all $j' \in V_D$ and $\rho^* = 1$. The triple is said to satisfy the CRP condition if the dual static planning problem has a nonnegative, unique optimal solution $(\mathbf{y}^*, \mathbf{z}^*)$.

Proposition 7. For primitives $(\hat{\phi}, G)$, define $\boldsymbol{\lambda}, \boldsymbol{\mu}$ according to (93) and (94). If $(\boldsymbol{\lambda}, \boldsymbol{\mu}, G)$ satisfy Assumption 6, then $(\hat{\phi}, G)$ satisfy Assumption 3 in our paper.

Proof of Proposition 7. Consider $(\hat{\phi}, G)$ such that $(\boldsymbol{\lambda}, \boldsymbol{\mu}, G)$ satisfy Assumption 6. Let $(\mathbf{y}^*, \mathbf{z}^*)$ be the unique optimal solution to the dual static planning problem. Applying Corollary A.1 in Dai and Lin (2008), we have that \mathbf{y}^* is the unique vector which satisfies

$$\max_{\mathbf{v} \in V} \mathbf{y}^* \cdot \mathbf{v} = 0, \quad (93)$$

$$\sum_{i \in V_S} \lambda_i y_i^* = 1. \quad (94)$$

Here V is defined as

$$V \triangleq \left\{ \mathbf{v} \in \mathbb{R}^m \left| \begin{array}{l} v_i = \sum_{j' \in \partial(i)} d_{ij'} \mu_{j'} - \lambda_i, \quad \forall i \in V_S \\ \sum_{i \in \partial(j')} d_{ij'} \leq 1, \quad d_{ij'} \geq 0, \quad \forall i \in V_S, j' \in V_D \end{array} \right. \right\},$$

which is the set of all possible flow rates out of the queues.

Let $\tilde{\mathbf{y}} = \frac{1}{\sum_{i \in V_S} \lambda_i} \mathbf{1}$, we will show that $\tilde{\mathbf{y}}$ satisfies (93) and (94), and hence $\mathbf{y}^* = \tilde{\mathbf{y}}$. Eq. (94) is easy to verify. For (93), because $(\boldsymbol{\lambda}, \boldsymbol{\mu}, G)$ satisfy Assumption 6, we have

$$\tilde{\mathbf{y}} \cdot \mathbf{v} = \frac{1}{\sum_{i \in V_S} \lambda_i} \left(\sum_{i \in V_S} \sum_{j' \in \partial(i)} d_{ij'} \mu_{j'} - \sum_{i \in V_S} \lambda_i \right) = \frac{1}{\sum_{i \in V_S} \lambda_i} \left(\sum_{j' \in V_D} \mu_{j'} \sum_{i \in \partial(j')} d_{ij'} - \sum_{i \in V_S} \lambda_i \right).$$

According to the definition of V , we have $\sum_{i \in \partial(i)} d_{ij'} \leq 1$, hence

$$\tilde{\mathbf{y}} \cdot \mathbf{v} \leq \frac{1}{\sum_{i \in V_S} \lambda_i} \left(\sum_{j' \in V_D} \mu_{j'} - \sum_{i \in V_S} \lambda_i \right),$$

where the equality can be achieved. Applying Assumption 6, we have

$$\sum_{i \in V_S} \lambda_i = \sum_{i \in V_S} \sum_{j' \in \partial(i)} \mu_{j'} x_{ij'}^* = \sum_{j' \in V_D} \mu_{j'} \sum_{i \in \partial(j')} x_{ij'}^* = \sum_{j' \in V_D} \mu_{j'}.$$

Hence $\tilde{\mathbf{y}}$ satisfies (93).

We now prove that $\sum_{i \in I} \lambda_i < \sum_{j' \in \partial(I)} \mu_{j'}$ for all $I \subsetneq V_S, I \neq \emptyset$. For any $I \subsetneq V_S, I \neq \emptyset$,

consider the vector $\mathbf{v} \in \mathbb{R}^m$ where

$$\begin{aligned} v_i &= 1 && \text{for } i \in I, \\ v_i &= -\frac{|I|}{m - |I|} && \text{for } i \in V_S \setminus I. \end{aligned}$$

Because $(\boldsymbol{\lambda}, \boldsymbol{\mu}, G)$ satisfy Assumption 6, by applying Lemma 5 in Dai and Lin (2008) we have: for $V^o \triangleq \{\mathbf{v} \in \mathbb{R}^m : \mathbf{1}^\top \mathbf{v} = 0\}$, there exists $\delta > 0$ such that $\{\mathbf{v} \in V^o : \|\mathbf{v}\|_2 \leq \delta\} \subset V$. It can be easily verified that $\mathbf{v} \in V^o$. As a result, there exists $\delta > 0$ such that $\delta \mathbf{v} \in V$. We have:

$$0 < \delta |I| = \sum_{i \in I} \delta v_i \leq \sum_{j' \in \partial(I)} \mu_{j'} - \sum_{i \in I} \lambda_i,$$

We have so far proved that $\sum_{i \in I} \lambda_i < \sum_{j' \in \partial(I)} \mu_{j'}$ for all $I \subsetneq V_S, I \neq \emptyset$, we now show that this implies Assumption 3. First, we establish

$$\sum_{i \in \partial(J)} \lambda_i > \sum_{j' \in J} \mu_{j'} \quad \forall J \subsetneq V_D, J \neq \emptyset, \quad (95)$$

because, for $I \triangleq V_S \setminus \partial(J)$, we know

$$\begin{aligned} & \sum_{i \in I} \lambda_i < \sum_{j' \in \partial(I)} \mu_{j'} \\ \Rightarrow \sum_{i \in V_S} \lambda_i - \sum_{i \in \partial(J)} \lambda_i &< \sum_{j' \in \partial(I)} \mu_{j'} \leq \sum_{j' \in V_D} \mu_{j'} - \sum_{j' \in J} \mu_{j'} \\ & \Rightarrow \sum_{i \in \partial(J)} \lambda_i > \sum_{j' \in J} \mu_{j'}. \end{aligned}$$

where we used $\partial(I) \cap J = \emptyset$ by definition of I in the second line, and we used $\sum_{i \in V_S} \lambda_i = \sum_{j' \in V_D} \mu_{j'}$ to get the third line. Our Assumption 3 follows by restricting attention to limited-flexibility subsets J and cancelling the terms which are common on the two sides of the inequality. This concludes the proof. \square

J Simulation experiments (full description)

In this appendix, we provide a full description of our simulations in an environment that resembles ride-hailing in Manhattan, New York City. We use demand estimates from Buchholz (2015) (the estimates are based on NYC yellow cab data) and Google Maps to estimate travel times, and simulate SMW-based dispatch policies.

J.1 The Data, Simulation Environment and Benchmark

Throughout this section, we use the following set of model primitives.

- *Graph topology.* We consider a 30-location model of Manhattan below 110-th street excluding Central Park (see Figure 6), based on Buchholz (2015). We let pairs of regions which share a non-trivial boundary be compatible with each other.
- *Demand arrival process, Pickup/service times, and number of cars.* Throughout this section, we consider a stationary demand arrival rate²⁸ that satisfies the CRP condition, which

²⁸We leave the cases where demand is time-varying for future research. Our numerical study in Section J.4 regarding transient performance may be seen as a first step towards the time-varying case.

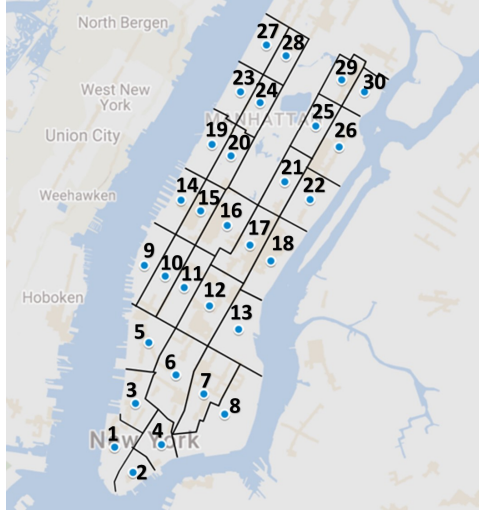


Figure 6: A 30 location model of Manhattan below 110-th street, excluding the Central Park. (Source: tessellation is based on Buchholz (2015), the figure is generated using Google Maps.)

is obtained by “symmetrizing”²⁹ the decensored demand estimated in Buchholz (2015) (see subsection J.5 for a full description). We estimate travel times between location pairs using Google Maps, and use as a baseline the fluid requirement K_{fl} on number of cars needed to meet demand. We use K_{tot} (not K) to denote the total number of cars, and $K_{\text{slack}} = K_{\text{tot}} - K_{\text{fl}}$ to denote the excess over the fluid requirement. Here K_{slack} is similar to the K in our theory since it is the average number of free cars assuming all demand is met.

Simulation Design. We consider the following simulation settings:

1. *Stationary performance with Service time.* We investigate steady state performance; steady state is reached in ~ 1 -2 hours under SMW policies.
2. *Stationary performance with Service+Pickup time.* Same as above.
3. *Transient performance with Service + Pickup time.* We investigate performance over a short horizon (below 2 hours) for different initial configurations.

Benchmark policy: fluid-based policy. The benchmark policy we consider is a static randomization based on the solution to the fluid problem (Banerjee et al. 2016, Ozkan and Ward 2016). See subsection J.5 for details.

Learning the optimal parameters. We use MATLAB’s built-in `particleswarm` solver to learn the optimal SMW scaling parameters via simulation-based optimization in each setting.

J.2 Steady state with Service times

A preliminary simulation of the setting in our paper (i.e., pickup and service are both instantaneous) showed that under vanilla MaxWeight policy we only need $K_{\text{slack}} = 120$ to obtain a demand-loss rate below 1%, under SMW(α) with α defined in Theorem 1 the number further reduces to 80. However, the demand-loss rate stays above 5% under the fluid-based policy even when $K_{\text{slack}} = 200$.³⁰ We then proceeded to simulate the Service time setting, and obtained

²⁹Instead of symmetrizing, an alternative would be to consider an “empty” relocation rule (see Section 8) such that CRP holds. We obtained similar results under this alternative (we omit those results in the interest of space).

³⁰The results remain similar when service time is included, hence we only include the graph of the latter case.

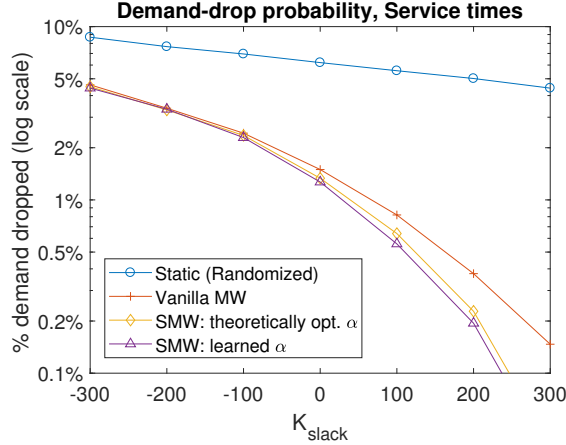


Figure 7: *Service times* setting: Stationary demand-loss probability under the static fluid-based policy, vanilla MaxWeight policy, SMW policy with theoretically optimal α , and SMW policy with learned α . Note that the y-axis is in log-scale. Here $K_{\text{fl}} = 7,061$. The plots indicate significant separation between fluid and SMW policies at all values of K , and separation between vanilla MaxWeight and optimized SMW. For each data point we run 200 trials and take the average.

similarly encouraging results. In this setting, the average trip time is 13.2 minutes, and the fluid requirement is $K_{\text{fl}} = 7,061$ cars.

Results. The simulation results on performance³¹ are shown in Figure 7, and the theoretical and learned α are shown in Figure 8. Figure 7 confirms that SMW policies including vanilla MaxWeight outperform the fluid-based policy; in fact only $K_{\text{slack}} = 100$ extra cars ($< 1.5\%$ of K_{tot} , or < 4 free cars per location on average if all demand is met) in the system lead to a negligible fraction of demand lost. The demand loss probability decays rapidly with K_{slack} under SMW policies, while it decays much slower under the fluid-based policy. SMW with parameters chosen based on Theorem 1 performs nearly as well as the learned SMW policy, despite small $K_{\text{slack}} = 100$. Figure 8 shows that the learned α is very similar to the theoretically optimal α structurally. Both policies allocate larger parameters (i.e., give more protection to the supply) in the Upper West Side area which has a small Hall’s gap (i.e., small slack in the CRP condition).

J.3 Steady state with Service and Pickup times

In the following experiment we further incorporate pickup times. The average pickup time is 5.5 minutes, and the fluid requirement increases to $K_{\text{fl}} = 10,002$ cars. Our objective here is to show that SMW policies can be heuristically adapted to more general settings, and retain their good performance. We propose the following SMW-based heuristic policy. Intuitively, pickup times need to be taken into consideration when making dispatch decisions, because every minute spent on picking up a customer leads to an opportunity cost. We consider policies of the following form. When demand arrives at location j , dispatch from

$$\operatorname{argmax}_{i \in \partial(j)} \frac{x_i}{\alpha_i} - zD_{ij},$$

³¹We also tested stochastic service times and found no significant difference in performance.

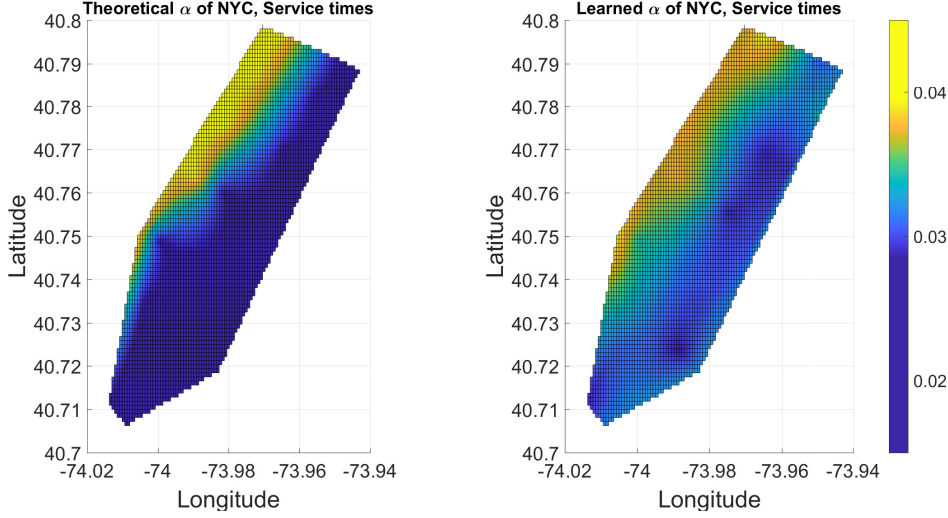


Figure 8: *Service times* setting: Theoretically optimal α derived from Theorem 1 (left) and the α learned via simulation-based optimization (right), both for the NYC dataset with $K_{\text{slack}} = 200$. Darker shades indicate smaller values of α_i , while lighter shades correspond to larger values.

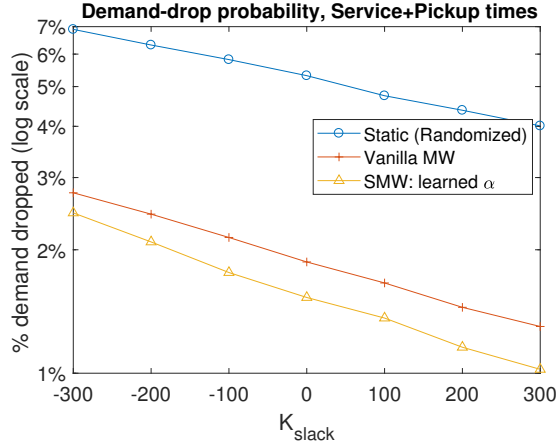


Figure 9: *Service+Pickup times* setting: Stationary demand-loss probability under the fluid-based policy, the vanilla MaxWeight policy, and the SMW policy with α learned via simulation optimization. Here $K_{\text{fl}} = 10,002$ cars. For each data point we average over 200 trials.

where x_i is the number of free cars at i , and D_{ij} is the pickup time between i and j . In addition to scaling parameters α , we have an additional parameter z which captures the importance given to pickup delay in making dispatch decisions.

Results. Simulation results are shown in Figure 9. We observe that the SMW-based policies including vanilla MaxWeight significantly outperform the fluid-based policy. A few hundred extra cars ($< 3\%$ of K_{tot}) in the system suffice to ensure that only $\sim 1\%$ of demand is lost.

J.4 Transient Behavior with Service and Pickup times

In the last experiment, we consider transient behavior instead of steady state performance. We fix K_{slack} to be 200. For initial configurations, we sample 4 initial queue-length vectors uniformly from the simplex $\{\mathbf{x} : x_1 + \dots + x_{30} = 200\}$, and the cars initially in transit are based on picking

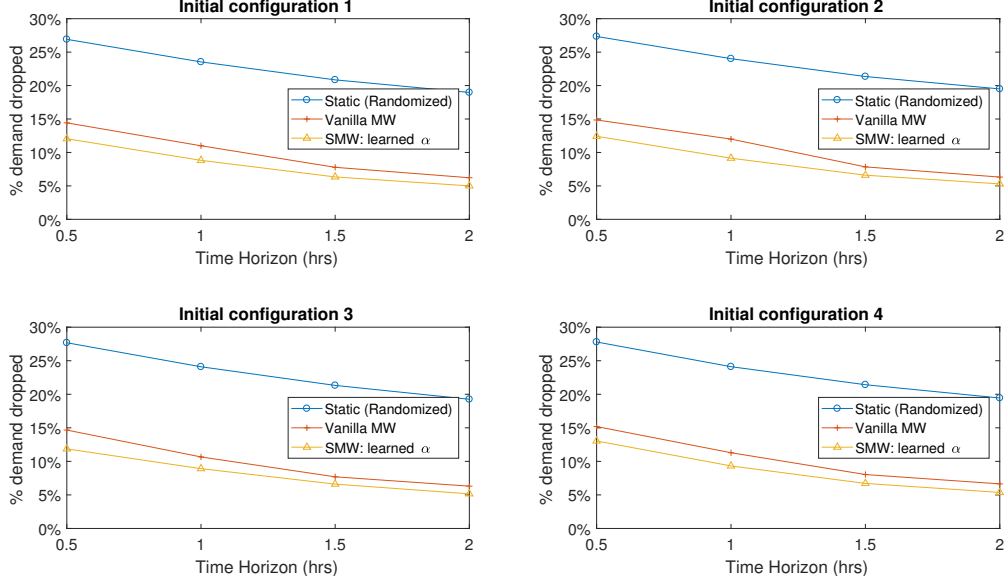


Figure 10: *Transient Performance with Service+Pickup times*: The plots show the demand-loss probability under the fluid-based policy, the vanilla MaxWeight policy, and the SMW policy with learned α , with 4 different initial configurations, chosen randomly on the simplex. We fix $K_{\text{slack}} = 200$, and consider time horizons ranging from 0.5 to 2 hours. For each data point we run 200 trials and take the average.

up all demand that arose in the last hour. For each initial state we consider 4 time horizons: 0.5, 1, 1.5 and 2 hours. We learn the optimal SMW parameters for each initial state and time horizon pair to minimize the fraction of demand lost and then compare the performance of SMW policies, vanilla MaxWeight and the fluid-based policy. The results are shown in Figure 10. It turns out that SMW policies outperform the fluid-based policy by an even larger margin in this case since they are able to quickly (in under an hour) spread the supply out across locations.

J.5 Simulation Settings

In this subsection, we fill in the missing details in the previous subsections.

Model Primitives.

- *Demand arrival process* (ϕ). Using the estimation in Buchholz (2015), which is based on Manhattan’s taxi trip data during August and September in 2012, we obtain the (average) demand arrival rates for each origin-destination pair during the day (7 a.m. to 4 p.m.) denoted by $\tilde{\phi}_{ij}$ ($i, j = 1, \dots, 30$). However, we find that $\tilde{\phi}_{ij}$ violates CRP (there are a lot more rides *to* Midtown than *from* Midtown). We consider the following “symmetrization” of $\tilde{\phi} \triangleq (\tilde{\phi}_{ij})_{30 \times 30}$ to ensure that CRP holds (ride-hailing platforms may use spatially varying prices and repositioning to obtain CRP, see Section 1):

$$\phi(\eta) \triangleq \eta \tilde{\phi} + (1 - \eta) \frac{1}{2} (\tilde{\phi} + \tilde{\phi}^T), \quad \eta \in (0, 1). \quad (96)$$

Figure 11 shows how the Hall’s gap of $\phi(\eta)$ varies with η . We pick $\eta = 0.21$ such that CRP is “almost violated”³². The subset of locations with smallest Hall’s gap is then the Upper West

³²We also ran simulations for $\eta = 0.15$ such that Hall’s gap is large. There is no significant difference in the policies’ relative performances, so we didn’t include it here.

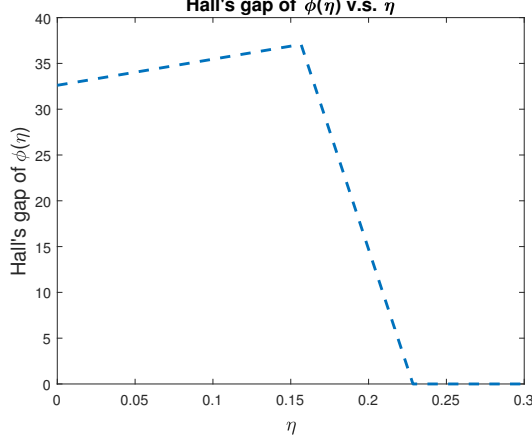


Figure 11: Hall’s gap of symmetrized matrix $\phi(\eta)$ (see Eq. (96)) versus parameter η , based on the demand arrival rates $\tilde{\phi}$ computed from the Manhattan taxi data. Our simulations use $\eta = 0.21$, which corresponds to a small but non-zero Hall’s gap (< 10).

Side (locations 19, 23, 24, 27, 28 in Figure 6).

- *Pickup/service times (D/\tilde{D})*. We extract the pairwise travel time between region centroids (marked by the dots in Figure 6) using Google Maps, denoted by D_{ij} ’s ($i, j = 1, \dots, 30$). We use D_{ij} as service time for customers traveling from i to j . For each customer at i who is picked up by a supply from k we add a pickup time³³ of $\tilde{D}_{ki} = \max\{\frac{3}{2}D_{ki}, 3 \text{ minutes}\}$.

Benchmark policy: fluid-based policy. We consider the fluid-based randomized policy (Banerjee et al. 2016, Ozkan and Ward 2016) as a benchmark. Let \mathcal{X} be the solution set of the feasibility problem

$$\sum_{j \in \partial(i)} x_{ij} = \lambda_i \quad \forall i, \quad \sum_{i \in \partial(j)} x_{ij} = \mu_j \quad \forall j.$$

Since CRP holds, $\mathcal{X} \neq \emptyset$. Let $\mathbf{x}^* \triangleq \operatorname{argmin}_{\mathbf{x} \in \mathcal{X}} \sum_{(i,j) \in E} \tilde{D}_{ij} x_{ij}$. When demand arrives at location j , the randomized fluid-based policy dispatches from location $i \in \partial(j)$ with probability x_{ij}^*/μ_j . Then x_{ij}^* is the rate of dispatching cars from i to serve demand at j . From Banerjee et al. (2016), we know that \mathbf{x}^* leads to a zero demand-loss as $K \rightarrow \infty$ with and without pickup times (assuming demand remains constant). Moreover, with pickup times, Little’s Law gives that the fluid-based policy minimizes the expected number of cars on-route to pick up customers.

Benchmark fleet-size. In the *Service time* setting, a fraction of cars are in transit under the stationary distribution; in the *Service+Pickup time* setting, there is an additional fraction of cars on-route to pick up customers. A simple workload conservation argument (using Little’s Law) gives the benchmark fleet-sizes as follows.

- *Service time*. Assuming no demand is lost, the mean number of cars in transit is: $K_{\text{fl}} = \sum_{i,j} \phi_{ij} D_{ij}$. In our setting, we have $K_{\text{in-transit}} \approx 7,061$. Since CRP holds and demand-loss probability goes to 0 under both fluid-based policy and SMW policies, $K_{\text{in-transit}}$ is a reasonable benchmark fleet-size K_{fl} . We will vary the number of cars in the system denoted by $K_{\text{tot}} = K_{\text{fl}} + K_{\text{slack}}$ and compare the performance of different policies. Here K_{slack} is the number of free cars in the system when no demand is lost.

³³We use the inflated D_{ij} ’s as pickup times to account for delays in finding or waiting for the customer.

- *Service+Pickup time.* Applying Little's Law, if no demand is lost, the mean number of cars picking up customers is at least $K_{\text{pickup}} = \min_{\mathbf{x} \in \mathcal{X}} \tilde{D}_{ij} x_{ij}$. In our case, we have $K_{\text{pickup}} \approx 2,941$. Hence, the benchmark fleet size is $K_{\text{fl}} = K_{\text{in-transit}} + K_{\text{pickup}} = 10,002$. Note that this number is close to the real-world fleet size: there were approximately 11,500 active medallions when Buchholz (2015) was written.