1st- and 2nd-order Time Integrators

A general form of a family of time integration schemes:
$v^{n+1} = v^n + h((1-\gamma)a^n + \gamma a^{n+1})$
$x^{n+1} = x^n + hv^n + \alpha h^2((1-2\beta)a^n + 2\beta a^{n+1})$
where $a^{n+1} = M^{-1}f(x^{n+1})$

Newmark:
- explicit central diff: $\alpha = 0.5, \beta = 0, \gamma = 0.5$
- middle point rule: $\alpha = 0.5, \beta = 0.25, \gamma = 0.5$

Euler:
- forward: $\alpha = 0, \beta = ?, \gamma = 0$
- symplectic: $\alpha = 1, \beta = 0, \gamma = 0$
- backward: $\alpha = 1, \beta = 0.5, \gamma = 1$

Static:
no inertia term, $2\alpha\beta h^2 \neq 0$

since
$x^{n+1} = x^n + hv^n + \alpha h^2 M^{-1}((1-2\beta)f(x^n) + 2\beta f(x^{n+1}))$
$Mx^{n+1} = M(x^n + hv^n + \alpha h^2 M^{-1}(1-2\beta)f(x^n)) + 2\beta\alpha h^2 f(x^{n+1})$
we have the incremental potential
$E(x) = \frac{1}{2}||x - \tilde{x}^n||_M + 2\alpha\beta h^2 P(x)$
where
$\tilde{x}^n = x^n + hv^n + \alpha(1-2\beta)h^2 a^n$

after solving for x^{n+1}, we can update
$a^{n+1} = (x^{n+1} - \tilde{x}^n)/(2\alpha\beta h^2)$ for implicit schemes, or generally $a^{n+1} = M^{-1}f(x^{n+1})$
$v^{n+1} = v^n + h((1-\gamma)a^n + \gamma a^{n+1})$
$$
\begin{aligned}
v^{n+1} &= v^n + h((1-\gamma)a^n + \gamma(x^{n+1} - \tilde{x}^n)/(2\alpha\beta h^2)) \\
&= \tfrac{\gamma}{2\alpha\beta h}x^{n+1} + v^n + h(1-\gamma)a^n - \tfrac{\gamma}{2\alpha\beta h}\tilde{x} \\
&= \tfrac{\gamma}{2\alpha\beta h}x^{n+1} - \hat{v}^n
\end{aligned}
$$

MPM

Extra or different steps than BE:

- ~~p2g~~ ma^n ~~and get grid~~ a^n compute a_i^n as $a_i^n = f_i^n / m_i$
- grid $\widetilde{x}^n$ and the coefficient in front of each potential energy is different in the incremental potential
- for elasticity, as currently we use grid x^{n+1} as variable, it should be consistent as $F^{n+1} = (\sum_i x_i^{n+1} (\bigtriangledown \omega_{ip}^n)^T) F^n$ still holds
- $J^{n+1} = (1 + \bigtriangledown \cdot (x^{n+1} - x^n)) J^n = (1 + (Tr \bigtriangledown x^{n+1} - dim)) J^n$
- compute updated a^{n+1} and v^{n+1} as derived above (same with FEM)
- ~~g2p~~ a^{n+1} (not needed then)

Make sure that after modifying the implementation to the above general form, BE still works.